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by

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1. In (/3/-/5/) we introduced and studied the normed almost linear space (nals), a concept which generalizes the concept of the normed linear space . Roughly speaking such a space satisfies some of the axioms of a linear space but the number of the axioms of the norm is increased . To support the idea that this concept is a good one, we introduced the "dual" space of a nals X which is also a nals. Here the functionals, are no longer linear, but "almost linear" , and when X is a normed linear space then the dual space defined by us is the usual dual space of X . A typical example of a nals is the set X of all nonempty, bounded, convex (and closed) subsets A of a normed linear space E , for the addition $A_1 + A_2 = \{ a_1+a_2 ; a_1 \in A_1 , a_2 \in A_2 \}$ (for the addition $\dot{+}$ defined by $A_1 \dot{+} A_2 = cl(A_1 + A_2)$), the element zero of X the set $\{ 0 \}$, the multiplication by scalars $\lambda A = \{ \lambda a ; a \in A \}$ and the norm $|||A||| = \sup_{a \in A} \| a \|$ (see /3/,/4/ for other examples).

The aim of this paper is to supply what we consider to be the main tool for the theory of nals . Namely in Theorem 3.2 we show that any nals can be "embedded" in a normed linear space . Though the embedding mapping is not one-to-one, it has enough properties which permit us to use normed linear spaces techniques to study certain problems in a nals . As a consequence, we can now answer all the questions raised in /4/, among which the main one, whether the dual space of a nals can be $\{0\}$, the answer being in the negative. Here we note that the algebraic dual of an almost linear space can be $\{0\}$ (see examples in /4/). We also mention two consequences of Theorem 3.2 . The first (Corollary 3.3) shows that the last axiom of the norm in /3/ and /5/ is superfluous and the second (Corollary 3.4) states that on any nals there exists a semi-metric with good properties. (As we remarked in (/3/,/4/) the function $\rho(x,y) = \|x-y\|$, $x,y \in X$ is never a semi-metric on the nals X, when X is not a normed linear space). Taking into account Corollary 3.4, Theorem 3.2 (except for iv)) can be also regarded as a generalization of Theorem 1 in /7/. Finally, we recall that embedding theorems for certain subclasses of the space of bounded, closed and convex sets of a normed linear space, endowed with the Hausdorff metric were given in /7/ (see /6/, /8/ and the bibliography cited there for generalizations and applications).

All spaces involved in this paper are over the real field R.

2. Besides notation, in this section we recall the definitions of a nals and its dual space (/3/,/4/), as well as some known results used in the next section.

We denote by R^n_+ the set $\{(\alpha_1,\dots,\alpha_n) : \alpha_i \geq 0,\ 1 \leq i \leq n\}$ and for n=1 we write the corresponding set by R_+ .

An almost linear space (als) is a set X together with two mappings $s: X\times X \longrightarrow X$ and $m: R\times X \longrightarrow X$ satisfying (L_1)-(L_8) below. We denote s(x,y) by x+y (or $x\dot{+}y$) and $m(\lambda,x)$ by λx (or $\lambda\circ x$). Let $x,y,z \in X$ and $\lambda,\mu \in R$. (L_1) x+(y+z) = (x+y)+z ; (L_2) x+y = y+x ; (L_3) There exists an element $0 \in X$ such that x+0 = x for each $x \in X$; (L_4) 1x = x ;

(L_5) $0x = 0$; (L_6) $\lambda(x+y) = \lambda x + \lambda y$; (L_7) $\lambda(\mu x) = (\lambda\mu)x$; (L_8) $(\lambda+\mu)x = \lambda x + \mu x$ for $\lambda, \mu \in R_+$. We denote $-1x$ by $-x$, and $x-y$ means $x+(-y)$. For an als X we introduce the following two sets:

$$V_X = \{ x \in X : x-x = 0 \}$$
$$W_X = \{ x \in X : x=-x \}$$

V_X and W_X are almost linear subspaces of X (i.e., closed under addition and multiplication by scalars), and by (L_1)-(L_8), V_X is a linear space. Clearly an als X is a linear space iff $X = V_X$, iff $W_X = \{0\}$.

A <u>norm</u> on the als X is a functional $\|\cdot\| : X \longrightarrow R$ satisfying (N_1)-(N_4) below. Let $x,y \in X$, $w \in W_X$ and $\lambda \in R$. (N_1) $\|x+y\| \leq \|x\| + \|y\|$; (N_2) $\|x\| \leq \|x+w\|$; (N_3) $\|x\| = 0$ iff $x=0$; (N_4) $\|\lambda x\| = |\lambda| \|x\|$. By (N_1)-(N_4) it follows that $\|x\| \geq 0$, $x \in X$. An als X together with $\|\cdot\| : X \longrightarrow R$ satisfying (N_1)-(N_4) is called a <u>normed almost linear space</u> (nals). For a nals X we denote by B_X and S_X the sets $\{x \in X : \|x\| \leq 1\}$ and $\{x \in X : \|x\| = 1\}$ respectively.

2.1. REMARK. In (/3/-/5/) we used instead of (N_1) and (N_2) the only axiom $(N_{1'})$ $\|x-y\| \leq \|x-z\| + \|z-y\|$, $x,y,z \in X$. It is easy to show that conditions (N_1)-(N_4) are equivalent with $(N_{1'})$, (N_3) and (N_4) .

In the next section we need some results of /4/ which we collect them in a lemma .

2.2. LEMMA. <u>Let</u> X <u>be a nals and</u> $x,y,z \in X$, $w_i \in W_X$,

$v_i \in V_X$, i=1,2. *We have* :

(i) *If* $x+y = x+z$ *then* $\|y\| = \|z\|$

(ii) *If* $w_1+v_1 = w_2+v_2$ *then* $w_1=w_2$ *and* $v_1=v_2$.

Let X be an als . A functional $f: X \longrightarrow R$ is called an *almost linear functional* if f is additive, positively homogeneous and $f(w) \geqslant 0$ for each $w \in W_X$. Let $X^\#$ be the set of all almost linear functionals on X . Define the addition in $X^\#$ by $(f_1+f_2)(x) = f_1(x)+f_2(x)$, $x \in X$ and the multiplication by scalars, denoted $\lambda \circ f$, by $(\lambda \circ f)(x)=f(\lambda x)$, $x \in X$. The element $0 \in X^\#$ is the functional which is 0 at each $x \in X$. Then $X^\#$ is an als . When X is a nals, for $f \in X^\#$ define $\|f\| = \sup\{|f(x)| : x \in B_X\}$ and let $X^* = \{f \in X^\#: \|f\| < \infty\}$. Then X^* is a nals (/3/), called the *dual* space of the nals X.

Following /8/, we say that a commutative semigroup S with zero 0 (i.e., satisfying (L_1)-(L_3)) is an *abstract convex cone* if there is also given a mapping $(\lambda, s) \longrightarrow \lambda s$ of $R_+ \times S$ into S such that (L_4),(L_6)-(L_8) hold for $x,y \in S$ and $\lambda, \mu \in R_+$. S satisfies the *law of cancellation* if the relations $s_1,s_2,s_3 \in S$, $s_1+s_2=s_1+s_3$ imply that $s_2=s_3$. Clearly, an abstract convex cone S satisfying the law of cancellation can be organized as an als $X = W_X$, defining in addition the multiplication by negative reals λ by $\lambda s = |\lambda| s$, $s \in S$.

The next result can be found in /7/ (see also /8/).

2.3. PROPOSITION. *Let* S *be an abstract convex cone satisfying the law of cancellation. Then there exist a linear space* L *and a one-to-one additive and positively homogeneous mapping* $h: S \longrightarrow L$ *such that* $L = h(S) - h(S)$.

In what follows, a cone in a linear space L is always with vertex at $0 \in L$ and it can contain lines.

Now let E be a normed linear space. For $x,y \in E$ we denote by $\tau(x,y)$ the one sided Gateaux differential of the norm at x in the direction y, i.e.,

$$\tau(x,y) = \lim_{t \to 0^+} t^{-1}(\|x+ty\| - \|x\|)$$

Let $A_E(x) = \{ f \in S_{E^*} : f(x) = \|x\| \}$. Then by (/2/, p. 447)

$$(2.1) \qquad \tau(x,y) = \max \{ f(y) : f \in A_E(x) \}$$

3. Let X and Y be two almost linear spaces. A mapping $T: X \longrightarrow Y$ is called a linear operator if $T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2)$ for all $\alpha_i \in R$ and $x_i \in X$, $i=1,2$.

3.1. REMARKS. (i) $T(0) = 0$; (ii) $T(V_X) \subset V_Y$; (iii) $T(W_X) = W_Y \cap T(X)$. If X and Y are normed almost linear spaces and in addition $\|T(x)\| = \|x\|$ for each $x \in X$ then we have: (iv) The restriction $T|V_X : V_X \longrightarrow V_Y$ is one-to-one ; (v) The singleton $T^{-1}(v)$ belongs to V_X for each $v \in V_Y \cap T(X)$, and so $T(V_X) = V_Y \cap T(X)$.

From now on we denote the multiplication by scalars in an als X by $\lambda \circ x$ and $-1 \circ x$ is never replaced by $-x$, the notation λx and $-x$ being used only in a linear space. Consequently for the elements $v \in V_X$ (which is a linear space) we can use $-v$ instead of $-1 \circ v$.

The main result of this paper is the following:

3.2. THEOREM. For any nals $(X, |||\cdot|||)$ there exist a normed linear space $(E, \|\cdot\|)$ and a mapping $T: X \longrightarrow E$ with the following properties:

i) The set $T(X)$ is a convex cone of E such that $E = T(X) - T(X)$, and $T(X)$ can be organized as an als where the addition and the multiplication by non-negative scalars are the same as in E.

ii) For each $z \in E$ we have:

$$(3.1) \qquad \|z\| = \inf \left\{ |||x_1||| + |||x_2||| : x_1, x_2 \in X,\ z = T(x_1) - T(x_2) \right\}$$

and the als $T(X)$ together with this norm is a nals.

iii) The mapping T from X onto the nals $T(X)$ is a linear operator and $\|T(x)\| = |||x|||$ for each $x \in X$.

iv) For the dual space X^* the following formula hold:

$$(3.2) \qquad S_{X^*} = \left\{ fT : f \in S_{E^*},\ f|W_{T(X)} \geqslant 0 \right\}$$

Proof. Define an equivalent relation $\sim$ on X in the following way: $x \sim y$ if there exists an element $u \in X$ such that $x+u = y+u$. It is straightforward to show (using Lemma 2.2 (i) for (3.5) below) that if $x \sim y$ then the following hold:

$$(3.3) \qquad x+x_1 \sim y+y_1 \quad \text{if } x_1 \sim y_1$$

$$(3.4) \qquad \lambda \circ x \sim \lambda \circ y \quad \text{for } \lambda \in R$$

$$(3.5) \qquad |||x||| = |||y|||$$

$$(3.6) \qquad f(x) = f(y) \quad \text{for each } f \in X^{\#}$$

Let $X^{\wedge} = X/\sim$ and let T_1 be the canonical mapping of X onto $X^\wedge$, i.e., $T_1(x) = x^\wedge = \{ y \in X : y \sim x \}$. Using (3.3)-(3.5) it is easy to show that $X^\wedge$ is a nals if the addition and the multiplication by scalars are defined by $x^\wedge \dot{+} y^\wedge = (x+y)^\wedge$, $x \in x^\wedge$, $y \in y^\wedge$ and $\lambda \circ x^\wedge = (\lambda \circ x)^\wedge$, $x \in x^\wedge$ respectively, the zero element of $X^\wedge$ is $0^\wedge$ and $|||x^\wedge||| = |||x|||$, $x \in x^\wedge$. Then $T_1 : X \longrightarrow X^\wedge$ is a linear operator from X onto $X^\wedge$ and $|||T_1(x)||| = |||x|||$ for each $x \in X$.

Now $X^\wedge$ is also an abstract convex cone for the addition and the multiplication by non-negative real scalars as in $X^\wedge$, satisfying the law of cancellation. By Proposition 2.3 there exist a linear space L and a one-to-one additive and positively homogeneous mapping $T_2\ X^\wedge \longrightarrow L$ such that $L = T_2(X^\wedge) - T_2(X^\wedge)$. Clearly $T_2(X^\wedge)$ is a nals if we define the addition and the multiplication by non-negative real scalars as in L, while for $\lambda < 0$ we define $\lambda \circ T_2(x^\wedge) = T_2(\lambda \circ x^\wedge)$, and also $|||T_2(x^\wedge)||| = |||x^\wedge|||$. Then $T_2 : X^\wedge \longrightarrow T_2(X^\wedge)$ is a linear operator. For $l \in L$ define

$$s(l) = \inf \left\{ |||x_1{}^\wedge||| + |||x_2{}^\wedge||| : l = T_2(x_1{}^\wedge) - T_2(x_2{}^\wedge), x_i{}^\wedge \in X^\wedge \right\}$$

Then s is a semi-norm on L and for $x^\wedge \in X^\wedge$ we have $s(T_2(x^\wedge)) = |||x^\wedge|||$. Let $E = L/\ker s$ and let T_3 be the canonical mapping of L onto E. Then $E = T_3T_2(X^\wedge) - T_3T_2(X^\wedge)$ and $T_3(T_2(X^\wedge))$ can be organized as an als for the addition as in E and the multiplication by scalars defined for $x^\wedge \in X^\wedge$ and $\lambda \in R$ by $\lambda \circ T_3(T_2(x^\wedge)) = T_3(\lambda \circ T_2(x^\wedge))$. Clearly for $\lambda \geq 0$ this is well defined and we must show that if $x^\wedge, y^\wedge \in X^\wedge$ are such that $T_3(T_2(x^\wedge)) = T_3(T_2(y^\wedge))$ then

(3.7) $\quad T_3(-1\circ T_2(x^\wedge)) = T_3(-1\circ T_2(y^\wedge))$

From our assumption we get that $s(T_2(x^\wedge)-T_2(y^\wedge)) = 0$ and so for each $\varepsilon > 0$ there exist $x_1{}^\wedge, y_1{}^\wedge \in X^\wedge$ such that $T_2(x^\wedge)-T_2(y^\wedge) = T_2(x_1{}^\wedge)-T_2(y_1{}^\wedge))$ and $|||x_1{}^\wedge||| + |||y_1{}^\wedge||| < \varepsilon$. Then

$$-1\circ T_2(x^\wedge)-(-1\circ T_2(y^\wedge)) = -1\circ T_2(x_1{}^\wedge)-(-1\circ T_2(y_1{}^\wedge))$$

and so

$$s(-1\circ T_2(x^\wedge)-(-1\circ T_2(y^\wedge)) \leq |||-1\circ T_2(x_1{}^\wedge)||| + |||-1\circ T_2(y_1{}^\wedge)||| = \\ = |||x_1{}^\wedge||| + |||y_1{}^\wedge|||$$

whence $s(-1\circ T_2(x^\wedge)-(-1\circ T_2(y^\wedge)) = 0$, i.e., we have (3.7). Note that $T_3 : T_2(X^\wedge) \longrightarrow T_3(T_2(X^\wedge))$ is a linear operator.

For $z \in E$, $z = T_3(1)$, $1 \in L$, define $\| z \| = s(1)$, which does not depend on the representative $1 \in L$. The normed linear space $(E, \|\cdot\|)$ and the mapping $T : X \longrightarrow E$ defined by $T = T_3T_2T_1$ satisfy all the conditions required in (i)-(iii).

For the proof of (iv) we need the facts (I)-(IV) below.

(I) If $T(x) = T(y)$, $x, y \in X$, then $f(x) = f(y)$ for each $f \in S_{X^*}$. Indeed, if $T(x) = T(y)$ then

$$s(T_2(T_1(x))-T_2(T_1(y))) = 0$$

and so for each $\varepsilon > 0$ there exist $x_1, y_1 \in X$ such that

$$T_2(T_1(x))-T_2(T_1(y)) = T_2(T_1(x_1))-T_2(T_1(y_1))$$
$$|||x_1||| + |||y_1||| < \varepsilon$$

Since T_2 is linear and one-to-one it follows that $T_1(x+y_1) = T_1(y+x_1)$, i.e., $(x+y_1) \sim (y+x_1)$, whence by (3.6) , $f(x+y_1) = f(y+x_1)$. Then $|f(x)-f(y)| = |f(x_1)-f(y_1)| \leq \leq |||x_1||| + |||y_1||| < \varepsilon$.

(II) We have $S_{X^*} = \{ fT : f \in S_{T(X)^*} \}$. Indeed, let $f_o \in S_{X^*}$ and define the functional f on $T(X)$ by $f(T(x)) = f_o(x)$, $x \in X$. By (I) this is well defined and by Remark 3.1, $f|W_{T(X)} \geq 0$. Hence by (iii) we have $f \in T(X)^{\#}$ and $\| f \| = |||f_o||| = 1$. Conversely, let $f \in S_{T(X)^*}$ and define the functional f_o on X by $f_o(x) = f(T(x))$, $x \in X$. By Remark 3.1 and (iii), it follows that $f_o \in S_{X^*}$.

(III) If $f_o \in S_{T(X)^*}$ and $f \in E^{\#}$ is such that $f|T(X) = f_o$ then $f \in S_{E^*}$. Indeed, let $z \in E$, $\| z \| < 1$. By (3.1) there exist $x, y \in X$ such that $z = T(x)-T(y)$ and $|||x||| + |||y||| \leq 1$. Then $|f(z)| = |f(T(x))-f(T(y))| = |f_o(T(x))-f_o(T(y))| \leq |||x||| + |||y||| \leq 1$.

(IV) We have $S_{T(X)^*} = \{ f|T(X) ; f \in S_{E^*} , f|W_{T(X)} \geq 0 \}$. Indeed, let $f_o \in S_{T(X)^*}$ and let $z \in E$. Since $z = T(x)-T(y)$ for some $x, y \in X$, define $f(z) = f_o(T(x))-f_o(T(y))$, which does not depend on the choice of $x, y \in X$. Then $f \in E^{\#}$ and by (III), $f \in S_{E^*}$. We have $f|T(X) = f_o$ and so $f|W_{T(X)} \geq 0$. The other inclusion is obvious by (III).

Now (3.2) is an immediate consequence of (II) and (IV), which completes the proof of Theorem 3.2.

If $(X, |||\cdot|||)$ is a nals then $(V_X, |||\cdot|||)$ is a normed linear space and so the weak convergence, denoted by $\rightharpoonup$, can be defined in V_X . The next result is an immediate consequence of Theorem 3.2 and Remark 3.1, being the last axiom of the norm in (/3/,/5/).

3.3. COROLLARY. Let $(X, |||\cdot|||)$ be a nals . If (v_α) is a net in V_X , $v_\alpha \longrightarrow v \in V_X$ then for each $x \in X$ we have that $|||x-v||| \leq \liminf |||x-v_\alpha|||$.

3.4. COROLLARY. For any nals $(X, |||\cdot|||)$ there exists a semi-metric ρ on X with the following properties :

$$(3.7) \qquad \big| |||x||| - |||y||| \big| \leq \rho(x,y) , \qquad x,y \in X$$

$$(3.8) \qquad \rho(x,v) = |||x-v||| , \qquad x \in X,\ v \in V_X$$

$$(3.9) \qquad \rho(x+z,y+z) = \rho(x,y), \qquad x,y,z \in X$$

$$(3.10) \qquad \rho(\lambda \circ x, \lambda \circ y) = |\lambda| \rho(x,y), \qquad x,y \in X,\ \lambda \in R$$

$$(3.11) \qquad \lim_{\lambda \to \lambda_0} \rho(\lambda \circ x,x) = \rho(\lambda_0 \circ x,x),\ x \in X,\ \lambda_0 \in R_+ \setminus \{0\}$$

Proof. Let $(E, \|\cdot\|)$ and $T:X \longrightarrow E$ be as in Theorem 3.2. Define for $x,y \in X$

$$(3.12) \qquad \rho(x,y) = \| T(x)-T(y) \|$$

Then ρ is a semi-metric on X and all the above properties are obvious except for (3.10) when $\lambda < 0$. To show this it is enough the following inequality holds

$$(3.13) \qquad \rho(x,y) \leq \rho(-1\circ x,-1\circ y), \qquad x,y \in X$$

For $\varepsilon > 0$ let $x_1,y_1 \in X$ such that

$$T(-1\circ x)-T(-1\circ y) = T(x_1)-T(y_1)$$
$$|||x_1||| + |||y_1||| \leq \rho(-1\circ x,-1\circ y) + \varepsilon$$

Then $T(-1\circ x+y_1) = T(-1\circ x_1+y)$ and as in the proof of (3.7) , this implies that $T(-1\circ y_1+x) = T(-1\circ x_1+y)$. Hence $T(x)-T(y) =$

$= T(-1\circ x_1) - T(-1\circ y_1)$ and so

$$\rho(x,y) = \| T(-1\circ x_1) - T(-1\circ y_1) \| \leq ||| x_1 ||| + ||| y_1 ||| \leq$$
$$\leq \rho(-1\circ x, -1\circ y) + \varepsilon ,$$

whence (3.13) follows.

3.5. REMARK. In /3/-/5/ we have also studied the notion of a strong normed almost linear space (snals), which is a nals X together with a semi-metric ρ on X satisfying conditions (3.9) with = replaced by $\leq$, (3.11) only for $\lambda_0 = 1$, (3.7) and (3.14) below.

(3.14) $\rho(x,y) \leq ||| x + (-1\circ y) |||$, $x, y \in X$

The semi-metric defined by (3.12) does not always satisfy (3.14) as simple examples show. In a snals X condition (3.8) also holds (/3/) and if we replace in the definition of a snals (3.14) by (3.8) then all results given for a snals in /3/-/5/ are true with the same proofs. Moreover, it is not difficult to show that if (X, ρ_1) is a snals such that ρ_1 is a metric on X then ρ defined by (3.12) is also a metric on X. Taking all this into account, By Corollary 3.5 we can always replace snals by nals in /3/-/5/.

3.6. REMARK. Let $(E, \|\cdot\|)$ be a normed linear space and $X \subset E$ an als such that the addition and the multiplication by non-negative reals are as in E . Then $(X, \|\cdot\|)$ is a nals iff the following two conditions hold; (i) $\| -1\circ x \| = \| x \|$, $x \in X$; (ii) $\tau(x,w) \geq 0$, $x \in X$, $w \in W_X$. The proof is obvious using (N_2) and the well known fact (/2/) that the function $\varphi(t) = t^{-1}(\|x+ty\| - \|x\|)$ is non-decreasing.

The next consequence of Theorem 3.2 is a generalization of a corollary of Hahn-Banach Theorem.

3.7. COROLLARY. <u>Let</u> $(X, |||\cdot|||)$ <u>be a nals. For each</u> $x \in X$ <u>there exists</u> $f \in S_{X^*}$ <u>such that</u> $f(x) = |||x|||$.

<u>Proof.</u> Let $(E, \|\cdot\|)$, $T: X \longrightarrow E$ and the nals $(T(X), \|\cdot\|)$ be given by Theorem 3.2, and let $Y = T(X)$. By Theorem 3.2 the corollary is proved if we show that for each $y \in S_Y$ there exists an $f \in A_E(y)$ such that $f|W_Y \geq 0$. Suppose that for each $f \in A_E(y)$ there is $w_f \in W_Y$ with $f(w_f) < 0$. Since $A_E(y)$ is a w^*-compact subset of E^*, there exists a finite number of elements, say, $w_1, \ldots, w_n \in W_Y$, with the property that for each $f \in A_E(y)$ there exists w_i, $1 \leq i \leq n$, with $f(w_i) < 0$. Let $\Psi: A_E(y) \longrightarrow R^n$ be defined by

$$\Psi(f) = (f(w_1), \ldots, f(w_n))$$

Then $\Psi(A_E(y))$ is a compact and convex subset of R^n and $\Psi(A_E(y)) \cap R^n_+ = \emptyset$. By the strict separation theorem there exists $\Phi \in (R^n)^*$, say, $\Phi = (\alpha_1, \ldots, \alpha_n)$ such that

$$(3.15) \qquad \sup \Phi(A_E(y)) < \inf \Phi(R^n_+) = 0$$

Since $\alpha_i \geq 0$, $1 \leq i \leq n$, the element $w = \sum_{i=1}^n \alpha_i w_i$ belongs to W_Y and by (2.1) and (3.15) we get $\tau(y,w) < 0$, contradicting Remark 3.6.

We observed in /4/ that in contrast with the case of a normed linear space, when Y is an almost linear subspace of a nals X and $\varphi \in Y^*$ then it is possible that for no $f \in X^{\#}$ to have $f|Y = \varphi$. On the other hand the almost linear subspace W_X of X has the property that for each $\varphi \in (W_X)^*$ there exists $f \in X^*$ with $f|W_X = \varphi$ and $|||f||| = |||\varphi|||$ (/4/) and we raised the question whether V_X has the same property. The affirmative answer is given in the next result where we show that the extension f can be chosen in V_{X^*}, i.e., f is linear on X.

3.8. COROLLARY. <u>If</u> $(X, |||\cdot|||)$ <u>is a nals and</u> $\varphi \in (V_X)^*$, <u>there exists</u> $f \in V_{X^*}$ <u>such that</u> $f|V_X = \varphi$ <u>and</u> $|||f||| = |||\varphi|||$.

<u>Proof</u>. Let $(E, \|\cdot\|)$ and $T: X \longrightarrow E$ be given by Theorem 3.2 and let $Y = T(X)$. Then $(Y, \|\cdot\|)$ is a nals and by Remark 3.1 and Theorem 3.2 it is enough to show that for each $\varphi \in (V_Y)^*$ there exists $f \in V_{Y^*}$ such that $f|V_Y = \varphi$ and $\|f\| = \|\varphi\|$. Let E_1 be the linear subspace of E defined by

$$E_1 = (W_Y + V_Y) - (W_Y + V_Y)$$

For $z \in E_1$, $z = (w_1+v_1)-(w_2+v_2)$, $w_i \in W_Y$, $v_i \in V_Y$, $i=1,2$, define $f_1(z) = \varphi(v_1 - v_2)$. By Lemma 2.2 (ii), f_1 is well defined. Clearly $f_1 \in E_1^{\#}$, $f_1|V_Y = \varphi$, $f_1|W_Y = 0$ and we have $|f_1(z)| \leq \|\varphi\| \| v_1 - v_2 \|$. We claim that $\| v_1 - v_2 \| \leq \| z \|$. Indeed, for $\varepsilon > 0$ let $x_1, x_2 \in X$ and $y_i = T(x_i)$, $i=1,2$, such that $z = y_1 - y_2$ and $\| y_1 \| + \| y_2 \| \leq \| z \| + \varepsilon$. Then $y_1 + w_2 + v_2 = y_2 + w_1 + v_1$ and so $y_1 + (-1 \circ y_2) + w_2 = y_2 + (-1 \circ y_2) +$

$+w_1+(v_1-v_2)$. Let us set :

$$y = y_1+(-1\circ y_2)$$
$$w = y_2+(-1\circ y_2) + w_1 \quad (\in W_Y)$$
$$v = v_1-v_2 \quad (\in V_Y)$$

Then $y+w_2 = w+v$ and so $-1\circ y+w_2 = w-v$, whence

$$y+(-1\circ y)+2w_2 = 2w = 2y+2w_2-2v$$

from which we get $y = w'+v$, where $w'=(y+(-1\circ y))/2 \in W_Y$. Hence using (N_2), we get $\| v_1-v_2\| = \|v\| \leq \|y\| \leq \|y_1\| + \|y_2\| \leq \leq \|z\| + \varepsilon$ and the claim is proved. Consequently $\|f_1\| = \|\varphi\|$. Let f_o be a norm-preserving extension of f_1 to E. Then $f = f_o|Y \in V_{Y^*}$, $f|V_Y = \varphi$ and $\|f\| = \|\varphi\|$.

Theorem 3.2 suggest the following question. Given a normed linear space $(E, \|\cdot\|)$, determine those convex cones $X \subset E$ satisfying the following three conditions :

(1) X can be organized as an als such that the addition and the multiplication by non-negative reals are as in E.

(2) $(X, \|\cdot\|)$ is a nals .

(3) $E = X-X$.

A partial answer to this question is given in the next result where we obtain normed almost linear spaces X with the property that $X = W_X$.

3.9. PROPOSITION. Let $(E, \|\cdot\|)$ be a normed linear space and $x \in S_E$. There exists a (maximal) convex cone $X \subset E$ satisfying (1)-(3) above, and such that $x \in X$ and $X = W_X$.

Proof. Let $Y = \{ y \in E : y = \lambda z,\ z \in B_E(x,1/3),\ \lambda \in R_+ \}$. Then Y is a convex cone of E such that $E = Y - Y$. Clearly Y is an als if we define the addition as in E and the multiplication by scalars by $\lambda \circ y = |\lambda| y$, $y \in Y$, $\lambda \in R$. Then $Y = W_Y$ and we show that $(Y, \|\cdot\|)$ is a nals. By Remark 3.6 it is enough to show that $\tau(y_1,y_2) \geq 0$, $y_i \in S_Y$, $i=1,2$. Suppose $y_i = \lambda_i z_i$, $\lambda_i > 0$, $z_i \in B_E(x,1/3)$, $i=1,2$ and let $f \in A_E(y_1)$. We have

$$\|z_1\| - f(z_2) = f(z_1) - f(z_2) \leq \|z_1 - z_2\| \leq \frac{2}{3}$$

and so

$$f(z_2) \geq \|z_1\| - \frac{2}{3} \geq \|x\| - \|x - z_1\| - \frac{2}{3} \geq 1 - \frac{1}{3} - \frac{2}{3} = 0$$

Then by (2.1) we get $\tau(y_1,y_2) \geq f(y_2) = \lambda_2 f(z_2) \geq 0$.

Let $\mathcal{F}_x$ be the set of all convex cones $Y \subset E$ satisfying (1)-(3) and such that $x \in Y$, $Y = W_Y$. $\mathcal{F}_x$ is a partially ordered set, ordered by set-inclusion. If (Y_α) is a totally ordered family of $\mathcal{F}_x$, then $Y = \bigcup Y_\alpha \in \mathcal{F}_x$. By Zorn's Lemma there exists a maximal element $X \in \mathcal{F}_x$.

We conclude this paper with the observation that the proofs of some results of /3/-/5/ can be simplified using Theorem 3.2.

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