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ITERATIVE SYSTEMS OF EQUATIONS

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In some branches of theoretical computer science we meet systems of equations whose solutions are obtained by a fixed-point technique. We quote three of them. a) Every context-free grammar has its system. The least solution gives the languages generated by nonterminals. b) Every flowchart program has its rational system. The least solution gives the unfoldment. The interpretation of the unfoldment gives the program behaviour. c) Every recursive program has its context-free system. The least solution gives the unfoldment. The interpretation of the unfoldment gives the program behaviour.

In the sequel we present, in the category theoretical language, the common methodology to solve such systems. It is founded on a small number of axioms and it presents in a unified manner the main results about such systems. For example, we define an ω-continuous algebraic theory such that every finite system is a morphism whose iterate gives the least solution of the system.

1. Solving equations in an F-algebra

Let $\mathbb{C}$ be an ω-continuous category, i.e. the category $\mathbb{C}$ has the following properties:

a) for every A,B in $|\mathbb{C}|$, the set $\mathbb{C}(A,B)$ is an ω-complete poset having a least element $\perp_{A,B}$,

b) the composition of morphisms is ω-continuous and left strict ($\perp_{A,B}g = \perp_{A,C}$ for every $g \in \mathbb{C}(B,C)$).

Let $F:\mathbb{C} \longrightarrow \mathbb{C}$ be a locally ω-continuous functor, that is for every A,B in $|\mathbb{C}|$ the restriction of F to $\mathbb{C}(A,B)$ is an ω-continuous function.

An F-algebra is an ordered pair (A,α) where $A \in |\mathbb{C}|$ and $\alpha : F(A) \longrightarrow A$ is a morphism in $\mathbb{C}$. An F-algebra morphism from (A,α) to (B,β) is a morphism $g:A \longrightarrow B$ such that $F(g)\beta = \alpha g$.

Let $s:X \longrightarrow F(X)$ be a morphism of $\mathbb{C}$. We shall say that s is a system of equations.

We shall solve the system s in every F-algebra. We say that the morphism $f:X \longrightarrow A$ is a solution of s in the F-algebra (A,α) if

$$f = sF(f)\alpha .$$

1.1. Proposition. The equation $s:X \longrightarrow F(X)$ has in every F-algebra (A,α) a smallest solution s_α.

Proof. Let $j_\alpha : \mathbb{C}(X,A) \longrightarrow \mathbb{C}(X,A)$ be the function defined by $j_\alpha(f) = sF(f)\alpha$ for every $f \in \mathbb{C}(X,A)$. Let us notice that $f \in \mathbb{C}(X,A)$ is a solution of s in (A,α) if and only if f is a fixed-point of j_α. As F is locally ω-continuous and the composition of $\mathbb{C}$ is ω-continuous, we deduce that j_α is an ω-continuous function. It follows from the Kleene fixed-point theorem that

$$s_\alpha = \bigvee \{ j_\alpha^n(\perp_{X,A}) \mid n \in \omega \}$$

is the least solution of s. ∎

If $f:X \longrightarrow A$ is a solution of s in the F-algebra (A,α) and $g:(A,\alpha) \longrightarrow (B,\beta)$ is an F-algebra morphism then fg is a solution of s in (B,β). Indeed, $j_\beta(fg) = sF(fg)\beta = sF(f)F(g)\beta = sF(f)\alpha g = fg$.

1.2. Proposition. If $g:(A,\alpha) \longrightarrow (B,\beta)$ is an F-algebra morphism then $s_\beta = s_\alpha g$.

Proof. We prove by induction that $j_\beta^n(\perp_{X,B}) = j_\alpha^n(\perp_{X,A})g$. As the composition of morphisms is left strict we deduce $\perp_{X,B} = \perp_{X,A}g$. If the equality holds for $n \in \omega$ then

$$j_\beta^{n+1}(\perp_{X,B}) = sF(j_\beta^n(\perp_{X,B}))\beta = sF(j_\alpha^n(\perp_{X,A}))F(g)\,\beta =$$

$$= sF(j_\alpha^n(\perp_{X,A}))\alpha\, g = j_\alpha^{n+1}(\perp_{X,A})g \,.$$

Therefore $s_\alpha\, g = (\bigvee\{j_\alpha^n(\perp_{X,A}) \mid n \in \omega\})g =$

$$= \bigvee\{j_\alpha^n(\perp_{X,A})g \mid n \in \omega\} = \bigvee\{j_\beta^n(\perp_{X,B}) \mid n \in \omega\} = s_\beta \,. \bullet$$

Let us suppose that there exists an initial F-algebra (I,i). The above proposition allows us to compute the smallest solution of s in every F-algebra (A,α) from the smallest solution of s in (I,i). Indeed, if $g_\alpha:(I,i) \longrightarrow (A,\alpha)$ is the unique F-algebra morphism then $s_\alpha = s_i g_\alpha$.

Let $U:\mathcal{D} \longrightarrow \mathcal{C}$ be a functor such that there exists a natural transformation $\emptyset:UF \longrightarrow U$. We may "solve" the system $s:X \longrightarrow F(X)$ in every $D \in |\mathcal{D}|$.

Let $D \in |\mathcal{D}|$. A morphism $f:X \longrightarrow U(D)$ is said to be a solution of s in D if f is a solution of s in the F-algebra $(U(D),\emptyset_D)$. It follows from proposition 1.1 that there exists a smallest solution, s_D, of s in D.

Let us notice that if $g \in \mathcal{D}(D,C)$ then U(g) is a morphism of F-algebras $(U(D),\emptyset_D)$ and $(U(C),\emptyset_C)$, therefore $s_C = s_D U(g)$ and $fU(g)$ is a solution of s in C for every solution f of s in D. Moreover if I in an initial object of $\mathcal{D}$ and α_C the unique morphism of $\mathcal{D}$ from I to $C \in |\mathcal{D}|$ then $s_C = s_I U(\alpha_C)$.

2. Solving equations in a category

We study another case that is closer to the practice, where the category $\mathcal{C}$ is not always an ω-continuous one.

Let $\mathcal{C}$ be a category and let $A \in |\mathcal{C}|$. Let us denote by $h_A:\mathcal{C} \longrightarrow$ Set the functor defined by

a) $h_A(B) = \mathcal{C}(A,B)$ for every $B \in |\mathcal{C}|$

b) $h_A(f)(g) = gf$ for every $f \in \mathbb{C}(B,C)$ and $g \in \mathbb{C}(A,B)$.

Let $U: \mathbb{D} \longrightarrow \mathbb{C}$ be a functor. Our hypotheses are:

(2.1) for every $X \in |\mathbb{C}|$ and $A \in |\mathbb{D}|$ the set $\mathbb{C}(X,U(A))$ is an ω-complete poset and has a least element $\perp_A$,

(2.2) for every $X \in |\mathbb{C}|$ and $f \in \mathbb{D}(A,B)$ the function $h_X(U(f))$ is ω-continuous and strict,

(2.3) for every $X \in |\mathbb{C}|$ there exist $VX \in |\mathbb{D}|$ and a natural transformation

$$\varphi_X : Uh_X \longrightarrow h_{VX}$$

such that for every $s \in \mathbb{C}(X,U(VX))$ and for every $A \in |\mathbb{D}|$ the function

$$\varrho_A : \mathbb{C}(X,U(A)) \longrightarrow \mathbb{C}(X,U(A))$$

defined for each $f \in \mathbb{C}(X,U(A))$ by $\varrho_A(f) = sU(\varphi_{X,A}(f))$, is ω-continuous.

The morphism $s: X \longrightarrow U(VX)$ from $\mathbb{C}$ is said to be a <u>system</u>.

Let $A \in |\mathbb{D}|$. A morphism $f \in \mathbb{C}(X,U(A))$ is said to be a <u>solution</u> of s in A if $s\,U(\varphi_{X,A}(f)) = f$. Let us notice that $f \in \mathbb{C}(X,U(A))$ is a solution of s in A if and only if f is a fixed-point of ϱ_A. The hypotheses (2.1) and (2.3) show that we may apply the Kleene fixed-point theorem to prove the following proposition.

2.1. <u>Proposition</u>. The equation $s \in \mathbb{C}(X,U(VX))$ has in every $A \in |\mathbb{D}|$ a least solution

$$s_A = \bigvee \{ \varrho_A^n(\perp_A) \mid n \in \omega \} . \bullet$$

2.2. <u>Proposition</u>. Let $g \in \mathbb{D}(A,B)$. If f is a solution of s in A, then $fU(g)$ is a solution of s in B.

<u>Proof</u>. As φ_X is a natural transformation we deduce

$s\,U(\varphi_{X,B}(fU(g))) = s\,U(\varphi_{X,A}(f)g) =$

$= s\,U(\varphi_{X,A}(f))U(g)$.

As f is a solution of s in A it follows that fU(g) is a solution of s in B. ●

2.3. Proposition. If $g \in \mathbb{D}(A,B)$ then $s_B = s_A U(g)$.

Proof. We prove by induction on $n \in \omega$ that

$$\rho_A^n(\perp_A)U(g) = \rho_B^n(\perp_B) .$$

As $h_X(U(g))$ is strict $\perp_B = h_X(U(g))(\perp_A) = \perp_A U(g)$. If the above equality is true for $n \in \omega$ then

$$\rho_A^{n+1}(\perp_A)U(g) = s\ U(\varphi_{X,A}(\rho_A^n(\perp_A)))U(g) =$$

$$= s\ U(\varphi_{X,A}(\rho_A^n(\perp_A))g) = s\ U(\varphi_{X,B}(\rho_A^n(\perp_A)U(g))) =$$

$$= s\ U(\varphi_{X,B}(\rho_B^n(\perp_B))) = \rho_B^{n+1}(\perp_B) .$$

As $h_X(U(g))$ is ω-continuous we deduce

$$s_A U(g) = h_X(U(g))(\bigvee\{\rho_A^n(\perp_A) \mid n \in \omega\}) =$$

$$= \bigvee\{\rho_A^n(\perp_A)U(g) \mid n \in \omega\} = \bigvee\{\rho_B^n(\perp_B) \mid n \in \omega\} = s_B . ●$$

Let I be an initial object of $\mathbb{D}$. For every $A \in |\mathbb{D}|$ let us denote by $\alpha_A : I \longrightarrow A$ the unique morphism of $\mathbb{D}$ from I to A.

2.4. Corollary. If $\mathbb{D}$ has an initial object I then $s_A = s_I U(\alpha_A)$ for every $A \in |\mathbb{D}|$.

2.5. Proposition. Let I be an initial object of $\mathbb{D}$ and let $u : X \longrightarrow U(I)$. The system $s = uU(\alpha_{VX}) : X \longrightarrow U(VX)$ has in every $A \in |\mathbb{D}|$ a unique solution $uU(\alpha_A)$.

Proof. Let $A \in |\mathbb{D}|$. If $f \in \mathbb{C}(X, U(A))$ then

$$\rho_A(f) = s\ U(\varphi_{X,A}(f)) = u\ U(\alpha_{VX}\varphi_{X,A}(f)) = u\ U(\alpha_A)$$

therefore f is a solution of s in A if and only if $f = u\ U(\alpha_A)$. ●

We feel that the present context is too general to study systems of equations with parameters. We will do it in the following sections.

3. The least solution is an iterate

We are going to study in the sequel a more particular case. We shall prove that we may obtain an ω-continuous algebraic theory such that the least solution of a finite system of equations is the iterate of the system.

Let $U: \mathbb{D} \longrightarrow Set_S$ be a functor. We assume that U has a left adjoint, that is

(3.1) for every $X \in |Set_S|$ there exist $VX \in |\mathbb{D}|$ and $\varepsilon_X \in Set_S(X, U(VX))$ such that for every $A \in |\mathbb{D}|$ and every $f \in Set_S(X, U(A))$ there exist a unique $f^{\#} \in \mathbb{D}(VX, A)$ such that $\varepsilon_X U(f^{\#}) = f$.

Let us first notice that $f \in Set_S(X, U(A))$ and $g \in \mathbb{D}(A, B)$ imply $f^{\#}g = (fU(g))^{\#}$. Indeed, the conclusion follows from the uniqueness part of (3.1) because

$$\varepsilon_X\, U(f^{\#}g) = \varepsilon_X\, U(f^{\#})U(g) = f\, U(g) .$$

If we write $\varphi_{X,A}(f) = f^{\#}$ for every $f \in Set_S(X, U(A))$ we deduce that $\varphi_X : Uh_X \longrightarrow h_{VX}$ is a natural transformation for every $X \in |Set_S|$.

For every $a \in S^{*}$ we denote

$$Xa = \{x_{a,1}, x_{a,2}, \ldots, x_{a,|a|}\}$$

This set is S-sorted by sort $(x_{a,i}) = a_i$ for every $i \in [|a|]$.

Let $T(a,b) = Set_S(Xa, U(VXb))$ for every a,b in S^{*}.

We define the composition by

$$f \circ g = fU(g^{\#})$$

for every $f \in T(a,b)$ and $g \in T(b,c)$. If $f \in T(a,b)$, $g \in T(b,c)$ and $h \in T(c,d)$ then

$$f \circ (g \circ h) = fU((gU(h^{\#}))^{\#}) = fU(g^{\#}h^{\#}) = fU(g^{\#})U(h^{\#}) = (f \circ g) \circ h$$

therefore the composition is associative. We prove that $1_a = \varepsilon_{Xa}$ is the identity morphism of $a \in S^{*}$. If $g \in T(a,b)$ then $1_a \circ g = \varepsilon_{Xa} U(g^{\#}) = g$.

If $f \in T(b,a)$ then $f \circ 1_a = fU(\varepsilon^{\#}_{Xa}) = fU(1_{VXa}) = f$. Therefore T is an S-sorted category.

For $a \in S^*$ and $i \in [|a|]$ let $j_i \in Set_S(Xa_i, Xa)$ be the function defined by $j_i(x_{a_i,1}) = x_{a,i}$.

Let $a \in S^*$. For $i \in [|a|]$ we define $x^a_i \in T(a_i, a)$ by $x^a_i = j_i \varepsilon_{Xa}$. If $f_i \in T(a_i, b)$ for every $i \in [|a|]$ then there exists a unique $f \in T(a,b)$ such that $x^a_i \circ f = f_i$ for every $i \in [|a|]$. Therefore T is an S-sorted algebraic theory.

Remark. If U is the forgetful functor from the category of $\sum$-algebras to the category of S-sorted sets then T is the free algebraic theory generated by $\sum$.

Our second hypothesis is:

(3.2) for every $A \in |D|$, each component of the S-sorted set U(A) is a strict ω-complete poset.

Let $X \in |Set_S|$ and $A \in |D|$. The set $Set_S(X, U(A))$ may be ordered pointwise : $f \leq g$ if and only if $f(x) \leq g(x)$ for every x in X. If follows from (3.2) that $Set_S(X, U(A))$ is a strict ω-complete poset, therefore (2.1) holds and $T(a,b) = Set_S(Xa, U(VXb))$ is a strict ω-complete poset.

The tupling in T in increasing. Let a,b in S^* and for every $i \in [|a|]$ let $f_i \leq g_i$ in $T(a_i, b)$. We deduce that for every $i \in [|a|]$

$$\langle f_1, f_2, \ldots, f_n \rangle (x_{a,i}) = f_i(x_{a_i,1}) \leq g_i(x_{a_i,1}) = \langle g_1, \ldots, g_n \rangle (x_{a,i})$$

therefore $\langle f_1, f_2, \ldots, f_n \rangle \leq \langle g_1, g_2, \ldots, g_n \rangle$.

Our third hypothesis is:

(3.3) for every $f \in D(A,B)$, each component of the S-sorted function $U(f): U(A) \longrightarrow U(B)$ is ω-continuous and strict.

We prove that (2.2) holds. Let $f \in D(A,B)$. If $\{f_n\}_{n \in \omega}$ is an increasing sequence from $Set_S(X, U(A))$ then for every x in X

$$h_X(U(f))\,(\bigvee_{n\in\omega} f_n)(x) = U(f)(\bigvee_{n\in\omega} f_n(x)) = \bigvee_{n\in\omega} U(f)(f_n(x)) =$$

$= \bigvee_{n\in\omega} h_X(U(f))(f_n)(x)$. If $\perp_A$ is the least element of $Set_S(X,U(A))$ then for each x in X

$$h_X(U(f))(\perp_A)(x) = U(f)(\perp_A(x)) = \perp_B(x) .$$

We deduce that the composition in T is left ω-continuous and left strict. If $f \in T(b,c)$ then $\perp_{a,b} \circ f = \perp_{a,b} U(f^{\#}) =$ $= h_{Xa}(U(f^{\#}))(\perp_{a,b}) = \perp_{a,c}$. If $f \in T(b,c)$ and $\{f_n\}_{n\in\omega}$ is an increasing sequence in $T(a,b)$ then $(\bigvee_{n\in\omega} f_n)\circ f = (\bigvee_{n\in\omega} f_n)U(f^{\#}) =$ $h_{Xa}(U(f^{\#}))(\bigvee_{n\in\omega} f_n) = \bigvee_{n\in\omega} h_{Xa}(U(f^{\#}))(f_n) = \bigvee_{n\in\omega} f_n U(f^{\#}) = \bigvee_{n\in\omega} f_n \circ f$.

Our last hyphotesis is:

(3.4) for every $X \in |Set_S|$ and $A \in |D|$ the function from $Set_S(X,U(A))$ to $Set_S(U(VX),U(A))$ which maps $f \in Set_S(X,U(A))$ in $U(f^{\#})$, is ω-continuous.

It follows from it that (2.3) holds. Let $X \in |Set_S|$, $A \in |D|$ and $s \in Set_S(X,U(VX))$. If $\{f_n\}_{n\in\omega}$ is an increasing sequence from $Set_S(X,U(A))$ then for every x in X

$$\varrho_A(\bigvee_{n\in\omega} f_n)(x) = U((\bigvee_{n\in\omega} f_n)^{\#})(s(x)) = (\bigvee_{n\in\omega} U(f_n^{\#}))(s(x)) =$$

$$= \bigvee_{n\in\omega} U(f_n^{\#})(s(x)) = \bigvee_{n\in\omega} \varrho_A(f_n)(x) = (\bigvee_{n\in\omega} \varrho_A(f_n))(x).$$

We deduce that the composition in T is right ω-continuous. If $f \in T(a,b)$ and $\{f_n\}_{n\in\omega}$ is an increasing sequence in $T(b,c)$ then for every $x \in Xa$

$$(f\circ(\bigvee_{n\in\omega} f_n))(x) = U((\bigvee_{n\in\omega} f_n)^{\#})(f(x)) = (\bigvee_{n\in\omega} U(f_n^{\#}))(f(x)) =$$

$= \bigvee_{n\in\omega} U(f_n^{\#})(f(x)) = \bigvee_{n\in\omega} (f\circ f_n)(x) = (\bigvee_{n\in\omega} f\circ f_n)(x)$. Therefore T is an ω-continuous S-sorted algebraic theory.

<u>Remark</u>. If U is the forgetful functor from the category of ω-continuous $\sum$-algebras to the category of S-sorted sets then T

is the free ω-continuous algebraic theory generated by Σ, which is as usual denoted by CT_{Σ}. ●

We have left out above the ordered case. If we omit "ω-complete" in (3.2) and we replace "ω-continuous" by increasing in (3.3) and in (3.4) then we may prove that T is an ordered S-sorted algebraic theory. If U is the forgetful functor from the category of ordered Σ-algebras to the category of S-sorted sets then T is the free ordered algebraic theory generated by Σ.

As $X\lambda$ is the initial object of Set_S we deduce from (3.1) that $VX\lambda$ is an initial object of D.

Let us remark that we may apply all the propositions of the previous section to solve systems of equations.

We shall study only finite systems of equations. The system $s:X \longrightarrow U(VX)$ is said to be finite if the set X is finite. Let $X = \{x_1, x_2, \ldots, x_n\}$ and let a_i be the sort of x_i. As the S-sorted set X is isomorphic to Xa, we may define, without loss of the generality, a finite system as a function $s:Xa \longrightarrow U(VXa)$ that is a morphism $s \in T(a,a)$. It follows from proposition 2.1 that

$$s_{VX\lambda} = \bigvee\{\varrho^n_{VX\lambda}(\perp_{VX\lambda}) \mid n \in \omega\}.$$

Let us notice that $\perp_{VX\lambda} = \perp_{a,\lambda}$. If $\varrho^n_{VX\lambda}(\perp_{VX\lambda}) = s^n\perp_{a,\lambda}$ then $\varrho^{n+1}_{VX\lambda}(\perp_{VX\lambda}) = \varrho_{VX\lambda}(s^n\perp_{a,\lambda}) = sU((s^n\perp_{a,\lambda})^{\#}) = s^{n+1}\perp_{a,\lambda}$.

Therefore $s_{VX\lambda} = \bigvee\{s^n\perp_{a,\lambda} \mid n \in \omega\} = s^{\dagger}$ by the definition of the iterate is an ω-continuous theory. Corollary 2.4 shows that using $s^{\dagger}$ we may compute the least solution of s in every $A \in |D|$.

3.1. Example. The rational systems. Let $r:\Sigma \longrightarrow S \times S^{*}$ be a signature. Let $U: \omega Alg_{\Sigma} \longrightarrow Set_S$ be the forgetful functor from the category of ω-continuous Σ-algebras to the category of S-sorted sets.

Let us notice that the functor U fulfills the conditions (3.1), (3.2), (3.3) and (3.4). Indeed, it is very well known that U has a left adjoint $V:Set_S \longrightarrow \omega Alg_\Sigma$ which maps every S-sorted set X into VX, the ω-continuous Σ-algebra freely generated by X. If Ⓐ $= (A, \sigma_A, \leq, \bot_A)$ is an ω-continuous Σ-algebra then each component of the S-sorted set A is a strict ω-complete poset. If $f:(A,\sigma_A,\leq,\bot_A) \longrightarrow (B,\sigma_B,\leq,\bot_B)$ is a morphism of ω-continuous Σ-algebras then each component of $f:(A,\leq,\bot_A) \longrightarrow (B,\leq,\bot_B)$ is ω-continuous and strict. Let Ⓐ $= (A,\sigma_A,\leq,\bot_A)$ be an ω-continuous Σ-algebra. If $f:X \longrightarrow A$ is a function let us denote by $f^{\#}:VX \longrightarrow A$ the unique morphism of ω-continuous Σ-algebras such that $f^{\#}(x) = f(x)$ for every x in X. Let $f_n:X \longrightarrow A$ be an increasing sequence of functions $(f_n(x) \leq f_{n+1}(x), \forall x \in X, \forall n \in \omega)$. It is easy to prove that the function $h:VX \longrightarrow A$ defined by $h(y) = \bigvee\{f_n^{\#}(y) \mid n \in \omega\}$ for every y in VX is an ω-continuous Σ-algebra morphism, therefore $h = (\bigvee\{f_n \mid n \in \omega\})^{\#}$. Hence condition (3.4) is also fulfilled.

It is known that VX contains as a subalgebra the free Σ-algebra generated by X.

A system is an S-sorted function $s:X \longrightarrow VX$. The system s is said to be <u>rational</u> if s assigns to every x in X, an element from the free Σ-algebra generated by X.

Let Ⓐ $= (A,\sigma_A,\leq,\bot_A)$ be an ω-continous Σ-algebra. It follows from proposition 2.1 that s has a least solution in Ⓐ

$$s_{Ⓐ} = \bigvee\{\varrho_{Ⓐ}^{n}(\bot_{Ⓐ}) \mid n \in \omega\}$$

where $\bot_{Ⓐ}(x) = \bot_A$ for every x in X and the function $\varrho_{Ⓐ} : Set_S(X,A) \longrightarrow Set_S(X,A)$ is defined for every $f:X \longrightarrow A$ by $\varrho_{Ⓐ}(f) = sf^{\#}$. As $V\emptyset$ is the initial object ωAlg_Σ it follows from corollary 2.4 that $s_{Ⓐ} = s_{V\emptyset}h_{Ⓐ}$ where $h_{Ⓐ}:V\emptyset \longrightarrow$ Ⓐ is the unique ω-continuous Σ-algebra morphism from $V\emptyset$ to Ⓐ. ●

3.2. Example. The context-free systems

Let $r:\Sigma \longrightarrow S\times S^{*}$ be a signature.

Let D be the category whose objects are the interpretation of Σ in an arbitrary ω-continuous algebraic theory. If $I:\Sigma$ — and $I':\Sigma \longrightarrow T'$ are objects of D then $D(I,I')$ is the set of all ω-continuous algebraic theory morphisms $F:T \longrightarrow T'$ such that $IF = I'$ (i.e. $F(I(\sigma)) = I'(\sigma)$ for every $\sigma \in \Sigma$).

The functor

$$U: D \longrightarrow Set_{S\times S^{*}}$$

is defined by:

a) $U(I:\Sigma \longrightarrow T) = \{T(s,a)\}_{(s,a)\in S\times S^{*}}$

b) if $F\in D(I,I')$ then $U(F)_{(s,a)}$ is the restriction of F to $T(s,a)$.

It can be proved that U satisfies (3.1), (3.2), (3.3) and (3.4). Let us mention that if X is an $S\times S^{*}$-sorted set then VX and ε_X are the restrictions of the standard interpretation of $\Sigma\cup X$ in $CT_{\Sigma\cup X}$ to Σ and X, respectively. ●

4. Finite systems of equations with parameters

We use the same hypotheses and notations as in the previous section.

4.1. Definition. A morphism $s\in T(a,ab)$ is said to be a finite system of equations with parameters. ●

If $g\in Set_S(Xa,X)$ and $f\in Set_S(Xb,X)$ we denote by $\langle g,f\rangle \in Set_S(Xab,X)$ the function defined by $\langle g,f\rangle(x_{ab,i}) =$ $=$ if $i\in[|a|]$ then $g(x_{a,i})$ else $f(x_{b,i-|a|})$.

To solve the system $s\in T(a,ab)$ we fix $A\in|D|$ and $f\in Set_S(Xb,U(A))$. The function

$$\rho_{A,f} : Set_S(Xa,U(A)) \longrightarrow Set_S(Xa,U(A))$$

is defined for every $g \in Set_S(Xa, U(A))$ by

$$\varrho_{A,f}(g) = s\ U(\langle g, f\rangle^{\#}).$$

A fixed-point of $\varrho_{A,f}$ is said to be a solution of s in A for f. If f_n is an increasing sequence from $Set_S(Xa, U(A))$ it follows from (3.4) that

$$U(\langle g, \bigvee_{n\in\omega} f_n\rangle^{\#}) = U((\bigvee_{n\in\omega}\langle g, f_n\rangle)^{\#}) = \bigvee_{n\in\omega} U(\langle g, f_n\rangle^{\#})$$

therefore $\varrho_{A,f}$ is an ω-continuous function. The Kleene fixed-point theorem tells us that

$$s_A(f) = \bigvee_{n\in\omega} \varrho^n_{A,f}(\bot_A)$$

is the least solution of s in A for f.

4.2. Proposition. Let $h \in \underline{D}(A,B)$ and $f \in Set_S(Xb, U(A))$. If g is a solution of s in A for f then $gU(h)$ is a solution of s in B for $fU(h)$.

Proof. If $g \in Set_S(Xa, U(A))$ and $\varrho_{A,f}(g) = g$ then

$$\varrho_{B,fU(h)}(gU(h)) = s\ U(\langle gU(h), fU(h)\rangle^{\#}) =$$

$$= s\ U((\langle g, f\rangle\ U(h))^{\#}) = s\ U(\langle g, f\rangle^{\#} h) = \varrho_{A,f}(g)U(h) = gU(h).$$ ∎

4.3. Proposition. If $h \in \underline{D}(A,B)$ then

$$s_A(f)U(h) = s_B(fU(h)).$$

Proof. We show by induction on n that

$$\varrho^n_{A,f}(\bot_A)U(h) = \varrho^n_{B,fU(h)}(\bot_B).$$

For n=0, $\bot_A U(h) = \bot_B$ by (3.3). If the above equality holds then

$$\varrho^{n+1}_{A,f}(\bot_A)U(h) = s\ U(\langle \varrho^n_{A,f}(\bot_A), f\rangle^{\#})U(h) =$$

$$= s\ U((\langle \varrho^n_{A,f}(\bot_A), f\rangle\ U(h))^{\#}) =$$

$$= s\ U(\langle \varrho^n_{B,fU(h)}(\bot_B), fU(h)\rangle^{\#}) = \varrho^{n+1}_{B,fU(h)}(\bot_B).$$

Using again (3.3) we deduce for every $i \in [|a|]$ that

$$(s_A(f)U(h))(x_{a,i}) = U(h)(\bigvee_{n\in\omega} \varrho^n_{A,f}(\bot_A)(x_{a,i})) =$$

$$= \bigvee_{n\in\omega} (\varrho^n_{A,f}(\bot_A)U(h))(x_{a,i}) = \bigvee_{n\in\omega} \varrho^n_{B,fU(h)}(\bot_B)(x_{a,i}) =$$

$$= s_B(fU(h))(x_{a,i}) ,$$

therefore $s_A(f)U(h) = s_B(fU(h))$. ●

4.4. Remark. Let $s' \in Set_S(Xa, U(VXb))$ and $j: Xb \longrightarrow Xab$ be the function defined by $j(x_{b,i}) = x_{ab,|a|+i}$ for $i \in [|a|]$. The system $s = s'U((j\varepsilon_{Xab})^{\#})$ has in every $A \in |\mathbb{D}|$ and for every $f \in Set_S(Xb, U(A))$ a unique solution $s'U(f^{\#})$.

Proof. Let $A \in |\mathbb{D}|$. If $g \in Set_S(Xa, U(A))$ then

$$\varrho_{A,f}(g) = s'U((j\varepsilon_{Xab})^{\#})U(\langle g,f\rangle^{\#}) =$$

$$= s'U((j\varepsilon_{Xab}\langle g,f\rangle^{\#})^{\#}) = s'U((j\langle g,f\rangle)^{\#}) =$$

$$= s'U(f^{\#})$$

therefore g is a solution of s in A for f if and only if $g = s'U(f^{\#})$.

4.5. Proposition. If $s \in T(a,ab)$ then $s^{\dagger} = s_{VXb}(\varepsilon_{Xb})$.

Proof. By definition $s^{\dagger} = \bigvee_{n\in\omega} s^{(n)}$ where $s^{(o)} = \bot_{a,b}$ and $s^{(n+1)} = s \circ \langle s^{(n)}, 1_b\rangle$ for $n \in \omega$.

We show by induction on $n \in \omega$ that

$$s^{(n)} = \varrho^n_{VXb,\varepsilon_{Xb}}(\bot_{VXb}).$$

For n=0 we remark that $\bot_{a,b} = \bot_{VXb}$ by definitions. For the inductive step we need the following computation where we omit the indices of ϱ:

$$\varrho^{n+1}(\bot_{VXb}) = \varrho(\varrho^n(\bot_{VXb})) = s\ U(\langle \varrho^n(\bot_{VXb}), \varepsilon_{Xb}\rangle^{\#}) =$$

$$= s\ U(\langle s^{(n)}, 1_b\rangle^{\#}) = s \circ \langle s^{(n)}, 1_b\rangle = s^{(n+1)}.$$

Therefore

$$s_{VXb}(\varepsilon_{Xb}) = \bigvee_{n\in\omega} \varrho^n_{VXb,\varepsilon_{Xb}}(\bot_{VXb}) = \bigvee_{n\in\omega} s^{(n)} = s^{\dagger} .●$$

4.6. Proposition. If $s \in T(a,ab)$, $A \in |\mathcal{D}|$ and $f \in Set_S(Xb,U(A))$ then

$$s_A(f) = s^{\dagger}U(f^{\#})$$

Proof. As $f^{\#} \in \mathcal{D}(VXb,A)$ it follows from proposition 4.3 that

$$s_{VXb}(\varepsilon_{Xb})U(f^{\#}) = s_A(\varepsilon_{Xb}U(f^{\#})) = s_A(f) .$$

The conclusion is obtained using the previous proposition.●

At this point a comment is welcome. Proposition 4.6 gives another method to solve the system s in A for f. We compute the formal solution $s^{\dagger}$ and then we interpret it, i.e. we compute $s^{\dagger}U(f^{\#})$. We say the solution $s^{\dagger}$ is formal because we need not know A and f to compute it. In other words, we may compute $s^{\dagger}$ without interpreting the system s and the parameters.

We may try two other ways to solve the system s. The first one is to interpret the system s in $A \in |\mathcal{D}|$, to solve it and then to interpret the parameters. The second way is to interpret the parameters in the system before solving it. Both ways give the least solution.

Let $A \in |\mathcal{D}|$. We assume that the coproduct $A + VXb$ exists in $\mathcal{D}$ and we denote by $i_A : A \longrightarrow A+VXb$ and $i : VXb \longrightarrow A+VXb$ its structural morphisms. If $f \in Set_S(Xb,U(A))$ let us denote by $\langle 1_A, f^{\#} \rangle \in \mathcal{D}(A+VXb,A)$ the unique morphism such that $i_A \langle 1_A, f^{\#} \rangle = 1_A$ and $i \langle 1_A, f^{\#} \rangle = f^{\#}$.

4.7. Proposition. If $s \in T(a,ab)$, $A \in |\mathcal{D}|$ and $f \in Set_S(Xb,U(A))$ then

$$s_A(f) = s_{A+VXb}(\varepsilon_{Xb}U(i))U(\langle 1_A, f^{\#} \rangle).$$

Proof. It follows from proposition 4.3 that

$$s_{A+VXb}(\varepsilon_{Xb}U(i))U(\langle 1_A, f^{\#}\rangle) =$$

$$= s_A(\varepsilon_{Xb}U(i)U(\langle 1_A, f^{\#}\rangle)) = s_A(\varepsilon_{Xb}U(f^{\#})) = s_A(f) \ .\bullet$$

Let $A \in |\mathbb{D}|$. We assume that the coproduct A+VXa exists in $\mathbb{D}$ and we denote by $i_A : A \longrightarrow A+VXa$ and $i : VXa \longrightarrow A+VXa$ its structural morphisms.

Let $f \in Set_S(Xb, U(A))$. If we interpret the parameters in the system s we obtain the function

$$s' = s\ U(\langle \varepsilon_{Xa}U(i)\ , fU(i_A)\rangle^{\#}\ ).$$

The function $s' : Xa \longrightarrow U(A+VXa)$ may be thought as another kind of system and we may try to solve it by the fixed-point technique applied to a function

$$\eta : Set_S(Xa, U(A)) \longrightarrow Set_S(Xa, U(A)).$$

Let $g \in Set_S(Xa, U(A))$. Let $\langle 1_A, g^{\#}\rangle \in \mathbb{D}(A+VXa, A)$ be the unique morphism such that $i_A \langle 1_A, g^{\#}\rangle = 1_A$ and $i \langle 1_A, g^{\#}\rangle = g^{\#}$. By definition $\eta(g) = s'U(\langle 1_A, g^{\#}\rangle)$. The point is that the function η is equal to $\varrho_{A,f}$. Indeed

$$\eta(g) = s\ U(\langle \varepsilon_{Xa}U(i), fU(i_A)\rangle^{\#}\langle 1_A, g^{\#}\rangle) =$$

$$= s\ U((\langle \varepsilon_{Xa}U(i), fU(i_A)\rangle\ U(\langle 1_A, g^{\#}\rangle))^{\#}) =$$

$$= s\ U(\langle \varepsilon_{Xa}U(i\langle 1_A, g^{\#}\rangle), fU(i_A\langle 1_A, g^{\#}\rangle)\rangle^{\#}) =$$

$$= s\ U(\langle \varepsilon_{Xa}U(g^{\#}), fU(1_A)\rangle^{\#}) =$$

$$= s\ U(\langle g, f\rangle^{\#}) = \varrho_{A,f}(g)\ .$$

Therefore $s_A(f) = \bigvee_{n\in\omega} \eta^n(\bot_A)$. Let us mention that even if the functions $\varrho_{A,f}$ and η are equal the difficulties in computing $\varrho_{A,f}(g)$ and $\eta(g)$ for a given $g \in Set_S(Xa, U(A))$ are different.

We come back to proposition 4.6. It suggests the definition of a functor from T to $\omega Pow_{U(A)}$. First we prefer to explain the last notation. Let $M = \{M_s\}_{s\in S}$ be an S-sorted set such that each M_s is endowed with a strict ω-complete order. We prefer this time to denote by $\omega Pow_M(a,b)$ the set of all ω-continuous functions from $Set_S(Xb,M)$ to $Set_S(Xa,M)$. As usual $Set_S(Xa,M)$ is ordered pointwise. It is very well known that ωPow_M is an ω-continuous algebraic theory. As our notation is not the usual one we recall some details. If $f \in \omega Pow_M(a,b)$ and $g \in \omega Pow_M(b,c)$ as usual $f \circ g = gf$. Let us recall that $\perp_{a,b}$ is the constant function mapping the set $Set_S(Xb,M)$ to the least element of $Set_S(Xa,M)$. If $\{f_n\}_{n\in\omega}$ is an increasing sequence from $\omega Pow_M(a,b)$ then $(\bigvee_n f_n)(h) = \bigvee_n f_n(h)$ for every $h \in Set_S(Xb,M)$. For every $i \in [|a|]$ and $h \in Set_S(Xa,M)$ we have $x_i^a(h)(x_{a_i,1}) = h(x_{a,i})$. If $f \in \omega Pow_M(a,c)$ and $g \in \omega Pow_M(b,c)$ then for every $h \in Set_S(Xc,M)$ and every $i \in [|ab|]$

$$\langle f,g\rangle (h)(x_{ab,i}) = \text{if } i \leq |a| \text{ then } f(h)(x_{a,i}) \text{ else } g(h)(x_{b,i-|a|})$$

For every $A \in |D|$ we define the functor

$$F(A): T \longrightarrow \omega Pow_{U(A)}$$

by

$$F(A)(f)(h) = fU(h^{\#})$$

for every $f \in T(a,b)$ and $h \in Set_S(Xb,U(A))$.

4.8. <u>Proposition.</u> F(A) is a morphism of ω-continuous S-sorted algebraic theories.

<u>Proof.</u> If $f \in T(a,b)$ and $g \in T(b,c)$ then for every $h \in Set_S(Xc,U(A))$

$$(F(A)(f) \circ F(A)(g))(h) = F(A)(f)(F(A)(g)(h)) =$$
$$= f\,U((gU(h^{\#}))^{\#}) = f\,U(g^{\#}h^{\#}) = f\,U(g^{\#})U(h^{\#}) =$$
$$= (f \circ g)U(h^{\#}) = F(A)(f \circ g)(h) \text{ therefore } F(A)(f) \circ F(A)(g) = F(A)(f \circ g).$$

If $i \in [|a|]$ then for every $h \in Set_S(Xa, U(A))$

$$F(A)(x_i^a)(h)(x_{a_i,1}) = U(h^{\#})(x_i^a(x_{a_i,1})) =$$

$$U(h^{\#})(\varepsilon_{Xa}(x_{a,i})) = h(x_{a,i}) = x_i^a(h)(x_{a_i,1})$$

therefore $F(A)(x_i^a) = x_i^a$.

For every $h \in Set_S(Xb, U(A))$ and for every $i \in [|a|]$, $F(A)(\perp_{a,b})(h)(x_{a,i}) = U(h^{\#})(\perp_{a,b}(x_{a,i}))$ is the least element of sort a_i from $U(A)$ therefore $F(A)(\perp_{a,b}) = \perp_{a,b}$.

If $\{f_n\}_{n\in\omega}$ is an increasing sequence from $T(a,b)$ then for every $h \in Set_S(Xb, U(A))$ and $i \in [|a|]$

$$F(A)(\bigvee_n f_n)(h)(x_{a,i}) = U(h^{\#})(\bigvee_n f_n(x_{a,i})) =$$

$$= \bigvee_n U(h^{\#})(f_n(x_{a,i})) = \bigvee_n F(A)(f_n)(h)(x_{a,i})$$

therefore $F(A)(\bigvee_n f_n) = \bigvee_n F(A)(f_n)$. ●

Another form of proposition 4.6 is:

4.9. <u>Corollary</u>. If $s \in T(a,ab)$ and $A \in |\mathcal{D}|$ then

$$s_A = F(A)(s^{\dagger}) .$$

4.10. <u>Example</u>. We continue example 3.1 where $U: \omega Alg_{\Sigma} \longrightarrow Set_S$ is the forgetful functor.

Let M be an S-sorted set having each component a strict ω-complete poset. It is well known that the concept of ω-continuous Σ-algebra having M as carrier is equivalent to the concept of interpretation of Σ in ωPow_M.

4.10.1. <u>Proposition</u>. If $\mathcal{A} = (A, \{\sigma_A\}_{\sigma\in\Sigma}, \leq, \perp_s)$ is an ω-continuous Σ-algebra and $I: \Sigma \longrightarrow \omega Pow_A$ is the equivalent interpretation then $F(\mathcal{A}): CT_{\Sigma} \longrightarrow \omega Pow_A$ is the unique extension of I to an ω-continuous algebraic theory morphism.

Proof. We have to prove that $F(\textcircled{A})(I_{\Sigma}(\sigma)) = I(\sigma)$ for every $\sigma \in \Sigma$.

Let $\sigma \in \Sigma$ and let $r(\sigma) = (s,a)$. We recall that if $f \in Set_S(Xa,A)$ then by definition

$$I(\sigma)(f)(x_{s,1}) = \sigma_A(f(x_{a,1}), f(x_{a,2}), \ldots, f(x_{a,|a|})) .$$

We recall that if $I_{\Sigma} : \Sigma \longrightarrow CT_{\Sigma}$ is the standard interpretation then

$$I_{\Sigma}(\sigma)(x_{s,1}) = \sigma(x_{a,1}, x_{a,2}, \ldots, x_{a,|a|}) .$$

If $f \in Set_S(Xa,A)$ then

$$F(A)(I_{\Sigma}(\sigma))(f)(x_{s,1}) = (I_{\Sigma}(\sigma)U(f^{\#}))(x_{s,1}) =$$

$$= f^{\#}(\sigma(x_{a,1}, x_{a,2}, \ldots, x_{a,|a|})) = \sigma_A(f(x_{a,1}), \ldots, f(x_{a,|a|})) =$$

$$= I(\sigma)(f)(x_{s,1})$$

therefore $F(A)(I_{\Sigma}(\sigma)) = I(\sigma)$. ●

Let $\textcircled{A} = (A, \sigma_A, \leq, \perp_A)$ be an ω-continuous Σ-algebra and let $d \in U(VXa)_s$. It is very well known that d induces an operation $d_{\textcircled{A}}$ on A of type (s,a). The derived operation $d_{\textcircled{A}}$ is defined for every $h \in Set_S(Xa,A)$ by $d_{\textcircled{A}}(h) = h^{\#}(d)$. If $f \in CT_{\Sigma}(s,a)$ and $h \in Set_S(Xa,A)$ then $F(\textcircled{A})(f)(h)(x_{s,1}) = h^{\#}(f(x_{s,1})) = f(x_{s,1})_{\textcircled{A}}(h)$, therefore ignoring the standard bijection between $Set_S(Xs,A)$ and A_s we may write

$$F(\textcircled{A})(f) = f(x_{s,1})_{\textcircled{A}} . \bullet$$

We interrupt for a while example 4.10 to give another result.

4.11. Proposition. Let R be an ω-continuous S-sorted algebraic theory and let $c \in S^{*}$. Let Rc denote the S-sorted set $\{R(s,c)\}_{s \in S}$. If we define

$$\underline{c}(f)(h)(x_{a,i}) = x_i^a f\langle h(x_{b,1}), \ldots, h(x_{b,|b|})\rangle$$

for every $f \in R(a,b)$, $h \in Set_S(Xb,Rc)$ and $i \in [|a|]$ then

$$\underline{c} : R \longrightarrow \omega Pow_{Rc}$$

is an ω-continuous algebraic theory morphism.

Proof. If $f \in R(a,b)$, $g \in R(b,d)$, $h \in Set_S(Xd,Rc)$ and $i \in [|a|]$ then

$$(\underline{c}(f) \circ \underline{c}(g))(h)(x_{a,i}) = \underline{c}(f)(\underline{c}(g)(h))(x_{a,i}) =$$

$$= x_i^a f < \underline{c}(g)(h)(x_{b,1}),\ldots,\underline{c}(g)(h)(x_{b,|b|})> =$$

$$= x_i^a f < x_1^b g,\ldots,x_{|b|}^b g >< h(x_{d,1}),\ldots,h(x_{d,|d|})> =$$

$$= x_i^a fg < h(x_{d,1}),\ldots,h(x_{d,|d|})> = \underline{c}(fg)(h)(x_{a,i})$$

therefore $\underline{c}(f) \circ \underline{c}(g) = \underline{c}(fg)$.

If $h \in Set_S(Xa,Rc)$ and $i \in [|a|]$ then

$$\underline{c}(x_i^a)(h)(x_{a_i,1}) = x_1^{a_i} x_i^a < h(x_{a,1}),\ldots,h(x_{a,|a|})> = h(x_{a,i})$$

therefore $\underline{c}(x_i^a) = x_i^a$.

If $i \in [|a|]$ and $h \in Set_S(Xb,Rc)$ then

$$\underline{c}(\perp_{a,b})(h)(x_{a,i}) = x_i^a \perp_{a,b} < h(x_{b,1}),\ldots,h(x_{b,|b|})> =$$

$$= x_i^a \perp_{a,c} = \perp_{a_i,c}$$

therefore $\underline{c}(\perp_{a,b}) = \perp_{a,b}$.

Let $\{f_n\}_{n\in\omega}$ be an increasing sequence of morphisms from $R(a,b)$. If $h \in Set_S(Xb,Rc)$ and $i \in [|a|]$ then

$$\underline{c}(\bigvee_n f_n)(h)(x_{a,i}) = x_i^a(\bigvee_n f_n) < h(x_{b,1}),\ldots,h(x_{b,|b|})> =$$

$$= \bigvee_{n\in\omega} x_i^a f_n < h(x_{b,1}),\ldots,h(x_{b,|b|})> = \bigvee_{n\in\omega} \underline{c}(f_n)(h)(x_{a,i})$$

therefore $\underline{c}(\bigvee_n f_n) = \bigvee_n \underline{c}(f_n)$. ●

4.10. Example (continued) Proposition 4.10.1 suggests us to replace ωPow_A by an arbitrary ω-continuous algebraic theory R.

Let $I:\Sigma \longrightarrow R$ an interpretation of Σ in R and let $I^{\#}:CT_{\Sigma} \longrightarrow R$ be its unique extension to an ω-continuous algebraic theory morphism. If $s \in T(a,ab)$ then $I^{\#}(s^{\dagger})$ is said to be the solution of s in R.

Let $\sigma \in \Sigma$ with $r(\sigma) = (s,a)$ and $c \in S^*$. We define on Rc an operation σ_c^I of type $r(\sigma)$ by

$$\sigma_c^I(f_1,f_2,\dots,f_{|a|}) = I(\sigma) < f_1,f_2,\dots,f_{|a|} >$$

where $f_i \in R(a_i,c)$ for every $i \in [|a|]$. It is easy to show that

$$\overset{I}{Rc} = (Rc, \{\sigma_c^I\}_{\sigma\in\Sigma}, \leq, \perp_{s,c})$$

is an ω-continuous Σ-algebra.

4.10.2. Proposition. $F(\overset{I}{Rc}) = I^{\#}\underline{c}$ for every $c \in S^*$ and every interpretation I of Σ in the ω-continuous algebraic theory R.

Proof. Both sides of the above equality are ω-continuous algebraic theory morphisms defined on CT_Σ. As CT_Σ is freely generated by Σ it suffices to show $F(\overset{I}{Rc})(I_\Sigma(\sigma)) = (I^{\#}\underline{c})(I_\Sigma(\sigma))$ for every $\sigma \in \Sigma$, where $I_\Sigma : \Sigma \longrightarrow CT_\Sigma$ is the standard interpretation. If $\sigma \in \Sigma$, $r(\sigma) = (s,a)$ and $h \in Set_S(Xa,Rc)$ then

$$F(\overset{I}{Rc})(I_\Sigma(\sigma))(h)(x_{s,1}) = h^{\#}(I_\Sigma(\sigma)(x_{s,1})) =$$

$$= h^{\#}(\sigma(x_{a,1},\dots,x_{a,|a|})) = \sigma_c^I(h(x_{a,1}),\dots,h(x_{a,|a|})) =$$

$$= x_1^s I(\sigma) < h(x_{a,1}),\dots,h(x_{a,|a|}) > = \underline{c}(I(\sigma))(h)(x_{s,1})$$

therefore

$$F(\overset{I}{Rc})(I_\Sigma(\sigma)) = \underline{c}(I^{\#}(I_\Sigma(\sigma))) = (I^{\#}\underline{c})(I_\Sigma(\sigma)) \ . \bullet$$

4.10.3. Corollary. If $s \in CT_\Sigma(a,ab)$, I is an interpretation of Σ in the ω-continuous algebraic theory R and $c \in S^*$ then

$$s_{\overset{I}{Rc}} = \underline{c}(I^{\#}(s^{\dagger})) \ . \bullet$$

4.10.4. Corollary. Let $s \in T(a,ab)$ and let I be an interpretation of Σ in the ω-continuous algebraic theory R. If $f \in Set_S(Xb,Rb)$ is defined by $f(x_{b,i}) = x_i^b$ for every $i \in [|b|]$ then

$$x_i^a I^{\#}(s^{\dagger}) = s_{\overset{I}{Rb}}(f)(x_{a,i})$$

for every $i \in [|a|]$.

Proof. It follows from the previous corollary that

$$s_{\underset{Rb}{I}}(f)(x_{a,i}) = \underline{b}(I^{\#}(s^{\dagger}))(f)(x_{a,i}) =$$

$$= x_i^a I^{\#}(s^{\dagger}) < f(x_{b,1}),\dots,f(x_{b,|b|}) > = x_i^a I^{\#}(s^{\dagger}). \bullet$$

The previous corollary gives a method to compute $I^{\#}(s^{\dagger})$. ●

4.12. Example. Context-free subsets of a monoid

A complete semilattice is a poset in which every subset has a least upper bound (lub). A complete semilattice monoid (cslm) is an algebraic structure which is both a complete semilattice and a monoid whose product is distributive with respect to lubs.

Let M be a monoid. Let us denote by P(M) the set of all subsets of M. The product in P(M) is defined by $AB = \{m'm'' \mid m' \in A, m'' \in B\}$. If we order the monoid P(M) by inclusion then P(M) becomes a cslm.

Given two complete semilattice monoids M and M', a monoid morphism $h: M \longrightarrow M'$ is called a cslm morphism if $h(\bigvee A) = \bigvee\{h(m) \mid m \in A\}$ for every subset A of M.

Given a monoid M and a set X we denote by $M[X]$ the coproduct of M and X^*.

Given two sets X and Y, a monoid M and a function $f: X \longrightarrow P(M[Y])$ we denote by $f^{\#}: P(M[X]) \longrightarrow P(M[Y])$ the unique cslm morphism such that $f^{\#}(A) = A$ for every $A \subseteq M$ and $f^{\#}(\{x\}) = f(x)$ for every $x \in X$.

Let M be a monoid and let X be a set. A system is a function $s: X \longrightarrow P(M[X])$. A solution of s in a function $f: X \longrightarrow P(M)$ such that $sf^{\#} = f$. It is well known that every system has a least solution. If the set X is finite and s(x) is finite for every $x \in X$ then every component of the least solution of s is called a context-free subset of M.

To apply the general theory to the above system we define the category Ⓓ and the functor U:Ⓓ ⟶ Set. The objects of Ⓓ are all the sets. Given the sets X and Y we say that $f \in \mathcal{D}(X,Y)$ if f is a cslm morphism from $P(M[X])$ to $P(M[Y])$ such that $f(A) = A$ for every subset A of M. The composition of $f \in \mathcal{D}(X,Y)$ by $g \in \mathcal{D}(Y,Z)$ in $\mathcal{D}$ is just their composition as functions. The functor $U:\mathcal{D} \longrightarrow Set$ is defined by $U(X) = P(M[X])$ for every set X and $U(f) = f$ for every $f \in \mathcal{D}(X,Y)$. It is not very difficult to prove that hypotheses 3.1, 3.2, 3.3 and 3.4 are fulfilled for the above U. If $s:X \longrightarrow P(M[X])$ is a system then its least solution is just the solution of s in the empty object of D.

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