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ABSTRACT

We discuss the problem of the existence of periodic and stationary solutions of affine stochastic differential equations. We prove that under a controllability condition the system has a periodic solution if and only if the linear part is eyponentially stable in mean square.

It is also shown that the controllability assumption is necessary for the existence of a "unique" weakly periodic solution with nondegenerate covariance.

1. NOTATIONS AND PRELIMINARIES

The following notation will be used throughout this paper. R^n is the real n-dimensional space. $B(R^n)$ is the σ-algebra of Borel sets of R^n. If A is a matrix (or a vector) A^* is the transposed.

$H>0 (H\geq 0)$ means that H is a positive (semi) definite matrix. I is the identity matrix in R^n; $\lambda(A)$ is the spectrum of the matrix A and $\rho(A)$ is the spectral radius of A.

Throughout this paper $\{\Omega,F,P\}$ is a given probability field; the argument $\omega\in\Omega$ will not be written. By Ex we denote the mean value of the random variable (random vector) x.

If x is a random vector, by cov(x,x) we denote the covariance of x; $cov(x,x)=E(x-Ex)(x^*-Ex^*)$. An n-dimensional random vector x is said to be Gaussian if there exist $a\in R^n$ and $H\geq 0$ such that $Ee^{iu^*x}=e^{iu^*a-1/2u^*Hu}$ for all $u\in R^n$ $(i=\sqrt{-1})$. If in the above equality H is a positive definite matrix, we say that x is a nondegenerate Gaussian random vector.

An n-dimensional stochastic process x(t), $t\geq 0$ is said to be a θ-periodic process if for all $t_1,\dots,t_m$ and all $A_1\in B(R^n),\dots,A_m\in B(R^n)$, $m\geq 1$, we have

$$P\{x(t_1+\theta)\in A_1,\dots,x(t_m+\theta)\in A_m\}=P\{x(t_1)\in A_1,\dots,x(t_m)\in A_m\}$$

If the above equality holds for all t_i, A_i and all $\theta\geq 0$ then we say that x(t), $t\geq 0$ is a stationary process. The process x(t) is weakly θ-periodic if $E|x(t)|<\infty$ and $Ex(t+\theta)=Ex(t)$, $Ex(t+\theta)x^*(t+\theta)=Ex(t)x^*(t)$ for all $t\geq 0$.

2. LIAPUNOV EQUATIONS. STABILITY AND CONTROLLABILITY

In this preliminary section we shall prove some results concerning Liapunov type equations in the space $\mathcal{H}$ of all nxn symmetric matrices.

A linear operator $T:\mathcal{H}\to\mathcal{H}$ is positive if $H\geq 0$ implies $T(H)\geq 0$.

Lemma 1

Let T be a linear positive operator and $G\geq 0$. Assume that there exists $n_o\geq 1$ such that $\sum_{i=0}^{n_o-1} T^i(G)>0$. If the Liapunov equation

(1) $H-T(H)=G$

<u>has a positive semidefinite solution H then</u>

$$\lim_{i\to\infty} ||T^i||=0 \quad \underline{\text{and}} \quad H>0,\ H=\sum_{i=0}^{\infty} T^i(G)$$

<u>Proof</u>

Let H be a positive semidefinite solution of (1). We have

(2) $$H-T^i(H)=\sum_{j=0}^{i-1} T^j(G)\ ,\quad i\geq 1$$

Since T is a linear positive operator and $\sum_{i=0}^{n_o-1} T^i(G)>0$ we get

(3) $$\delta_1 I\leq \sum_{j=0}^{n_o-1} T^j(G)\leq H,\qquad \delta_1>0$$

Using again (2) we obtain

$$T^{n_o}(H)\leq H-\delta I=(1-\delta)H$$

with $0<\delta<1$

$$T^{2n_o}(H)\leq(1-\delta)T^{n_o}(H)\leq(1-\delta)^2 H$$

$$\delta_2 T^{jn_o}(I)\leq T^{jn_o}(H)\leq(1-\delta)^j H$$

Therefore

$$\delta_2 |T^{jn_o}(I)|\leq(1-\delta)^j |H|$$

Since T^i is a linear positive operator we have

$$||T^i||=|T^i(I)|$$

and from the above inequality it follows that

$$\lim_{i\to\infty} ||T^i||=0$$

The last assertion follows directly from (2).

We remark that if $\lim_{i\to\infty}||T^i||=0$ and if there exists $n_o\geq 1$ such that $\sum_{j=0}^{n_o-1} T^j(G)>0$ then (1) has a unique positive definite solution, namely $H=\sum_{j=0}^{\infty} T^j(G)$.

Now, we prove

Lemma 2

Let $T:H\to H$ be a linear positive operator and $G\geq 0$. If the equation (1) has a unique positive definite solution then $\lim_{i\to\infty}||T^i||=0$ and there exists $n_o\geq 1$ such that $\sum_{j=0}^{n_o-1} T^j(G)$ is a positive definite matrix.

Proof

Let H be the unique positive definite solution of (1). As in the proof of Lemma 1 (see (3)) it follows that the series $\sum_{j=0}^{\infty} T^j(G)$ is convergent. A simple calculation shows that $H_1=\sum_{j=0}^{\infty} T^j(G)$ is a positive semidefinite solution of (1). From (3) it follows that $H\geq H_1$.

Since $H+(H-H_1)$ is a positive definite solution of (1) we get $H_1=H$; thus $\sum_{i=0}^{\infty} T^i(G)>0$. Hence there exists $n_o\geq 1$ such that $\sum_{i=0}^{n_o-1} T^i(G)>0$. According to Lemma 1 the proof is complete.

We apply Lemma 1 and Lemma 2 to the well known Liapunov equations.

Application 1

Let A be an nxn matrix and $G \geq 0$.

Applying Lemmas 1 and 2 for the operator $T:\mathcal{H}\rightarrow\mathcal{H}$ defined by $T(H)=AHA^*$ we can conclude that:

a) if the Liapunov equation

(4) $H-AHA^*=G$

has a positive semidefinite solution H and if the pair (A,G) is completely controllable then $\rho(A)<1$ and $H>0$, $H=\sum_{i=0}^{\infty} A^i G(A^*)^i$.

b) if the equation (4) has a unique positive definite solution then $\rho(A)<1$ and (A,G) is completely controllable.

Application 2

Consider the Liapunov equation

(5) $AH+HA^*=-G$

It is easy to prove that H verifies (5) if and only if

$$e^{At}He^{A^*t}-H=-\int_0^t e^{As}Ge^{A^*s}ds \text{ for all } t\geq 0$$

Let t_o be a positive number. Since the pair (A,G) is completely controllable if and only if $\int_0^{t_o} e^{At}Ge^{A^*t}dt$ is a positive definite matrix we can use the result in Application 1 to conclude that:

a) if (5) has a solution $H\geq 0$ and if (A,G) is completely controllable then A is a stable matrix (i.e. $\max \mathrm{Re}\lambda(A)<0$) and (5) has a unique positive definite solution, namely $H=\int_0^{\infty} e^{At}Ge^{A^*t}dt$;

b) If the equation (5) has a unique positive definite solution then A is a stable matrix and (A,G) is completely controllable.

3. PERIODIC SOLUTIONS OF A CLASS OF AFFINE DIFFERENTIAL EQUATIONS IN THE SPACE $\mathcal{H}$

Let us consider the following equation in the space $\mathcal{H}$

$$\frac{dM(t)}{dt}=A(t)M(t)+M(t)A^*(t)+\sum_{j=1}^{m} B_j(t)M(t)B_j^*(t)+G(t) \tag{6}$$

where $A(t)$, $B_j(t)$ and $G(t)$ are nxn θ-periodic continuous matrices. In addition we suppose that $G(t)\geq 0$ for all $t\geq 0$.

By $M(t,s,H)$, $H\in\mathcal{H}$ we denote the solution of (6) which verifies $M(s,s,H)=H$. We shall use the notation

$$M_o(t)=M(t,0,0)$$

Let $C(t,s)$ be the fundamental matrix associated with the system $\frac{dx}{dt}=A(t)x$.

It is easy to verify that

$$M(t,s,H)=C(t,s)HC^*(t,s)+\sum_{j=1}^{m}\int_s^t C(t,u)B_j(u)M(u,s,H)B_j^*(u)C^*(t,u)du+ \tag{7}$$
$$+\int_s^t C(t,u)G(u)C^*(t,u)du.$$

Remark 1

Using the method of succesive approximations for Volterra equations one sees easily that if $H\geq 0$ then $M(t,s,H)\geq 0$ for all $t\geq s$ and thus according (7) we have $M(t,s,H)>0$ for all $t\geq s$ if $H>0$.

Consider now the following linear equation in the space $\mathcal{H}$.

$$\frac{dR(t)}{dt}=A(t)R(t)+R(t)A^*(t)+\sum_{j=1}^{m} B_j(t)R(t)B_j^*(t) \tag{8}$$

Denote by $R(t,s,H)$, $H\in\mathcal{H}$ the solution of (8) with $R(s,s,H)=H$. Define the following linear operators

$$T(t,s):\mathcal{H}\to\mathcal{H}\ ,\quad T(t,s)(H)=R(t,s,H)$$

and $S=T(\theta,0)$.

It is easy to verify that

$$(9)\quad T(t+\theta,s+\theta)=T(t,s),\ \ T(t+\theta,0)=T(t,0)S,\ \ T(t,s)=T(t,u)T(u,s)$$

for all t,s and u

$$(10)\quad M(t,s,H)=T(t,s)(H)+\int_s^t T(t,u)(G(u))du$$

$$(11)\quad M_o(t)=\int_o^t T(t,u)(G(u))du$$

From Remark 1 it follows that if $H\geq 0$ $(H>0)$ then $R(t,s,H)\geq 0$ $(R(t,s,H)>0)$ for all $t\geq s$.

Thus $T(t,s)$, $t\geq s$ are linear positive operators.

Remark 2

It is easy to prove that

$$M_o(j\theta)=\sum_{i=0}^{j-1} S^i(M_o(\theta))\ \text{ for all } j\geq 1$$

Remark 3

Using (9) one proves easily that the zero solution of (8) is exponentially stable if and only if

$$\lim_{i\to\infty} ||S^i||=0$$

Remark 4

It is easy to prove that $M(t,0,H)$, $t\geq 0$ is a θ-periodic solution of (6) if and only if $M(\theta,0,H)=H$, and thus by (10) we conclude that $M(t,0,H)$, $t\geq 0$ is a θ-periodic solution of (6) if and only if H verifies

(12) $H-S(H)=M_o(\theta)$

Thus, according to Remark 3 we can conclude that if the zero solution of (8) is exponentially stable then the equation (6) has a unique θ-periodic positive semidefinite solution.

From Remarks 1-4 it also follows that if the zero solution of (8) is exponentially stable and if there exists $n_o\geq 1$ such that $M_o(n_o\theta)>0$ then there exists a unique θ-periodic positive definite solution of (6), namely $M(t,0,H_o)$ with

$$H_o=\sum_{i=0}^{\infty} S^i(M_o(\theta))$$

Proposition 1

Assume that the equation (6) has a θ-periodic positive semidefinite solution $M(t,0,H)$, $t\geq 0$ and there exists $n_o\geq 1$ such that $M_o(n_o\theta)$ is positive definite. Then the zero solution of (8) is exponentially stable and $H>0$, $H=\sum_{i=0}^{\infty} S^i(M_o(\theta))$.

Proof

Let $M(t,0,H)$, $H\geq 0$ be a θ-periodic solution of (6). Then H verifies (12). According to Remmark 2 and applying Lemma 1 for $T=S$, $G=M_o(\theta)$ we can conclude than $\lim_{i\to\infty} ||S^i||=0$ and $H>0$, $H=\sum_{i=0}^{\infty} S^i(M_o(\theta))$.

Proposition 2

If the equation (6) has a unique θ-periodic positive definite solution then the zero solution of (8) is exponentially stable and there exists $n_o\geq 1$ such that

$M_o(n_o\theta)>0$

Proof

Suppose that the equation (6) has a unique θ-periodic solution $M(t,0,H)$, $t\geq 0$ with $H>0$. From Remark 4 it follows that the equation (12), has a unique positive definite solution. Thus Proposition 2 follows directly from Lemma 2 and Remarks 2 and 3.

4. PERIODIC SOLUTIONS OF AFFINE STOCHASTIC DIFFERENTIAL EQUATIONS

Let us consider the system

$$(13)\quad dx(t)=(A(t)x(t)+f(t))dt+\sum_{j=1}^{m}(B_j(t)x(t)+h_j(t))dw_j(t)$$

where A,f,B_j,h_j are θ-periodic, and w_j are standard independent Wiener processes.

By $x(t,z)$, $t\geq 0$ we denote the solution of (13) with $x(0,z)=z$.

F_t, $t\geq 0$ will be the σ-algebra generated by

$$\{w(u)-w(s),\ 0\leq s\leq u\leq t\}$$

We consider only solutions $x(t,z)$ where z is independent of $\{F_t, t\geq 0\}$ and $E|z|^2<\infty$.

It is well known [1], [2] that $x(t,z)$ is a Markov process and the transition probability function $p(s,x,t,A)$ associated with system (13) satisfies $p(s+\theta,x,t+\theta,A)=$ $=p(s,x,t,A)$ for all $0\leq s<t$, $x\in R^n$, $A\in B(R^n)$; we shall call this property θ-periodicity.

Let $x(t)$, $t\geq 0$ be a solution of (13).

Let $m(t)=Ex(t)$ and $N(t)=cov(x(t),x(t))$.

It is easy to prove that

(14) $\frac{dN}{dt}=A(t)N(t)+N(t)A^*(t)+\sum_{j=1}^{m} B_j(t)N(t)B_j^*(t)+$

$$+\sum_{j=1}^{m}(h_j(t)+B_j(t)m(t))(h_j^*(t)+m^*(t)B_j^*(t))$$

This equation will be considered in the space H. We denote by $N_{m(.)}(t)$ the solution of (14) corresponding to the function m(t) and which verifies $N_{m(.)}(0)=0$.

Remark 5

From Remark (4) it follows that if $H\geq 0$ ($H>0$) then $N(t,0,H)\geq 0$ ($N(t,0,H)>0$) for all $t\geq 0$; thus, since $N(t,0,H_o)=cov(x(t,z),x(t,z))$, ($H_o=cov(z,z)$; $m(t)=Ex(t,z)$) we conclude that if $cov(z,z)>0$ then $cov(x(t,z),x(t,z))>0$ for all $t\geq 0$.

We remark that if $x(t,z)$ is a solution of (13) and $m(t)=Ex(t,z)$ then $N_{m(.)}(t)=cov(x(t,Ez),x(t,Ez))$, $t\geq 0$.

Remark 6

It is obvious that the solution of (13) $x(t,z)$, $t\geq 0$ is weakly θ-periodic if and only if $Ex(\theta,z)=Ez$ and $N(t,0,H)$ (with $H=cov(z,z)$) is a θ-periodic positive semidefinite solution of (14) with $m(t)=Ex(t,z)$.

Consider now the following linear stochastic differential system

(15) $dy(t)=A(t)y(t)dt+\sum_{j=1}^{m} B_j(t)y(t)dw_j(t)$

In order to prove the main results in this section we shall use the next stability lemma

Lemma 3

The following two assertions are equivalent:

(i) The zero solution of (8) is exponentially stable.

(ii) The zero solution of (15) is exponentially stable in mean square.

Proof

Suppose (i) holds; hence there exist $\alpha>0$, $\beta>0$ such that

$$|R(t,s,H)|\leq\beta e^{-\alpha(t-s)}|H|, \quad t\geq s, \quad H\in H$$

Let $y(t,s,x)$, $t\geq s$ be the solution of (15) with $x(s,s,x)=$ $=x$; $x\in R^n$.

We can easily verify that $R(t)=Ey(t,s,x)y^*(t,s,x)$ is a solution of (8). Hence $Ey(t,s,x)y^*(t,s,x)=R(t,s,xx^*)$

We get

$$E|y(s,t,x)|^2\leq\gamma e^{-\alpha(t-s)}|x|^2, \quad t\geq s\geq 0, \quad x\in R^n$$

Thus (i) $\Rightarrow$ (ii).

Suppose that (ii) holds. We have

$$|R(t,s,xx^*)|\leq\gamma_1 e^{-\alpha(t-s)}|xx^*| \quad \text{for all } x\in R^n$$

Since every $H\in H$ can be written $H=\sum_{i=1}^{n} c_i e_i e_i^*$, where c_i are real numbers and $e_i\in R^n$, $e_i^* e_j=0$, $|e_i|=1$, $j\neq i$, we can conclude that (ii) $\Rightarrow$ (i).

Theorem 1

Suppose that the zero solution of (15) is exponentially stable in mean square. Then

(i) There exists a θ-periodic solution of system (13).

(ii) If $x_1(t)$ and $x_2(t)$ are two θ-periodic solutions of system (13) then they have the same set of joint distribution functions.

(iii) If $x(t,z)$ is a θ-periodic solution of (13) then $Ex(t,z)=m_o(t)$, $cov(z,z)=H_o$ where $m_o(t)$ is the unique θ-periodic solution of

(16) $$\frac{dm(t)}{dt}=A(t)m(t)+f(t)$$

and

$$H_o=\sum_{i=0}^{\infty} S^i(N_{m_o(.)}(\theta))$$

Proof

Let

$$V(t,x)=x^*P(t)x=\int_t^{\infty}E|y(s,t,x)|^2ds, \quad t\geq 0, \quad x\in R^n$$

Since the transition probability function associated with system (15) is θ-periodic one sees easily that $P(t)$ is θ-periodic.

From the hypothesis it follows that [3] that $P(t)\geq\gamma I$, $(\gamma>0)$ and

$$L_oV(t,x)=-|x|^2, \quad t\geq 0, \ x\in R^n$$

where L_o is the parabolic operator associated with system (15)

$$L_oV(t,x)=x^*\frac{dP(t)}{dt}x+x^*A^*(t)P(t)x+x^*P(t)A(t)x+$$

$$+\sum_{j=1}^{m} x^*B_j^*(t)P(t)B_j(t)x$$

Let L_1 be the parabolic operator associated with system (13). We have

$$L_1V(t,x)=L_oV(t,x)+2x^*P(t)f(t)+2x^*\sum_{j=1}^{m}B_j^*(t)P(t)h_j(t)+$$

$$+\sum_{j=1}^{m}h_j^*(t)P(t)h_j(t)$$

Obviously

$$\lim_{R\to\infty}\sup_{|x|\geq R}L_1V(t,x)=-\infty,\quad \lim_{R\to\infty}\inf_{|x|\geq R}V(t,x)=\infty$$

From Theorem 5.2 in [3] it follows that the system (13) has a θ-periodic solution. Thus the assertion (i) is proved. We prove now the assertion (iii).

Let $X(t,s)$, $t\geq s\geq 0$ be the (random) fundamental matrix associated with system (15). From the stability assumption of the theorem it follows that $E|X(t,s)|^2\leq \leq\beta e^{-\alpha(t-s)}$, ($\alpha,\beta$ being positive numbers).

We have

$$y(t,z)=X(t,0)z$$

Since $X(t,0)$ is measurable with respect to the σ-algebra F_t and z is independent of $\{F_t, t\geq 0\}$ it follows that z is independent of $X(t,0)$, $t\geq 0$ and thus we get

(17) $\lim_{t\to\infty} E|y(t,z)|^2=0$

Hence $\lim_{t\to\infty} E|y(t,x)|^2=0$ for all $x\in R^n$.

But $Ey(t,x)=C(t,0)x$, where $C(t,s)$ is the fundamental matrix associated with the matrix $A(.)$.

Therefore

(18) $\lim_{t\to\infty} |C(t,0)|=0$

Let $x(t,z)$, $t\geq 0$ be a θ-periodic solution of (13).

Since $Ex(t,z)$ is a θ-periodic solution of (16), from (18) we conclude that $Ex(t,z)=m_o(t)$, where $m_o(t)$ is the unique θ-periodic solution of (16).

Using Remarks 6, 4 and 3 and Lemma 3 we can conclude that $cov(z,z)=H_o=\sum_{i=0}^{\infty} S^i(M_{m_o(.)}(\theta))$.

We prove now the assertion (ii).

Let $x_1(t)$ and $x_2(t)$ be two θ-periodic solutions of (13).

Since $x_1(t)-x_2(t)$ is a solution of (15), from (17) it follows that $\lim_{t\to\infty} E|x_1(t)-x_2(t)|=0$.

Further, we have

$$|Ee^{iu^*x_1(t)}-Ee^{iu^*x_2(t)}|\le E|e^{iu^*x_1(t)}-e^{iu^*x_2(t)}|\le$$

$$\le|u|E|x_1(t)-x_2(t)|, \quad \text{for all } u\in R^n, \quad t\ge 0$$

Hence

$$\lim_{t\to\infty}(Ee^{iu^*x_1(t)}-Ee^{iu^*x_2(t)})=0, \quad u\in R^n, \quad (i=\sqrt{-1})$$

But $Ee^{iu^*x_1(t)}$, $Ee^{iu^*x_2(t)}$ are θ-periodic functions. Thus

$$Ee^{iu^*x_1(t)}=Ee^{iu^*x_2(t)} \quad \text{for all } u\in R^n \text{ and } t\ge 0$$

Therefore $x_1(t)$ and $x_2(t)$ have the same distribution functions, and using the Markov property of x_1 and x_2 we can conclude that the assertion (ii) holds, and thus the theorem is proved.

Remark 7

Under the assumption of Theorem 1, by using of Remark 2 we can conclude that if $x(t,z)$ is a θ-periodic solution of (13) then $cov(z,z)>0$ if and only if there exists $n_o\geq 1$ such that $N_{m_o(.)}(n_o\theta)>0$, where m_o is the unique θ-periodic solution of (16).

Proposition 3

If there exists a weakly θ-periodic solution $x(t,z)$ of system (13) with the property that there exists $n_o\geq 1$ such that $N_{m_o(.)}(n_o\theta)>0$ $(m_o(t)=Ex(t,z))$ then the zero solution of system (15) is exponentially stable in mean square and $cov(x(t,z),x(t,z))>0$, $t\geq 0$, $m_o(t)$ is the unique θ-periodic solution of (16);

$$cov(z,z)=\sum_{i=0}^{\infty} S^i(N_{m_o(.)}(\theta)).$$

Proof

From Remark 4, Proposition 1 and Lemma 3 it follows that the zero solution of system (15) is exponentially stable in mean square and $cov(z,z)=\sum_{i=0}^{\infty} S^i(N_{m_o(.)}(\theta))>0$, where $m_o(t)=Ex(t,z)$. But $m_o(t)$ is a θ-periodic solution of (16). Since the zero solution of (15) is exponentially stable in mean square it follows (see (18)) that $m_o(t)$ is the unique θ-periodic solution of (16). From Remark 5 it follows that $cov(x(t,z),x(t,z))>0$ for all $t\geq 0$, thus the proposition is proved.

Theorem 2

Suppose that the following two assertions hold:

(i) there exists a weakly θ-periodic solution $x(t,z_o)$ of (13) with $cov(z_o,z_o)>0$.

(ii) If $x(t,z_1)$ and $x(t,z_2)$ are two weakly θ-periodic solutions of (13) with $cov(z_1,z_2)>0$, $cov(z_2,z_2)>0$

then

$$Ez_1=Ez_2 \quad \text{and} \quad Ez_1z_1^*=Ez_2z_2^*$$

Then the zero solution of (13) is exponentially stable in mean square and there exists $n_o\geq 1$ such that $N_{m_o(.)}(n_o\theta)>0$ where $m_o(t)$ is the unique θ-periodic solution of (16).

Proof

Let $x(t,z_o)$ be a weakly θ-periodic solution of (13) with $cov(z_o,z_o)>0$. Let $H_o=cov(z_o,z_o)$ and $m_1(t)=$ $=Ex(t,z_o)$. Let $N_1(t)$, $t\geq 0$ be the solution of (14) corresponding to $m(t)=m_1(t)$ and which verifies $N_1(0)=H_o$. From Remark 6 it follows that N_1 is θ-periodic.

We prove that N_1 is the unique θ-periodic positive definite solution of (14), (with $m(t)=m_1(t)$). Indeed, let $N_2(t)$ be an another θ-periodic positive definite solution of this equation.

Let z_1 be a random vector independent of $\{F_t,t\geq 0\}$ and such that $Ez_1=Ez_o$, $cov(z_1,z_1)=H_1=N_2(0)$.

It is easy to prove that $Ex(t,z_1)=Ex(t,z_o)$, $t\geq 0$; thus $Ex(t,z_1)$ is θ-periodic. Hence by Remark 6, $x(t,z_1)$ is a weakly θ-periodic solution of (13). According to (ii) we conclude that $H_1=H_o$; hence by Proposition 2 and Lemma 3 we deduce that the zero solution of system (15) is exponentially stable in mean square and there exists $n_o\geq 1$ such that $N_{m_1(.)}(n_o\theta)>0$. As in the proof of Proposition 3, we get that $m_1(t)=m_o(t)$ where $m_o(t)$ is the unique θ-periodic solution of (16). The theorem is proved. The next result follows directly from Theorem 2.

Proposition 4

If $f(t)=0$, $t\geq 0$, under the assumption of Theorem 2 it follows that the zero solution of system (15) is exponentially stable in mean square and there exists $n_o\geq 1$ such that $N_o(n_o\theta)>0$ where $N_o(t)$ is the solution of (6) with $G(t)=\sum_j h_j(t)h_j^*(t)$ and $N_o(0)=0$.

5. PERIODIC SOLUTIONS OF A PARTICULAR CLASS OF AFFINE STOCHASTIC DIFFERENTIAL EQUATIONS

Consider the following system

$$(19)\quad dx(t)=(A(t)x(t)+f(t))dt+\sum_{j=1}^{m} h_j(t)dw_j(t)$$

where A, f, h_j are θ-periodic.

Throughout this section, the following notation will be used

$C(t,s)$ is the fundamental matrix associated with $A(.)$,

$U(\theta)=C(\theta,0)$ $m(\theta)=\int_o^\theta C(\theta,s)f(s),ds$,

$F(t)=(h_1(t),\ldots,h_m(t))$, $G(\theta)=\int_o^\theta C(\theta,s)F(s)F^*(s)C^*(\theta,s)ds$

Since the transition probability function associated with (19) is θ-periodic and the solution $x(t,z)$ of (19) is a Markov process, using the Markov property one proves easily (see [3,p.98]) that $x(t,z)$ is θ-periodic if and only if

$$Ee^{iu^*x(\theta,z)}=Ee^{iu^*z} \quad \text{for all } u\in R^n \quad (i=\sqrt{-1})$$

In this section we use the following elementary lemma which can be proved easily using Ito's formula.

Lemma 4

Let L(.) be an nxm deterministic continuous matrix. We have

$$Ee^{iu^*\int_0^t L(s)dw(s)} = e^{-1/2u^*\int_0^t L(s)L^*(s)ds\, u} \quad u \in R^n, \quad (i=\sqrt{-1})$$

$(w(t)=(w_1(t),\ldots,w_m(t))$

Theorem 3

The system (19) has a θ-periodic solution if and only if there exist $b \in R^n$ and $H \geq 0$ such that

(20) $(I-U(\theta))b=m(\theta)$

(21) $U(\theta)HU^*(\theta)-H=-G(\theta)$

If the equation (20) has a solution b and if the equation (21) has a solution $H \geq 0$ $(H>0)$ then there exists a θ-periodic solution $x(t,z)$ of (19) with the property that for each $t \geq 0$ $x(t,z)$ is a Gaussian (nondegenerate Gaussian) random vector.

Proof

Let $x(t,z)$, $t \geq 0$ be a θ-periodic solution of (19). We can write

$$x(t,z)=C(t,0)z+\int_0^t C(t,s)f(s)ds+C(t,0)\int_0^t C(0,s)F(s)dw(s)$$

Since z is independent of $\{F_t, t \geq 0\}$, applying Lemma 4 we get

(22) $Ee^{iu^*x(t,z)}=$

$$=Ee^{iu^*C(t,0)z}e^{iu^*\int_0^t C(t,s)f(s)ds}e^{iu^*\int_0^t C(t,s)F(s)F^*(s)C^*(t,s)ds\, u}$$

Hence

$$Ee^{iu^*x(\theta,z)}=Ee^{iu^*U(\theta)z}e^{iu^*m(\theta)-1/2u^*G(\theta)u}\quad , \ u\in R^n$$

But $x(t,z)$ is θ-periodic. Then

$$(23)\quad Ee^{iu^*z}=Ee^{iu^*U(\theta)z}e^{iu^*m(\theta)-1/2u^*G(\theta)u},\ \text{for all } u\in R^n$$

Let us denote by $g_1(u)$ and $g_2(u)$ the left hand side and respectively the right hand side in (23).

The relation $\frac{\partial g_1}{\partial u}\Big|_{u=0}=\frac{\partial g_2}{\partial u}\Big|_{u=0}$ implies (20) with $b=Ez$; from $\frac{\partial^2 g_1}{\partial u^2}\Big|_{u=0}=\frac{\partial^2 g_2}{\partial u^2}\Big|_{u=0}$ one obtains (21) where H is the covariance of z.

Suppose, now that $b\in R^n$ verifies (20) and $H\geq 0$ verifies (21).

Let z be a Gaussian random vector such that $Ez=b$, $cov(z,z)=H$ and z is independent of $\{F_t, t\geq 0\}$. Therefore

$$(24)\quad Ee^{iu^*z}=e^{iu^*b-1/2u^*Hu}\quad ,\quad u\in R^n$$

By a simple calculation one may show that z verifies (23). Thus $x(t,z)$ is a θ-periodic solution of (19). The last assertion follows directly from (24) and (22).

In the case of system (19) the equation (14) becomes

$$(25)\quad \frac{dN(t)}{dt}=A(t)N(t)+N(t)A^*(t)+F(t)F^*(t)$$

From Remarks 6 and 4 and from the proof of Theorem 3 it follows that the next proposition holds.

<u>Proposition 5</u>

<u>We have</u>

(i) $x(t,z)$ *is a weakly* θ*-periodic solution of* (19) *if and only if* $Ez=b$ *verifies* (20) *and* $H=cov(z,z)$ *verifies* (21).

(ii) *If* $x(t,z)$ *is a weakly* θ*-periodic solution of* (19) *and if* z *is a Gaussian random vector then* $x(t,z)$ *is a* θ*-periodic solution of* (19).

We denote by $K(t)$ the solution of (25) with $K(0)=0$; ($K(t) = cov(x(t,0),x(t,0))$, $x(t,0)$ being the solution of (19) with $x(0)=0$). Since $K(t)=\int_0^t C(t,s)F(s)F^*(s)C^*(t,s)ds$ we get by a simple calculation that $K(j\theta)=$

$$=\sum_{i=0}^{j-1} (U(\theta))^i G(\theta)(U^*(\theta))^i, \quad j\geq 1.$$

Thus, by Proposition 3 we can conclude that the following result holds.

Theorem 4

If the system (19) *has a weakly* θ*-periodic solution* $x(t,z)$ *and if the pair* $(U(\theta),G(\theta))$ *is completely controllable then* $\rho(U(\theta))<1$ *and* $Ez=(I-U(\theta))^{-1}m(\theta)$,

$$cov(z,z)=\sum_{i=0}^{\infty} U^i(\theta)G(\theta)(U^*(\theta))^i.$$

From the proofs of Theorems 2 and 3 it follows that the following theorem holds

Theorem 5

Suppose that the following two assertions hold:

(i) *There exists a* θ*-periodic solution* $x(t,z_o)$ *of* (19) *with* $cov(z_o,z_o)>0$.

(ii) *If* $x(t,z_1)$ *and* $x(t,z_2)$ *are two* θ*-periodic solutions of* (19) *with* $cov(z_1,z_1)>0$, $cov(z_2,z_2)>0$ *then* $Ez_1=Ez_2$ *and* $Ez_1z_1^*=Ez_2z_2^*$.

Then $\rho(U(\theta))<1$ *and the pair* $(U(\theta),G(\theta))$ *is completely controllable.*

Remark 8

It is easy to prove that $(U(\theta),G(\theta))$ is completely controllable if and only if the following system

$$\frac{dx}{dt}=A(t)x(t)+F(t)u(t)$$

is controllable, i.e. for every $x\in R^n$, $x\neq 0$ there exists a piecewise continuous function $u:[0,n\theta]\to R^m$ such that $x_u(n\theta,0)=x$ (n is the dimension of the system)

The next result follows directly from Theorems 1 and 3.

Corollary 1

We have:

(i) If $\rho(U(\theta))<1$ then every θ-periodic solution $x(t,z)$ of (19) has the property that for each $t\geq 0$, $x(t,z)$ is a Gaussian random vector.

(ii) If $\rho(U(\theta))<1$ and if $(U(\theta),G(\theta))$ is completely controllable then every θ-periodic solution $x(t,z)$ of (19) has the property that for each $t\geq 0$, $x(t,z)$ is a nondegenerate Gaussian random vector.

6. INVARIANT PROBABILITY MEASURES

In this section we suppose that the stochastic differential equations considered in the preceding sections have constant coefficients and we shall discuss the problem of the existence of a stationary solution.

Let us consider the following stochastic differential equations with constant coefficients

$$(26)\quad dx=(Ax+f)dt+\sum_{j=1}^{m}(B_jx+h_j)dw_j(t)$$

$$(27)\quad dy=Aydt+\sum_{j=1}^{m}B_jydw_j(t)$$

(28) $dx=(Ax+f)dt+\sum_j h_j dw_j(t)$

It is known [2] that the transition probability function p associated with system (26) is stationary, i.e. $p(s,x,t,A)=p(0,x,t-s,A)$. We can prove easily that the solution $x(t,z)$ of (26) is a stationary process (stationary solution) if and only if

$$\mu(A)=\int p(0,x,t,A)\mu(dx) \text{ for all } t>0 \text{ and } A\in B(R^n)$$

$$(\mu(A)=P\{z\in A\})$$

A probability measure μ on $B(R^n)$ which has the above property is said to be an invariant probability measure of the system (26).

We consider only invariant probability measures which have second moments.

Let $L:H \to H$ be the linear operator defined by

$$L(H)=AH+HA^*+\sum_{j=1}^{m} B_j HB_j^*$$

From Lemma 3 it follows that the operator L is stable (i.e. $\lim_{t\to\infty} e^{Lt}=0$) if and only if the zero solution of (27) is exponentially stable in mean square.

For every $a\in R^n$ we consider the Liapunov equation

(29) $L(H)=-Q(a)$

where $Q(a)=\sum_{j=1}^{m}(h_j+B_j a)(h_j^*+a^*B_j^*)$

Denote

$$M_a(t)=\int_0^t e^{Ls}(Q(a))ds$$

It is obvious that if L is stable then the equation (29) has a unique positive semidefinite solution H, namely $H=\int_0^\infty e^{Lt}(Q(a))dt$.

It is easy to verify that a symmetric matrix H verifies (29) if and only if

$$e^{Lt}(H)-H=-\int_0^t e^{Ls}(Q(a))ds \quad \text{for all} \quad t>0$$

Thus, the next corollary follows directly from Lemmas 1 and 2.

Corollary 2

(i) If the equation (29) has a positive semidefinite solution $H(a)$ and if there exists $t_o>0$ such that $M_a(t_o)>0$ then the operator L is stable and $H(a)>0$, $H(a)=\int_0^\infty e^{Ls}(Q(a))dt$.

(ii) If the equation (29) has a unique positive definite solution then L is stable and there exists $t_o>0$ such that $M_a(t_o)>0$.

Assertion (i) was proved, in a different way, by Kleinman [4].

Using the same reasoning as in the proof of results concerning periodic solutions we can conclude that the following propositions hold

Proposition 6

The system (28) has an invariant probability measure if and only if there exist $b\in R^n$, $H\geq 0$ such that $Ab+f=0$, $AH+HA^*=-FF^*$.

If b and $H\geq 0$ $(H>0)$ are the solutions of the above equations then there exists a Gaussian (nondegenerate Gaussian) invariant probability measure of system (28).

Proposition 7

The system (28) admits a unique nondegenerate

Gaussian invariant probability measure if and only if $\max \mathrm{Re}\lambda(A)<0$ and (A,F) is completely controllable.

Proposition 8

If the zero solution of (27) is exponentially stable in mean square then there exists a unique invariant probability measure μ_o of system (26); μ_o has the properties:

$$\int x\mu_o(dx)=a_o \;, \quad \int (x-a_o)(x-a_o)^*\mu_o(dx)=\int_o^\infty e^{Lt}(Q(a_o))dt,$$

where $a_o=-A^{-1}f$.

Proposition 9

If the system (26) has an invariant probability measure μ_1 with the property that there exists $t_1>0$ such that $M_{a_1}(t_1)>0$ $(a_1=\int x\mu_1(dx))$ then the zero solution of (27) is exponentially stable in mean square and $a_1=-A^{-1}f$,

$$\int (x-a_1)(a-a_1)^*\mu_1(dx)=\int_o^\infty e^{Lt}(Q(a_1))dt$$

In the case $f=0$ Proposition 6 was proved by Zakai and Snyders [5]. Necessary and sufficient conditions for the existence of a unique nondegenerate Gaussian invariant probability measure of the system of the following type

$$dx=Axdt+\sum_{j=1}^{m} B_j x dw_j(t)+\sum_{j=1}^{r} h_j dv_j(t)$$

were given by Brockett [6].

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