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ISSN 0250 3638

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PREPRINT SERIES IN MATHEMATICS
No.3/1985

BUCUREȘTI

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January 1985

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ON THE FIELD OF DEFINITION OF ALGEBRAIC VARIETIES IN CHARACTERISTIC ZERO

A. Buium

1. Introduction

Let K be an algebraically closed field of characteristic zero and V a K-variety (by this we mean an irreducible reduced quasi-projective scheme over K). A subfield K_1 of K will be called a field of definition for V if there exists a K_1-variety V_1 such that V is K-isomorphic to $V_1 \otimes_{K_1} K$. The aim of this paper is to show how one can compute fields of definition for V with the help of derivations on the function field K(V) of V.

For any set Δ of derivations on K(V) define

$$K^{\Delta} = \{\lambda \in K ; \delta\lambda = 0 \quad \text{for all} \quad \delta \in \Delta\}$$

Clearly K^{Δ} is an algebraically closed subfield of K. An important role will be played by the set $\Delta(V)$ of all derivations δ on K(V) which are integral on V in the sense that $\delta(\mathcal{O}_{V,p}) \subset \mathcal{O}_{V,p}$ for all $p \in V$ (here $\mathcal{O}_{V,p}$ denotes the local ring of V at p). Indeed our main result is:

Theorem 1. Suppose V is smooth and projective over K. Then $K^{\Delta(V)}$ is a field of definition for V and any other algebraically closed field of definition for V must contain $K^{\Delta(V)}$.

An immediate consequence of Theorem 1 is the following criterion. Suppose k is an algebraically closed subfield of K, V a smooth projective K-variety and $\{t_\alpha\}_\alpha$ a transcendence basis of K/k; then k is a field of definition for V if and only if the derivations $\partial/\partial t_\alpha : K \to K$ lift to derivations $\delta_\alpha : K(V) \to K(V)$ which are integral on V.

Theorem 1 will be proved in Section 3.

In Section 4 we shall discuss the possibility of extending Theorem 1 to singular and to open varieties. We would like to note that in the case of open varieties the right substitute for $\Delta(V)$ will be the set $\Delta(V, \log)$ of all "logarithmic" (instead of "integral") derivations (see Section 4 for precise definitions and results).

In Section 5 we shall discuss the problem of finding the smallest algebraically closed "field of definition" for a complete local ring (again we send to Section 5 for definitions and results

The main motivation for our work concerns algebraic differential equations without movable singularities (cf. [8], [1]). More precisely Theorem 1 may be taken as a starting point for a generalisation of the "one variable theory" from [8] to the case of several variables (see [1] for the case of two variables). We shall achieve this program in a separate paper [2].

Our proof of Theorem 1 is not purely algebro-geometric it will involve a "reduction to the complex field $\mathbb{C}$". Then the main step towards Theorem 1 will be the following result which has an interest in itself and which will be proved in Section 2:

Theorem 2. Let $f: X \longrightarrow S$ be a smooth projective morphism of smooth $\mathbb{C}$- varieties . Then there is a diagram with cartesian squares:

$$\begin{array}{ccccc} X & \longleftarrow & X' & \longrightarrow & X'' \\ {\scriptstyle f}\downarrow & & \downarrow & & \downarrow{\scriptstyle f''} \\ S & \overset{\alpha}{\longleftarrow} & S' & \overset{\beta}{\longrightarrow} & S'' \end{array}$$

such that β is a surjective map of $\mathbb{C}$- varieties, S" is smooth, α is an étale covering of a Zariski open set of S, f" is a smooth projective morphism and for any $t \in S''$ the Kodaira-Spencer map

$$\rho_t : T_t S'' \longrightarrow H^1(X''_t, T_{X''_t/\mathbb{C}})$$

is injective (where $T_t S''$=tangent space of S'' at t, $X''_t=(f'')^{-1}(t)$, $T_{X''_t/\mathbb{C}}$ = tangent bundle of X''_t).

We would like to note that Theorem 2 was proved in [13], p.574 under a very restrictive assumption on the local Torelli map of f at the generic point of S.

2. Proof of Theorem 2

In this section we prove Theorem 2. Points of $\mathbb{C}$-varieties will always mean closed points. Choose an invertible sheaf $\mathcal{L}$ on X which is ample relative to f, put $\mathcal{L}_t=\mathcal{L}|_{X_t}$ ($X_t=f^{-1}(t)$) and let $\lambda_t \in \mathrm{Pic}(X_t)/\mathrm{Pic}^{\tau}(X_t)$ be the class of $\mathcal{L}_t$ modulo numerical equivalence.

Claim 1. The set

$$R=\{(t,s)\in S\times S;\ (X_t,\lambda_t)\simeq(X_s,\lambda_s)\}$$

is constructible in $S\times S$ (note that if no X_t was ruled then R would be Zariski closed in $S\times S$; this follows from [9]).

An argument for this goes as follows. Let $p_i : Z=S\times S \longrightarrow S$, $i=1,2$, be the canonical projections and let $Y_i \longrightarrow Z$ be obtained from $X \longrightarrow S$ by base change with p_i. Let U be the Z-scheme representing the functor $Z' \longrightarrow \mathrm{Isom}_{Z'}(Y_1\times_Z Z',\ Y_2\times_Z Z')$ [4]; recall that U is a countable disjoint union of Z-schemes U_n of finite type. Let $\mathcal{L}_i$ be the pull-back of $\mathcal{L}$ on $V_i=Y_i\times_Z U$ and let $F:V_1\longrightarrow V_2$ be the universal isomorphism. Clearly the sets

$$U'_n=\{u\in U_n;\ ((F^*\mathcal{L}_2)\otimes\mathcal{L}_1^{-1})_u\equiv 0\}$$

are closed in U_n (here "$\equiv$" denotes the numerical equivalence) and we have $R=\mathrm{Im}(U'\to Z)$ where U' is the union of all U'_n for $n\geq 1$. So, by Chevalley's constructibility theorem, we shall be done if we prove that U'_n are empty for all except a finite number of n's. Now for any $u\in U'$ let $z(u)=(t(u),\ s(u))$ denote the image of u under $U\to Z$ and let $\Gamma_u\subset X_{t(u)}\times X_{s(u)}$ be the graph of the corresponding isomorphism which we denote also by $u:X_{t(u)}\to X_{s(u)}$. Consider on $Y_1\times_Z Y_2$ the sheaf $q_1^*\mathcal{L}\otimes q_2^*\mathcal{L}$ ($q_i:Y_i\to X$ being the canonical projections); this sheaf is ample relative to Z and denote by $\mathcal{O}_{\Gamma_u}(1)$ its restriction to Γ_u. Now if $1\times u:X_{t(u)}\to\Gamma_u\subset X_{t(u)}\times X_{s(u)}$ is the graph map then:

$$(1\times u)^*(\mathcal{O}_{\Gamma_u}(m))=\mathcal{L}^m_{t(u)}\otimes u^*(\mathcal{L}^m_{s(u)})\cong\mathcal{L}^{2m}_{t(u)}.$$

Hence the Hilbert polynomial $m\to\chi(\Gamma_u,\mathcal{O}_{\Gamma_u}(m))$ equals to a polynomial $m\to\chi(X_{t(u)},\mathcal{L}^{2m}_{t(u)})$ which does not depend on u. This implies that U'_n is empty for sufficiently big n.

Claim 2. Replacing S by a Zariski open subset of it we may suppose there exists a morphism $\psi:S\to M$ into a $\mathbb{C}$- variety M such that for any $s\in S$ we have

$$\psi^{-1}\psi(s)=\{t\in S;\ (s,t)\in R\}$$

This can be done by standard manipulation of Chow varieties (see [11] p.406 for similar arguments). The idea is the embed S as a locally closed subset of a projective space $\mathbb{P}$ and to take the Zariski closure $\bar{R}$ of R in $\mathbb{P}\times S$; by Claim 1, for each irreducible component $\bar{R}_j$ of $\bar{R}$ the projection $\bar{R}_j\to S$ will give a family of cycles of codimension m_j and degree d_j in $\mathbb{P}$ (m_j, d_j being some integers) and hence a rational map from S to the corresponding Chow variety $C(m_j,d_j)$. Using constructibility of R one can make an elementary analysis showing that, after shrinking S in the Zariski topology, the resulting morphism

$$\psi : S \longrightarrow \prod_j C(m_j, d_j) \longrightarrow \prod_q C(q, \sum_{m_j = q} d_j)$$

has the property required in Claim 2.

Claim 3. Replacing S by an étale open set of it one can find a morphism $\eta : S \longrightarrow N$ onto a variety N such that η has a section and such that for any $t \in S$ the set

$$S_t = \{ s \in S;\ X_s \simeq X_t \}$$

is a union of at most countably many fibres of η.

Indeed, since the set of classes of numerically equivalent divisors on a fixed variety is countable, S_t is a union of at most countably many fibres of the map ψ from Claim 2. Now we are done by replacing M by an étale open set N of $\psi(S)$ and replacing S by $S \times_M N$.

Claim 4. We may suppose in Claim 3 that in addition there exists a smooth projective morphism $g : Y \longrightarrow N$ such that X is S-isomorphic to $Y \times_N S$; in particular we shall have that for any $u \in N$ the set

$$N_u = \{ v \in N;\ Y_v \simeq Y_u \}$$

is at most countable.

The argument in this step is similar to the one in [13], p.576. Take $\gamma : N \longrightarrow T \subset S$ a section of $S \to N$, put $X_T = X \times_S T$, $X_N = X_T \times_T N$, $X' = X_N \times_N S$. Then for any $t \in S$, the fibres of $X \to S$ and $X' \longrightarrow S$ above t are isomorphic; this means that the S-scheme $U = U_1 \cup U_2 \cup \ldots$ representing

$$S' \longrightarrow \mathrm{Isom}_{S'}(X \times_S S', X' \times_S S')$$

maps onto S. By Baire's theorem there is at least a finite type piece U_n of U dominating S. Now we are done by replacing S by some locally closed irreducible subscheme of U_n which is étale over S.

Claim 5. For any t in a Zariski open set of N (notations being as in Claim 4) the Kodaira-Spencer map ρ_t associated to $g:Y \longrightarrow N$ at t is injective (this will of course close the proof of Theorem 2 !).

Indeed if the morphism $\rho : T_{N/\mathbb{C}} \longrightarrow R^1 g_*(T_{Y/N})$ is injective at the generic point of N we are done. If not, we may choose, after shrinking N in the Zariski topology, a line bundle L contained in Ker(ρ). By Frobenius there is a germ of analytic curve C whose analytic tangent bundle T_C equals to the restriction of L to C. By [7] 6.2, the family $Y \times_N C \rightarrow C$ must be analytically locally trivial, contracting Claim 4 which states that N_u is at most countable for $u \in N$.

3. <u>Proof of Theorem 1</u>

The fact that any algebraically closed field of definition K_1 for V contains $K^{\Delta(V)}$ is quite easy and general (it does not require smoothness or projectivity of V). Indeed it will be sufficient to prove that any K_1-derivation θ on K must vanish on $K^{\Delta(V)}$. But if $V \simeq V_1 \otimes_{K_1} K$ (V_1 being some K_1-variety) we see that θ extends to a derivation $\delta : K(V) \longrightarrow K(V)$ defined by

$$\delta(\lambda \otimes y) = \lambda \otimes (\theta y) \quad \text{for all } \lambda \in K_1(V_1), y \in K$$

Now δ is integral on V, hence will vanish on $K^{\Delta(V)}$ and we are done. So in the remainder of this section we concentrate ourselves on proving that $K^{\Delta(V)}$ is a field of definition for V. This is of course equivalent to proving that K^{Δ} is a field of definition for V whenever Δ is a subset of $\Delta(V)$.

We assume first that K^{Δ} is uncountable. Consequently K will contain a subfield k which is isomorphic to $\mathbb{C}$. One can easily construct a smooth projective morphism of k-varieties $f:X\longrightarrow S$ such that the function field $k(S)$ of S is contained in K and $V\cong X\times_S Spec($
Apply Theorem 2 to f and put $K'=k(S')$, $K''=k(S'')$. Since K' is a finite extension of $k(S)$, there is an embedding $K'\longrightarrow K$ extending the inclusion $k(S)\rightarrow K$. Put $V''=X''\times_{S''}Spec(K'')$. We have a field extension $K''\rightarrow K'\longrightarrow K$ and V is K-isomorphic to $V''\otimes_{K''}K$ so we shall be done if we prove that K^{Δ} contains K''. Now there is standard exact sequence [3] Ch.0, 20.5.7:

$$(*)\qquad 0\rightarrow \pi^{*}\Omega_{K/k}\rightarrow\Omega_{V/k}\rightarrow\Omega_{V/K}\rightarrow 0$$

where $\pi:V\longrightarrow Spec(K)$ is the canonical structure morphism. A similar sequence exists for $V''\longrightarrow Spec(K'')$. These sequences plus the injectivity of the Kodaira-Spencer maps associated to f" at the points of S" yeld a diagram with exact rows and colomns:

$$\begin{array}{ccccc}
 & & T_{K/K''} & & \\
 & & \downarrow\psi & & \\
H^0(V,T_{V/k}) & \xrightarrow{\ \varphi\ } & T_{K/k} & \longrightarrow & H^1(V,T_{V/K}) \\
 & & \downarrow & & \\
0 & \longrightarrow & T_{K''/k}\otimes_{K''}K & \longrightarrow & H^1(V,T_{V''/K''}\otimes_{K''}K)
\end{array}$$

(where for any scheme W over a field L we denote by $T_{W/L}$ the sheaf $Hom_{\mathcal{O}_W}(\Omega_{W/Spec(L)},\mathcal{O}_W)$ of L-derivations from $\mathcal{O}_W$ into $\mathcal{O}_W$; if furthermore we have $W=Spec(A)$ then we put $T_{A/L}=H^0(W,T_{W/L})$). A diagram chase shows that φ and ψ have the same image in $T_{K/k}$. Since

Since Δ' is a subset of $H^0(V, T_{V/k})$ we get in particular that $K'' \subset K^\Delta$.

Theorem 1 is proved in the case K^Δ uncountable.

Suppose now K^Δ is countable. Then there is an embedding $K^\Delta \to \mathbb{C}$; the ring $K \otimes_{K^\Delta} \mathbb{C}$ will be a domain and denote by L its field of quotients.

Now it is easy to see (use the exact sequence (*) with $k = \mathbb{Q}$) that for any $\delta \in \Delta$ we have $\delta(K) \subset K$ so one can define a derivation δ' on L by the formula

$$\delta'(\lambda \otimes y) = (\delta\lambda) \otimes y \qquad \text{for all } \lambda \in K \text{ and } y \in \mathbb{C}$$

Moreover one can define a derivation δ'' on $L(V \otimes_K L)$ by the formula

$$\delta''(u \otimes v) = (\delta u) \otimes v + u \otimes (\delta' v) \qquad \text{for all } u \in K(V), v \in L$$

Clearly δ'' is integral on $V \otimes_K L$ and let Δ'' be the set of all such δ'' as δ runs through Δ. Now $L^{\Delta''}$ contains $1 \otimes \mathbb{C}$ hence it is uncountable so by the first part of our proof $L^{\Delta''}$ is a field of definition for $V \otimes_K L$. We have four fields

$$\begin{array}{ccc} K^\Delta & \hookrightarrow & K \\ \downarrow & & \downarrow \\ L^{\Delta''} & \hookrightarrow & L \end{array}$$

and note that K and $L^{\Delta''}$ are linearly disjoint over K^Δ (this may be proved exactly as in [6], p.87 using the Wronskian argument). So we shall be done if we prove the following general fact:

Lemma 1. Let V be a smooth projective K-variety and let K_0, K_1 and K_2 be algebraically closed subfields of K such that

$$\begin{array}{ccc} K_o & \hookrightarrow & K_1 \\ \big\downarrow & & \big\downarrow \\ K_2 & \hookrightarrow & K \end{array}$$

and such that K_1 and K_2 are linearly disjoint over K_o.

Suppose K_1 and K_2 are fileds of definition for V. Then K_o is also a field of definition for V.

Proof. Choose an ample $\mathcal{L}\in \mathrm{Pic}(V)$. Suppose V is K-isomorphic to $V_i\otimes_{K_i}K$, $i=1,2$. Then there exist $\mathcal{L}_i\in \mathrm{Pic}(V_i)$ such that $\mathcal{L}_i\otimes_{K_i}K\simeq$ clearly $\mathcal{L}_i$ are still ample. One can find projective morphisms $f_i:X_i\to S_i$ of K_o-varieties such that $K_o(S_i)\subset K_i$, V_i is K_i-isomorphic to $X_i\times_{S_i}\mathrm{Spec}(K_i)$ and such that $\mathcal{L}_i$ is the pull back of some $\mathcal{M}_i\in \mathrm{Pic}(X_i)$ with $\mathcal{M}_i$ ample relative to f_i. Put $T=S_1\times S_2$, $Y_i=X_i\times_{S_i}T$. By linear disjointness of K_1 and K_2 over K_o the morphis $K_1\otimes_{K_o}K_2\to K$ is injective, hence $\mathrm{Spec}(K)\to T$ is dominant. Since $Y_1\times_T K$ is K-isomorphic to $Y_2\times_T K$, it follows that $\mathrm{Spec}(K)\to T$ facto through some finite type component U_n of the object U representin the functor $T'\to \mathrm{Isom}_{T'}(Y_1\times_T T', Y_2\times_T T')$. But since the isomorphism $Y_1\times_T K\simeq Y_2\times_T K$ preserves the polarisations induced by $\mathcal{M}_1$ and $\mathcal{M}_2$ we conclude that the image of $\mathrm{Spec}(K)\to U_n$ is contained in $U'_n=U'\cap U_n$ where U' is the closed subset of U whose geometric poin are precisely those points for which the corresponding isomorphis preserves polarisations (see the proof of Claim 1 in Section 2).

Now the image of $U'_n\to T$ contains an open subset T_o of T in other words for any $(s_1,s_2)\in T_o$ the fibres of $Y_1\to T$ and $Y_2\to T$ above (s_1,s_2) are isomorphic as polarized varieties. But these fibres identify with $f_1^{-1}(s_1)$ and $f_2^{-1}(s_2)$ respectively with polarisations given by $\mathcal{M}_1,\mathcal{M}_2$. Now fix $(s_1^o,s_2^o)\in T_o$ and put $S'_2=\{s_2\in S_2;\ (s_1^o,s_2)\in T_o\}$; then $X'_2:=X_2\times_{S_2}S'_2\to S'_2$ has all its closed fibres isomorphic as polarized varieties (with polarisation given by $\mathcal{M}_2$).

Let X_2'' be $F\times S_2'$, $F=f_2^{-1}(s_2^o)$ and let H be the object representing the functor $B\longrightarrow \mathrm{Isom}_B(X_2'\times_{S_2'}B,\ X_2''\times_{S_2'}B)$. Then let H' be the closed subset of H whose geometric points correspond to those isomorphisms which preserve polarisations (we take on $X_2''\longrightarrow S_2'$ the polarisation induced from that of F). As noted in Claim 1, Section 2, H' is of finite type over S_2' (and not only locally of finite type). Since the map $H'\longrightarrow S_2'$ is surjective, we can find a component of H' dominating S_2' and hence an étale map $\tilde{S}_2\longrightarrow S_2'$ such that $\tilde{X}_2=X_2'\times_{S_2'}\tilde{S}_2\longrightarrow$ $\longrightarrow\tilde{S}_2$ is $\tilde{S}_2$-isomorphic to $\tilde{S}_2\times_{K_o}F$. Since K is algebraically closed we may embed $K_o(\tilde{S}_2)$ in K and we get

$$V\simeq X_2\times_{S_2}K=\tilde{X}_2\times_{\tilde{S}_2}K=F\otimes_{K_o}K \quad \text{(over } K\text{)}.$$

4. Singular varieties and open varieties

A general strategy of treating singular varieties and open varieties is to treat first pairs consisting of a smooth projective variety plus an effective divisor (sometimes supposed with normal crossings). As a general principle too, global objects have to be replaced by objects with a logarithmic behaviour along the divisor.

This is precisely what we shall do now; namely we shall give a variant of our theory from §§1-3 for pairs (V,D) where V is a smooth projective K-variety (K being as usual algebraically closed of characteristic zero) and D is an effective Cartier divisor on V. A subfield K_1 of K will be called a field of definition for (V,D) if there exists a K_1-variety V_1, a divisor D_1 on V_1 and a K-isomorphism $V\simeq V_1\otimes_{K_1}K$ such that $q^*D_1=D$ where $q:V\to V_1$ is the projection. Clearly if K_1 is a field of definition for (V,D) it is also a field of definition for the open variety $V\setminus D$. Now for (V,D) as above we say that a derivation δ on $K(V)$ is logarithmic on (V,D) if it is integral on V and if for any $p\in V$ and any local equation $f\in\mathcal{O}_{V,p}$ of D at p we have

$$f^{-1}\delta f \in \mathcal{O}_{V,P}$$

(this is the same as to say that δ takes the ideal sheaf $\mathcal{O}_V(-D)$ into itself!). Denote by $\Delta(V,D)$ the set of logarithmic derivations on (V,D); note that $\Delta(V,D)\subset\Delta(V)$ and that $\Delta(V,D_1)=\Delta(V,D_2)$ provided D_1 and D_2 have the same support; this follows from the fact that primes associated to differential ideals in a differential ring are differential [10], p. 232.

Now denote by $T_{V/K}$ (log D) the subsheaf of the tangent sheaf $T_{V/K}$ of V consisting of those derivations which take $\mathcal{O}_X(-D)$ into itself (see also [5]).

The following Theorem reduces to Theorem 1 if D=0.

<u>Theorem 3.</u> Let V be a smooth projective K-variety and D an effective divisor on V. Suppose the injective map

$$H^o(V,T_{V/K}(\log D))\to H^o(V,T_{V/K})$$

it also surjective. Then $K^{\Delta(V,D)}$ is the smallest algebraically closed field of definition for (V,D).

Note that the surjectivity of the map above occurs in each of the following cases:

a) D=0

b) $H^o(V,T_{V/K})=0$

c) $D=\sum D_i$, D_i are smooth subvarieties of V crossing normally and $H^o(D_i,N_{D_i})=0$ (where N_{D_i} is the normal sheaf of D_i). Indeed in this case the cokernel of the map from Theorem 3 injects into $\bigoplus_i H^o(D_i,N_{D_i})$ cf [5].

Proof of Theorem 3. The only non-trivial fact to prove it that $K_o=K^{\Delta(V,D)}$ is a field of definition for (V,D). Since $K^{\Delta(V)}\subset K^{\Delta(V,D)}$ we get by Theorem 1 that K_o is a field of definition for V i.e. V is K-isomorphic to $V_o\otimes_{K_o}K$ for some V_o. For any

$\delta\in\Delta(V,D)$ we have $\delta(K)\subset K$ so we may consider the derivation $\delta^*\in\Delta(V)$ defined by

$$\delta^*(\lambda\otimes y)=\lambda\otimes\delta y \quad \text{for all } \lambda\in K_o(V_o), y\in K$$

Then $\delta-\delta^*\in H^o(V,T_{V/K})$. By hypothesis $(\delta-\delta^*)(\mathcal{O}_V(-D))\subset\mathcal{O}_V(-D)$. Since $\delta(\mathcal{O}_V(-D))\subset\mathcal{O}_V(-D)$ we get $\delta^*(\mathcal{O}_V(-D))\subset\mathcal{O}_V(-D)$. Now we may conclude by the following general:

<u>Lemma 2.</u> Let K be a field, Δ a set of derivations on K, $K_o=\{\lambda\in K,\ \delta\lambda=0 \text{ for all } \delta\in\Delta\}$ and let A_o be a K_o-algebra. Put $A=A_o\otimes_{K_o}K$ and define for any $\delta\in\Delta$ a derivation $\delta^*:A\longrightarrow A$ by the rule $\delta^*(\lambda\otimes y)=\lambda\otimes\delta y$ for all $\lambda\in A_o$, $y\in K$. Suppose I is an ideal in A such that $\delta^*(I)\subset I$ for all $\delta\in\Delta$. Then $I=I_o\otimes_{K_o}K$ for some ideal I_o in A_o.

Proof. Put $I_o=I\cap A_o$ and $J=I_o\otimes_{K_o}K$. Suppose $I\setminus J\neq\emptyset$. Let $(e_k)_k$ be a basis of A_o as a K_o-vector space and take an element $a=\sum e_k\otimes a_k\in I\setminus J$ $(a_k\in K)$ for which the number

$$\#\{k;\ a_k\neq 0\}$$

is minimal. We may of course assume there is an index k_o such that $a_{k_o}=1$. Now for all $\delta\in\Delta$,

$$\sum e_k\otimes\delta a_k=\delta^*(\sum e_k\otimes a_k)\in I$$

so by minimality of a we have that $\sum e_k\otimes\delta a_k\in J$. Since $a\notin J$ there is at least an index k_1 and there is a derivation $\delta\in\Delta$ such that $\delta a_{k_1}\neq 0$. By minimality of a we get that

$$a-a_{k_1}(\delta a_{k_1})^{-1}(\sum e_k\otimes\delta a_k)\in J$$

from which we get $a\in J$, contradiction. The lemma is proved.

Using Theorem 3 we shall prove the following:

Theorem 4. Let V be a normal projective K-variety of dimension two. Then $K^{\Delta(V)}$ is the smallest algebraically closed field of definition for V.

Proof. Let $f:W\longrightarrow V$ be Zariski's canonical resolution; so f is obtained as a composition $W=V_n\longrightarrow V_{n-1}\longrightarrow\ldots\longrightarrow V_1=V$ where V_i is obtained from V_{i-1} by first normalizing V_{i-1} and then blowing up the (reduced) ideal of the singular locus $\sum_{i-1}$ of $(V_{i-1})^{nor}$. By a theorem of Seidenberg [12] $\Delta(V_{i-1})\subset\Delta((V_{i-1})^{nor})$. By another theorem of Seidenberg [10], p.233 for any $y\in\sum_{i-1}$ and for any $\delta\in\Delta((V_{i-1})^{nor})$ we have $\delta(m_y)\subset m_y$ (here m_y=maximal ideal of $\mathcal{O}_y$). An elementary local computation shows then that $\Delta(V_{i-1})^{nor})\subset\Delta(V_i)$. So after all we deduce that $\Delta(V)\subset\Delta(W)$. Put $D=f^{-1}(\sum_1)$ set-theoretically; then D is the support of a reduced divisor which we still call D. Since $W\setminus D\simeq V\setminus\sum_1$ we immediately get that $\Delta(W)\subset\Delta(V)$ so we get $\Delta(V)=\Delta(W)$. We claim that $\Delta(W)=\Delta(W,D)$.

Indeed if $\delta\in\Delta(W)$ then $\delta\in\Delta(V)$ so by Seinberg's theorem $\delta(m_y)\subset m_y$ for all $y\in\sum_1$. Consequently $\delta(m_y\mathcal{O}_W)\subset m_y\mathcal{O}_W$. We conclude using the fact that the radical of a differential ideal in a differential ring is still a differential ideal [10], p.232. Now the equality $\Delta(W)=\Delta(W,D)$ implies in particular that the map $H^o(W,T_{W/K}(\log D))\longrightarrow H^o(W,T_{W/K})$ is an isomorphism. Applying Theorem 3 we get that $K_o=K^{\Delta(V)}$ is a field of definition for (W,D) so there is a smooth projective K_o-variety W_o such that $W\simeq W_o\otimes_{K_o}K$ and there is a divisor D_o on W_o with $D=q^*D_o$, $(q:W\longrightarrow W_o)$. Then we claim that there is a birational morphism $f_o:W_o\longrightarrow V_o$ onto a normal surface V_o which is an isomorphism above $V\setminus f_o(D_o)$ and such that $f_o(D_o)$

is a finite set.

Indeed there exist projective morphisms $f_S:X\longrightarrow Y$, $g:X\longrightarrow S$, $h:Y\longrightarrow S$, $g=f_S h$ where g and h are projective, S is an affine algebraic K_o-scheme with $K_o(S)\subseteq K$ and $f_S\times_S \mathrm{Spec}(K):X\times_S \mathrm{Spec}(K)\longrightarrow Y\times_S \mathrm{Spec}(K)$ identifies with $f:W\longrightarrow V$. Then the desired $f_o:W_o\longrightarrow V_o$ may be obtained by taking the morphism $g^{-1}(s)\longrightarrow(h^{-1}(s))^{nor}$ induced from f_S where $s\in S$ is a sufficiently general K_o-point of S. Now it is easy to see that V is K-isomorphic to $V_o\otimes_{K_o}K$ and we are done.

The following seems quite plausible:

<u>Conjecture 1.</u> If V is a normal projective K-variety then $K^{\Delta(V)}$ is the smallest algebraically closed field of definition for V.

Now we close by discussing the case of open non-singular varieties. Let U be a non-singular K-variety. By a compactification of U we mean a triple (V,D,φ) with V non-singular and projective, D a divisor on V and φ a K-isomorphism $U\simeq V\setminus D$.

For any such compactification, $\Delta(V,D)$ identifies via φ with a set of derivations on K(U). Define

$$\Delta(U,\log)=\bigcup\Delta(V,D)$$

the union being taken after all possible compactifactions (V,D,φ) of U. It is easy to see that $K^{\Delta(U,\log)}$ is contained in any algebraically closed field of definition for U. We hope the following to be true:

<u>Conjecture 2.</u> If U is a non-singular K-variety, $K^{\Delta(U,\log)}$ is the smallest algebraically closed field of definition for U.

We can prove *Conjecture* 2 in various special cases. For instance:

Theorem 5. Conjecture 2 holds in any of the following cases:

1) U is an affine curve

2) U is an affine surface of general type.

To prove Theorem 5 we need some preparation.

We say that $(V_1,D_1,\varphi_1)\leq(V_2,D_2,\varphi_2)$ for two compactifications of U if the rational map $\varphi_1\varphi_2^{-1}:V_2 \dashrightarrow V_1$ is everywhere defined. It is easy to see that in this situation $\Delta(V_2,D_2)\subset\Delta(V_1,D_1)$ as subsets in $K\Delta(U)$. So if the set of compactifications of U has a smallest element (V_1,D_1,φ_1) we have

$$\Delta(U, \log)=\Delta(V_1,D_1)$$

Note that a smallest element as above does not necessarily exist (compare with [5]).

Now for a smooth projective K-variety V, let $\sigma:V\longrightarrow V$ be a K-automorphism and let $\sigma^*:K(V)\longrightarrow K(V)$ the corresponding K-automorphism of K(V). Take D an effective divisor on V. Furthermore consider a set Δ of derivations on K(V). Denote by Δ^{σ} the set $\{(\sigma^*)^{-1}\delta\sigma^*;\ \delta\in\Delta\}$ .Then it is easy to check that:

a) $K^{\Delta}=K^{\Delta^{\sigma}}$.

b) $(\Delta(V,D))^{\sigma}=\Delta(V,\sigma(D))$

In particular $K^{\Delta(V,D)}$ is a field of definition for (V,D) if and only if $K^{\Delta(V,\sigma(D))}$ is a field of definition for $(V,\sigma(D))$.

Now let's start the proof of Theorem 5.

Suppose U is an affine curve.

In this case there is essentially a unique compactification

(V,D,φ) with D reduced so $\Delta(U,\log)=\Delta(V,D)$.

Put g=genus of V. If $g\geq 2$, $H^0(V,T_{V/K})=0$ and we conclude by Theorem 3. Suppose $g=1$.

Put $K_0=K^{\Delta(V,D)}$; by Theorem 1, there is a K-isomorphism $V\simeq V_0\otimes_{K_0}K$ with V_0 an elliptic curve over K_0. Let $p_0\in V_0(K_0)$ be a K_0-point of V_0 and $p\in V(K)$ the unique K-point of V lying over p_0. By transitivity of $\mathrm{Aut}_K(V)$ on V and by the preparation above, we may suppose $p\in D$. For any $\delta\in\Delta(V,D)$ let $\delta^*\in\Delta(V)$ be the derivation defined as in the proof of Theorem 3 (so $\delta^*(\lambda\otimes y)=\lambda\otimes\delta y$ for $\lambda\in K_0(V_0)$, $y\in K$). Since $\delta-\delta^*\in H^0(V,T_{V/K})=H^0(V_0,T_{V_0/K_0})\otimes_{K_0}K$ we get $\delta-\delta^*=f\theta$ with $f\in K$, θ= a generator of $H^0(V_0,T_{V_0/K_0})$. Now if t is a parameter of the maximal ideal m_{p_0} of $\mathcal{O}_{V_0,p_0}$ then $\theta t\notin m_{p_0}$. On the other hand $\delta^*(m_p)\subset m_p$ because $m_p=m_{p_0}\otimes K$ hence $f\theta(m_p)\subset m_p$. In particular $\theta t\otimes f=f\theta(t\otimes 1)\in m_{p_0}\otimes K$ which implies $f=0$, hence $\delta=\delta^*$. Now we may conclude by Lemma 2.

Suppose now $g=0$. If $\#D\leq 3$, $\mathbb{Q}$ is a field of definition for $(\mathbb{P}^1_K,D)$ and we are done. Suppose $\#D\geq 4$ and take $p_1,p_2,p_3\in D$. Since $\mathrm{Aut}_K(\mathbb{P}^1_K)$ is triply transitive we may assume that each p_i $(i=1,2,3)$ lies over a K_0-point p_i^0 of $\mathbb{P}^1_{K_0}$ $(K_0=K^{\Delta(V,D)})$. For any $\delta\in\Delta(V,D)$ define δ^* as above; then we have $\delta-\delta^*=a_0\theta_0+a_1\theta_1+a_2\theta_2$ with $a_0,a_1,a_2\in K$ and $\theta_0,\theta_1,\theta_2\in H^0(\mathbb{P}^1_{K_0},T_{\mathbb{P}^1_{K_0}/K_0})$,

$$\theta_0 t = 1$$

$$\theta_1 t = t$$

$$\theta_2 t = t^2$$

where $\mathbb{P}^1_{K_0}=\mathrm{Proj}\,K_0[t_0,t_1]$, $t=t_1/t_0$. Once again $(\delta-\delta^*)(m_{p_i})\subset m_{p_i}$ and if $m_{p_i}=(t-\lambda_i)$ for $\lambda_i\in K_0$ we get

$$a_0+a_1\lambda_i+a_2\lambda_i^2=0 \quad \text{for } i=1,2,3$$

This implies $a_0=a_1=a_2$ and we conclude again by Lemma 2.

We would like to note that in a similar vein but using some additional tricks one can treat complements of divisors in projective spaces and abelian varieties of dimension ≥ 2 (cf. [2]).

Let's consider the case when U is as in 2) and embed U in a smooth projective surface V. Contracting succesively the exceptional curves of the first kind in $V \setminus U$ we may suppose $V \setminus U$ does not contain such curves.

Since U is affine, $D = V \setminus U$ is a divisor and one can easily see that if $i: U \longrightarrow V$ is the inclusion then (V, D, i) is the smallest compactification of U. By our preparation and since $H^0(V, T_{V/K}) = 0$ we may conclude by Theorem 3. Clearly, the same argument works for a large class of surfaces U, not necessarily of general type.

5. Complete local rings

In this section we discuss the local analog of our theory. As in §1, let K be an algebraically closed field of characteristic zero. A K-singularity will mean any local noetherian complete K-algebra whose residue field is a trivial extension of K; so A is K-isomorphic to $K[[X_1, \ldots, X_n]]/J$ for some $n \geq 1$ and some ideal J. A subfield K_1 of K will be called a field of definition for A if there exists a K-isomorphism as above with J generated by elements of $K_1[[X_1, \ldots, X_n]]$.

Now let $\Delta(A)$ be the set of all derivations $\delta: A \longrightarrow A$ for which $\delta(K) \subset K$ and define

$$K^{\Delta(A)} = \{\lambda \in K;\ \delta\lambda = 0 \quad \text{for all } \delta \in \Delta(A)\}$$

Clearly $K^{\Delta(A)}$ is an algebraically closed subfield of K. We hope the following to be true:

<u>Conjecture 3.</u> If A is a normal isolated K-singularity,

$K^{\Delta(A)}$ is the smallest algebraically closed field of definition for A.

Now it is easy to see (using an argument analog to that given in the beginning of Section 3) that $K^{\Delta(A)}$ is always contained in any algebraically closed field of definition for A; so the hard part of Conjecture 3 says that $K^{\Delta(A)}$ is a field of definition for A. Note also that, exactly as in §1, if Conjecture 3 holds for A and if k is an algebraically closed subfield of K and $\{t_\alpha\}_\alpha$ is a transcendence basis of K/k then k is a field of definition for A if and only if $\partial/\partial t_\alpha : K \longrightarrow K$ lift to derivations $\delta_\alpha : A \longrightarrow A$.

We are able to prove Conjecture 3 in two special cases:

Theorem 6. Conjecture 3 holds in each of the following cases:

1) A is a homogeneous singularity

2) A is a quasi-homogeneous surface singularity .

Recall that a K-singularity is called homogeneous (quasi-homogeneous respectively) if there is a K-isomorphism $A \simeq K[[X_1, \ldots, X_n]]/J$ with J generated by homogeneous polynomials (respectively by polynomials which are quasi-homogeneous with respect to some weights $w_1, \ldots, w_n$ associated to $X_1, \ldots, X_n$).

Theorem 6 will be proved by reduction to the global case.

Suppose first A is a quasi-homogeneous surface singularity, $A = K[[X_1, \ldots, X_n]]/(F_1, \ldots, F_m)$, F_j being quasihomogeneous with respect to the weights $w_1, \ldots, w_n$. Put $B = K[X_1, \ldots, X_n]/(F_1, \ldots, F_m) = \bigoplus_{k=0} B_k$ where B_k is the piece of degree k with respect to the weights. Now there are natural K-linear maps $\varphi_k : A \longrightarrow B_k$ which take the class of a series $f \in K[[X_1, \ldots, X_n]]$ into the class of the polynomial f_k, where f_k is the sum of all monomials of f having degree k (with respect to $w_1, \ldots, w_n$). For any derivation

$\delta\in\Delta(A)$ one can construct in a canonical way a derivation $\tilde{\delta}:B\to B$ with $\tilde{\delta}(B_k)\subset B_k$ and such that δ and $\tilde{\delta}$ coincide on K; indeed for any $b\in B$ write $b=\sum b_k$, $b_k\in B_k$ and put

$$\tilde{\delta}(b)=\sum \varphi_k(\delta b_k)$$

It is trivial to check that $\tilde{\delta}$ has the desired properties. Put $W=\mathrm{Proj}(B[T])$ where weight(T)=1 and extend $\tilde{\delta}$ to a derivation still denoted by $\tilde{\delta}$ on $B[T]$ such that $\tilde{\delta}T=0$. Now W is a projective surface and we consider its normalisation $V=W^{nor}$. Clearly $\tilde{\delta}$ induces a derivation (still denoted by $\tilde{\delta}$) which belongs to $\Delta(W)$. By Seidenberg's theorem [12] this derivation induces a derivation $\tilde{\delta}\in\Delta(V)$. But now $K^{\Delta(V)}\subset K^{(A)}$ so by Theorem 4, $K_o=K^{\Delta(A)}$ is a field of definition for V hence V is K-isomorphic to $V_o\otimes_{K_o}K$ where V_o is some projective normal K_o - surface. So there exists a K_o-point $p_o\in V_o$ such that the only K-point of V lying above it is the isolated singular point p corresponding to the irrelevant ideal of B. Let U_o be an open affine neighbourhood of p_o in V_o, $U_o=\mathrm{Spec}(K_o[X_1,\dots,X_N]/(G_1,\dots,G\ ))$, $p_o=(X_1-\lambda_1,\dots,X_N-\lambda_N)$, $\lambda_j\in K_o$. Then we have K-isomorphisms

$$A\simeq\hat{\mathcal{O}}_{V,p}\simeq((K[X_1,\dots,X_N]/(G_1,\dots,G_M))_{(X_1-\lambda_1,\dots,X_N-\lambda_n)})^{\wedge}\simeq$$

$$\simeq K[[X_1,\dots,X_N]]/(\sigma G_1,\dots,\sigma G_M)$$

where $\sigma:K[[X_1,\dots,X_N]]\to K[[X_1,\dots,X_N]]$ takes X_j into $X_j+\lambda_j$ and we are done because $\sigma G_j\in K_o[[X_1,\dots,X_N]]$.

The proof of Theorem 6 in the homogeneous case is similar and we omit it; istead of using Theorem 4 one has to blow up the vertex of the projective cone W associated to the graded ring of A and to apply Theorem 1 to this blown up cone.

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