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ON EXISTENCE AND BEHAVIOUR OF THE SOLUTION OF A QUASISTATIC ELASTIC-VISCO-PLASTIC PROBLEM

by

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0. Introduction

An initial and boundary value problem for materials with a constitutive equation of the form

$$\dot{T} = \mathcal{E}\dot{E} + F(T,E) \tag{0.1}$$

is considered. Such type of equations are used in order to describe the behaviour of real bodies like rubbers,metals,rocks and so on. Various results and mechanical interpretations concerning constitutive equations of the form (0.1) may be found for instance in the papers of Freudenthal and Geringer[7], Cristescu and Suluciu[2], Gurtin,Williams and Suliciu[9], Suliciu[18], Podio-Guidugli and Suliciu[17].

In particular cases, equation (0.1) reduces to some classical models used in elasticity and viscoplasticity ; thus ,for $F = 0$ (0.1) becomes the linear elastic model ; for $F(T,E) = \frac{1}{2\mu}(T-P_K T)$, (0.1) is an elastic - viscoplastic model ; for $\mathcal{E} = 0$ and $F(T,E) = \frac{1}{2\mu}(T-P_K T)$, (0.1) represents the rigid viscoplastic Bingham model (here K is the von Mises convex, P_K is the projector map on K and $\mu > 0$ is a viscosity coefficient). Existence results for mixte problems for these classical models may be found in the book of Duvaut and Lions [5] both in dynamic and quasistatic case. Howhever, in [5] ch.5 it is implied that the tractions at the boundary are constant in time.

Other results concerning problems for materials of the form (0.1) in which F does not depend on E may be found in the works of Suquet [19],[20] where $F(T) \in \partial\varphi(T)$ and in

Djaoua and Suquet [4] for viscoelastic Maxwell-Norton materials, where $F(T) = \lambda |T^D|^{p-2} T^D$, $\lambda > 0$, $T^D = T - (trT)\mathbb{1}$ and $p \geq 2$.

In this paper F depends both on T and E and it is assumed to be a Lipschitz function. No monotony properties of F are required, but only quasistatic case is considered. An existence and uniqueness result for a solution of the class C^1 defined on the time interval $[0, +\infty)$ is obtained (theorem 3.1).

For finite time intervals the continuous dependence of the solution upon initial and boundary data is given (theorem 4.1)

In the papers of Suliciu [18], Podio-Guidugli and Suliciu [17] it is assumed that there exists a strong monotone function G such that $F(T,E) = 0$ iff $T = G(E)$. Starting from this assumption in order to get a better insight on the model and on its connection to the elasticity, for viscoelastic materials we particularize (0.1) as

$$F(T,E) = -k(T - G(E)) \tag{0.2}$$

where $k > 0$ is a viscosity coefficient. In this case the assymptotic stability of every solution is obtained (corollary (4.2).

For periodic external data, the existence of a unique periodic solution is proved (theorem 5.1).

Using the energy function, Suliciu [18] and Podio-Guidugli and Suliciu [17] obtained for isolated bodies the following estimation :

$$\int_0^t \int_\Omega (T - G(E))^2 \leq \frac{C}{K} \tag{0.3}$$

for all $t > 0$ where $C > 0$.

The above inequality suggested us the comparation of the solution of viscoelastic problem with the solution of the elastic problem defined with the same external data and the constitutive equation

$$T = G(E) \quad . \tag{0.4}$$

Thus, for every fixed $t > 0$, the convergence of the solution of viscoelastic problem to the solution of elastic problem when $k \longrightarrow +\infty$ is proved (theorem 6.1).

Finnally it is proved that the solution of elastic problem can characterize in some cases the large time behaviour of the solution of viscoelastic problem (theorem 7.1).

1. Problem statement

Let Ω be a bounded domain in R^n $(n=1,2,3)$ with a smooth boundary $\Gamma = \partial\Omega$ and let Γ_1 be a open subset of Γ and $\Gamma_2 = \Gamma - \overline{\Gamma}_1$. We suppose $\text{mes}\,\Gamma_1 > 0$. Let us consider the following mixt problem :

Find the displacement function $u : R_+ \times \Omega \longrightarrow R^n$ and the stress function $T : R_+ \times \Omega \longrightarrow \mathcal{S}$ such that

$$\text{div}\, T(t) + b(t) = 0 \quad , \tag{1.1}$$

$$E\, u(t) = \frac{1}{2}(\nabla u(t) + \nabla^T u(t)) \quad , \tag{1.2}$$

$$\dot{T}(t) = \mathcal{E}\, E\dot{u}(t) + F(T(t), Eu(t)) \quad , \tag{1.3}$$

in Ω ,

(1.4) $u(t)\big|_{\Gamma_1} = g(t)$,

(1.5) $T(t)\nu\big|_{\Gamma_2} = f(t)$,

for all $t > 0$ and

(1.6) $u(0) = u_0$,

(1.7) $T(0) = T_0$, in Ω ,

where $\mathcal{S}$ is the set of second order symmetric tensors on R^n, ν is the exterior unit normal at Γ . The equations (1.1) are the Cauchy's equilibrium equations in which $b : R_+ \times \Omega \longrightarrow R^n$ is the given body force and (1.2) defines the strain tensor of small deformations. (1.3) represents a rate-type viscoelastic or viscoplastic constitutive equation in which $\mathcal{E}$ is a forth order tensor and $F : \Omega \times \mathcal{S} \times \mathcal{S} \longrightarrow \mathcal{S}$ is a constitutive function. The functions u_0 and T_0 are the initial data and f , g are the given boundary data .

2. Notations and preliminaries

We denote by $\cdot$ the inner product on the spaces R^n , $\mathcal{S}$ and by $|\cdot|$ the euclidian norms on these spaces. The following notations are used :

$$L = \left\{ T = (T_{ij}) \mid T_{ij} = T_{ji} \in L^2(\Omega) , i,j = \overline{1,n} \right\} ,$$

$$L_u = \left\{ u = (u_i) \mid u_i \in L^2(\Omega) , i = \overline{1,n} \right\} ,$$

$$L_d = \left\{ T \in L \mid \operatorname{div} T \in L_u \right\} ,$$

$$H = \{ u = (u_i) \mid u_i \in H^1(\Omega) \quad , \quad i = \overline{1,n} \} \quad .$$

The spaces L, L_u, L_d, H are Hilbert spaces with respect to the cannonical inner products given by :

(2.1) $$(T,E) = \int_\Omega T \cdot E \, dx \quad ,$$

(2.2) $$((u,v)) = \int_\Omega u \cdot v \, dx \quad ,$$

(2.3) $$(T,E)_d = (T,E) + ((\operatorname{div} T, \operatorname{div} E)) \, ,$$

(2.4) $$(u,v)_H = ((u,v)) + (\nabla u, \nabla v) \; .$$

The norms induced by (2.1)-(2.4) will be denoted by $\| \cdot \|, ||| \cdot |||, \| \; \|_d, \| \; \|_H$ respectively .

Let $\gamma_0 : H \longrightarrow H_\Gamma$ be the trace map (see for instance Nečas [45] chap. II) where we denote by H_Γ the space $(H^{1/2}(\Gamma))^n$ and its norm by $\| \; \|_\Gamma$. Let V_1 be the subspace of H given by

$$V_1 = \{ u \in H \mid \gamma_0(u) = 0 \quad \text{on } \Gamma_1 \} \quad ,$$

and let H_1 be the subspace of H defined by

$$H_1 = \gamma_0(V_1) = \{ \xi \in H_\Gamma \mid \xi = 0 \quad \text{on } \Gamma_1 \} \quad .$$

The operator $E : H \longrightarrow L$ given by

(2.5) $$Eu = \frac{1}{2} (\nabla u + \nabla^T u)$$

is linear and continuous. Moreover, since $\operatorname{mes} \Gamma_1 > 0$, the Korn's inequality holds :

(2.6) $$\| Eu \| \geq C \| u \|_H \qquad \text{for all } u \in V_1$$

where $C > 0$ is a positive constant which depends only on Ω.

and Γ_1 (see for instance Hlavacek and Nečas [11], ch. 6).

Everywhere in this paper $C, \overline{C}, C_i$, $i \in \mathbb{N}$ will represent strictely positive generic constants which depend on $\mathcal{E}$, F , Ω , Γ_1 , Γ_2 and do not depend on time and on input data .

If $T \in L_d$ then there exists $\gamma_\gamma T \in H'_\Gamma$ (where $(H'_\Gamma , \|\cdot\|_{\Gamma})$ is the strong dual of H_Γ) such that

$$(2.7) \qquad < \gamma_\gamma T, \; \gamma_0 v > = (T, E\, v) + ((\operatorname{div} T, v))$$

for all $v \in H$,

$$(2.8) \qquad \| \gamma_\gamma T \|_{\Gamma} \leq C \; \| T \|_d$$

(see for instance Léné [12]). By $T\nu|_{\Gamma_2}$ we shall understand the element of H'_1 (the dual of H_1) which is the restriction of $\gamma_\gamma T$ on H_1. We shall denote by $\|\cdot\|_1$ the norm on H'_1 .

Let us denote by V_2 the following subspace of L_d :

$$V_2 = \left\{ T \in L_d \mid \operatorname{div} T = 0 , \quad T\nu|_{\Gamma_2} = 0 \right\} .$$

As it follows from Geymonat and Suquet [8], $E(V_1)$ is the orthogonal complement of V_2 in L . Hence ,

$$(2.9) \qquad (T, Ev) = 0 \quad \text{for all } v \in V_1 , \; T \in V_2 .$$

Let X be one of the above Hilbert spaces and let us define the following spaces

$$C^0(R_+, X) = \left\{ z : R_+ \longrightarrow X \mid z \text{ is continuous} \right\} ,$$

$$C^1(R_+, X) = \left\{ z : R_+ \longrightarrow X \mid \text{there exists } \dot{z} \in C^0(R_+, X) \text{ the derivative of } z \right\} ,$$

where $R_+ = [0, +\infty)$. In a similar way the spaces $C^i(0, T, X)$,

$i = 0,1$ can be defined and the norms on these spaces are given by

$$\|z\|_{\mathcal{T},X,0} = \max_{t\in[0,\mathcal{T}]} \|z(t)\|_X \ , \quad \|z\|_{\mathcal{T},X,1} = \|z\|_{\mathcal{T},X,0} + \|\dot{z}\|_{\mathcal{T},X,0} \ .$$

3. An existence and uniqueness result

The following hypotheses are made :

$\mathcal{E}$ is symmetric and positively definite, i.e.

$$(3.1)\begin{cases} \text{(a)} \quad |\mathcal{E}_{ijkh}(x)| \leqslant Q \quad \text{for all } i,j,k,h = \overline{1,n},\ x\in\Omega, \\ \text{(b)} \quad \mathcal{E}(x)T\cdot E = T\cdot\mathcal{E}(x)E \quad \text{for all } T,E \in \mathcal{S},\ x\in\Omega, \\ \text{(c)} \quad \text{there exists a strictly positive constant } d \text{ such} \\ \qquad \text{that for all } T\in\mathcal{S},\ x\in\Omega \text{ we have } \mathcal{E}(x)T\cdot T \geqslant \\ \qquad \geqslant d\,|T|^2 . \end{cases}$$

$$(3.2)\begin{cases} \text{(a)} \quad F \text{ is a Lipschitz function i.e. there exists } L>0 \\ \qquad \text{such that} \\ \qquad |F(x,T_1,E_1)-F(x,T_2,E_2)| \leqslant L(|T_1-T_2| + |E_1-E_2|) \\ \qquad \text{for all } T_i,E_i \in \mathcal{S},\ i=1,2,\ x\in\Omega, \\ \text{(b)} \quad F(x,0,0) = 0 \quad \text{for all } x\in\Omega . \end{cases}$$

$$(3.3)\begin{cases} \text{(a)} \quad b\in C^1(R_+,L_u) \ , \quad f\in C^1(R_+,H_1') \ , \\ \text{(b)} \quad \text{there exists } h\in C^1(R_+,H_\Gamma) \text{ such that } h = g \text{ on } \Gamma_1 . \end{cases}$$

$$(3.4) \qquad u_0\in H \ , \quad T_0\in L_d \ ;$$

The initial condition fit with the boundary data , i.e. :

$$(3.5)\begin{cases}(a) & \operatorname{div} T_o + b(0) = 0 ,\\ (b) & T_o\nu|_{\Gamma_2} = f(0) ,\\ (c) & u_o|_{\Gamma_1} = g(0) .\end{cases}$$

The main result of this section is given by :

Theorem 3.1. Suppose that the hypothesis (3.1)-(3.5) are fulfiled. Then there exists a unique solution $u \in C^1(R_+,H)$, $T \in C^1(R_+,L_d)$ of the problem (1.1)-(1.7).

Remark 3.1. Let us observe that if the problem (1.1)-(1.7) has a solution (u,T) such that $u \in C^1(R_+,H)$, $T \in C^1(R_+,L_d)$ then the hypothesis (3.3)-(3.5) are fulfiled. Indeed , if we put $h = \gamma_o(u)$, from the continuity of γ_o we get $(3.3)_b$. Since $T \in C^1(R_+,L_d)$, from (1.1) we get $b \in C^1(R_+,L_u)$ and from (1.5), (2.8) and $\|T\nu|_{\Gamma_2}\|_{\Gamma}\leq\|T\|_{\Gamma'}$ we obtain $(3.3)_a$. Taking the limit in (1.1),(1.4) and (1.5) when $t \longrightarrow 0$ we get (3.5).

Remark 3.2. Let us consider $K \subset \mathcal{S}$ a convex closed set, $0 \in K$, $P_K : \mathcal{S} \longrightarrow K$ the projector on K and $\mu > 0$. If we put $F(T,E) = \frac{1}{2\mu}(T-P_K T)$ then (3.2) holds and problem (1.1)-(1.7) describe a quasistatic process for elastic-visco-plastic bodies. For f constant in time, an existence and uniqueness result for this problem was given by Duvaut and Lions [5] ,ch.5 ,p.240 . However, the technique used in [5] cannot be applied in the case when F has no monotony properties and depends also on E.

In order to prove theorem 3.1. we need some preliminary results.

Lemma 3.1. Let (3.1),(3.3) hold. Then there exists a unique couple of functions $\tilde{u} \in C^1(R_+;H)$, $\tilde{T} \in C^1(R_+;L_d)$ such that

$$(3.6) \qquad \operatorname{div} \tilde{T}(t) + b(t) = 0 \quad ;$$

$$(3.7) \qquad \tilde{T}(t) = \mathscr{E}\, E\tilde{u}(t) \quad ;$$

$$(3.8) \qquad \tilde{u}(t)|_{\Gamma_1} = g(t) \quad ;$$

$$(3.9) \qquad \tilde{T}(t)\nu|_{\Gamma_2} = f(t) \quad ;$$

for all $t \in R_+$. Moreover, we have

$$(3.10) \qquad \dot{\tilde{T}}(t) = \mathscr{E}\, E\, \dot{\tilde{u}}(t) \quad ; \text{ for all } t \in R_+ \; ;$$

and if we denote by $\tilde{u}_i$, $\tilde{T}_i$, $i=1,2$ the solution of (3.6)-(3.9) for the data b_i, f_i, g_i , the following inequalities hold :

$$(3.11) \qquad \|\tilde{u}_1(t)-\tilde{u}_2(t)\|_H + \|\tilde{T}_1(t)-\tilde{T}_2(t)\|_d \leq$$
$$\leq C\left[|||b_1(t)-b_2(t)||| + \|f_1(t)-f_2(t)\|_1 + \|h_1(t)-h_2(t)\|_\Gamma\right];$$

$$(3.12) \qquad \|\dot{\tilde{u}}_1(t)-\dot{\tilde{u}}_2(t)\|_H + \|\dot{\tilde{T}}_1(t)-\dot{\tilde{T}}_2(t)\|_d \leq$$
$$\leq C\left[|||\dot{b}_1(t)-\dot{b}_2(t)||| + \|\dot{f}_1(t)-\dot{f}_2(t)\|_1 + \|\dot{h}_1(t)-\dot{h}_2(t)\|_\Gamma\right],$$

for all $t \in R_+$ (the constant C depend only on Ω, Γ_1, Q and d).

Proof. The statement of the above lemma can be easily obtained using standard existence theorems for linear elasticity. ∎

Denoting by $\bar{u}_0 = u_0 - \tilde{u}(0)$; $\bar{T}_0 = T_0 - \tilde{T}(0)$, let us homogenize the boundary conditions of (1.4)-(1.5) by considering the following problem :

Find $\bar{u} : R_+ \times \Omega \longrightarrow R^n$, $\bar{T} : R_+ \times \Omega \longrightarrow \mathcal{S}$ such that

$$\text{div}\, \bar{T}(t) = 0 \; ; \tag{3.13}$$

$$\dot{\bar{T}}(t) = \mathcal{E}\, E\dot{\bar{u}}(t) + F(\bar{T}(t)+\tilde{T}(t), E\bar{u}(t)+E\tilde{u}(t)) \; ; \tag{3.14}$$

$$\bar{u}(t)|_{\Gamma_1} = 0 \; ; \tag{3.15}$$

$$\bar{T}(t)\nu|_{\Gamma_2} = 0 \; ; \tag{3.16}$$

for all $t > 0$ and

$$\bar{u}(0) = \bar{u}_0 \; ; \tag{3.17}$$

$$\bar{T}(0) = \bar{T}_0 \; . \tag{3.18}$$

The following lemma can be easily obtained :

<u>Lemma 3.2.</u> The pair (u,T) is a solution of (1.1)-(1.7) iff the pair $(\bar{u},\bar{T})$ defined by $\bar{u} = u-\tilde{u}$; $\bar{T} = T-\tilde{T}$ is a solution of (3.13)-(3.18) .

Let $V = V_1 \times V_2$ be the product space with the norm denoted by $\|\cdot\|_V$ which is given by the following inner product :

$$(x,y)_V = (\mathcal{E}\, Eu, Ev) + (\mathcal{E}^{-1}T, P) \; ; \tag{3.19}$$

for all $x \, ; \, y \in V$; $x = (u,T)$; $y = (v,P)$.

Using (3.1) and (2.6) we observe that $\|\cdot\|_V$ is equivalent with the natural norm on V . Let $A : R_+ \times V \longrightarrow V$ be the operator defined by Riesz representation theorem as follows :

$$(A(t,x),y)_V = -(F(T+\tilde{T}(t), Eu+E\tilde{u}(t)), Ev) + \tag{3.20}$$
$$+(\mathcal{E}^{-1}F(T+\tilde{T}(t), Eu+E\tilde{u}(t)), P) \; ;$$

for all $x,y \in V$; $x=(u,T)$; $y=(v,P)$; $t \in R_+$.

We denote by $x_0 = (\bar{u}_0, \bar{T}_0)$; and using (3.5) we get $x_0 \in V$.

<u>Lemma 3.3.</u> The pair $(u, T) \in C^1(R_+, H \times L_d)$ is a solution of (3.13)-(3.18) iff $x = (\bar{u}, \bar{T}) \in C^1(R_+, V)$ is a solution of the following Cauchy problem

(3.21) $$\dot{x}(t) = A(t, x(t)) \quad \text{for all } t > 0 \; ;$$

(3.22) $$x(0) = x_0 \; .$$

<u>Proof.</u> Let $y = (v, P) \in V$. Multiplying (3.14) by Ev ; after integrating on Ω and using (2.9) we get

(3.23) $$0 = (\mathcal{C}\, E\dot{\bar{u}}(t), Ev) + (F(\bar{T}(t) + \tilde{T}(t), E\bar{u}(t) + E\tilde{u}(t)), Ev) \; .$$

Multiplying (3.14) by $\mathcal{C}^{-1}$ at the left and by P at the right and integrating on Ω ; from (2.9) we obtain

(3.24) $$(\mathcal{C}^{-1}\dot{\bar{T}}(t), P) = (\mathcal{C}^{-1}F(\bar{T}(t) + \tilde{T}(t), E\bar{u}(t) + E\tilde{u}(t)), P) \; .$$

From (3.23), (3.24), (3.20) and (3.19) we get

(3.25) $$(\dot{x}(t), y)_V = (A(t, x(t)), y)_V \; ;$$

for all $y \in V$.

Conversely, let $x = (\bar{u}, \bar{T}) \in C^1(R_+, V)$ be a solution of (3.21)-(3.22). Let $B(t) \in L$ be given by

(3.26) $$B(t) = \dot{\bar{T}}(t) - \mathcal{C}(E\dot{\bar{u}}(t)) - F(\bar{T}(t) + \tilde{T}(t), E\bar{u}(t) + E\tilde{u}(t)) \; .$$

Taking $y = (v, 0)$ in (3.25) and using (2.9) we get

(3.27) $$(B(t), Ev) = 0 \; ; \quad \text{for all } v \in V_1 \; .$$

If we put $y = (0, P)$ in (3.25) and we use (2.9) we obtain

(3.28) $\qquad (\mathcal{E}^{-1}B(t),P) = 0 \quad$ for all $P \in V_2$.

Since the orthogonal complement of $E(V_1)$ in L is V_2 ; from (3.27) we get $B(t) \in V_2$; thus we may put $P = B(t)$ in (3.28) and from (3.1) we deduce $B(t) = 0$, for all $t > 0$; hence (3.9) holds. ■

Using lemma 3.3. the problem (3.13)-(3.18) was replaced by the Cauchy problem (3.21)-(3.22) in the Hilbert space V . In order to prove the existence and the uniqueness of the solution of (3.21)-(3.22) we recall the following result (see Lovelady and Martin [13] and Pavel and Ursescu [16]) :

<u>Lemma 3.4.</u> Let V be a Hilbert space and $A : R_+ \times V \longrightarrow V$ a continuous operator such that there exists $D > 0$ with

(3.29) $\qquad (A(t,x_1)-A(t,x_2),x_1-x_2)_V \leq D \, \|x_1-x_2\|_V^2$;

for all $t > 0$ and all $x_1,x_2 \in V$. Then, for all $x_0 \in V$, there exists a unique solution $x \in C^1(R_+,V)$ of the problem (3.21) - (3.22) .

<u>Lemma 3.5.</u> The operator A given by (3.20) is continuous and satisfies (3.29) .

<u>Proof.</u> Let $t_1,t_2 > 0$, $x_1=(u_1,T_1)$, $x_2 = (u_2,T_2) \in V$. For all $y = (v,P) \in V$ we have

$$|(A(t_1,x_1)-A(t_2,x_2),y)_V| \leq |(F(T_1+\tilde{T}(t_1),Eu_1+E\tilde{u}(t)),Ev) -$$
$$- F(T_2+\tilde{T}(t_2),Eu_2+E\tilde{u}(t_2)), Ev)| \quad +$$
$$+|(\mathcal{E}^{-1}F(T_1+\tilde{T}(t_1),Eu_1+E\tilde{u}(t_1))-\mathcal{E}^{-1}F(T_2+\tilde{T}(t_2),Eu_2+E\tilde{u}(t_2)),P)| \ .$$

Using (3.2) in the above inequality, after some estimations we get

$$(A(t_1,x_1)-A(t_2,x_2), y)_V \leq C\left(\|x_1-x_2\|_V+\|\tilde{u}(t_1)-\tilde{u}(t_2)\|_H + \right.$$
$$\left. + \|\tilde{T}(t_1)-\tilde{T}(t_2)\|_d\right)\|y\|_V .$$

Therefore,

$$\|A(t_1,x_1)-A(t_2,x_2)\|_V \leq \tag{3.30}$$
$$\leq C\left(\|x_1-x_2\|_V+\|\tilde{u}(t_1)-\tilde{u}(t_2)\|_H+\|\tilde{T}(t_1)-\tilde{T}(t_2)\|_d\right) .$$

Using (3.30) and lemma 3.1 , the continuity of A from $R_+ \times V$ into V follows. For $t_1 = t_2 = t > 0$, (3.30) becomes

$$\|A(t,x_1)-A(t,x_2)\|_V \leq C\|x_1-x_2\|_V \tag{3.31}$$

for all $x_1,x_2 \in V$; hence (3.29) holds . ■

Proof of theorem 3.1. follows from lemma 3.1-3.5. ■

Remark 3.3. Let us observe that (3.2) was used in order to obtain $F(T,E)\in L$, for all $T,E \in L$ and also in the proof of lemma 3.5. Howhever, (3.2) may be replaced by a weaker assumption , namely :

(3.2')
(a) F is a continuous function ;
(b) There exists $L_1,L_2 > 0$ such that $|F(T,E)|^2 \leq$ $\leq L_1 + L_2(|E|^2 + |T|^2)$ for all $T,E \in \mathcal{S}$;
(c) There exists $\bar{L}>0$ such that for all $E_i,T_i \in \mathcal{S}$; $i = 1,2$, the following inequality holds

$$-(F(T_1,E_1)-F)T_2,E_2))\cdot(E_1-E_2) +$$
$$+(\mathcal{E}^{-1}(F(T_1,E_1) - F(T_2,E_2)))\cdot(T_1-T_2) \leq$$
$$\leq \bar{L}(|E_1-E_2|^2 + |T_1-T_2|^2) .$$

Indeed, $(3.2')_b$ is a sufficient condition for $F(T,E) \in L$ for all $T,E \in L$; $(3.2')_a$ assures the continuity of A and from $(3.2')_c$ we get (3.29) . Hence, the statement of theorem 3.1. holds also in the hypothesis (3.1),(3.2')-(3.5) .

4. The continuous dependence of the solution upon the input data

In this section two solution of the problem (1.1)-(1.7) for two different input data $b_i, g_i, f_i, u_{oi}, T_{oi}$; $i=1,2$ are considered. An estimation of the difference of these solutions is given for finite time intervals which give the continuous dependence of the solution upon all input data (theorem 4.1.). In this way, the finite time stability of the solution is obtained (corollary 4.1.).

In order to get a better insight on the model and on its connections to the elasticity, the constitutive equation (1.3) is particularized taking $F(T,E) = -k(T-G(E))$, with G a strongly monotone function and $k>0$. In this case, the assymptotic stability of the solution is deduced (corollary 4.2.).

Theorem 4.1. Let (3.1)-(3.2) hold and let (u_i,T_i) be the solution of (1.1)-(1.7) for the data $b_i, f_i, g_i, u_{oi}, T_{oi}$; $i=1,2$ such that (3.3)-(3.5) hold. For all $\mathcal{T}>0$ there exists $C(\mathcal{T})>0$ (which generally depends on Ω, Γ_1, $\mathcal{T}$, Q, d, $\mathcal{L}$) such that

$$\|u_1-u_2\|_{T,H,j} + \|T_1-T_2\|_{T,L_d,j} \leq$$

$$\leq C(T)\Big(\|b_1-b_2\|_{T,L_u,j} + \|f_1-f_2\|_{T,H_1',j} + \tag{4.1}$$

$$+ \|h_1-h_2\|_{T,H_\Gamma,j} + \|u_{01}-u_{02}\|_H + \|T_{01}-T_{02}\|_d\Big);$$

for $j = 0,1$.

Remark 4.1. It may be usefull to observe that if we suppose in addition to the assumptions of theorem 4.1. that there exists $\bar{h} \in C^1(0,T;H_\Gamma)$ such that $\bar{h} = g_1-g_2$ on Γ_1 and $\bar{h} = 0$ on Γ_2, then one may choose $h_2 = h_1 - \bar{h}$. Hence $\|h_1-h_2\|_{T,H_\Gamma,j}$ in (4.1) can be replaced by $\|g_1-g_2\|_{T,H_\Gamma,j}$.

Corollary 4.1. Let the hypotheses of theorem 4.1. hold. If $b_1=b_2$, $f_1=f_2$, $g_1=g_2$, then

$$\|u_1-u_2\|_{T,H,j} + \|T_1-T_2\|_{T,L_d,j} \leq \tag{4.2}$$

$$\leq C(T)\left(\|u_{01}-u_{02}\|_H + \|T_{01}-T_{02}\|_d\right).$$

In the next we give the definition of stability, finite time stability and assymptotic stability for the problem (1.1)-(1.7) (following for instance Hahn[10], ch V). A solution (u,T) of the problem (1.1)-(1.7) will be called :

i) stable, if there exists $m : R_+ \longrightarrow R_+$ a continuous increassing function with $m(0) = 0$ such that

$$(4.3) \qquad \|u(t)-u_1(t)\|_H + \|T(t)-T_1(t)\|_d \leq$$

$$\leq m(\|u_o-u_{o1}\|_H + \|T_o-T_{o1}\|_d)$$

for all $t \in R_+$ and all (u_{o1}, T_{o1}) satisfying (3.5) ;

ii) <u>finite time stable</u> if (4.3) holds for a finite time interval ;

iii) <u>assymptotic stable</u> if there exist m as in i) and $n : R_+ \longrightarrow R_+$ a descreasing continuous function with $\lim_{t \to +\infty} n(t) = 0$ such that

$$\|u(t)-u_1(t)\|_H + \|T(t)-T_1(t)\|_d \leq m(\|u_o-u_{o1}\|_H + \|T_o-T_{o1}\|_d) n(t)$$

for all $t \in R_+$ and all (u_{o1}, T_{o1}) satisfying (3.5) where (u_1, T_1) is the solution of (1.1)-(1.7) for the data u_{o1}, T_{o1} .

<u>Remark 4.2.</u> From (4.2) we deduce the finite time stability of every solution of (1.1)-(1.7). Generally, stability does not hold (see remark 4.4.).

In order to prove theorem 4.1., the following lemma will be usefull :

<u>Lemma 4.1.</u> Let $\ell : [0,T] \longrightarrow R_+$ be a positive continuous function and $\alpha \in R$. If $\theta : [0,T] \longrightarrow R_+$ is a absolutely continuous positive function such that

$$\dot{\theta}(t) \leq 2\alpha\,\theta(t) + 2\sqrt{\theta(t)}\;\ell(t) \quad \text{a.e. } t \in [0,T]$$

then ;

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$$(4.4) \qquad \sqrt{\theta(t)} \leq e^{\alpha t}\sqrt{\theta(0)} + \int_0^t \ell(s) e^{\alpha(t-s)} ds \quad ;$$

for all $t \in [0,T]$.

<u>Proof of theorem 4.1.</u> Let $(\tilde{u}_i, \tilde{T}_i)$ $i = 1,2$ be the functions given by lemma 3.1. for the data f_i, b_i, g_i , $i=1,2$. We denote by $\bar{u}_i = u_i - \tilde{u}_i$, $\bar{T}_i = T_i - \tilde{T}_i$ and $x_i = (\bar{u}_i, \bar{T}_i) \in V$; $i=1,2$. Let A_i be given by (3.20) (replacing $\tilde{u}$, $\tilde{T}$ by $\tilde{u}_i$, $\tilde{T}_i$).

For all $y_i \in V$, $i=1,2$ and all $t \in R_+$; from (3.11), (3.2) it results that :

$$(4.5) \qquad \| A_1(t,y_1) - A_2(t,y_2) \|_V \leq$$

$$\leq C(\|y_1-y_2\|_V + ||| b_1(t)-b_2(t) ||| + \|f_1(t)-f_2(t)\|_1 + \|h_1(t)-h_2(t)\|_\Gamma)$$

Let $x_{oi} = (u_{oi} - \tilde{u}_i(0), T_{oi} - \tilde{T}_i(0)) \in V$. From lemma 3.3. we have

$$(4.6) \qquad \dot{x}_i(t) = A_i(t, x_i(t)) \qquad \text{for all} \quad t > 0 \; ,$$

$$(4.7) \qquad x_i(0) = x_{oi} \qquad ; \quad i=1,2 .$$

From (4.5)-(4.6) we get :

$$(\dot{x}_1(t)-\dot{x}_2(t), x_1(t)-x_2(t))_V = (A_1(t,x_1(t))-A_2(t,x_2(t)), x_1(t)-x_2(t))_V$$

$$\leq C \; \|x_1(t)-x_2(t)\|_V \Big(\|x_1(t)-x_2(t)\|_V + \|b_1-b_2\|_{T,L_u,0} + $$

$$+ \|f_1-f_2\|_{T,\mathring{H}_1,0} + \|h_1-h_2\|_{T,H_\Gamma,0} \Big) \quad ;$$

for all $t \in [0,\mathcal{T}]$. Taking $\theta(t) = \|x_1(t)-x_2(t)\|_V^2$ and having in mind that $\dot{\theta}(t) = 2(\dot{x}_1(t)-\dot{x}_2(t), x_1(t)-x_2(t))_V$; from lemma 4.1. and the above inequality we obtain

$$(4.8) \qquad \|x_1(t)-x_2(t)\|_V \leq e^{Ct} \left(\|x_{01}-x_{02}\|_V + \|b_1-b_2\|_{\mathcal{T},L_u,0} + \right.$$

$$\left. + \|f_1-f_2\|_{\mathcal{T},H_1,0} + \|h_1-h_2\|_{\mathcal{T},H_\Gamma,0} \right) ;$$

for all $t \in [0,\mathcal{T}]$.

From (3.19), (3.1) and (2.6) we get

$$\|x_1(t)-x_2(t)\|_V \geqslant C \Big(\|u_1(t)-u_2(t)\|_H + \|T_1(t)-T_2(t)\|_d -$$

$$- \|\tilde{u}_1(t)-\tilde{u}_2(t)\|_H - \|\tilde{T}_1(t)-\tilde{T}_2(t)\|_d \Big) ;$$

for all $t \in R_+$.

Using (4.8), the above inequality and (3.11) we deduce (4.1) for $j=0$ with $C(\mathcal{T}) = C(1+e^{C\mathcal{T}})$.

In order to obtain (4.1) for $j = 1$, let us observe that

$$\|\dot{x}_1(t)-\dot{x}_2(t)\|_V = \|A_1(t,x_1(t))-A_2(t,x_2(t))\|_V \leqslant$$

$$\leqslant C(\|x_1(t)-x_2(t)\|_V + |||b_1(t)-b_2(t)||| + \|f_1(t)-f_2(t)\|_1 +$$

$$+ \|h_1(t)-h_2(t)\|_\Gamma) ;$$

and from (4.8), (2.6) and (3.12) we get (4.1) for $j=1$. ∎

Further on we particularize the function F from (1.3) s

$$(4.9) \qquad F(T,E) = -k(T-G(E)) \quad ;$$

for all $T, E \in \mathcal{S}$ where $k > 0$ is a viscosity coefficient and $G : \Omega \times \mathcal{S} \rightarrow \mathcal{S}$. In order to satisfy (3.2) we suppose that:

G is a Lipshitz function i.e. there exists $\bar{L} > 0$ such that

$$(4.10) \begin{cases} |G(x,E_1) - G(x,E_2)| \leq \bar{L}\,|E_1-E_2| & \text{for all } x \in \Omega \text{ and} \\ E_1 \, ; \, E_2 \in \mathcal{S} \quad ; \\ \text{and} \\ G(x,0) = 0 \quad ; \text{ for all } x \in \Omega \, . \end{cases}$$

Moreover, we consider the following assumption :

G is a strongly monotone function , i.e. there exists a constant $a > 0$ such that

$$(4.11) \qquad (G(x,E_1)-G(x,E_2))\cdot(E_1-E_2) \geq a\,|E_1-E_2|^2$$

for all $x \in \Omega$ and $E_1 , E_2 \in \mathcal{S}$.

<u>Theorem 4.2.</u> Let (3.1), (4.9)-(4.11) hold and let (u_i, T_i) be two solutions of (1.1)-(1.7) for the data $b_i, f_i, g_i, u_{oi}, T_{oi}$, $i=1,2$ for which (3.3)-(3.5) hold. Then there exist two constants C ; $\bar{C} > 0$ (depending only on Ω, Γ_1 ; Q , d , $\bar{L}$ and a) such that for all $\mathcal{T} \geq 0$ we have

$$\|u_1(t)-u_2(t)\|_H + \|T_1(t)-T_2(t)\|_d \leq$$

(4.12) $$\leq \bar{C}\Big[(\|u_{o1}-u_{o2}\|_H + \|T_{o1}-T_{o2}\|_d)e^{-Ckt} +$$

$$+ \|b_1-b_2\|_{\mathcal{T},L_u,0} + \|f_1-f_2\|_{\mathcal{T},H_1,0} + \|h_1-h_2\|_{\mathcal{T},H_\Gamma,0} \Big] ;$$

for all $t \in [0,\mathcal{T}]$.

Remark 4.3. If in theorem 4.1. we suppose in addition that (4.9)-(4.11) hold then from (4.12) one can obtain (4.1) with $C(\mathcal{T})$ independent of T .

Corollary 4.2. Let hypotheses of theorem 4.2. hold. If $b_1 = b_2$, $f_1 = f_2$, $g_1 = g_2$ then

$$\|u_1(t)-u_2(t)\|_H + \|T_1(t)-T_2(t)\|_d \leq$$

(4.13)

$$\leq \bar{C}(\|u_{o1}-u_{o2}\|_H + \|T_{o1}-T_{o2}\|_d)e^{-Ckt} ,$$

for all $t \in R_+$.

Remark 4.4. From (4.13) we deduce the assymptotic stability of every solution of the problem (1.1)-(1.7),(4.9) .

If G is not a monotone function the stability generally cannot hold. Some one dimensional examples can be given with the property that there exists $M > 0$ such that for all $\varepsilon > 0$ there exists $\mathcal{T}$, (u_{o1},T_{o1}) , (u_{o2},T_{o2}) satisfying (3.5) with

$\|u_{o1}-u_{o2}\|_H + \|T_{o1}-T_{o2}\|_d < \varepsilon$ and $\|u_1(t)-u_2(t)\|_H + \|T_1(t)-T_2(t)\|_d$

$> M$; for all $t \geq T$.

In order to prove theorem 4.2., the following lemma is usefull :

Lemma 4.2. Let $T > 0$ and $\theta \in C^1([0,T],R)$ be a positive function and $\beta, \gamma \geq 0$; $\alpha > 0$ be constants. If

(4.14) $$\dot{\theta}(t) \leq -2\alpha\,\theta(t) + 2\beta\sqrt{\theta(t)} + 2\gamma \quad \text{for all } t \in [0,T]$$

then

(4.15) $$\sqrt{\theta(t)} \leq \sqrt{\theta(0)}\; e^{-\alpha t} + \frac{\beta + \sqrt{\beta^2 + 4\alpha\gamma}}{2\alpha}$$

for all $t \in [0,T]$.

Proof. Suppose $\gamma \neq 0$ (if $\gamma = 0$ lemma 4.1. can be used) ; let $\varepsilon > 0$ and let M_ε be the set defined by

$$M_\varepsilon = \left\{ t \in [0,T] \;\middle|\; \sqrt{\theta(t)} \geq \varepsilon + \sqrt{\theta(0)}\, e^{-\alpha t} + \frac{\beta + \sqrt{\beta^2 + 4\alpha\gamma}}{2\alpha} \right\}.$$

If M_ε is not empty ; since $0 \notin M_\varepsilon$ and M_ε is a closed set we get $0 < t_\varepsilon = \inf M_\varepsilon$ and $\sqrt{\theta(t_\varepsilon)} = \varepsilon + \sqrt{\theta(0)}\, e^{-\alpha t} + \frac{\beta + \sqrt{\beta^2 + 4\alpha\gamma}}{2}$. Using (4.14) and the above equality we get

$$\frac{d}{dt}\left(\sqrt{\theta(t)} - \sqrt{\theta(0)}\, e^{-\alpha t}\right)\Big|_{t=t_\varepsilon} \leq \frac{-\alpha\varepsilon^2}{\sqrt{\theta(t_\varepsilon)}} \quad ; \text{ for all}$$

$\varepsilon > 0$. Hence $M_\varepsilon = \emptyset$. ■

Proof of theorem 4.2. The same notations as in the proof of theorem 4.1. are used. From (3.14) and (4.9) we get

(4.16) $\dot{\overline{T}}_i(t) = \mathscr{E} E\dot{\tilde{u}}_i(t) - k\left[\overline{T}_i(t)+\tilde{T}_i(t)-G(E\,\overline{u}_i(t)+E\tilde{u}_i(t))\right]$

for all $t \in R_+$, $i=1,2$.

If we take the difference in (4.16) for $i=1$ and $i=2$ and we put $\overline{u} = \overline{u}_1-\overline{u}_2$, $\overline{T} = \overline{T}_1-\overline{T}_2$, $\tilde{T} = \tilde{T}_1-\tilde{T}_2$, $\tilde{u} = \tilde{u}_1-\tilde{u}_2$; after multiplying by $E\overline{u}(t)$ and integrating the result on Ω, from (2.9) we get

(4.17) $$(\mathscr{E} E\dot{\overline{u}}(t), E\overline{u}(t)) = k(\tilde{T}(t), E\overline{u}(t)) -$$

$$- k(G(Eu_1(t))-G(Eu_2(t)), E\overline{u}(t)) .$$

We denote $\theta(t) = (\mathscr{E} E\overline{u}(t), E\overline{u}(t))$ and $\beta(\mathcal{T}) = \|b_1-b_2\|_{\mathcal{T},L_u,0} + \|f_1-f_2\|_{\mathcal{T},H_1',0} + \|h_1-h_2\|_{\mathcal{T},H_\Gamma,0}$; then from (4.17), (4.10)-(4.11) and (3.11), after some algebra we obtain

(4.18) $$\dot{\theta}(t) \leq k(-2C_1\theta(t)+2C_2\beta(\mathcal{T})\sqrt{\theta(t)}+2C_2\beta^2(\mathcal{T}))$$

for all $t \in [0,\mathcal{T}]$ with $C_1, C_2 > 0$, $C_1 < 1$.

From (4.18) and lemma 4.2. it follows

(4.19) $$\sqrt{\theta(t)} \leq \sqrt{\theta(0)}\, e^{-kC_1 t} + \beta(\mathcal{T})\frac{C_2+\sqrt{C_1^2+4C_1C_2}}{2C_1}$$;

for all $t \in [0,\mathcal{T}]$. Using (4.19), (3.1) and (2.6) in (4.19) we get

(4.20) $$\|u_1(t)-u_2(t)\|_H \leq C_3\left(\|u_{01}-u_{02}\|_H\, e^{-kC_1 t}+\beta(\mathcal{T})\right) .$$

In a similar way, if we take the difference in (4.16) for $i=1$ and $i=2$ and we multiply by $\mathcal{C}^{-1}$ at the left and by $\overline{T}(t)$ at the right and integrate the result on Ω, using (2.9) we obtain

$$(\mathcal{C}^{-1}\dot{\overline{T}}(t),\overline{T}(t)) = -k(\mathcal{C}^{-1}\overline{T}(t),\overline{T}(t))-k(\mathcal{C}^{-1}\tilde{\overline{T}}(t),\overline{T}(t)) + \tag{4.21}$$
$$+ k(\mathcal{C}^{-1}\sigma(Tu_1(t))-\mathcal{C}^{-1}\sigma(Tu_2(t)),\overline{T}(t)) ,$$

for all $t\in R_+$.

Let $\theta(t) = (\mathcal{C}^{-1}\overline{T}(t),\overline{T}(t))$. From (4.21), (3.11) and (4.10) we obtain

$$\dot{\theta}(t) \leq -2k\,\theta(t) + 2kC_4(\beta(\mathcal{T})+ \|u_1(t)-u_2(t)\|_H)\sqrt{\theta(t)}$$

for all $t\in[0,\mathcal{T}]$. Using (4.20) in the above inequality we have

$$\dot{\theta}(t) \leq -2k\,\theta(t) + 2kC_5(\beta(\mathcal{T})+ \|u_{o1}-u_{o2}\|_H e^{-C_1kt}) \sqrt{\theta(t)}$$

for all $t\in[0,\mathcal{T}]$ and from lemma 4.1 we get

$$\sqrt{\theta(t)} \leq \sqrt{\theta(0)}\, e^{-kt}+ C_5\beta(\mathcal{T}) + \frac{C_5}{1-C_1}\cdot e^{-C_1kt}\|u_{o1}-u_{o2}\|_H . \tag{4.22}$$

Using (3.1), (4.22) and (3.11) we get

$$\|T_1(t)-T_2(t)\|_d \leq C_6(\|u_{o1}-u_{o2}\|_H+ \|T_{o1}-T_{o2}\|_d)e^{-C_1kt} + \tag{4.23}$$
$$+ C_6\beta(\mathcal{T}) \quad ; \quad \text{for all } t\in[0,\mathcal{T}].$$

From (4.20) and (4.23) we obtain (4.12). ∎

5. Periodic solutions

In this section we are interested in periodic solutions of the problem (1.1)-(1.7),(4.9). The main result of this section is :

Theorem 5.1. Let (3.1),(4.9)-(4.11) hold and let the data b,f,g satisfying (3.3) be periodic functions with the same period ω. Then, there exists a unique initial data (u_o,T_o) satisfying (3.4)-(3.5) such that the solution of (1.1)-(1.7) (4.9) is a periodic function with the same period ω.

Remark 5.1. In the hypotheses of theorem 5.1., using corrollary 4.2. we deduce that for all initial data the solution of the problem (1.1)-(1.7),(4.9) approaches to a unique periodic function when $t \longrightarrow +\infty$. In other words, if the external data are oscilating then the body will "begin" after a while to oscilate too.

Proof of theorem 5.1.

Let the family of sets $\{\mathscr{C}(t)\}_{t\in R_+}$ be given by

$$(5.1)\qquad \mathscr{C}(t) = \{ v \in H \mid v|_{\Gamma_1} = g(t) \} \times$$

$$\times \{ P \in L_d \mid \operatorname{div} P + b(t) = 0 , P\nu|_{\Gamma_2} = f(t) \} .$$

Let $\mathscr{L}(t,.) : \mathscr{C}(0) \longrightarrow \mathscr{C}(t)$ for all $t \in R_+$ be given by $\mathscr{L}(t,x_o) = x(t)$ where $x(t) = (u(t),T(t))$ is the solution of (1.1)-(1.7), (4.9) for the initial data $x_o = (u_o,T_o)$.

From (4.13) we have

$$(5.2) \qquad \|\mathcal{L}(t,x_1)-\mathcal{L}(t,x_2)\|_W \leq \bar{C}e^{-Ckt}\|x_1-x_2\|_W$$

for all $x_1, x_2 \in \mathcal{C}(0)$. The notation $W = H \times L_d$, $\|x\|_W = \|u\|_H + \|T\|_d$ for all $x = (u,T) \in W$ have been used.

Since b,f,g are periodic functions of period ω we obtain $\mathcal{C}(0) = \mathcal{C}(m\omega)$ for all $m \in N$.

Let $n_0 \in N$ be such that $\bar{C}e^{-Ckn_0} = \lambda < 1$. We deduce from (5.2) that $\mathcal{L}(n_0\omega, \cdot)$ is a contraction defined on $\mathcal{C}(0)$. Hence there exists a unique element $x^* = (u_0^*, T_0^*) \in \mathcal{C}(0)$ such that

$$(5.3) \qquad \mathcal{L}(n_0\omega, x^*) = x^* .$$

Since b,f,h, are periodic functions with the period $m\omega$, for all $m \in N$, using (5.3) we deduce

$$(5.4) \qquad \mathcal{L}(pn_0\omega+t, x^*) = \mathcal{L}(t,x^*) ,$$

for all $p \in N$, $t \in R_+$. The same arguments can be used to prove that there exists $x_1^* = (u_1^*, T_1^*) \in \mathcal{C}(0)$ such that

$$(5.5) \qquad \mathcal{L}(p(n_0+1)\omega+t, x_1^*) = \mathcal{L}(t,x_1^*) ,$$

for all $p \in N$, $t \in R_+$.

We put $p = m(n_0+1)$, $t = 0$ in (5.4) and $p = mn_0$, $t = 0$ in (5.5), and from (5.2) we get $\|x^*-x_1^*\|_W \leq$ $\leq \bar{C}e^{-Ckmn_0(n_0+1)}\|x^*-x_1^*\|_W$, for all $m \in N$. Hence $x^* = x_1^*$ and using (5.5) and (5.4) it results that $\mathcal{L}(t+\omega, x^*) = \mathcal{L}(t,x^*)$, for all $t \in R_+$.

■

6. Approach to elasticity

The purpose of this section is to prove the convergence when $k \longrightarrow +\infty$ of the solution $(u_k(t), T_k(t))$ of (1.1)-(1.7), (4.9) for all $t > 0$ to the solution of the following boundary value problem for an elastic body :

Find the displacement function $\hat{u} : R_+ \times \Omega \longrightarrow R^n$ and the stress function $\hat{T} : R_+ \times \Omega \longrightarrow R^n$ such that

(6.1) $\quad \operatorname{div} \hat{T}(t) + b(t) = 0$;

(6.2) $\quad E\hat{u}(t) = \frac{1}{2}(\nabla \hat{u}(t) + \nabla^T \hat{u}(t))$;

(6.3) $\quad \hat{T}(t) = G(E\hat{u}(t))$;

in Ω ;

(6.4) $\quad \hat{u}(t)|_{\Gamma_1} = g(t)$;

(6.5) $\quad \hat{T}(t)\nu|_{\Gamma_2} = f(t)$;

for all $t \in R_+$.

Lemma 6.1. Let us suppose that (3.3),(4.10),(4.11) hold. Then, the problem (6.1)-(6.5) has a unique solution $\hat{u} \in C^0(R_+, H)$, $\hat{T} \in C^0(R_+, L_d)$. Moreover, for all $\mathcal{T} > 0$, u , T are absolutely continuous functions on $[0, \mathcal{T}]$ and there exists $C > 0$ (which depends only on Ω, Γ_1, $\mathcal{Z}$ and a) such that

(6.6) $\quad \|\dot{\hat{u}}(t)\|_H + \|\dot{\hat{T}}(t)\|_d \leq C(\|\dot{f}(t)\|_1 + |||\dot{b}(t)||| + \|\dot{h}(t)\|_\Gamma)$ a.e. in R_+

Remark 6.1. The elastic problem (6.1)-(6.5) was considered by many authors with differents assumptions on the function G. See for instance Fichera[6], Duvaut and Lions[5], Dincă[3], Léné[12], Mazilu and Sburlan[14]. However we sketch here a proof of this lemma.

Proof of Lemma 6.1.

Let $t \in R_+$ be fixed and let $\bar{\bar{u}}(t) \in H$ be such that $\gamma_0(\bar{\bar{u}}(t)) = h(t)$. We see that $(\hat{u}(t), \hat{T}(t))$ is a solution for (6.1)-(6.5) iff $\hat{T}(t) = G(E\,\hat{u}(t))$ and $B_t(\hat{u}(t) - \bar{\bar{u}}(t)) = L_t$ where $B_t : V_1 \longrightarrow V_1'$, $L_t \in V_1'$ are given by

$$\langle\langle B_t u, v \rangle\rangle = (G(Eu + E\bar{\bar{u}}(t), Ev) \quad ;$$

$$L_t(v) = \langle f(t), \gamma_0(v) \rangle + ((b(t), v)) \quad ;$$

for all $u, v \in V_1$ (here $\langle\langle\ ,\ \rangle\rangle$ denotes the duality between V_1' and V_1 and $\langle\ ,\ \rangle$ denotes the duality between H_1' and H_1).

The existence of the solution $(\hat{u}(t), \hat{T}(t))$ is deduced from Browder's surjectivity theorem (see Browder[1]). From the strong monotony of B_t we get the uniqueness of the solution and, since in addition B_t is a Lipsitz operator, for all $t_1, t_2 \in R_+$ we get

$$\|\hat{u}(t_1) - \hat{u}(t_2)\|_H + \|\hat{T}(t_1) - \hat{T}(t_2)\|_d \leqslant$$

(6.7)

$$\leqslant C\big(|||b(t_1) - b(t_2)||| + \|f(t_1) - f(t_2)\|_1 + \|h(t_1) - h(t_2)\|_\Gamma \big) \ ;$$

We obtain from the continuity of b, f, h and the above inequality that $\hat{u} \in C^0(R_+, H)$, $\hat{T} \in C^0(R_+, L_d)$.

Let $\mathcal{T} > 0$. It follows by using (3.3) that b,f,h are absolutely continuous functions on $[0,\mathcal{T}]$ and it also results from (6.7) that $\hat{u}$ and $\hat{T}$ are absolutely continuous functions defined on $[0,\mathcal{T}]$. Hence $\hat{u}$ and $\hat{T}$ are almost everywhere derivable on $[0,\mathcal{T}]$ for all $\mathcal{T} > 0$. Therefore we deduce (6.6) from (6.7). ∎

The following lemma (which will also be useful in section 7) evaluate the difference between the solutions of (1.1)-(1.7), (4.9) and of (6.1)-(6.5) for the same data b,f,g.

Lemma 6.2. Let (3.1),(3.3), (4.11) hold and let (u,T) be the solution of (1.1)-(1.7),(4.9) and $(\hat{u},\hat{T})$ be the solution of (6.1)-(6.5). For all $t \in R_+$ we have

$$\text{(6.8)} \quad \begin{aligned} \|u(t)-\hat{u}(t)\|_H \leq \overline{C}\Big[& \|u_0-u(0)\|_H e^{-Ckt} + \\ & + \int_0^t \left(|||\dot{b}(s)||| + \|\dot{f}(s)\|_1 + \|\dot{h}(s)\|_\Gamma\right) e^{-Ck(t-s)} ds\Big], \end{aligned}$$

$$\text{(6.9)} \quad \begin{aligned} \|T(t)-\hat{T}(t)\|_d \leq \overline{C}\Big[& \|T_0-T(0)\|_d e^{-kt} + \\ & + \int_0^t \left(k\|u(s)-\hat{u}(s)\|_H + |||\dot{b}(s)||| + \|\dot{f}(s)\|_1 + \|\dot{h}(s)\|_\Gamma\right) e^{k(s-t)} ds\Big] \end{aligned}$$

where the strictely positive constants $\overline{C}$, $C > 0$, $C < 1$ depend only on Ω, Γ_1, Q, d, $\mathcal{Z}$ and a.

Proof. We denote $\overline{u} = u-\hat{u}$, $\overline{T} = T-\hat{T}$ and from (1.3),(4.9) and lemma 6.1 we get

$$\dot{\hat{T}}(t)+\dot{\bar{T}}(t) = \mathcal{E}\, E\dot{\hat{u}}(t) + \mathcal{E}\, E\dot{\bar{u}}(t) - \tag{6.10}$$
$$-k\left[\bar{T}(t)+G(E\hat{u}(t))-G(E\bar{u}(t)+E\hat{u}(t))\right]$$

a.e. $t\in R_+$;

Multiplying (6.10) by $E\bar{u}(t)$, after integration on Ω and use of (2.9) we get :

$$(\dot{\hat{T}}(t)-\mathcal{E}E\dot{\hat{u}}(t),E\bar{u}(t)) = (\mathcal{E}\, E\dot{\bar{u}}(t),E\bar{u}(t)) +$$
$$+ k[G(E\bar{u}(t)+E\hat{u}(t))-G(E\bar{u}(t)),E\bar{u}(t)] \quad ; \text{ a.e. } t\in R_+ \; .$$

Hence, from (6.6), (3.1) and (4.11) we get

$$(\mathcal{E}\, E\dot{\bar{u}}(t),E\bar{u}(t)) \leq -C_1 k(\mathcal{E}\, E\bar{u}(t),E\bar{u}(t)) +$$
$$+ C_2(|||\dot{b}(t)||| + \|\dot{f}(t)\|_1 + \|\dot{h}(t)\|_\Gamma)\cdot(\mathcal{E}E\bar{u}(t),E\bar{u}(t))^{1/2} \text{ a.e. } t\in R_+ .$$

We denote $\theta(t) = (\mathcal{E}\, E\bar{u}(t),E\bar{u}(t))$ and use lemma 4.1., (2.6) and (3.1) to get (6.8).

Applying $\mathcal{E}^{-1}$ to the left of (6.10) and taking the scalar product of it with $\bar{T}(t)$, integrating the result on Ω and using (2.9) we get :

$$(\mathcal{E}^{-1}\dot{\bar{T}}(t),\bar{T}(t)) = -k(\mathcal{E}^{-1}\bar{T}(t),\bar{T}(t))-(\mathcal{E}^{-1}\dot{\hat{T}}(t)-E\dot{\hat{u}}(t),\bar{T}(t)) +$$
$$+ k(\mathcal{E}^{-1}G(E\bar{u}(t)+E\hat{u}(t))-\mathcal{E}^{-1}G(E\hat{u}(t)),\ \bar{T}(t)) \; ; \text{ for all } t\in R_+ \text{ a.e.}$$

Using above the lemma 6.1., (4.10) and (2.6) we have

$$(\mathcal{E}^{-1}\dot{\bar{T}}(t),\bar{T}(t)) \leq -k(\mathcal{E}^{-1}\bar{T}(t),\bar{T}(t)) +$$
$$+ C_3(\mathcal{E}^{-1}\,\bar{T}(t),\bar{T}(t))^{1/2}\left[\, k\,\|\bar{u}(t)\|_H + |||\dot{b}(t)||| + \|\dot{f}(t)\|_1 + \|\dot{h}(t)\|_\Gamma\right] \; ,$$

for $t \in R_+$ a.e.

Using here again lemma 4.1 for $\theta(t) = (\mathcal{E}^{-1}\bar{T}(t), \bar{T}(t))$, we obtain (6.9).

■

Theorem 6.1. Suppose (3.1)-(3.5), (4.9)-(4.11) hold. Let (u_k, T_k) be the solution of the problem (1.1)-(1.7) for any $k>0$ and let $(\hat{u}, \hat{T})$ be the solution of (6.1)-(6.5). Then, for all $t>0$ we have $\|u_k(t)-\hat{u}(t)\|_H \longrightarrow 0$; $\|T_k(t)-T(t)\|_d \longrightarrow 0$ when $k \longrightarrow \infty$.

Remark 6.1. The behaviour of (u_k, T_k) when $k \longrightarrow +\infty$ was also studied (in the dynamical case) in the papers of Suliciu [18] and Podio-Guidugli and Suliciu [17] where $F(T,E) = -\mathcal{K}(T,E)(T-G(E))$ with $\mathcal{K}(T,E)P \cdot P \geq k\,|P|^2$ for all $P \in \mathcal{S}$, $k = \text{constant} > 0$.

For isolated bodies, assuming the existence and the smoothness of the solution and using the energy function, in [17], [18] is proved that $\|T_k(t)-G(E_k(t))\| \longrightarrow 0$ when $k \longrightarrow +\infty$, for all $t>0$. In [18] E_k represents the small strain tensor ($E_k = Eu_k$) and in [17] E_k represents the finite strain tensor.

Proof of theorem 6.1.

We get from (6.8), for $t>0$

$$\text{(6.11)} \quad \|u_k(t)-\hat{u}(t)\|_H \leq \bar{C}\Big[\|u_o-\hat{u}(0)\|_H e^{-Ckt} + \frac{1}{Ck}\big(\|\dot{b}\|_{t,L_u,0} + \|\dot{f}\|_{t,H_1',0} + \|\dot{h}\|_{t,H_\Gamma,0}\big)\Big];$$

From (6.9) and (6.11) we obtain

$$(6.12)\quad \begin{aligned} \|T_k(t)-\hat{T}(t)\|_d \le \bar{C}\Big[&\|T_o-\hat{T}(0)\|_d e^{-kt} + \\ &+ \frac{\bar{C}}{1-C} e^{-Ckt} \|u_o-\hat{u}(0)\|_H + \\ &+ \frac{C+1}{Ck}\left(|||\dot{b}|||_{t,L_u,0} + \|\dot{f}\|_{t,H_1,0} + \|\dot{h}\|_{t,H_\Gamma,0} \right)\Big] \end{aligned}$$

and from (6.11), (6.12) the theorem follows. ∎

7. Large time behaviour of the solution

In this section we consider the problem (1.1)-(1.7),(4.9) for a fixed $k>0$, and we study the behaviour of the solution when $t \to +\infty$. The main result is the following :

Theorem 7.1. Let (3.1),(3.3)-(3.5), (4.9)-(4.11) hold ; we denote by (u,T) the solution of (1.1)-(1.7) and by $(\hat{u},\hat{T})$ the solution of (6.1)-(6.5) . If $\lim_{t\to\infty} (|||\dot{b}(t)||| + \|\dot{h}(t)\|_\Gamma + \|\dot{f}(t)\|_1) = 0$ then

$$(7.1)\qquad \lim_{t\to\infty} (\|u(t)-\hat{u}(t)\|_H + \|T(t)-\hat{T}(t)\|_d) = 0 \quad .$$

Remark 7.1. Let us observe that for all $t \geq 0$ the functions $\hat{u}(t),\hat{T}(t)$ are uniquely determined by the data $b(t), f(t)$ and $g(t)$. From theorem 6.1. we get that if $|||\dot{b}(t)||| + \|\dot{f}(t)\|_1 + \|\dot{h}(t)\| \to 0$ when $t \to +\infty$ then after a large enough time the solution (u,T) will be "determined" only by the present values of b,f,g. Hence, in this case the initial data and the history of external data have "no influence" upon the large time behaviour of the solution.

Remark 7.2. If $\lim_{t\to+\infty} (|||\dot{b}(t)||| + \|\dot{f}(t)\|_1 + \|\dot{h}(t)\|_\Gamma) \neq 0$ the statement of theorem 6.1. cannot generally hold. For example, let b, f, g be periodic functions with the same period. Then $\hat{u}$, $\hat{T}$ are periodic functions and from theorem 5.1. we get that ~~there exists an initial data (u_0, T_0) for which the solution~~ (u, T) of (1.1)-(1.7), (4.9) is periodic. If we suppose that $\|u(t) - \hat{u}(t)\|_H \to 0$ and $\|T(t) - \hat{T}(t)\|_d \to 0$ when $t \to +\infty$ we get $u = \hat{u}$, $T = \hat{T}$ and from (1.3) we obtain $\dot{\hat{T}}(t) = \mathcal{G}(E\dot{\hat{u}}(t))$ for all $t \in R_+$; this equality is generaly false if $\dot{\hat{T}} \neq 0$, $\dot{\hat{u}} \neq 0$ and $G(E) \neq \mathcal{G}E$. Hence, if the external data are periodic then there exists a phase shift between the periodic solutions (u, T) and $(\hat{u}, \hat{T})$.

Corollary 7.1. Let the hyppotheses of theorem 7.1. hold. If in addition we suppose that there exist $\bar{b} \in L_u$, $\bar{f} \in \mathring{H}_1$ and $\bar{h} \in H_\Gamma$ such that

$$(7.2) \qquad \lim_{t\to\infty} (|||b(t) - \bar{b}||| + \|f(t) - \bar{f}\|_1 + \|h(t) - \bar{h}\|_\Gamma) = 0$$

and if we denote by $(\hat{\bar{u}}, \hat{\bar{T}})$ the solution of (6.1)-(6.5) for the data $\bar{b}$, $\bar{f}$ and $\bar{h}$, then

$$(7.3) \qquad \lim_{t\to\infty} (\|u(t) - \hat{\bar{u}}\|_H + \|T(t) - \hat{\bar{T}}\|_d) = 0 .$$

Proof. From the continuous dependence of the solution of (6.1)-(6.5) upon the data f, b, h and (7.2) we get $\lim_{t\to+\infty} (\|\hat{u}(t) - \hat{\bar{u}}\|_H + \|\hat{T}(t) - \hat{\bar{T}}\|_d) = 0$ and from (7.1) we deduce (7.3). ∎

Remark 7.3. Let the data b ; f ; h be constant in time and let $(\hat{u},\hat{T})$ be the solution of (1.1)-(1.7), (4.9) and (u,T) be the solution of (6.1)-(6.5). In this case the differential equation (3.21), (3.22) is an autonomous one and $(\hat{u},\hat{T})$ is a stationnary point of A. From theorem 4.2. we can obtain

$$(7.4)\qquad \begin{aligned}&\|u(t)-\hat{u}\|_H + \|T(t)-\hat{T}\|_d \leq \\ &\leq \bar{C}e^{-Ckt}(\|u_o-\hat{u}\|_H + \|T_o-\hat{T}\|_d)\end{aligned}$$

where C and $\bar{C}$ are defined in (6.8).

Indeed, we observe that $u_1(t) = \hat{u}$; $T_1(t) = \hat{T}$ for all $t \in R_+$ is a solution of (1.1)-(1.7), (4.9) for initial data $u_o = \hat{u}$; $T_o = \hat{T}$. Hence ; from (4.12) we can deduce (7.4) .

In order to give the proof of theorem 6.1., the following lemma is usefull :

Lemma 7.1. Let $r : R_+ \longrightarrow R_+$ be a continuous function such that $\lim\limits_{t\to+\infty} r(t) = 0$ and $p : R_+ \longrightarrow R_+$ given by

$$p(t) = \int_0^t r(s)\, e^{-C(t-s)}\, ds \qquad \text{with } C>0 . \quad \text{Then } \lim_{t\to\infty} p(t)= 0.$$

Proof of theorem 7.1.

The proof easely follows from (6.8)-(6.9) and lemma 7.1. ∎

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References

[1] BROWDER, F.E. ; Les problèmes non-linéaires, Les Presses de l'Université de Montreal, 1964.

[2] CRISTESCU, N. and SULICIU, I. ; Viscoplasticity, Martinus Nijhoff, The Netherlands, Ed. Tehnică, Bucharest 1982.

[3] DINCĂ, G. ; Sur la monotonie d'après Minty-Browder de l'opérateur de la théorie de plasticité, C.R. Acad. Sci., tome 269(1969), p.535-538.

[4] DJAOUA, M and SUQUET, P.; Évolution quasi-statique des milieu visco-plastique de Maxwell-Norton, Math. Meth. in the Appl. Sci. 6(1984) p.192-205.

[5] DUVAUT, G. and LIONS J.L.; Les inéquations en mécanique et en physique, Dunod, Paris, 1972.

[6] FICHERA, G., Existence theorems in elasticity, Handbuch der Physik, C. Truesdell ed., VI/a 2, Berlin, Springer, 1972.

[7] GEIRINGER,H. and FREUDENTHAL,A.M., The Mathematical Theories of the Inelastic Continuum,Handbuch der Physik, Springer-Verlag ,Berlin,1958.

[8] GEYMONAT,G. and SUQUET,P.,Fonctional Spaces for Norton-Hoff Materials,Preprint N° 84-6,Lab.de Méc Générale des Milieux Continus,Montpellier(1984).

[9] GURTIN,M.E.,WILLIAMS,W.O. and SULICIU,I.,On Rate-Type Constitutive Equations and the Energy of Viscoelastic and Viscoplastic Materials,Int.J.Solids Structures, 16(1884),p.607-617.

[10] HAHN, W.,Stability of Motion,Springer-Verlag,Berlin,1967.

[11] HLAVÁČEK,I.,and NEČAS,J.,Mathematical theory of elastic and elasto-plastic bodies :an introduction Els.Sci.Pub. Comp.,1981.

[12] LÉNÉ,F. Sur les materiaux élastiques à énergie de déformation non quadratique,J.Mécanique,13,3(1974)p.499-534.

[13] LOVELADY,D.L. and MARTIN R.H.jr.,A global existence theorem for a nonautonomous differential equations in Banach spaces,Proc.Amer.Math.Soc.,35(1972),p.445-449.

[14] MAZILU,P and SBURLAN,S.,Metode funcţionale în rezolvarea ecuaţiilor teoriei elasticităţii,Ed.Acad. R.S.R., Bucureşti,1973(in romanian).

[15] NEČAS,J.,Les Méthodes directes enthéorie des équations Elliptiques,Academia,Praha,1967.

[16] PAVEL,N and URSESCU,C.,Existence and uniqueness for some nonlinear functional equations in a Banach space,Anal. St.Univ.Iaşi,f.I.Mat,XX(1974)p.53-58.

[17] PODIO-GUIDUGLI,P.and SULICIU,I.,On Rate-Type Viscoelasticity and the Second Law of Thermodynamics,Int.J.Non-Linear Mechanics,19,6(1984)p.545-564.

[18] SULICIU,I.,Some energetic Properties of smooth solutions in rate-type Viscoelasticity, Int.J.Non-Linear Mechanics,19,6(1984)p.525-544.

[19] SUQUET,P.,Existence and regularity of solutions for plasticity problems,Variational Methods in Solid Mechanics, S.Nemat.Nasser Ed.,Pergamon Press,1980.

[20] SUQUET,P.,Sur les équations de la Plasticité :existence et régularité des solutions,J.Mécanique,20(1981)p.3-39.