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INTRODUCTION

The aim of this paper is to deep the concept of potential on a standard H-cone introduced in [1]. We suppose that the standard H-cone S and its dual S^* are reprezented as H-cone of functions on the same Green set Ω and we denote by $(x,y)\rightarrow G(x,y)$ the Green function on $\Omega x \Omega$ associated with S and S^* ([3]). An element $p\in S$ is called Green-potential if there exists a positive measure μ on Ω such that

$$p(x)=G_{\mu}(x):=\int G(x,y)d\mu(y)$$

We give various charaterizations of Green potential, particularly an element $p\in S$ is a Green potential iff for any $q\in S, q\lesssim p, q\neq 0$ the harmonic carrier of q with respect to the co-natural topology on Ω is non empty. We show that if $p\in S$ is such that and $\bigwedge_n B^{X\setminus D_n}p=0$ for any increasing convering $(D_n)_n$ of Ω with co-natural open subset D_n, then p is a Green potential. The same property fails if D_n is natural open subset of Ω. Also we charaterize the elements $s\in S$ which are orthogonal on the set of all Green potentials.

The above results extend the similar ones given by B. Fuglede ([4],[5]) in the framework of fine potential theory on a harmonic space for which the natural and co-natural topology (as well fine and co-fine topology) coincide and the axiom D is fulfilled.

1. Preliminaries and first results

a) Standard H-cones and their representations

Giving a standard H-cone S and noting by S^* its dual, then for any weak unit $u\in S$ the set

$K_u := \{ t \in S \mid t(u) \leq 1 \}$

endowed with the natural topology is a compact convex set such that for any $s_1, s_2 \in S$ we have

$$s_1 \leq s_2 \Leftrightarrow t(s_1) \leq t(s_2) \quad , \quad (\forall)\ t \in K_u .$$

If we denote by X_u the set of all nonzero extreme points of K_u then X_u is a G_δ-set (with respect to the natural topology). For any $s \in S$ the function

$$K_u \ni t \to \tilde{s}(t) := t(s) \in \overline{R}_+$$

is a lower semicontinuous affine function on K_u. Obviously for any $s_1, s_2 \in S$ we have

$$s_1 \leq s_2 \Leftrightarrow \tilde{s}_1|_{X_u} \leq \tilde{s}_2|_{X_u} .$$

The set

$$\mathcal{S} := \{ \tilde{s}|_{X_u} ; s \in S \}$$

is a standard H-cone of functions on X_u which is isomorphic with S if we identify any element $s \in S$ with the function $\tilde{s}|_{X_u}$. Often the function $\tilde{s}|_{X_u}$ will be noted simply by s.

The paire $(\mathcal{S}, X_u)$ is called the <u>saturated representation</u> of S <u>associated</u> with the weak <u>unit u</u> [3]. This representation has the property that for any element $t \in S^*$ such that $t(u) < \infty$ there exists a pozitive finite measure μ_t on X_u for which we have

$$t(s) = \int \tilde{s}(x) d\mu_t(x) = \int s(x) d\mu_t(x)$$

for any $s \in S$.

In an analogous manner, choosing a weak unit u^* in S^*, the set

$$K^*_{u^*} = \{ s \in S \mid u^*(S) \leq 1 \}$$

is convex and compact with resepct to the natural topology given by S (this topology is usually termed "co-natural").

Noting by $X^*_{u^*}$ the set of all nonzero extreme points of $K^*_{u^*}$ and identifiyng any element $t \in S^*$ with the function

$$X^*_{u^*} \ni \xi \to t(\xi) \in \overline{R}_+$$

we obtain a standard H-cone of functions $\mathcal{S}^*$ on $X^*_{u^*}$ which is isomorphic with S^* and the pair $(\mathcal{S}^*, X^*_{u^*})$ is the saturated representation of S^* associated with the weak unit u^*. As usually we denote by $\langle , \rangle$, the duality between $\mathcal{S}$ and $\mathcal{S}^*$ i.e the map

$\mathcal{S}^* \times \mathcal{S} \ni (t,s) \to \langle t,s \rangle := t(s)$

and we denote by G the map from $X_u \times X^*_{u^*}$ into $\overline{R}_+$ defined by

$$(x,\xi) \to G(x,\xi) := \langle x,\xi \rangle .$$

It is known ([3], Proposition 5.5.1) that this map is lower semicontinuous with respect to the product of the natural and co-natural topologies on X_u and $X^*_{u^*}$ respectively and moreover for any positive measure μ (resp ν) on X_u (resp $X^*_{u^*}$ such that the function ${}^*G_\mu$ (resp G_ν) defined by

$$X^*_{u^*} \ni \xi \to {}^*G_\mu(\xi) := \int G(x,\xi)\, d\mu(x)$$

$$(\text{resp } X_u \ni x \to G_\nu(x) := \int G(x,\xi)\, d\nu(\xi))$$

is finite on a dense subset of $X^*_{u^*}$ (resp X_u) we have ${}^*G_\mu \in \mathcal{S}^*$ (resp. $G_\nu \in \mathcal{S}$) and

$$\langle {}^*G_\mu , G_\nu \rangle = \int^* G_\mu(\xi)\, d\nu(\xi) = \int G_\nu(x)\, d\mu(x) .$$

Also if A is a subset of X_u (resp $X^*_{u^*}$) and μ, ν are as above we have

$$\langle t, B^A G_\nu \rangle = \int \langle t, B^A \xi \rangle\, d\nu(\xi), \quad (\forall) t \in S^*$$

$$(\text{resp.} \langle {}^*B^{A*} G_\mu , s \rangle = \int \langle {}^*B^A x, s \rangle\, d\mu(x), \quad (\forall) s \in S)$$

and therefore,

$$\int B^A G_\nu(x)\, d\mu(x) = \int B^A \xi(x)\, d\nu \otimes \mu(\xi, x)$$

$$(\text{resp.} \int^* B^{A*} G_\mu(\xi)\, d\nu(\xi) = \int^* B^A x(\xi)\, d\mu \otimes \nu(x,\xi)) .$$

Let now v^* be another weak unit of S^* such that v^* is u^*-continuous. We may consider now S^* as a standard H-cone of functions on the set $X^*_{v^*}$. In fact there exists an injective map denoted by φ from $X^*_{u^*}$ into $X^*_{v^*}$ defined by

$$\varphi(\xi) = \frac{\xi}{v(\xi)} .$$

We remark that the previous map is continuous if we endow $X^*_{u^*}$ and $X^*_{v^*}$ with the co-natural topologies and therefore φ is a

$$X^*_{v^*} \setminus \varphi(X^*_{u^*}) = \{ \xi \in X^*_{v^*} \mid \text{Ш}(\xi) = \infty \}$$

i.e. $X^*_{v^*} \setminus \varphi(X^*_{u^*})$ is a polar G_δ-subset of $X^*_{v^*}$. If we denote by G' the map from $X_u \times X^*_{v^*}$ into $\overline{R}_+$ defined by

$$G'(x,\eta) = \langle x, \eta \rangle$$

we remark that for any $\xi \in X^*_{u^*}$ we have

$$G'(x, \varphi(\xi)) = \langle x, \varphi(\xi) \rangle = \frac{G(x,\xi)}{v^*(\xi)} .$$

If μ is a positive measure on $X^*_{u^*}$ such that $G_\mu \in \mathcal{S}$ then there exists a measure μ' on $\varphi(X^*_{u^*})$ such that $G_\mu = G'_{\mu'}$. Indeed it is sufficient to take $\mu' = (v^* \cdot \mu) \circ \varphi^{-1}$. Conversely for any positive measure ν on $X^*_{v^*}$ such that $\nu(X^*_{v^*} \setminus \varphi(X^*_{u^*})) = 0$ and such that $G'_\nu \in \mathcal{S}$ there exists a measure μ on $X^*_{u^*}$ such that $G_\mu = G'_\nu$. Indeed it is sufficient to take μ of the form

$$\mu(A) = \int_A \frac{1}{v^*(\xi)} d(\nu \circ \varphi)(\xi) .$$

Hence for a pozitive measure μ on $X^*_{v^*}$ such that $G'_\mu \in \mathcal{S}$ we have $G'_\mu \wedge G_\lambda = 0$ for any H-measure λ on $X^*_{u^*}$ iff $\mu(\varphi(X^*_{u^*})) = 0$.

Further, if u^* is a weak unit of S^* and $(\mathcal{S}, X^*_{u^*})$ is the saturated representation of S^* associated with the weak unit u^* and if we consider a subset $\Upsilon \subset X^*_{u^*}$ which is nearly saturated with respect to S^* then we denote by $S_\Gamma(\Upsilon)$ the set of all elements $s \in S$ for which there exists a positive measure μ_s on Υ which represents s i.e.

$$\langle t,s \rangle = \int \langle t,y \rangle \, d\mu(y) = \langle t, G_\mu \rangle$$

We remark that $S_\Gamma(\Upsilon)$ is a band in the convex cone S with respect to the specific order and moreover Υ being nearly saturated we have $p \in S_\Gamma(\Upsilon)$ for any universally continuous element p of S.

We remember also (see [1]) that an element $h \in S$ is <u>substractible</u> if for any $s \in S$ we have

$$h \leq s \Rightarrow h \precsim s$$

The set of all substractible elements of an H-come is band in S with respect to the specific order. An element $p \in S$ is called a pure potential of S if for any substractible element h of S we have

$$h \leq p \Rightarrow h=0 .$$

Theorem 1.1. Let $\Upsilon \subset X^*_{u^*}$ a nearly saturated subset of $X^*_{u^*}$ with respect to S^* . Then the following assertions are equivalent:

1) $S_r(\Upsilon)$ is solid in S with respect to the natural order "$\leq$" of S.

2) Any universally bounded element of S belongs to $S_r(\Upsilon)$.

3) Υ is semi-saturated with respect to S i.e. any subset K of $X^*_{u^*} \setminus \Upsilon$, K compact with respect to the co-natural topology, is a polar subset of $X^*_{u^*}$.

4) Any pure potential of S belong to $S_r(\Upsilon)$.

Proof. 1)$\Rightarrow$2) follows from the fact that any universally bounded element of S is naturally dominated by an universally continuous element of S.

2)$\Leftrightarrow$3) follows from [2], Proposition 1.2.

3)$\Rightarrow$4) Let p be a pure potential of S . Using the fact that $S_r(\Upsilon)$ is a band in S we consider $p', p'' \in S$ such that $p=p'+p''$, $p' \in S_r(\Upsilon)$ and p'' is such that $p'' \wedge q=0$ for any $q \in S_r(\Upsilon)$. We want to show that $p''=0$. Let us consider v^* a weak unit in S^*, $v^* \leq u^*$ such that $v^*(p'')<\infty$. On the saturated set $X^*_{v^*}$ of representation of S^* we consider a positive measure μ such that

$$\langle t,p'' \rangle = \int t(\xi) d\mu(\xi), \ (\forall) t \in S^* .$$

Considering now the map $\varphi : X^*_{u^*} \to X^*_{v^*}$ defined by

$$\varphi(\xi) \equiv \frac{1}{v^*(\xi)}$$

we deduce that for any $q \in S_r(\varphi(\Upsilon))$ we have $q'' \wedge q=0$ and therefore the measure μ does not charge the set $\varphi(\Upsilon)$ i.e. there exists a sequence $(K_n)_n$ of subsets of $X^*_{v^*} \setminus \varphi(\Upsilon)$ which are compact with respect to the co-natural topology on $X^*_{v^*}$ and such that

$$\mu = \sum_n \mu_n, \ \mu_n := \mu|_{K_n}$$

Since $\mathcal{Y}$ is semi-saturated in $X^*_{u^*}$ and since the set $X^*_{v^*} \setminus \varphi(X^*_{u^*})$ is a polar subset of $X^*_{v^*}$ we deduce that for any $n \in N$, K_n is polar (with respect to S^*).

If for any $n \in N$ we denote by p''_n the element of S defined as an H-integral on S^* in the following way

$$S^* \ni t \to q''_n(t) = \int t(\xi)\, d\mu_n(\xi)$$

we remark that μ_n being carrid by a polar set we have $q''_n \wedge q = 0$ for any $q \in S_o$.

Since $q''_n \prec p$, q''_n is a pure potential, and, using [1], Theorem 4.5, we deduce that $q''_n = 0$ for any $n \in N$, $q'' = 0$.

4) $\Rightarrow$ 1) Let p be an element of $S_r(\mathcal{Y})$ and let $q \in S$ be such that $q \leq p$. Since the set of all pure potentials of S is a band in S we may consider $q', q'' \in S$ such that $q = q' + q''$, q' is a pure potential and q'' is substractible. By hypothesis we have $q' \in S_r(\mathcal{Y})$. On the other hand the element q'' being substractible and $q'' \leq q \leq p$ we have $q'' \precsim p$ and therefore $q'' \in S_r(\mathcal{Y})$.

Hence $q \in S_r(\mathcal{Y})$.

b) <u>Harmonic carrier.</u> If $(\mathcal{G}, X_v)$ (resp $(\mathcal{G}^*, X^*_{u^*})$) is a saturated representation of S (resp S^*) with respect to the weak unit u (resp. u^*) then for any $s \in \mathcal{G}$ (resp $t \in \mathcal{G}^*$) the harmonic carrier of s (resp t) is the set

$$\text{carr } s = \{ x \in \overline{X_u} \mid B^{X_u \setminus V} s \neq s \text{ for any } V \in \mathcal{V}_x \}$$

$$(\text{resp carr } t = \{ \xi \in \overline{X^*_{u^*}} \mid {}^*B^{X^*_{u^*} \setminus V} t \neq t \text{ for any } V \in \mathcal{V}_\xi \}),$$

where $\overline{X_u}$ (resp $\overline{X^*_{u^*}}$) is the closure of X_u (resp $X^*_{u^*}$) in K_u resp $K^*_{u^*}$) and $\mathcal{V}_x$ (resp $\mathcal{V}_\xi$) is the set of all neighbourhoods of x (resp ξ).

If X is a set such that S and S^* are represented as standard H-cones of functions on X then sometimes is usefull to consider for any $s\in S$ the harmonic carrier of s with respect to the co-natural topology on X i.e. the coarsest topology on X which makes continuous any element of S_0^*.

This new harmonic carrier of s is denoted c.n-carr s and it consists from all points $x\in X$ such that for any co-natural open neighbourhood V of x we have

$$B^{X\setminus V} s\neq s.$$

If H (resp H^*) denote the band in $\mathcal{S}$ (resp $\mathcal{S}^*$) generated by t[he] set of all elements of $\mathcal{S}$ (resp $\mathcal{S}^*$) having an empty carrier and if we put

$$H^{\perp}=\{s\in\mathcal{S} \mid s\wedge h=0,\ \ (\forall)h\in H\}$$

$$(\text{resp } H^{*\perp}=\{t\in\mathcal{S}^* \mid t\wedge h=0\ \ (\forall)h\in H^*\})$$

then we have

$$X_u=X_h\cup X_p \ \ (\text{resp. } X^*_{u^*}=X^*_h\cup X^*_p)$$

where $X_h=X_u\cap H^*$, $X_p=X_u\cap H^{*\perp}$

$(\text{resp } X^*_h=X^*_{u^*}\cap H,\ \ X^*_p=X^*_{u^*}\cap H^{\perp})$.

We recall (see [3], Theorem 5.5.3-5.5.4, Proposition 5.5.[?]) that the sets X_p, X^*_p are Borel subsets of X_u respectively X^*_u and the maps θ,θ^* on X^*_p, respectively X_p defined by

$$\{\theta(\xi)\}=\text{carr}\,\xi,\ \ \{\theta^*(x)\}=\text{carr}\,x$$

are Borel measurable . Moreover if μ is a positive Borel measure on $X^*_{u^*}$ (resp X_u) such that G_μ (resp ${}^*G_\mu$) belongs to $\mathcal{S}$ (resp. $\mathcal{S}^*$) then $G_\mu\in H \Leftrightarrow \mu(X^*_p)=0$, $G_\mu\in H^{\perp}\Leftrightarrow\mu(X^*_h)=0$

(resp. ${}^*G_\mu\in H^*\Leftrightarrow\mu(X_p)=0$, ${}^*G_\mu\in H^{*\perp}\Leftrightarrow\mu(X_h)=0$)

<u>Proposition 1.2.</u> (R.Wittmann). For any bounded element $s\in\mathcal{S}$ we have carr $s\neq\emptyset$.

<u>Proof.</u> We consider the set X_u, the compact space $Y:=\overline{X_u}$ and $\mathcal{S}$ as a convex cone of positive functions on Y. Obviously $\mathcal{S}_0$ is a convex subcone of $\mathcal{S}$ such that any element of $\mathcal{S}_0$ is the restriction to X of a finite continuous function on Y. Also

the convex cone $\mathcal{S}_0+R_+$ separates linearly the points of $\mathcal{Y}$ and contains any positive constant function. Then, using a theorem of R.Wittmann ([6]), for any bounded function $s\in\mathcal{S}$ there exists the smallest compact subset K_s of $\mathcal{Y}$ such that for any natural neighbourhood V of K_s we have

$$B^{X_u\cap V}s=s.$$

From this fact it follows that if $s\neq 0$ then $K_s\neq\emptyset$. The assertion follows now since we have

$$K_s\subset \operatorname{carr} s.$$

<u>Theorem 1.3.</u> The set X_h (resp. X_h^*) is a polar subset of X_u (resp. $X^*_{u^*}$).

<u>Proof.</u> Let K be a compact subset of X_h and for any $t_0\in S_0^*$ let

Let t be element of S^* defined by

$$S\ni s\to\langle t,s\rangle:=\langle t_0,B^K s\rangle$$

Since $t\leq t_0$ and $\langle t_0,u\rangle<\infty$ there exists a positive measure μ on X_u such that

$$\langle t,s\rangle=\int s(x)\,d\mu(x)\qquad(\forall)s\in S.$$

From the preceding considerations we get

$$s_1,s_2\in S,\ s_1=s_2 \text{ on } K\Rightarrow\int s_1\,d\mu=\int s_2\,d\mu$$

i.e. μ is carried by K and therefore the element $t={}^*G\mu$ belongs to H^*. Hence any specific minorant t' of t belongs to H^* and in the same time being dominated by t_0 it has a non-empty carrier if $t'\neq 0$.

We conclude that $t=0$ and therefore the element $t_0\in S_0^*$ being arbitrary we have

$$s\in S\Rightarrow B^K s=0,\ K \text{ polar}.$$

<u>Theorem 1.4.</u> Any element $s\in H$ (resp. H^*) is substractible.

<u>Proof.</u> Let v^* be a weak unit in S^* such that $\langle v^*,s\rangle<\infty$ and let μ be a positive measure on $X^*_{v^*}$ such that

$$s(x)=\int\langle x,y\rangle\,d\mu(y).$$

We have

$X^*_{v^*} = (X^*_{v^*})_h \cup (X^*_{v^*})_p$.

Since $s \in H$ it follows that μ is carried by the set $(X^*_{v^*})_h$. From the previous theorem the set $(X^*_{v^*})_h$ is polar. Since $s \in H$ we have $s \wedge p = 0$ for any $p \in S_0$. The assertion follows now using [1], Theorem 4.3.

c) <u>Green set associated with a pair (u,u^*) of weak units.</u>

Taking u, u^* weak units in S respectively S^* and noting as above by $\mathcal{S}$ respectively $\mathcal{S}^*$ the standard H-cones of functions on X_u respectively $X^*_{u^*}$ it is known ([3], Theorem 5.5.8) that for the maps θ and θ^* defined as in the point b) on X^*_p respectively X_p there exists a Borel subset E^* of X^*_p and a Borel subset E of X_p such that $\theta^*(\theta(\xi)) = \xi$ for any $\xi \in E^*$, $\theta(\theta^*(x)) = x$ for any $x \in E$, θ (resp. θ^*) is Borel measurable and the set $X^*_p \setminus E^*$ (resp. $X_p \setminus E$) is semipolar (resp. co-semipolar). Moreover, for any subset A of E and any $s \in \mathcal{S}$, $t \in \mathcal{S}^*$ we have

$$\langle t, B^A s \rangle = \langle {}^*B^{\theta^*(A)} t, s \rangle$$

and particularly A is semipolar (resp. polar) iff $\theta^*(A)$ is co-semipolar (resp. co-polar). If we denote for any $x \in E$ by x^* the element $\theta^*(x)$ of E^* and then we identify the pair (x, x^*) with x then the set X of all these pairs (x, x^*) is the <u>saturated Green set associated</u> with the <u>weak units u and u^*</u>

Obviously $\mathcal{S}$ and $\mathcal{S}^*$ become H-cones of functions on X and X is nearly saturated with respect to $\mathcal{S}$ and $\mathcal{S}^*$. The function on $X \times X$ defined by

$$(x,y) \to G(x, y^*)$$

is called the Green function and we shall denote simply $G(x,y)$ instead of $G(x,y^*)$.

A subset Ω of X will be termed <u>a Green set for S</u> (corresponding to the weak units u and u^*) if $\mathcal{S}$ and $\mathcal{S}^*$ are standard H-cones of functions on Ω and Ω is still nearly saturated with respect to $\mathcal{S}$ and $\mathcal{S}^*$. The restriction of the Green function to $\Omega\times\Omega$ is denoted again by G. Sometimes we put $\Omega=\Omega(u,u^*)$ when we want to express that Ω is associated with the weak units u and u^*.

Let Ω be a Green set for S (corresponding to the weak units u,u^*), let G be the associated Green function on $\Omega\times\Omega$ and let v^* be another weak unit of S^* which is u^*-continuous.

We consider as in the point a) the map $\varphi : X^*_{u^*}\to X^*_{v^*}$, $\varphi(\xi)=\frac{\xi}{v^*(\xi)}$ which is continuous with respect to the co-natural topologies on $X^*_{u^*}$ and $X^*_{v^*}$. The restriction of φ to Ω give us a continuous map from Ω onto $\varphi(\Omega)$ if we consider on Ω and $\varphi(\Omega)$ the corresponding co-natural topologies.

<u>Proposition 1.5</u>. For any $x\in\Omega$, considered as an element of the standard H-cone of functions S^* on $X^*_{v^*}$, the harmonic carrier of x is the set $\{\varphi(x)\}$ and the harmonic carrier of the element $\varphi(x)$ considered as an element of the standard H-cone of functions S on X_u is the set $\{x\}$.

<u>Proof.</u> The second part is obvious because $\varphi(x)=\frac{x}{v^*(x)}$ and carr $x=\{x\}$. As for the first part let V be a co-natural neighbourhood of the point $\varphi(x)=\frac{x}{v^*(x)}$ from $X^*_{v^*}$. Since the map φ is continuous we deduce that $\varphi^{-1}(V)$ is a co-natural neighbourhood of x in Ω and therefore

$${}^*B_x^{\Omega\setminus\varphi^{-1}(V)}\neq x.$$

Any balayage B on S^* being representable on Ω and therefore on $\varphi(\Omega)$ we deduce that $\varphi(\Omega)$ is nearly saturated with respect to S^*. Hence $\varphi(\Omega)$ is dense in $X^*_{v^*}$ with respect to the co-fine topology. From the preceding considerations we get

$${}^*B_x^{X^*_{v^*}\setminus V}\leq {}^*B_x^{X^*_{v^*}\setminus\varphi(\varphi^{-1}(V))}={}^*B_x^{\varphi(\Omega)\setminus V}={}^*B_x^{\Omega\setminus\varphi^{-1}(V)}\neq x$$

and therefore the harmonic carrier of the extreme element x of S^* considered as a standard H-cone of functions on $X^*_{v^*}$ is the set $\varphi(x)$.

Remark 1.6. With the previous notations if we identify any element $x \in \Omega$ with the pair $(x, \varphi(x))$, Ω becomes a Green set for S but this time corresponding to the weak units u and v^*. Sometimes we mark this distinction noting Ω by $\Omega(u,v^*)$.

The natural topologies on $\Omega(u,u^*)$ or $\Omega(u,v^*)$ coincide whereas the co-natural topology on $\Omega(u,v^*)$ is weaker than the co-natural topology on $\Omega(u,u^*)$.

If we denote by G' the Green function on $\Omega(u,v^*)$ we have

$$G'(x,y) = \frac{G(x,y)}{v^*(y)} \ .$$

d) Green potentials. If $\Omega = \Omega(u,u^*)$ is a Green set for $\mathcal{S}$ and G is the Green function on $\Omega \times \Omega$ then we denote by $\mathcal{M}_{\mathcal{S}}(\Omega)$ (resp. $\mathcal{M}_{\mathcal{S}^*}(\Omega)$) the set of all positive Borel measures μ on Ω such that the function $G\mu$ (resp. ${}^*G\mu$) on Ω defined by

$$G_\mu(x) = \int G(x,y)\,d\mu(y) \ (\text{resp. } {}^*G_\mu(y) = \int G(x,y)\,d\mu(x))$$

belongs to $\mathcal{S}$ (resp. $\mathcal{S}^*$) or equivalently G_μ (resp. ${}^*G_\mu$) is finite on a dense subset of Ω. We put

$$P(\Omega) = \{G_\mu \mid \mu \in \mathcal{M}_{\mathcal{S}}(\Omega)\}$$

$$(\text{resp. } P^*(\Omega) = \{{}^*G_\mu \mid \mu \in \mathcal{M}_{\mathcal{S}^*}(\Omega)\}) \ .$$

The elements of $P(\Omega)$ (resp. $P^*(\Omega)$) are called Green potentials (resp. Green co-potentials) on Ω. We remember that any universally continuous elements s of $\mathcal{S}$ (resp. $\mathcal{S}^*$) belongs to $P(\Omega)$ (resp. $P^*(\Omega)$) and for any point $x_0 \in \Omega$ and any subset A of Ω we have

$$A \text{ is thin at } x_0 \Leftrightarrow {}^*B^A\, {}^*G_{x_0} \neq G_{x_0}$$

$$A \text{ is co-thin at } x_0 \Leftrightarrow B^A G_{x_0} \neq G_{x_0}$$

(see [3], Proposition 5.5.13) where G_{x_0} (resp. ${}^*G_{x_0}$) denote the Green potential (resp. Green co-potential) corresponding to the Dirac measure ε_{x_0}. Particularly we get that any natural (resp. co-natural) open subset of Ω is a co-fine (resp. fine) bands in $\mathcal{S}$, respectively $\mathcal{S}^*$ with respect to the specific order.

Proposition 1.7. If v^* is a u^*-continuous weak unit of S^* then the sets $P(\Omega(u,u^*))$ and $P(\Omega(u,v^*))$ considered as subset of S coincide.

Proof. Indeed, let $p \in P(\Omega(u,u^*))$ and let $\mu \in \mathcal{M}_{\mathcal{G}}(\Omega(u,u^*))$ be such that $p=G_\mu$. Since $\Omega(u,u^*)$ and $\Omega(u,v^*)$ have the same Borel structure, if we take the Borel measure μ' on $\Omega(u,v^*)$ defined by $\mu'=v^*\mu$, we have

$$G'_{\mu'}(x)=\int G'(x,y)\cdot v^*(y)d\mu(y)=$$
$$=\int G(x,y)d\mu(y)=G_\mu(x)=p\ .$$

Hence $P(\Omega(u,u^*))\subset P(\Omega(u,v^*))$. The converse inequality may be similarly shown.

Theorem 1.8. Let $\Omega(u,u^*)$ be a Green set for S corresponding to the units u and u^*.

The following assertions are equivalent.

1) $P(\Omega)$ is solid in S with respect to the natural order.

2) Any universally bounded element of S is a Green potential on Ω.

3) Ω is semi-saturated in $X^*_{u^*}$ with respect to S^*.

4) Any pure potential of S is a Green potential on Ω.

Proof. The assertion follows directly from Theorem 1.1. taking $Y=\Omega$.

2. Balayages on Green sets.

We suppose that Ω is a Green set for S (corresponding to the weak units u and u^*) and we suppose that G is the Green function on $\Omega \times \Omega$.

Theorem 2.1. Let λ be a positive measure on $X^*_{u^*}$ such that $G_\lambda \in S$ and let M be a subset of Ω. Then we have $B^M G_\lambda = G_\lambda$ iff λ is carried by the set $b^*(M)$ of all points $x \in X^*_{u^*}$ such that M is not co-thin at x.

Proof. It is known (see [3], Proposition 4.3.8) that $b^*(M)$ is a G_δ-set in X^*_u with respect to the co-natural topology on X_u and that $b^*(M) = \{x \in X^*_{u^*} \mid {}^*B^M t(x) = t(x)\}$ for any finite generator t of $\mathcal{S}^*$.

Let μ be a positive measure on Ω such that ${}^*G_\mu$ is a finite generator of $\mathcal{S}^*$ and such that $\mu(G_\lambda) < \infty$. Since $B^M p \leq p$ for any $p \in \mathcal{S}$ then the equality $B^M p = p$ is equivalent with the equality $\mu(B^M p) = \mu(p)$. Hence we have

$$B^M G_\lambda = G_\lambda \Leftrightarrow \mu(B^M G_\lambda) = \mu(G_\lambda).$$

From the relation

$$\mu(B^M G_\lambda) = \langle {}^*G_\mu, B^M G_\lambda \rangle = \langle {}^*B^{M\,*}G_\mu, G_\lambda \rangle = \int {}^*B^{M\,*}G_\mu \, d\lambda;$$

$$\mu(G_\lambda) = \langle {}^*G_\mu, G_\lambda \rangle = \int {}^*G_\mu \, d\lambda$$

we deduce that we have

$$B^M G_\lambda = G_\lambda \Leftrightarrow \int {}^*G_\mu(\xi) \, d\lambda(\xi) = \int {}^*B^{M\,*}G_\mu(\xi) \, d\lambda(\xi).$$

The last equality is equivalent with the fact that λ is carried by the Borel set

$$b^*(M) = \{\xi \in X^*_{u^*} \mid {}^*B^{M\,*}G_\mu(\xi) = {}^*G_\mu(\xi)\}.$$

Corollary 2.2. Let λ be a positive measure on $X^*_{u^*}$ such that $G_\lambda \in \mathcal{S}$ and let D be a co-fine open subset of $X^*_{u^*}$. Then we have $B^{\Omega \cap D} G_\lambda = G_\lambda$ iff λ is carried by the co-fine closure of the set D (or $\Omega \cap D$) in $X^*_{u^*}$.

Proof. Since $\mathcal{S}^*$ is a standard H-cone of functions on Ω we deduce that Ω is dense in $X^*_{u^*}$ with respect to the co-fine topology. Hence the co-fine closures in $X^*_{u^*}$ of the sets $\Omega \cap D$

theorem using the fact that $b^*(\mathcal{D})$ is nothing else than the co-fine closure of $\mathcal{D}$.

Corollary 2.3. Let $\lambda \in \mathcal{M}_\varphi(\Omega)$ and let M be a subset of Ω. Then we have $B^M G_\lambda = G_\lambda$ iff λ is carried by the set of all points $x \in \Omega$ such that M is not co-thin at x.

Proposition 2.4. a) For any $\lambda \in \mathcal{M}_\varphi(\Omega)$ we have

$$\operatorname{supp} \lambda = \Omega \cap \operatorname{carr} G_\lambda$$

where $\operatorname{supp}\lambda$ is the support of λ with respect to natural topology on Ω.

b) For any $\lambda \in \mathcal{M}_\varphi(\Omega)$ we have

$$\text{c.n.-supp}\,\lambda = \text{c.n.-carr}\, G_\lambda$$

where c.n.-supp λ denotes the support of λ with respect to the co-natural topology on Ω and c.n.-carr G_λ is the harmonic carrier of G_λ with respect to the co-natural topology on Ω. (see [3], 3.4).

Proof. Let $\mathcal{D}$ be a natural open subset of Ω such that $\mathcal{D} \supset \operatorname{supp}\lambda$. From Corollary 2.3 it follows that

$$B^D(G_\lambda) = G_\lambda$$

and therefore

$$\Omega \cap \operatorname{carr} G_\lambda \subset \overline{\mathcal{D}}$$

where $\overline{D}$ is the natural closure of D in Ω. The set D being arbitrary we get

$$\Omega \cap \operatorname{carr} G_\lambda \subset \operatorname{supp}\lambda .$$

Conservely, let $x_0 \in \Omega \setminus \operatorname{carr} G_\lambda$ and let V be a natural open neighbourhood of x_0 such that

$$B^{\Omega\setminus V} G_\lambda = G_\lambda .$$

Since $\Omega \setminus V$ is a co-fine closed subset of Ω, we get, using again Corollary 2.3, $\lambda(V)=0$ and therefore

$$\operatorname{supp}\lambda \subset \Omega \setminus V, \quad x_0 \notin \operatorname{supp}\lambda .$$

The point x_0 being arbitrary we deduce

$$\operatorname{supp}\lambda \subset \Omega \cap \operatorname{carr} G_\lambda .$$

b) Let $x_0 \in$ c.n.-supp λ and let V be an arbitrary co-natural open neighbourhood of x_0. We have $\lambda(V)>0$ and therefore, using Corollary 2.3

$$B^{\Omega\setminus V} G_\lambda \neq G_\lambda .$$

Hence $x_0 \in$ c.n.-carr G_λ .

Let now $x_0 \in$ c.n.-carr G_λ and let V be an arbitrary co-natural open neighbourhood of x_0. By definition of the set c.n.-carr G_λ we have

$$B^{\Omega\setminus V} G_\lambda \neq G_\lambda$$

and therefore, using again Corollary 2.3, $\lambda(V)>0$.

The co-natural open neighbourhood V of x_0 being arbitrary we get $x_0 \in$ c.n.-supp λ.

Theorem 2.5. Let $s \in S$ be such that for any H-measure λ on $X^*_{u^*}$ we have $s \wedge G_\lambda = 0$. Then, for any $t \in S^*$ such that $\langle t,s\rangle < \infty$ we have

$$B^{[t<1]} s=s$$

where

$$[t<1]:=\{x\in\Omega \mid t(x)<1\}$$

Proof. Let v^* be a weak unit of S^* which is u^*-continuous and such that $\langle v^*,s\rangle<\infty$. As in the introductory point c), Remark 1.6., Ω may be considered as a Green set for S corresponding to the weak units u and v^*. The Green function G' on $\Omega=\Omega(u,v^*)$ will be

$$G'(x,y)=\frac{G(x,y)}{v^*(y)}$$

Since $\langle v^*,s\rangle < \infty$ there exists a positive measure μ on $X^*_{v^*}$ such that

$$s(x)=\int_{X^*_{v^*}} \langle x,\eta\rangle \, d\mu(\eta) .$$

Further using the considerations from introductory point a) deduce that

$$\mu(X^*_{v^*} \setminus \varphi(X^*_{u^*}))=0$$

where φ is the map from $X^*_{u^*}$ into $X^*_{v^*}$ defined by

$$\varphi(\xi)=\frac{\xi}{v^*(\xi)}\,.$$

Since

$$X^*_{v^*}\setminus\varphi(X^*_{u^*})=\{\eta\in X^*_{v^*}\mid u^*(\eta)=+\infty\}$$

and since

$$\int t(\eta)\,d\mu(\eta)=\langle t,s\rangle<+\infty$$

it follows that μ is carried by the set

$$\{\eta\in X^*_{v^*}\setminus\varphi(X^*_{u^*})\mid t(\eta)<\infty\}\subset\{\eta\in X^*_{v^*}\mid t(\eta)<u^*(\eta)\}.$$

The assertion follows now from the relations:

$$B^{[t<1]}s=B^{\Omega\cap[t<u^*]}s=B^{\varphi(\Omega)\cap[t<u^*]}s=$$
$$=B^{X^*_{v^*}\cap[t<u^*]}s\,.$$

and from Corollary 2.2 using the fact that $[t<u^*]\cap X^*_{v^*}$ is a co-fine open subset of $X^*_{v^*}$.

Lemma 2.6. Let λ be an H-measure on $X^*_{u^*}$ such that its support is a compact subset K of $X^*_{u^*}\setminus\Omega$ and let A be a polar subset of Ω.

Then we have

$$\wedge\{B^V G_\lambda \mid V\in\mathcal{V}(A)\}=0$$

where $\mathcal{V}(A)$ denotes the set of all co-natural neighbourhood in Ω of the set A.

Proof. Let us denote by $(U_n)_n$ a decreasing sequence of co-natural neighbourhood in $X^*_{u^*}$ of K such that

$$U_n\supset\overline{U}_{n+1},\quad \bigcap_n U_n=K$$

and let μ be a measure on Ω such that ${}^*G_\mu$ is a u^*-continuous generator of S^* with $\mu(p)<\infty$, where $p:=G_\lambda$.

We show that for any $n\in N$ and any $\varepsilon>0$ there exists a co-natural open subset V_n^ε of Ω such that

$$V_n^\varepsilon\supset A\cap(\Omega\setminus U_n),\ \mu(B^{V_n^\varepsilon}p)<\varepsilon\,.$$

Indeed, since the set $A\cap(\Omega\setminus U_n)$ is polar we deduce that it is also co-polar subset of Ω and therefore, ${}^*G_\mu$ being finite and continuous with respect to the co-natural topology on Ω, there exists a decreasing sequence $(W_m)_m$ of co-natural open subsets of $X^*_{u^*}$ such that

$$W_m \cap \overline{U}_{n+1} = \emptyset,\ W_m \supset A \cap (\Omega \setminus U_n)\ (\forall)\, m \in N$$

$$^*B^{W_m \cap \Omega} * G_\mu = 0 .$$

The sequence $({}^*B^{W_m \cap \Omega} * G_\mu)_m$ being decreasing on $X^*_{u^*}$

and since

$${}^*B^{X \setminus \overline{U}_{n+1}}({}^*B^{W_m \cap \Omega} * G_\mu) = {}^*B^{W_n \cap \Omega} * G_\mu$$

for any $m \in N$, we deduce (see [3] , Corollary 5.1.7) that

$$\inf_m {}^*B^{W_m \cap \Omega} * G_\mu (y) = 0 \quad (\forall)\, y \in K$$

and therefore

$$0 = \inf_m \lambda({}^*B^{W_m \cap \Omega} * G_\mu) = \inf_m \int B^{W_m \cap \Omega} G_\lambda \, d\mu .$$

We choose $m_0 \in N$ sufficently large such that

$$\int B^{W_{m_0} \cap \Omega} G_\lambda \, d\mu < \varepsilon$$

and we put

$$V_n^\varepsilon := W_{m_0} \cap \Omega .$$

Taking now for any $n \in N$, $\varepsilon_n := \varepsilon/2^n$ and V_n a co-natural open subset of Ω such that

$$V_n \supset A \cap (\Omega \setminus U_n),\ \mu(B^{V_n} G_\lambda) < \varepsilon/2^n$$

we deduce

$$\mu(B^{V_\varepsilon} G_\lambda) \leq \mu(\sum_{n=1}^{\infty} B^{V_n} G_\lambda) < \varepsilon$$

where $V_\varepsilon := \bigcup_n V_n$. Since V_ε is a co-natural open neighbourhood of the set A and ε is arbitrary, we get

$$\mu(\bigwedge \{ B^V G_\lambda \mid V \in \vartheta(A) \}) = 0,$$

$$\bigwedge \{ B^V G_\lambda \mid V \in \vartheta(A) \} = 0 .$$

<u>Theorem 2.7.</u> Let p be an element of S such that for any $q \in P(\Omega)$ we have $p \curlywedge q = 0$. Then, for any polar subset A of Ω we have

$$\bigwedge \{ B^V p \mid V \in \vartheta(A) \} = 0$$

where $\vartheta(A)$ is the set of all co-natural neighbourhoods of A in Ω.

<u>Proof.</u> We consider v^* another weak unit of S^* which is u^*-continuous and such that

Med 21349

$$\langle v^*, p \rangle < \infty$$

In this case there exists an H-measure μ on $X^*_{v^*}$ such that

$$p(x)=\int_{X^*_{v^*}}\langle x,\eta\rangle\, d\mu(\eta).$$

Using the hypothesis and Proposition 1.6 we deduce that

$$p\wedge G'_\lambda=0$$

for any $\lambda\in\mathcal{M}_\varphi(\Omega(u,v^*))$ where G' is the Green function for S corresponding to weak units u,v^*. Hence μ is carried by $X^*_{v^*}\setminus\Omega(u,v^*)$ and therefore there exists a sequence $(K_n)_n$ of compact subset of $X^*_{v^*}\setminus\Omega(u,v^*)$ such that

$$\mu=\sum_n \mu|_{K_n},$$

$$p=\sum_n G'\mu|_{K_n}.$$

From the preceding lemma we have,

$$\bigwedge\{B^V(G'_{\mu|K_n})\mid V\in\mathcal{V}_1(A)\}=0,\ (\forall)n\in N$$

where $\mathcal{V}_1(A)$ is the set of all co-natural neighbourhoods of A in $\Omega(u,v^*)$. Since the co-natural topology on $\Omega(u,v^*)$ is smaller then the co-natural topology on $\Omega(u,u^*)$ we deduce that

$$\bigwedge\{B^V G'\mu|_{K_n}\mid V\in\mathcal{V}(A)\}=0.\ (\forall)n\in N.$$

Hence

$$\bigwedge\{B^V p\mid V\in\mathcal{V}(A)\}=0$$

<u>Corollary 2.8.</u> Let p be as in the above theorem. Then for any polar subset A of Ω we have

$$\inf_V B^V p(x)=0\quad(\forall)x\in\Omega\setminus A,\ p(x)<\infty,$$

where V runs the set of all co-natural neighbourhood of A.

3. Characterizations of Green potentials.

In this section we consider Ω a Green set for the standard H-cone S corresponding to the weak units u, u^* and G the associated Green function on $\Omega \times \Omega$.

We know that any element s (resp. t) form S (resp. $\overset{*}{S}$) may be thought as a function on Ω belonging to the standard H-cone of functions $\mathcal{S}$ (resp. $\mathcal{S}^*$) on Ω which is isomorphic with S (resp. $\overset{*}{S}$).

As function on Ω the element $s \in S$ (resp. $t \in \overset{*}{S}$) will be again denoted by s (resp. t).

Theorem 3.1. If $p \in S$ the following assertions are equivalent:

1) $p \in P(\Omega)$;

2) for any $q \in S$, $q \precsim p$ and any subset M of Ω such that M is not co-thin at any point $x_0 \in \Omega \cap \operatorname{carr} q$ we have

$B_q^M = q$;

3) for any $q \in S$, $q \precsim p$ and any co-fine neighbourhood V (in Ω) of the set $\Omega \cap \operatorname{carr} q$ we have

$B_q^V = q$;

4) for any $q \in S$, $q \precsim p$ and any co-natural neighbourhood V (in Ω) of the set $\Omega \cap \operatorname{carr} q$ we have

$B^V q = q$;

5) there exists a polar set A of Ω such that for any $q \in S$ $q \precsim p$ and any subset M of Ω which is not co-thin at any point x from $A \cup (\Omega \cap \operatorname{carr} q)$ we have

$B^M q = q$;

6) there exists a polar set A of Ω such that for any $q \in S$, $q \precsim p$ and any co-fine neighbourhood V (in Ω) of the set $A \cup (\Omega \cap \operatorname{carr} q)$ we have

$B^V q = q$;

7) there exists a polar set A of Ω such that for any $q \in S$, $q \precsim p$ and any co-natural neighbourhhod V (in Ω) of the set $A \cup (\Omega \cap \operatorname{carr} q)$ we have

8) for any $q\in S$, $q\neq 0$, $q\lesssim p$ we have c.n.-carr $q\neq\emptyset$ where c.n.-carr q means the harmonic carrier of q on Ω with respect to the co-natural topology (see [3], 3.4).

Proof. The following assertions are obvious

$$\begin{array}{ccccc} 2) & \Rightarrow & 3) & \Rightarrow & 4) \\ \Downarrow & & \Downarrow & & \Downarrow \\ 5) & \Rightarrow & 6) & \Rightarrow & 7) \end{array}$$

The assertion 1)$\Rightarrow$2) follows using Proposition 2.4. and Corollary 2.3. We show now that 7)$\Rightarrow$1). Since $P(\Omega)$ is a band in S we may write p as a sum of the form $p=p'+q$ where $p'\in P(\Omega)$ and $q\in S$ is such that $q\wedge u=0$ for any $u\in P(\Omega)$.

Our intention is to proove that $q=0$. For thus we remark that, from theorem 2.7, it follows that

$$\wedge\{B^{V'}q \mid V'\in\mathcal{V}(A)\}=0$$

where $\mathcal{V}(A)$ is the set of all co-natural neighbourhood (in Ω) of A. Let now V be a co-natural neighbourhood (in Ω) of the set $\Omega\cap$ carr q' where q' is an arbitrary element of S such that $q'\lesssim q$.

We have, using the hypothesis,

$$q'=B^{V\cup V'}q'\leq B^{V'}q+B^{V}q'$$

for any $V'\in\mathcal{V}(A)$ (the set of all co-natural neighbourhood of A) and therefore,

$$q'\leq B^{V}q'+\bigwedge_{V'\in\mathcal{V}(A)}B^{V'}q,\quad q'=B^{V}q'$$

Let v^* be a weak unit of S^* which is u^*-continuous and such that $\langle v^*,q\rangle<\infty$ and let $\Omega(u,v^*)$ be the Green set for S corresponding to the weak units u and v^*. Since $\langle v^*,q\rangle<\infty$ we deduce that there exists a positive measure μ on $X^*_{v^*}$ such that

$$q(x)=\int_{X^*_{v^*}}\langle x,\eta\rangle\, d\mu(\eta)\qquad (\forall)\,x\in\Omega .$$

From Proposition 1.7 we deduce that there exists a sequence $(K_n)_{n\in N}$ of compact subsets of $X^*_{v^*}\setminus\Omega(u,v^*)$ such that

$q=\sum_n q_n$ where, for any $n\in N$, q_n is the element of S defined by

$$q_n(x)=\int_{K_n} \langle x,\eta\rangle \, d\mu(\eta).$$

We fix now $n_0\in N$ and we consider a decreasing sequence $(V_n)_n$ of co-natural open neighbourhoods of K_{n_0} in $X^*_{v^*}$ such that $\overline{V}_{n+1}\subset V_n$ for any $n\in N$ and $\bigcap_n V_n=K_n$.

Obviously the sequence $(A_n)_n$ of subsets of Ω defined by $A_n=\Omega\setminus V_n$ is a sequence of Borel subsets of Ω which increases to Ω. From the preceding considerations we deduce that for any element $q'\in S$, $q'\precsim q_{n_0}$ and any co-natural neighbourhood V (in $\Omega(u,\overset{*}{u})$) of the set $\Omega\cap$ carr q' we have

$$B^V q'=q'$$

If V is a co-natural neighbourhood (in $\Omega(u,v^*)$) of the set $\Omega\cap$ carr q' we have also $B^V q'=q'$ since the co-natural topology on $\Omega(u,v^*)$ is weaker than the co-natural topology on $\Omega(u,u^*)$ (see Remark 1.6). Obviously $q'_{n_0}\in H^{\perp}$ and therefore (see [3], Theorem 3.4.6) we may consider the specific multiplication associated with q_{n_0} i.e. the map

$$A\to q_{n_0}(A)$$

from the set $\mathcal{B}$ of all Borel subsets of $\overline{X}_u$ into S having the following properties

$$A\in\mathcal{B}\Rightarrow q_{n_0}(A)=\curlyvee\{q'\in S \mid q'\precsim q_{n_0},\ \text{carr } q'\subset A\}$$

$$\curlyvee_n q_{n_0}(B_n)=q_{n_0}(\cup B_n) \text{ for any}$$

sequence $(B_n)_n$ from B. Above all, in our special case, we have for any two Borel subsets B_1, B_2 from $\mathcal{B}$

$$B_1\cap\Omega\subset B_2\cap\Omega\Rightarrow q_{n_0}(B_1)\precsim q_{n_0}(B_2).$$

Indeed, if we put $B=B_1\cap B_2$ we have $B\subset B_1$, $B\subset B_2$,

$$B\cap\Omega=B_1\cap\Omega,\ (B_1\setminus B)\cap\Omega=\emptyset,$$

and therefore, since

$$q_{m_0}(B_1\setminus B)=\curlyvee\{q'\in S \mid q'\precsim q_{n_0},\ \text{carr } q'\subset B_1\setminus B\}$$

since $B^V q'=q'$ for any $q'\in S$, $q'\precsim q_{n_0}$ and any co-natural nbhd V of $\Omega\cap$ carr q' we get

$q_{n_0}(B_1 \setminus B)=0$, $q_{n_0}(B_1)=q_{n_0}(B) \curlyvee q_{n_0}(B_1 \setminus B)=$

$=q_{n_0}(B)$, $q_{n_0}(B_1) \precsim q_{n_0}(B_2)$

Having in mind all these considerations we may consider an increasinng sequence $(B_n)_n$ of Borel subset from $\mathcal{B}$ such that for any $n \in N$, $B_n \cap \Omega = A_n = \Omega \setminus \mathcal{V}_n$.

Since $(\bigcup_n B_n) \cap \Omega = (\bigcup A_n) \cap \Omega = \Omega$ we deduce that

$$\curlyvee_n q_{n_0}(B_n) = q_{n_0}(\bigcup_n B_n) = q_{n_0}(\overline{X}_u) = q_{n_0}$$

Taking $q' \in S$, $q' \precsim q_{n_0}$, carr $q' \subset B_n$ we deduce that the set $\mathcal{V} := \Omega \setminus \overline{\mathcal{V}_{n+1}}$ is a co-natural open subset of Ω and

$\mathcal{V} \supset \Omega \setminus \mathcal{V}_n = A_n = B_n \cap \Omega \supset (\text{carr } q') \cap \Omega$.

Hence using again the hypothesis we deduce that there exists a positive measure ν carried by K_{n_0} such that

$$q'(x) = \int_{K_{n_0}} \langle x, \eta \rangle \, d\nu(\eta)$$

and moreover

$$B^{\mathcal{V}} q' = q'.$$

From Corollary 2.2 we deduce that ν is carried by the co-natural clasure in $X^*_{v^*}$ of the subset $\mathcal{V}$ and therefore ν is carried by $X^*_{v^*} \setminus \mathcal{V}_{n+1}$. Hence, ν being carried also by K_{n_0}, we have

$$\nu = 0, \; q' = 0, \; q_{n_0}(B_n) = 0$$

The numbers n and n_0 being arbitraries we get

$$q_{n_0} = \curlyvee_n q_{n_0}(B_n) = 0,$$

$$q = \curlyvee_{n_0} q_{n_0} = 0.$$

The relation 1)$\Rightarrow$8) follows directly from Proposition 2.4, point b). We show now that 8)$\Rightarrow$1). We consider the weak unit v^* in S^* which is u^*-continuous and such that $v^*(p) < \infty$. Let μ be a measure on $X^*_{v^*}$ such that $p(x) = \int_{X^*_{v^*}} \langle x, \xi \rangle \, d\mu(\xi)$. Since the co-natural topology on $\Omega(u, v^*)$ is smaller than the co-natural topology on $\Omega(u, u^*)$ we deduce, using the hypothesis, that for any $q \in S$, $q \precsim p$, $q \neq 0$ we have c.n.-carr $q \neq \emptyset$ where, in this case, the set c.n.-carr q is the harmonic carrier of q with respect to the co-natural topology on $\Omega=\Omega(u,v^*)$. Let $p=q+q'$ where $q\in P(\Omega)$ and $q'\in S$ is such that for any $t\in P(\Omega)$ we have $q'\wedge t=0$. Obvionsly q' satisfies again the assertion 8) and moreover the positive measure μ' on $X^*_{v^*}$ for which

$$q'(x)=\int_{X^*_{v^*}}\langle x,\xi\rangle\, d\mu'(\xi)$$

may be written as the from $\sum_{m=1}^{\infty}\mu'_m$, where for any $m\in N$, μ'_m is a positive measure with compact support K_m, $K_m\subset X^*_{v^*}\setminus\Omega$.

For each $m\in N$ let $(V_n)_{n\in N}$ be a sequence of co-natural open neighbourhood of K_m such that

$$\overline{V_{n+1}}\subset V_n \quad (\forall)n\in N \quad \text{and } K_m=\bigcap_n V_n.$$

Using Theorem 2.1 we get, for any n N,

B n q'_m where $q'_m(x)$:= x, d m( )

and therefore

c.n.-carr q'_n n, ( )n N

c.n.-carr $q'_m=\emptyset$, $q'_m=0$

The number $m\in N$ being arbitrary we deduce $q'=0$, $p=q$, $p\subset P(\Omega)$.

Corollary 3.2. Suppose that the natural topology on Ω is stronger than the co-natural topology on Ω. Then an element $p\in S$ belongs to $P(\Omega)$ iff for any $q\in S$, $q\lesssim p$, $q\neq 0$ we have

$\Omega\cap$ carr $q\neq 0$.

Proof. The assertion follows from the preceding theorem and from the fact that

$\Omega\cap$ carr $q\subset$ c.n.-carr q

for any $q\in S$.

Remark 3.3. We know that any natural open subset of Ω is also a co-fine open subset of Ω and therefore from previous theorem we deduce that for any $p\in P(\Omega)$ and any natural open neighbourhood V of the set $\Omega\cap$ carr p we have $B^V p=p$.

We may ask if the following assertion is true:

4') if $p\in S$ is such that for any $q\in S$, $q\lesssim p$ and any

The following example show that, generally, the above assertion does not hold.

Example 3.4. We consider on the set $\Omega=(-1,1)$ the standard H-cone S of all lower semicontinuous functions f on $(-1,1)$ such that $f|(-1,0]$ is increasing and concave and $f|[0,1)$ is concave. We remark that 1-continuous elements of S are exactly the continuous functions $f\in S$ with $\lim_{|x|\to 1} f(x)=0$, the natural topology on Ω coincides with the usual topology of the interval $(-1,1)$ of the real line, and the dual S^* contains is the convexe cone of all positive Radon measure μ on Ω such that

$$\int(1-x^2)\,d\mu<\infty.$$

Also it is easy to see that Ω is saturated with respect to S and a Radon measure μ on Ω is a weak unit in S^* iff $\mu((0,1))>0$. Also one can verify that for any $x\in\Omega$ the measure ε_x is an universally continuous element of S^*. Let u^* be the weak unit of S^*, $u^*=\varepsilon_{1/2}$. For any $t\in\Omega\setminus\{0\}$ we consider the extreme element $\xi_t\in S$ such that carr $\xi_t=t$ and $u^*(t)=1$ namely

$$\xi_t=\begin{cases}\frac{2}{t+1}(x+1) & \text{for } x\in(-1,t)\\ 2 & \text{for } x\in[t,0]\\ -2(x-1) & \text{for } x\in(0,1)\end{cases}$$

if $t\in(-1,0)$,

$$\xi_t=\begin{cases}0 & \text{for } x\in(-1,0]\\ \frac{2(1-t)}{t}x & \text{for } x\in(0;t)\\ -2(x-1) & \text{for } x\in[t,1)\end{cases}$$

if $t\in(0,\frac{1}{2}]$,

$$\xi_t=\begin{cases}0 & \text{for } x\in(-1,0]\\ 2x & \text{for } x\in(0,t)\\ \frac{-2t}{1-t}(x-1) & \text{for } x\in[t,1)\end{cases}$$

if $t\in(\frac{1}{2},1)$.

As for the element $0 \in \Omega$ there exist two extreme elements of S denoted by ξ_0^-, ξ_0^+ with carr $\xi_0^- =$ carr $\xi_0^+ = \{0\}$, namely

$$\xi_0^-(x) = \begin{cases} 2(x+1) & \text{for } x \in (-1,0] \\ 2(1-x) & \text{for } x \in (0,1) \end{cases}$$

$$\xi_0^+(x) = \begin{cases} 0 & \text{for } x \in (-1,0] \\ 2(1-x) & \text{for } x \in (0,1). \end{cases}$$

We remark also that the natural closure of Ω is the Alexandrof compactification of the locally compact space $(-1,1)$ and the only element which is added to $(-1,1)$ by this compactification, is the element μ_0 of S^* defined by

$$\mu_0(s) = 0 \qquad (\forall) s \in S.$$

The set $X^*_{u^*}$ consists from the elements ξ_t, $t \in (-1,0) \cup (0,1)$, ξ_0^-, ξ_0^+ described as above and from the following two elements denated by ξ_{-1}, ξ_{+1} where

$$\xi_{-1}(x) = \begin{cases} 2 & \text{if } x \in (-1,0] \\ 2(1-x) & \text{if } x \in (0,1) \end{cases}$$

$$\xi_{+1}(x) = \begin{cases} 0 & \text{if } x \in (-1,0] \\ 2x & \text{if } x \in (0,1) \end{cases}$$

Since the set $\{\varepsilon_x \mid x \in (-1,1)\}$ of universally continuous elements of S^* separates the compact set $K^*_{u^*}$, where

$$K^*_{u^*} = \{s \in S \mid s(\tfrac{1}{2}) \leq 1\}$$

we deduce that the space $X^*_{u^*}$ of all nonzero extreme elements of $K^*_{u^*}$ endowed with the co-natural topology may be identified with the topologic sum

$$[-1,0] \oplus [0,1]$$

where ξ_0^- (resp. ξ_0^+) is identified by the point 0 from $[-1,0]$ (resp., $[0,1]$), and Ω coincides with $(-1,0] \cup (0,1)$. One can easely verify that the element ξ_0^+ satisfies the condition 4') but it is not a Green potential.

Remark 3.5. The above example shows that if p is an element of S such that for any $q\in S$, $q\lesssim p$, $q\neq 0$ we have carr $q\neq\emptyset$ we can not deduce that p is a Green potential.

IV. Potentials on Green sets

In this section we deal with the study of those elements of an H-cone of functions S on a Green set Ω having a "potential" behaviour with respect to some covering of Ω and we give necessary and sufficient conditions under which an element of S is a Green potential iff it has this "potential" behaviour.

In the sequel Ω will be a Green set for the standard H-cone S corresponding to the weak units u and u^*.

Lemma 4.1. Let U_n be an increasing sequence of fine open subsets of Ω such that $\Omega=\bigcup_n U_n$. Then the following assetions are equivalent:

1) $\inf_n B^{\Omega\setminus U_n} p=0 \qquad (\forall)p\in S_0$

2) $\bigwedge_n B^{\Omega\setminus U_n} p=0 \qquad (\forall)p\in S_0$

3) $\bigwedge^* B^{\Omega\setminus U_n} q=0 \qquad (\forall)q\in S_0^*$

4) $\bigwedge^* B^{\Omega\setminus U_n} q=0 \qquad (\forall)p\in P^*(\Omega)$.

Proof. The assertions 1)$\Rightarrow$2), 4)$\Rightarrow$3) are obvious. We show now that 2)$\Leftrightarrow$3). Let λ and μ be two positive measure on Ω such that $G_\lambda\in S_0$ and ${}^*G_\mu\in S_0$. From the relations

$$\int B^{\Omega\setminus U_n} G_\lambda\, d\mu=\int {}^*B^{\Omega\setminus U_n}\, {}^*G_\mu\, d\lambda \qquad (\forall)n\in N$$

and using the fact that the semipolar subsets of Ω are λ and μ negligible we deduce

$$\int \bigwedge_n B^{\Omega\setminus U_n} G_\lambda\, d\mu=\int \inf_n B^{\Omega\setminus U_n} G_\lambda\, d\mu=$$

$$=\int \inf_n{}^* B^{\Omega\setminus U_n}\, {}^*G_\mu\, d\lambda=\int \bigwedge_n^* B^{\Omega\setminus U_n}\, {}^*G_\mu\, d\lambda$$

and therefore, λ and μ being arbitrary,

$$\bigwedge_n B^{\Omega\setminus U_n} G_\lambda=0 \Leftrightarrow \bigwedge_n^* B^{\Omega\setminus U_n}\, {}^*G_\mu=0$$

2)$\Rightarrow$1) Let us consider $x_0\in\Omega$ and let $n_0\in N$ be such that $x_0\in U_{n_0}$. Considerung now a fine open neighbourhood V of x_0 such that the natural closure of V is contained in U_{n_0} and using [2], Proposition 2.2. we have

$$(p-B^{\Omega\setminus U_n}p)|_V \in S'(V),\ (B^{\Omega\setminus U_n}p)|_V \in S'(V)$$

for any $n \geq n_0$. On the other hand the sequence $(B^{\Omega\setminus U_n}p)|_V$ $n \geq n_0$ is pointwise decreasing to a fine continuous function on V as it is shown by the following relation

$$p|_V = (p-B^{\Omega\setminus U_n}p)|_V + (B^{\Omega\setminus U_n}p)|_V$$

Hence

$$\inf_n B^{\Omega\setminus U_n}p(x_0) = (\bigwedge_n B^{\Omega\setminus U_n}p)(x_0)$$

1)$\Rightarrow$4). Let $\mu \in \mathcal{M}_S(\Omega)$ and let λ be an element of $\mathcal{M}_{S^*}(\Omega)$ such that $G_\lambda \in S_0$.

Since λ does not charge any semipolar subset of Ω we have

$$\lambda(\bigwedge_n{}^*B^{\Omega\setminus U_n *}G_\mu) = \lambda(\inf_n{}^*B^{\Omega\setminus U_n *}G_\mu) =$$
$$= \inf_n \lambda({}^*B^{\Omega\setminus U_n *}G_\mu) = \inf_n \mu(B^{\Omega\setminus U_n}G_\lambda) =$$
$$= \mu(\inf_n B^{\Omega\setminus U_n}G_\lambda) = 0$$

and therefore, λ being arbitrary, we get

$$\bigwedge_n{}^*B^{\Omega\setminus U_n *}G_\mu = 0$$

<u>Corolarry 4.2.</u> Let $(U_n)_n$ be an increasing sequence of co-fine open subset of Ω such that $\Omega = \bigcup_n U_n$. The following assertion are equivalent:

1) $\inf {}^*B^{\Omega\setminus U_n}q = 0$, $(\forall) q \in S_0^*$
2) $\bigwedge_n{}^*B^{\Omega\setminus U_n}q = 0$, $(\forall) q \in S_0^*$
3) $\bigwedge_n B^{\Omega\setminus U_n}p = 0$, $(\forall) p \in S_0$
4) $\bigwedge_n B^{\Omega\setminus U_n}p = 0$, $(\forall) p \in P(\Omega)$.

<u>Corollary 4.3.</u> Let $(U_n)_n$ be an increasing sequence of natural (resp. co-natural) open subsets of Ω such that $\bigcup_n U_n = \Omega$. The following assertion are equivalent:

1) $\bigwedge_n B^{\Omega\setminus U_n}p = 0$, $(\forall) p \in S_0$
2) $\bigwedge_n{}^*B^{\Omega\setminus U_n}q = 0$, $(\forall) q \in S_0^*$
3) $\bigwedge_n B^{\Omega\setminus U_n}p = 0$, $(\forall) p \in P(\Omega)$
4) $\bigwedge_n{}^*B^{\Omega\setminus U_n}q = 0$, $(\forall) q \in P^*(\Omega)$
5) $\inf B^{\Omega\setminus U_n}q = 0$, $(\forall) p \in S_0$

6) $\inf_m {}^*B^{\Omega\setminus U_n} q=0$, $(\forall) q\in S_0$.

Proof. The assertion follows directly from Lema 4.1. and Corollary 4.2. using the fact that any natural (resp.co-natural) open subset of Ω is both finaly and co-finely open.

Theorem 4.4. Let p be an element of S such that for any increasing sequence $(D_n)_n$ of co-natural open subsets of Ω with $\bigcup_m D_n = \Omega$ we have

$$\bigwedge_m B^{\Omega\setminus D_n} p=0$$

Then $p\in P(\Omega)$.

Proof. Let $p=p_1+p_2+p_3$ where $p_1\in P(\Omega)$, $p_2=G_\lambda$ for a suitable measure μ on $X^*_{u^*}$ and $p_3\in S$ is such that $p_3 \wedge G_\mu =0$ for any positive measure μ on $X^*_{u^*}$ with $G_\mu \in S$. We show that $p_2=p_3=0$. Let μ be a positive measure on Ω such that ${}^*G_\mu$ is a u^*- continuous generator of S^* and $\mu(p)<\infty$. Let $(D_n)_n$ be an increasing sequence of co-natural open subsets of Ω given by

$$D_n:=\left[{}^*G_\mu > \frac{1}{m}\right]$$

Obviously $\Omega = \bigcup_m D_n$ and by hypothesis we have

$$\bigwedge_m B^{\Omega\setminus U_n} p_3 \le \bigwedge_m B^{\Omega\setminus U_n} p=0.$$

On the other hand, from Theorem 2.5. we deduce

$$p_3=B^{\Omega\setminus U_n} p_3,\ p_3=\bigwedge_m B^{\Omega\setminus U_n} p_3=0$$

As for the element p_2 we consider an increasing sequence $(K_n)_n$ of compact subsets of $X^*_{u^*}\setminus\Omega$ (with respect to the co-natural topology) such that

$$p_2=G_\lambda=\bigvee_m G_{\lambda_m},\ \text{where } \lambda_m:=\lambda/_{K_m}$$

For any $m\in N$ we consider a decreasing sequence $(V_n)_n$ of co-natural open neighbourhoods of K_m such that

$$\overline{V}_{n+1}\subset V_n,\ (\forall)\ n\in N,\ \bigcap_m V_n=K_m$$

From hypothesis we have

$$\bigwedge_m B^{\Omega\cap \overline{V}_n} p_2 \le \bigwedge_m B^{\Omega\cap\overline{V}_n} p=0$$

Using now Theorem 2.1. we deduce for any $n\in N$,

$$^{*}G_{\lambda_m} = B^{\Omega\cap\overline{V}_n}\, {}^{*}G_{\lambda_m} \leq B^{\Omega\cap\overline{V}_n} p_2,$$

and therefore $^{*}G_{\lambda_m}=0$. The number $m\in N$ being arbitrary we get $p_2=0$.

Remark 4.5. We may ask if the above theorem still holds when we replace the sequences $(D_n)_n$ of co-natural open subsets of Ω by sequences of natural open subsets of Ω.

The example 3.4. shows that, generally the assertion is not true.

Indeed, let us consider the element ξ_0^+ of S defined on $\Omega=(-1,1)$ by

$$\xi_0^+(x) = \begin{cases} 0 \text{ if } x \;\; (-1,0] \\ 2(1-x) \text{ if } x\in(0,1) \end{cases}$$

and let $(D_n)_n$ be an increasing sequence of natural open subset of $(-1,1)$ such that $\bigcup_n D_n=(-1,1)$. Obviously for any $n\in N$ there exists a sufficiently large number $n'\in N$ such that $[0,1-\frac{1}{n}]\subset D_{n'}$ and therefore for any $K\geq n'$ we get

$$B^{\Omega\setminus D_K}\xi_0^+\leq \frac{2}{n}$$

Hence, we have

$$\bigwedge_n B^{\Omega\setminus D_n}\xi_0^+=0$$

but obviously $\xi_0^+\notin P(\Omega)$.

Theorem 4.6. For any $s\in S$ the following assertion are equivalent

1) $s\wedge p=0$, $(\forall)p\in P(\Omega)$

2) there exists a sequence $(s_k)_k$ in S such that $s=\sum_{k=1}^{\infty} s_k$ and such that for any $k\in N$ there exists an increasing sequence $(D_n)_n$ of co-natural open subsets of Ω with $\Omega=\bigcup_n D_n$ and with

$$B^{\Omega\setminus D_n}s_k=s_k \quad (\forall)n\in N.$$

3) there exists a sequence $(s_k)_k$ in S such that $s=\sum_{k=1}^{\infty} s_k$ and such that for any $k\in N$ there exists a sequence $(D_n)_n$ of co-fine open subsets of Ω with $\Omega=\bigcup D_n$ and with

$B^{\Omega \setminus D_n} s_k = s_k \quad (\forall)\ n \in N.$

Proof. Obviously 2)$\Rightarrow$3).

1)$\Rightarrow$2). Let $s_1, s_2 \in S$ be such that $s=s_1+s_2$, $s_1=G_\lambda$ for a suitable positive measure λ on $X^*_{u^*}$, $s_2 \wedge G_\mu = 0$ for any positive measure μ on $X^*_{u^*}$ with $G_\mu \in S$. Using Theorem 2.5 we deduce that taking a weak unit $t \in S^*$ such that $t(s_2) < \infty$ then for any $n \in N$ we have

$$B^{\Omega \setminus D_n} s_2 = B^{[t \leq \frac{1}{n}]} s_2 = s_2$$

where for any $n \in N$ we have denoted

$$D_n := [t > \frac{1}{n}].$$

Obviously $(D_n)_n$ is an increasing sequence of co-natural open subsets of Ω and $\Omega = \bigcup_n D_n$.

As for the element s_1, using the hypothesis, we deduce that there exists a sequence $(K_m)_m$ of compact (with respect to the co-natural topology) subsets of $X^*_{u^*} \setminus \Omega$ such that

$$s_1 = \sum_m G_{\lambda_m}, \text{ where } \lambda_m := \lambda / K_m$$

For any $m \in N$ we choose a sequence $(V_n)_n$ of co-natural open neighbourhoods of K_m such that

$$\overline{V_{n+1}} \subset V_n \quad (\forall)\ n \in N, \quad \bigcap_n V_n = K_m.$$

Obviusly the sequence $(D_n)_n$ defined by

$$D_n := \Omega \setminus \overline{V_n}$$

is an increasing sequence of co-natural open subsets of Ω with $\Omega = \bigcup_n D_n$.

Moreover, using Theorem 2.1., we have

$$B^{\Omega \setminus D_n} G_{\lambda_m} = B^{\Omega \cap \overline{V}_n} G_{\lambda_m} = G_{\lambda_m}$$

for any $n \in N$.

3)$\Rightarrow$1) Let $p \in S$ be such that there exists a covering $(D_n)_{n \in N}$ of Ω such that, for any $n \in N$, D_n is a co-fine open subset of Ω such that

$$B^{\Omega \setminus D_n} p = p.$$

We decompose p as a sum $s=s_1+s_2$ where $s_1 \in P(\Omega)$ and $s_2 \wedge q = 0$ for any $q \in P(\Omega)$.

we show that $p_1=0$. Let $\lambda \in \mathcal{M}_S(\Omega)$ be such that

$s_1=G_\lambda$.

Since $G_\lambda = s_1 \preceq p$ we have, for any $n \in N$,

$$B^{\Omega \setminus D_n} G_\lambda = G_\lambda$$

On the other hand using corollary 2.3. and the preceding relation we get

$$\lambda(D_n)=0 \quad (\forall)\, n \in N$$

and therefore $\lambda=0$, $s_1=0$.

From now on we suppose that (S,Ω) satisfies the following axiom:

(G-P) For any increasing sequence of co-natural open subsets $(D_n)_n$ of Ω such that $\bigcup D_n=\Omega$ and for any $p \in S_0$ we have

$$\bigwedge B^{\Omega \setminus D_n} p=0.$$

Remark 4.7. We may ask if in the preceding axiom we can replace the sequences $(D_n)_n$ of co-natural open subsets of Ω by sequences of natural (Resp. fine of co-fine) open subsets of Ω.

Generally the answer is negative.

a) Let us consider the H-cone S of all positive increasing and lower semicontinuous real functions on (-1.1). Its dual S^* may be identified with the cone of all positive decreasing and lower semicontinuous real functions an $(-1,1)$ such that if $f \in S$ and $g \in S^*$ we have

$$g(f)=\int f d\mu_g$$

where μ_g is the Lebesque-Stieltges positive measure on $(-1,1)$ associated with g. In this example the natural and co-natural topologies on $\Omega=(-1,1)$ coincide with the usual topology.

Taking $p \in S$

$$p(x)=\begin{cases} 0 & \text{if } x \in (-1,-\frac{1}{2}] \\ 2x+1 & \text{if } x \in (-\frac{1}{2}, 0] \\ 1 & \text{if } x > 1 \end{cases}$$

$D_n=(-1,0] \cup (1/n,1)$

we have $p \in S_0$, D_n is fin and co-fine open,

$D_n \subset D_{n+1}$, $(\forall) n \in N$,

$\bigcup_n D_n=(-1,1)$

and

$\bigwedge_n B^{(-1,1)\setminus D_n} p=1$ on $(o,1)$

On the other hand we see that (S,Ω) and (S^*,Ω) satisfy axiom G-P.

b) We consider the standard H-cone on $(-1,1)$ given in Exemple 3-4. Using corollary 4.3. is easy to see that the (S^*,Ω) satisfies axiom G.P. We show that (S,Ω) does not satisfy axiom G-P. Indeed taking $q \in S_0$, $q \neq 0$ and for any $n \in N$

$D_n=(-1,0] \cup (\frac{1}{n},1)$

we have: D_n is co-natural open subset of $\Omega=(-1.1)$, $(D_n)_n$ is increasing

$\bigcup_n D_n=\Omega$

and

$\bigwedge_n B^{X\setminus D_n} q=q(0)(1-x)$ on $(o,1)$

Theorem 4.8. Suppose that (S,Ω) satisfy axiom G-P. Then for any $p \in S$ the following assertions are equivalent :

1) $p \in P(\Omega)$

2) For any increasing sequence $(D_n)_n$ of co-natural open subsets of Ω such that $\bigcup_n D_n=\Omega$ we have

$\bigwedge_n B^{\Omega\setminus D_n} p=0$

3) For any sequence $(D_n)_n$ of co-natural open subsets of Ω such that $\overline{D_n} \subset D_{n+1}$ for any $n \in N$, we have

$\bigwedge_n B^{\Omega\setminus D_n} p=0.$

Proof. The assertion follows from Lemma 4.1. and Theorem 4.4.

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