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PARTIAL FLOWCHARTS, INFINITE FLOWCARTS AND RATIONAL FLOWCARTS

by

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by

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PARTIAL FLOWCHARTS

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The goal of this paper is to introduce the concept of ordered T-module with iterate and to prove that the partial flowcharts form a free commutative ordered T-module with iterate.

1. Introduction

Our study is based on the model of flowcharts over the algebraic theory $PStr_S$. These flowcharts are called partial. The reason for studying partial flowcharts is the same as the reason for studying partial trees.

The intuitive ideas which yield the order relation on partial flowcharts are similar to whose which yield the order relation on partial trees.

Let A and B be partial flowcharts having the same inputs and exits. We say that A is less than B if there exists an injective function f from the set of internal vertices of A to the set of internal vertices of B such that:

a) for every internal vertex n of A, the vertices n and f(n) are labeled by the same statement,

b1) if the input i of A is defined and equal to the exit j, then the input i of B is defined and equal to the exit j,

b2) if the input i of A is defined and is equal to the input j of the statement which labels the internal vertex n, then the input i of B is defined and is equal to the input j of the statement which labels the internal vertex f(n),

c1) if in A the k-th arrow going from the internal vertex n is defined and goes to the exit j, then in B the k-th arrow going from the internal vertex f(n) is defined and goes to the exit j, c2) if in A the k-th arrow going from the internal vertex n is defined and goes to the input j of the statement which labels the internal vertex v, then in B the k-th arrow going from the internal vertex f(n) is defined and goes to the input j of the statement which labels the internal vertex f(v).

These ideas yield definition 3.1 below.

From the semantic point of view, if an input or a successor of an internal vertex is not defined, then it may be interpreted as a loop without exit.

In this paper we work with the following fixed objects:

a) An S-sorted algebraic theory T which is also an algebraic theory with iterate and an ordered algebraic theory and fulfills the following conditions:

1) $\perp_{a,b} = 1_a^{\dagger}0_b$, that is $1_a^{\dagger}0_b$ is the smallest morphism of T(a,b),

2) the iterate is an increasing function;

b) An equidivisible monoid M endowed with two monoid morphisms $r : M \longrightarrow S^*$ and $r' : M \longrightarrow S^*$.

Let us remark that every rationally closed algebraic theory fulfills condition a).

2. Ordered T-modules with iterate

2.1. Definition. A T-module with iterate $H : T \longrightarrow Q$ is said to be ordered if the following conditions are fulfilled:

a) for every $a,b \in S^*$, the set Q(a,b) is partially ordered and $H(\perp_{a,b})$ is its first element,

b) $f \leq g$ in T(a,b) implies $H(f) \leq H(g)$ in Q(a,b),

c) the composition, the tupling and the iterate are increasing.

It is easy to see that $1_T : T \longrightarrow T$ is an ordered T-module with iterate.

As $PStr_S$ is initial both in the category of ordered theories and in the category of theories with iterate we deduce that every rationally closed S-sorted algebraic theory is an ordered $PStr_S$-module with iterate.

If $H : T \longrightarrow Q$ is an ordered T-module with iterate then:

1) $H(\perp_{a,b})+H(\perp_{c,d}) = H(\perp_{ac,bd})$,

2) the sum is increasing.

2.2. <u>Definition</u>. Let Q and Q' be ordered T-modules with iterate. A T-module with iterate morphism $F:Q \longrightarrow Q'$ is said to be a <u>morphism of ordered T-modules with iterate</u> if $\alpha \leq \beta$ in Q(a,b) implies $F(\alpha) \leq F(\beta)$ in Q'(a,b).

The composition of two morphisms of ordered T-modules with iterate is a morphism of ordered T-modules with iterate.

2.3. <u>Definition</u>. Let $H:T \longrightarrow Q$ be a T-module with iterate. We say that $\prec$ is an <u>admissible preorder</u> relation on Q if for every $a,b \in S^*$, $\prec$ is a preorder relation on Q(a,b) such that:

1) $f \leq g$ in T(a,b) implies $H(f) \prec H(g)$,

2) $H(\perp_{a,b}) \prec \alpha$ for each $\alpha \in Q(a,b)$,

3) $\alpha \prec \beta$ in Q(a,b) and $\alpha' \prec \beta'$ in Q(b,c) imply $\alpha\alpha' \prec \beta\beta'$ in Q(a,c),

4) $\alpha \prec \beta$ in Q(a,c) and $\alpha' \prec \beta'$ in Q(b,c) imply $\langle \alpha, \alpha' \rangle \prec \langle \beta, \beta' \rangle$

5) $\alpha \prec \beta$ in Q(a,ab) implies $\alpha^{\dagger} \prec \beta^{\dagger}$ in Q(a,b).

Let $\prec$ be an admissible preorder relation on the T-module with iterate $H : T \longrightarrow Q$. We introduce on Q(a,b) the corresponding equivalence relation

$$\alpha \equiv \beta \iff \alpha \prec \beta \text{ and } \beta \prec \alpha .$$

It is easy to see that $\equiv$ is a congruence on Q. Let $Q/_{\equiv}$ be the quotient T-module with iterate of Q by $\equiv$ and let $P:Q \longrightarrow Q/_{\equiv}$ be the naturale T-module with iterate morphism. We introduce in the usual way an order relation on $Q/\equiv (a,b)$:

$$P(\alpha) \leq P(\beta) \Longleftrightarrow \alpha \prec \beta .$$

$Q/\equiv$ endowed with this order relation becomes an ordered T-module with iterate which is called the quotient of Q by $\prec$ and is denoted by $Q/\prec$.

2.4. Proposition. Let $\prec$ be an admissible preorder relation on the T-module with iterate Q. If Q' is an ordered T-module with iterate and $F:Q \longrightarrow Q'$ a T-module with iterate morphism such that

$$\alpha \prec \beta \text{ in } Q(a,b) \text{ implies } F(\alpha) \leq F(\beta)$$

then there exists a unique ordered T-module with iterate morphism $G:Q/\prec \longrightarrow Q'$ such that $PG = F$.

3. The free ordered T-module with iterate

3.1. Definition. Let (i,t,m) and (i',t',m') be in $Fl_{M,T}(a,b)$. We write

$$(i,t,m) \prec (i',t',m')$$

if and only if there exists two positive integers n,p, an injection $f \in N(n,p)$ and $m_1,m_2,\ldots,m_n,m'_1,m'_2,\ldots,m'_p$ in M such that

$$m = m_1m_2\ldots m_n,$$
$$m' = m'_1m'_2\ldots m'_p,$$
$$m_i = m'_{f(i)} \text{ for every } i \in [n],$$
$$i(r(f;m'_1,m'_2,\ldots,m'_p)+1_b) \leq i'$$
$$t(r(f;m'_1,m'_2,\ldots,m'_p)+1_b) \leq r'(f;m'_1,m'_2,\ldots,m'_p)t'.$$

Let us remark that the congruence $\sim$ introduced in the previous paper (Căzănescu and Grama) is included in $\prec$.

In the above definition we may replace every element m_i or m'_i by a new product. It is useful to study two cases.

a) $m'_k = m^k_1m^k_2\ldots m^k_j$ and $k \notin Im(f)$. In this case we write

$$m' = m'_1m'_2\ldots m'_{k-1}m^k_1m^k_2\ldots m^k_jm'_{k+1}\ldots m'_p$$

and we define the injection $g \in N(n,p-1+j)$ by

$$g(u) = \begin{cases} f(u) & \text{if } f(u) < k \\ f(u)+j-1 & \text{if } f(u) > k\ . \end{cases}$$

As $r(f;m_1',\ldots,m_p')=r(g;m_1',\ldots,m_{j-1}',m_1^j,\ldots,m_k^j,m_{j+1}',\ldots,m_p')$ and similarly for r' it is easy to see that the conditions of the above definition are again fulfilled for the new decomposition of m' as a product and for the injection g.

b) $m_k= m_1^k m_2^k\ldots m_j^k$. As $m_k = m_{f(k)}'$, this case is equivalent to $m_{f(k)}'= m_1^k m_2^k\ldots m_j^k$. In this case we write

$$m = m_1 m_2\ldots m_{k-1} m_1^k m_2^k\ldots m_j^k\, m_{k+1}\ldots m_n\ ,$$

$$m'= m_1' m_2'\ldots m_{f(k)-1}' m_1^k m_2^k\ldots m_j^k m_{f(k)+1}'\ldots m_p'$$

and we define the injection $g \in N(n-1+j,\ p-1+j)$ by

$$g(u) = \begin{cases} f(u) & \text{if } u < k \quad \text{and } f(u) < f(k) \\ f(u)+j-1 & \text{if } u < k \quad \text{and } f(u) > f(k) \\ f(k)+u-k & \text{if } k \leq u < k+j \\ f(u-j+1) & \text{if } k+j \leq u \text{ and } f(u-j+1) < f(k) \\ f(u-j+1)+j-1 & \text{if } k+j \leq u \text{ and } f(u-j+1) > f(k) \end{cases}$$

As $r(f;m_1',\ldots,m_p')=r(g;m_1',\ldots,m_{f(k)-1}',m_1^k,\ldots,m_j^k,m_{f(k)+1}',\ldots,m_p')$ and similarly for r' it is easy to see that the conditions of the above definition are again fulfilled for the new decompositions of m and m' as products and for the injection g.

3.2. Proposition. The relation introduced by definition 3.1 is an admissible preordere relation on $Fl_{M,T}$.

Proof. As the reflexibility is obvious let us prove that our relation is transitive. Let us suppose $(i,t,m) \leq (i',t',m')$ with the same notations as in definition 2.1 and $(i',t',m') \leq (i'',t'',m'')$ in $Fl_{M,T}(a,b)$. From the remark following definition 3.1 and from the equidivisibility of M we may accept without loss of generality that we have only one decomposition of m' as a product. So there exist a positive integer q, an injection

$g\in N(p,q)$ and $m''_1,m''_2,\dots,m''_q$ in M such that

$$m''= m''_1m''_2\dots m''_q$$

$$m'_j= m''_{g(j)} \quad \text{for each} \quad j\in [p]$$

$$i'(r(g;m''_1,m''_2,\dots,m''_q)+1_b)\leq i''$$

$$t'(r(g;m''_1,m''_2,\dots,m''_q)+1_b)\leq r'(g;m''_1,m''_2,\dots,m''_q)t'' .$$

We deduce that the injection $fg\in N(m,q)$ has the following properties

$$m_j = m'_{f(j)}= m''_{(fg)(j)} \quad \text{for each} \quad j\in [n]$$

$$i(r(fg;m''_1,m''_2,\dots,m''_q)+1_b) =$$

$$= i(r(f;m'_1,m'_2,\dots,m'_p)+1_b)(r(g;m''_1,m''_2,\dots,m''_q)+1_b)\leq i''$$

$$t(r(fg;m''_1,m''_2,\dots,m''_q)+1_b) =$$

$$= t(r(f;m'_1,m'_2,\dots,m'_p)+1_b)(r(g;m''_1,m''_2,\dots,m''_q)+1_b)\leq$$

$$\leq r'(f;m'_1,m'_2,\dots,m'_p)r'(g;m''_1,m''_2,\dots,m''_q)t'' =$$

$$= r'(fg;m''_1,m''_2,\dots,m''_q)t''$$

therefore $(i,t,m)\prec(i'',t'',m'')$.

1) If $f\leq g$ in $T(a,b)$ then $St(f)\prec St(g)$ in $Fl_{M,T}(a,b)$. Indeed, as $St(f) = (f,0_b,e_M)$ and $St(g) = (g,0_b,e_M)$ we deduce

$$f(r(1_1;e_M)+1_b) = f(1_\lambda+1_b) = f\leq g \quad \text{and}$$

$$0_b(r(1_1;e_M)+1_b)= 0_b= r'(1_1;e_M)0_b .$$

2) If $(i,t,m)\in Fl_{T,M}(a,b)$ then $St(\perp_{a,b})\prec(i,t,m)$. We decompose e_M and m as e_M and e_Mm, respectively. Let $f\in N(1,2)$ be defined by $f(1) = 1$. We deduce that

$$\perp_{a,b}(r(f;e_M,m)+1_b) = \perp_{a,r(m)b}\leq i \quad \text{and}$$

$$0_b(r(f;e_M,m)+1_b) = 0_{r(m)b}= 0_{r'(m)}t = r'(f;e_M,m)t.$$

3) Let us suppose $(i,t,m)\prec(i',t',m')$ in $Fl_{M,T}(a,b)$ with the same notations as in definition 3.1 and $(j,u,v)\prec(j',u',v')$ in $Fl_{M,T}(b,c)$. We may write according to definition 3.1 that

$$v = m_{n+1}m_{n+2}\ldots m_{n+k}, \quad v' = m'_{p+1}m'_{p+2}\ldots m'_{p+q} \; ,$$

and that there exists an injection $g \in N(k,q)$ such that

$$m_{n+w} = m'_{p+g(w)} \quad \text{for each} \quad w \in [k]$$

$$j(r(g;m'_{p+1},m'_{p+2},\ldots,m'_{p+q})+1_c) \leq j' \quad \text{and}$$

$$u(r(g;m'_{p+1},\ldots,m'_{p+q})+1_c) \leq r'(g;m'_{p+1},\ldots,m'_{p+q})u' \; .$$

We deduce

$$mv = m_1m_2\ldots m_{n+k}, \; m'v' = m'_1m'_2\ldots m'_{p+q} \; ,$$

$f+g \in N(n+k,p+q)$ is an injection ,

$$m_w = m'_{(f+g)(w)} \quad \text{for each} \quad w \in [n+k] \; ,$$

$$i(1_{r(m)}+j)(r(f+g;m'_1,\ldots,m'_{p+q})+1_c) =$$

$$= i(r(f;m'_1,\ldots,m'_p)+1_b)(1_{r(m')}+j(r(g;m'_{p+1},\ldots,m'_{p+q})+1_c)) \leq$$

$$\leq i'(1_{r(m')}+j')$$

$$\langle t(1_{r(m)}+j),0_{r(m)}+u\rangle (r(f+g;m'_1,\ldots,m'_{p+q})+1_c) =$$

$$= \langle t(r(f;m'_1,\ldots,m'_p)+1_b)(1_{r(m')}+j(r(g;m'_{p+1},\ldots,m'_{p+q})+1_c)),$$

$$,0_{r(m')}+u(r(g;m'_{p+1},\ldots,m'_{p+q})+1_c)\rangle \leq$$

$$\leq \langle r'(f;m'_1,\ldots,m'_p)t'(1_{r(m')}+j'),0_{r(m')}+r'(g;m'_{p+1},\ldots,m'_{p+q})u'\rangle =$$

$$= r'(f+g;m'_1,\ldots,m'_{p+q})\langle t'(1_{r(m')}+j'),0_{r(m')}+u'\rangle$$

therefore $(i,t,m)(j,u,v) \leqslant (i',t',m')(j',u',v')$.

4) Let us suppose $(i,t,m) \leqslant (i',t',m')$ in $Fl_{M,T}(a,b)$ with the same notations as in definition 3.1 and $(j,u,v) \leqslant (j',u',v')$ in $Fl_{M,T}(c,b)$. We may write according to definition 3.1 that

$$v = m_{n+1}m_{n+2}\ldots m_{n+k}, \quad v' = m'_{p+1}m'_{p+2}\ldots m'_{p+q}$$

and that there exists an injection $g \in N(k,q)$ such that

$$j(r(g;m'_{p+1},m'_{p+2},\ldots,m'_{p+q})+1_b) \leq j'$$

$$u(r(g;m'_{p+1},m'_{p+2},\ldots,m'_{p+q})+1_b) \leq r'(g;m'_{p+1},\ldots,m'_{p+q})u' \; .$$

We deduce

$mv = m_1 m_2 \ldots m_{n+k}$, $m'v' = m'_1 m'_2 \ldots m'_{p+q}$

$f+g \in N(n+k,p+q)$ is an injection,

$m_w = m''_{(f+g)(w)}$ for each $w \in [n+k]$,

$\langle i(1_{r(m)}+0_{r(v)}+1_b), 0_{r(m)}+j\rangle (r(f+g;m'_1,\ldots,m'_{p+q})+1_b) =$

$= \langle i(r(f;m'_1,\ldots,m'_p)+1_b)(1_{r(m')}+0_{r(v')}+1_b),$

$0_{r(m')}+j(r(g;m'_{p+1},\ldots,m'_{p+q})+1_b)\rangle \leq$

$\leq \langle i'(1_{r(m')}+0_{r(v')}+1_b), 0_{r(m')}+j'\rangle$,

$\langle t(1_{r(m)}+0_{r(v)}+1_b), 0_{r(m)}+u\rangle (r(f+g;m'_1,\ldots,m'_{p+q})+1_b) =$

$= \langle t(r(f;m'_1,\ldots,m'_p)+1_b)(1_{r(m')}+0_{r(v')}+1_b),$

$0_{r(m')}+u(r(g;m'_{p+1},\ldots,m'_{p+q})+1_b)\rangle \leq$

$\leq \langle r'(f;m'_1,\ldots,m'_p)t'(1_{r(m')}+0_{r(v')}+1_b), 0_{r(m')}+r'(g;m'_{p+1},\ldots,m'_{p+q})u'\rangle$

$= r'(f+g;m'_1,\ldots,m'_{p+q})\langle t'(1_{r(m')}+0_{r(v')}+1_b), 0_{r(m')}+u'\rangle$

therefore $\langle (i,t,m),(j,u,v)\rangle \leqslant \langle (i',t',m'),(j',u',v')\rangle$.

5) Let us suppose $(i,t,m) \leqslant (i',t',m')$ in $Fl_{M,T}(a,ab)$. Let $m = m_1 m_2 \ldots m_n$, $m' = m'_1 m'_2 \ldots m'_p$ and $f \in N(m,p)$ an injection such that

$m_j = m'_{f(j)}$ for each $j \in [n]$,

$i(r(f;m'_1,m'_2,\ldots,m'_p)+1_{ab}) \leq i'$ and

$t(r(f;m'_1,m'_2,\ldots,m'_p)+1_{ab}) \leq r'(f;m'_1,m'_2,\ldots,m'_p)t'$.

We deduce

$(i(S^a_{r(m)}+1_b))^\dagger (r(f;m'_1,\ldots,m'_p)+1_b) =$

$= (i(S^a_{r(m)}+1_b)(1_a+r(f;m'_1,\ldots,m'_p)+1_b))^\dagger =$

$= (i(r(f;m'_1,m'_2,\ldots,m'_p)+1_{ab})(S^a_{r(m')}+1_b))^\dagger \leq$

$\leq (i'(S^a_{r(m')}+1_b))^\dagger$ and

$$t(S^a_{r(m)}+1_b)\langle(i(S^a_{r(m)}+1_b))^\dagger, 1_{r(m)b}\rangle(r(f;m'_1,\dots,m'_p)+1_b) =$$

$$= t(S^a_{r(m)}+1_b)(1_a+r(f;m'_1,\dots,m'_p)+1_b)$$

$$\langle(i(S^a_{r(m)}+1_b))^\dagger(r(f;m'_1,\dots,m'_p)+1_b), 1_{r(m')b}\rangle \leqslant$$

$$\leqslant t(r(f;m'_1,\dots,m'_p)+1_{ab})(S^a_{r(m')}+1_b)\langle(i'(S^a_{r(m')}+1_b))^\dagger, 1_{r(m')b}\rangle \leqslant$$

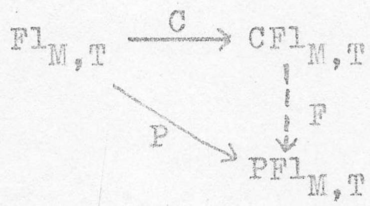
$$\leqslant r'(f;m'_1,\dots,m'_p)t'(S^a_{r(m')}+1_b)\langle(i'(S^a_{r(m')}+1_b))^\dagger, 1_{r(m')b}\rangle$$

therefore $(i,t,m)^\dagger \prec (i',t',m')^\dagger$.

Let $PFl_{M,T}$ be the quotient of $Fl_{M,T}$ by the admissible preorder relation introduced by definition 3.1, let $PSt:T \longrightarrow PFl_{M,T}$ be its structural functor and $P_M:M \longrightarrow PFl_{M,T}$ the standard interpretation of M in $PFl_{M,T}$ defined as the composition of I_M with the natural T-module with iterate morphism $P:Fl_{M,T} \longrightarrow PFl_{M,T}$.

3.3. Lemma. The ordered T-module with iterate $PFl_{M,T}$ is commutative.

Proof. We remarked that $\sim \subseteq \prec$, therefore $\sim \subseteq \equiv$. We deduce the existence of a unique T-module with iterate morphism $F:CFl_{M,T} \longrightarrow PFl_{M,T}$ such that $CF = P$, i.e. the following diagram is commutative



Let $\alpha \in Fl_{M,T}(a,c)$ and $\beta \in Fl_{M,T}(b,d)$. As $CFl_{M,T}$ is commutative we deduce

$$CSt(S^a_b)(C(\alpha)+C(\beta)) = (C(\beta)+C(\alpha))CSt(S^c_d) .$$

We deduce applying F that

$$PSt(S^a_b)(P(\alpha)+P(\beta)) = (P(\beta)+P(\alpha))PSt(S^c_d)$$

then $PFl_{M,T}$ is commutative.

We recall that in a T-module with iterate $H:T \longrightarrow Q$

a) $H(f)\alpha^{\dagger} = (H(f)\alpha H(f^{-1}+1_b))^{\dagger}$

for every $\alpha \in Q(a,ab)$ and every isomorphism $f \in T(c,a)$.

b) $H(1_a+0_b)\langle \alpha , \beta \rangle^{\dagger} = (\alpha\, H(S^b_a+1_c)\langle (\beta H(S^b_a+1_c))^{\dagger}, 1_{ac}\rangle)^{\dagger}$

for every $\alpha \in Q(a,abc)$ and $\beta \in Q(b,abc)$.

3.4. Theorem. The commutative ordered T-module with iterate $PFl_{M,T}$ is freely generated by M.

Proof. Let $H:T \longrightarrow Q$ be a commutative ordered T-module with iterate and let I be an interpretation of M in Q. As $Fl_{M,T}$ is the T-module with iterate freely generated by M there exists one and only one T-module with iterate morphism $F:Fl_{M,T} \longrightarrow Q$ such that $I_M F = I$. We recall that for every $(i,t,m) \in Fl_{M,T}(a,b)$ we have

$$F(i,t,m) = H(i)\langle (I(m)H(t))^{\dagger}, 1_b\rangle .$$

To apply proposition 2.4 it is enough to show that $(i,t,m) \prec (i',t',m')$ in $Fl_{M,T}(a,b)$ implies $F(i,t,m) \leq F(i',t',m')$.

Let us suppose $(i,t,m) \prec (i',t',m')$ in $Fl_{M,T}(a,b)$. Then there exists an injection $f \in N(n,p)$ and the products $m=m_1 m_2 \dots m_n$, $m'=m'_1 m'_2 \dots m'_p$ such that

$$m_i = m'_{f(i)} \quad \text{for every } i \in [n] ,$$

$$i(r(f;m'_1,m'_2,\dots,m'_p)+1_b) \leq i' \quad \text{and}$$

$$t(r(f;m'_1,m'_2,\dots,m'_p)+1_b) \leq r'(f;m'_1,m'_2,\dots,m'_p)t' .$$

As $f \in N(n,p)$ is an injection there exists another injection $g \in N(p-n,p)$ such that $\langle f,g\rangle \in N(p,p)$ is a bijection. Let us denote

$$m'' = m'_{g(1)} m'_{g(2)} \dots m'_{g(p-n)} \quad \text{and}$$

$$h = r(\langle f,g\rangle ; m'_1, m'_2, \dots, m'_p) .$$

We deduce that

$$r(f;m'_1,m'_2,\dots,m'_p) = (1_{r(m)}+0_{r(m'')})h \quad \text{and}$$

$$(r(f;m'_1,m'_2,\dots,m'_p)+1_b)(h^{-1}+1_b) = 1_{r(m)}+0_{r(m'')}+1_b .$$

As the T-module Q is commutative it follows that

$$H(h)I(m') = H(1_S \times (\langle f,g\rangle; r(m_1'), r(m_2'), \ldots, r(m_p')))$$

$$(I(n_1')+I(m_2')+\ldots+I(m_p')) =$$

$$= (I(m'_{f(1)})+\ldots+I(m'_{f(n)})+I(m'_{g(1)})+\ldots+I(m'_{g(p-n)}))$$

$$H(1_S \times (\langle f,g\rangle; r'(m_1'), \ldots, r'(m_p'))) =$$

$$= \langle I(m)H(r'(f;m_1',\ldots,m_p')), I(m'')H(r'(g;m_1',\ldots,m_p'))\rangle$$

Let $A=H(r(f;m_1',m_2',\ldots,m_p'))(I(m')H(t'))^{\dagger}$. It follows from the remarks before this theorem that

$$A = H(1_{r(m)}+0_{r(m'')})H(h)(I(m')H(t'))^{\dagger} =$$

$$= H(1_{r(m)}+0_{r(m'')})(H(h)I(m')H(t')H(h^{-1}+1_b))^{\dagger} =$$

$$= H(1_{r(m)}+0_{r(m'')}) \langle I(m)H(r'(f;m_1',\ldots,m_p')t'(h^{-1}+1_b)),$$

$$I(m'')H(r'(g;m_1',\ldots,m_p')t'(h^{-1}+1_b))\rangle^{\dagger} \geqslant$$

$$\geqslant H(1_{r(m)}+0_{r(m'')})\langle I(m)H(t(r(f;m_1',\ldots,m_p')+1_b)(h^{-1}+1_b)),$$

$$H(\perp_{r(m''),r(mm'')b})\rangle^{\dagger} =$$

$$= H(1_{r(m)}+0_{r(m'')})\langle I(m)H(t(1_{r(m)}+0_{r(m'')}+1_b)), H(\perp_{r(m''),r(mm'')b})\rangle^{\dagger} =$$

$$= (I(m)H(t(0_{r(m'')}+1_{r(m)b})\langle \perp_{r(m''),r(m)b}, 1_{r(m)b}\rangle))^{\dagger} =$$

$$= (I(m)H(t))^{\dagger} .$$

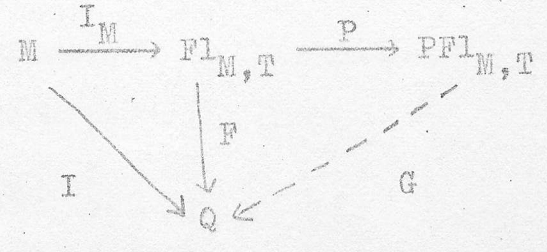
Therefore

$$F(i',t',m') = H(i')\langle (I(m')H(t'))^{\dagger}, 1_b\rangle \geqslant$$

$$\geqslant H(i)\langle H(r(f;m_1',m_2',\ldots,m_p'))(I(m')H(t'))^{\dagger}, 1_b\rangle \geqslant$$

$$\geqslant H(i)\langle (I(m)H(t))^{\dagger}, 1_b\rangle = F(i,t,m).$$

In the diagram



we deduce from proposition 2.4 that there exists a unique morphism of ordered T-modules with iterate $G:PFl_{M,T} \longrightarrow Q$ such that $PG = F$, therefore $P_M G = I_M PG = I_M F = I$.

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INFINITE FLOWCHARTS

The goal of this paper is to introduce the concept of Z-continuous T-module with iterate and to prove that the infinite partial flowcharts form a free commutative Z-continuous T-module with iterate.

1. Introduction

The reason for studying infinite flowcharts is the same as the reason to study infinite trees.

Nevertheless, the situation in the case of trees is better than in the case of flowcharts because we have a very nice mathematical object to represent an infinite tree. We shall work in the case of infinite flowcharts as in the pioneering papers on infinite trees, i.e. we shall model an infinite flowcharts as an adequate set of finite flowcharts.

From the algebraic view-point the problem is then to complete an ordered T-module with iterate. We know how to complete an ordered algebra and this technique will be adequate for our problem although a new difficulty arises. We have to prove that the axioms of T-modules with iterate are preserved by this completeness. We will do it in the next section.

2. Completeness and equations

Let Z be a union complete, crossed-down and crossed-up subset system on posets such that $\emptyset \notin Z[P]$ for every poset P.

We recall that for a crossed-down subset system Z on posets, a function of several variables is Z-continuous if and only if it is Z-continuous by components.

Let $r : \sum \longrightarrow S \times S^{*}$ be an S-sorted signature.

Let $A = (A_s, \sigma_A, \leq, \perp_s)$ be an ordered Σ-algebra. Let $\sigma \in \Sigma_{s,a}$ and $C_i \in I[A_{a_i}]$ for every $i \in [|a|]$. As Z is crossed-up we deduce that

$$\sigma_I(C_1, C_2, \ldots, C_{|a|}) = (\{\sigma_A(c_1, c_2, \ldots, c_{|a|}) \mid c_i \in C_i \text{ for } i \in [|a|]\}]$$

is a Z-ideal of A_s. The Σ-algebra

$$I[A] = (I[A_s], \sigma_I, \subseteq, \{\perp_s\})$$

is ordered. As Z is union complete its carriers are Z-complete and its operations are Z continuous by components. As Z is crossed-down its operations are Z-continuous. Therefore $I[A]$ is a Z-continuous Σ-algebra.

Let $i_A : A \longrightarrow I[A]$ be defined for every $s \in S$ and $a \in A_s$ by $i_{A,s}(a) = (a]$. As $\emptyset \notin Z[A_s]$ we deduce that $(\perp_s] = \{\perp_s\}$ is the smallest element of $I[A_s]$, therefore $i_{A,s}$ is strict. The S-sorted function i_A is a homomorphism of ordered Σ-algebras.

2.1. <u>Theorem.</u> Let Z be a union complete, crossed-down and crossed-up subset system such that $\emptyset \notin Z[P]$ for every posed P. Let A be an ordered Σ-algebra. For every Z-continuous Σ-algebra B and for every homomorphism of ordered Σ-algebras $f : A \longrightarrow B$ there is one and only one homomorphism of Z-continuous Σ-algebras $g : I[A] \longrightarrow B$ such that $i_A g = f$.

In the sequel whenever we shall use the notation $I[A]$ for an ordered algebra A then we shall suppose that the hypotheses on Z from theorem 2.1 are fulfilled.

Let X be an S-sorted set. An element of X_s is called a variable of sort s. Let $L = (L_s, \sigma_L)$ be the free Σ-algebra generated by X. A pair $e = (e_1, e_2)$ of elements from L_s is said to be a <u>Σ-equation</u> of sort s.

Let A be a Σ-algebra. For every S-sorted function f from X to the carrier of A we denote by $f^{\#} : L \longrightarrow A$ the unique Σ-homomorphism which extends f. The Σ-equation $e = (e_1, e_2)$ is valid in A, or e holds in A if $f^{\#}(e_1) = f^{\#}(e_2)$ for every S-sorted function f from X to the carrier of A.

Let $M = (\{\omega\}_{s\in S}, \sigma_M)$ be the Σ-algebra whose operations are defined by

$$\sigma_M(n_1, n_2, \ldots, n_{|a|}) = n_1 + n_2 + \ldots + n_{|a|}$$

for every $\sigma \in \Sigma_{s,a}$. Let x be a variable. Let

$$nx : L \longrightarrow M$$

the unique Σ-morphism such that

$$nx(y) = \begin{cases} 0 & y \neq x \\ 1 & y = x \end{cases}$$

If $p \in L_s$ then $nx_s(p)$ is the number of the occurrences of x in p.

2.2. Lemma. Let A be a Σ-algebra. Let f and g be functions from X to the carrier of A. If $e \in L_s$ and if $nx_s(e) > 0$ imply $f(x) = g(x)$ for every x in X then $f^{\#}_s(e) = g^{\#}_s(e)$.

2.3. Definition. An element $e \in L_s$ is said to be simple if $nx_s(e) \leq 1$ for every x from X. Un equation $e = (e_1, e_2)$ is said to be simple if e_1 and e_2 are simple.

Let us remark that the axioms of commutative T-modules with iterate are simple. Therefore we work with simple equations. If you deal with an algebraic structure whose axioms are not simple you may take the hypothesis $Z \subseteq \Delta$, i.e. every Z-set is a Δ-set (directed). The subsequent proofs may be easily modified.

2.4. Lemma. Let A be an ordeded Σ-algebra and let $e \in L_s$ be simple. If f is an S-sorted function from X to the carrier of $I[A]$ then

$$f^{\#}(e) = (\{g^{\#}(e) \mid g : X \longrightarrow A,\ g(x) \in f(x) \text{ for every } x \in X\}].$$

Proof. By structured induction.

If e is a variable then

$(\{g^{\#}(e) \mid g:X \longrightarrow A,\ g(x) \in f(x) \text{ for every } x \in X\}] =$
$= (f(e)] = f(e) = f\ (e).$

Let $e = \sigma_L(e_1, e_2, \ldots, e_n)$ with $r(\sigma) = (s, s_1 s_2 \ldots s_n)$. As e is simple it follows that e_i is simple for every $i \in [n]$, then we may use the inductive hypothesis for every e_i, $i \in [n]$.

Because $f^{\#}(e) = (B]$ where

$$B = \{\sigma_A(c_1, c_2, \ldots, c_n) \mid c_i \in f^{\#}(e_i) \text{ for } i \in [n]\}$$

and $(\{g^{\#}(e) \mid g:X \longrightarrow A,\ g(x) \in f(x) \text{ for } x \in X\}] = (C]$ where

$$C = \{\sigma_A(g^{\#}(e_1), g^{\#}(e_2), \ldots, g^{\#}(e_n)) \mid g:X \longrightarrow A, g(x) \in f(x) \text{ for } x \in X\}$$

it is enough to show that the sets B and C are mutually cofinal.

We deduce from the inductive hypothesis that $C \subseteq B$.

Let $b \in B$, i.e. $b = \sigma_A(c_1, c_2, \ldots, c_n)$ where $c_i \in f^{\#}(e_i)$ for every $i \in [n]$. It follows from the inductive hypothesis that for every $i \in [n]$ there exists $g_i : X \longrightarrow A$ with $g_i(x) \in f(x)$ for every $x \in X$ such that $c_i \leq g_i^{\#}(e_i)$. As e is simple we deduce, using lemma 2.2, that there exists $g : X \longrightarrow A$ with $g(x) \in f(x)$ for every $x \in X$ such that $g^{\#}(e_i) = g_i^{\#}(e_i)$ for every $i \in [n]$ (if $nx(e_i) > 0$ we take $g(x) = g_i(x)$). Therefore, since σ_A is increasing we deduce

$$b \leq \sigma_A(g_1^{\#}(e_1), \ldots, g_n^{\#}(e_n)) = \sigma_A(g\ e_1), \ldots, g^{\#}(e_n)) \in C.$$

2.5. <u>Theorem</u>. If the simple equation $e = (e_1, e_2)$ is valid in the ordered Σ-algebra A then e is valid in $I[A]$.

<u>Proof</u>. Let $f : X \longrightarrow I[A]$. As e holds in A we deduce $g^{\#}(e_1) = g^{\#}(e_2)$ for every $g : X \longrightarrow A$. It follows from lemma 2.4 that $f^{\#}(e_1) = f^{\#}(e_2)$ therefore e holds in $I[A]$.

3. Z-continuous T-modules with iterate

We suppose the S-sorted algebraic theory T and the monoid M satisfy the same hypotheses as in our previous paper "Partial flowcharts".

Let Z be a subset system on posets.

3.1. Definition. An ordered T-module with iterate Q is said to be Z-continuous if

1) Q(a,b) is a Z-complete poset for every $a,b \in S^*$,

2) the composition is Z-continuous by components,

3) the tupling and the iterate are Z-continuous.

3.2. Example. Let Z be a crossed-down subset system on posets such that $\omega \subseteq Z$. If T is a Z-continuous algebraic theory then

1) $1_T : T \longrightarrow T$ is a Z-continuous T-module with iterate and

2) the ordered $PStr_S$-module with iterate T is Z-continuous. ●

Let us remark that in every Z-continuous T-module with iterate Q the sum

$$+ : Q(a,b)\ Q(c,d) \longrightarrow Q(ac,bd)$$

is Z-continuous.

Let Q be a Z-continuous T-module with iterate. Let us denote by Int(M,Q) the set of all the interpretations of M in Q. We order Int(M,Q) componentwise, i.e. $I \leq I'$ if for every $m \in M$, $I(m) \leq I'(m)$ in $Q(r(m),r'(m))$. The set Int(M,Q) is Z-complete and for every $A \in Z[\mathrm{Int}(M,Q)]$ and $m \in M$, $(\bigvee A)(m) = \bigvee\{I(m) \mid I \in A\}$. The set Int(M,Q) has a smallest element $\perp$ defined by $\perp(m) = H(\perp_{r(m),r'(m)})$ for every $m \in M$.

3.3. Definition. Let Q and Q' be Z-continuous T-modules with iterate. An ordered T-module with iterate morphism $F : Q \longrightarrow Q'$ is said to be a morphism of Z-continuous T-modules with iterate if for every a,b in S^* the restriction of F to Q(a,b) is a Z-continuous function.

The composition of two morphisms of Z-continuous T-modules with iterate is a morphism of Z-continuous T-modules with iterate.

We suppose in the sequel that Z is union-complete, crossed-down and crossed-up; moreover $\emptyset \notin Z[P]$ for every poset P.

3.4. <u>Theorem</u>. Let $H : T \longrightarrow Q$ be an ordered (comutative) T-module with iterate. There exists a Z-continuous (comutative) T-module with iterate

$$I[H] : T \longrightarrow I[Q]$$

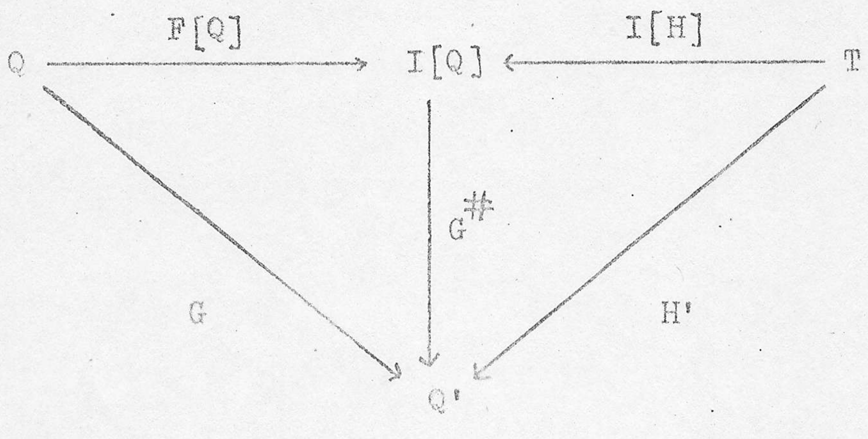
and an ordered T-module with iterate morphism

$$F[Q] : Q \longrightarrow I[Q]$$

such that:

for every Z-continuous T-module with iterate $H' : T \longrightarrow Q'$ and for every ordered T-modules with iterate morphism $G : Q \longrightarrow Q'$ there exists one and only one Z-continuous T-module with iterate morphism $G : I[Q] \longrightarrow Q'$ such that $F[Q]G^{\#} = G$.

<u>Proof</u>. The following diagram may help the reader:



Let $I[Q](a,b) = I[Q(a,b)]$ for every a,b in S^{*}, i.e. every morphism of I[Q] from a to b is a Z-ideal of Q(a,b). As Z is union complete, I[Q](a,b) ordered by inclusion is a Z-complete poset.

The operations of I[Q] are defined as follows:

1) composition: $AB = (\{\alpha\beta \mid \alpha \in A , \beta \in B\}]$

for every $A \in I[Q](a,b)$ and $B \in I[Q](b,c)$,

2) tupling: $\langle A,B\rangle = (\{\langle\alpha,\beta\rangle \mid \alpha\in A, \beta\in B\}]$

for every $A\in I[Q](a,c)$ and $B\in I[Q](b,c)$,

3) iterate: $A^\dagger = (\{\alpha^\dagger \mid \alpha\in A\}]$

for every $A\in I[Q](a,ab)$.

As Z is crossed-down these operations are Z-continuous.

Let us remark that the identity morphism of $a\in S^*$ in $I[Q]$ is $(1_a]$.

The functor $I[H] : T \longrightarrow I[Q]$ is defined for every $f\in T(a,b)$ by $I[H](f) = (H(f)]$.

As the axioms which define the concept of (commutative) T-module with iterate are simple equations it follows that $I[Q]$ is a Z-continuous (commutative) T-module with iterate.

The morphism $F[Q] : Q \longrightarrow I[Q]$ is defined for every $\alpha\in Q(a,b)$ by $F[Q](\alpha) = (\alpha]$.

Let $H' : T \longrightarrow Q'$ be a Z-continuous T-module with iterate. The morphism $G^{\#} : I[Q] \longrightarrow Q'$ is defined for every $A\in Z[Q(a,b)]$ by

$$G^{\#}((A]) = \bigvee\{G(\alpha) \mid \alpha\in A\}$$

and fulfills the conditions of our theorem.●

Let $ZF\ell_{M,T} = I[PF\ell_{M,T}]$ and let $Z_M : M \longrightarrow ZF\ell_{M,T}$ be the standard interpretation of M in $ZF\ell_{M,T}$ defined by $Z_M = P_M F[PF\ell_{M,T}]$. Let $ZSt : T \longrightarrow ZF\ell_{M,T}$ be the structural functor of $ZF\ell_{M,T}$.

3.5. Corollary. The commutative Z-continuous T-module with iterate $ZF\ell_{M,T}$ is freely generated by M.

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RATIONAL FLOWCHARTS

We introduce the concept of system of equations with flowcharts and we show that the methodology of our previous paper "Iterative Systems of Equations" may be used to solve them. A component of the smallest solution of a finite system of equations with finite flowcharts is said to be a rational flowchart. We prove that the unfoldment of a rational flowchart consists of context-free trees.

1. System of equations with flowcharts

Let M be the monoid of statements and let $r : M \longrightarrow S^*$ and $r' : M \longrightarrow S^*$ be two monoid morphisms.

Let X be a set. Let $rX : X \longrightarrow S^*$ and $r'X : X \longrightarrow S^*$ be two functions. In practice every $x \in X$ is the name of a recursive procedure without parameters; $rX(x)$ and $r'X(x)$ show the number and the sorts of the entries of x and the number and the sorts of the exits of x, respectively. We endow the monoid $M[X]$, the coproduct of M and X^*, with the monoid morphisms $\langle r, rX \rangle : M[X] \longrightarrow S^*$ and $\langle r', r'X \rangle : M[X] \longrightarrow S^*$, therefore we may work with T-M[X]-flowcharts.

1.1. Definition. We say that

$$s : X \longrightarrow \omega Fl_{M[X],T}$$

is a system of equations with flowcharts if $s(x) \in \omega Fl_{M[X],T}(rX(x), r'X(x))$ for every $x \in X$. The system s is rational if X is finite and for every x in X, s(x) is the image of an element from $Fl_{M[X],T}$. ∎

We shall show that we may use the method introduced in the previous paper for solving a system of equations with flowcharts.

Let us notice first that the triple $(X,rX,r'X)$ may be thought of as an object from $Set_{S^*\times S^*}$: the sort of $x\in X$ is $(rX(x),r'X(x))$.

Let Mod_ω be the category of ω-continuous commutative T-modules with iterate.

Let $Int_\omega(M,T)$ be the category whose objects are interpretations of M in an arbitrary object of Mod_ω and whose morphisms from $I : M \longrightarrow Q$ to $I' : M \longrightarrow Q'$ are the morphisms F from $Mod_\omega(Q,Q')$ with the property $IF = I'$.

Let Q be a T-module. Let us remark that the set of all morphism of Q becomes in a natural way an $S^*\times S^*$-sorted set: the sort of $\alpha \in Q(a,b)$ is (a,b).

Let $U : Int_\omega(M,T) \longrightarrow Set_{S^*\times S^*}$ be the forgetful functor. It assigns to each object $I : M \longrightarrow Q$ of $Int_\omega(M,T)$ the $S^*\times S^*$-sorted set of all the morphisms of Q. We prove that the functor U fulfills conditions (3.1), (3.2), (3.3) and (3.4) from the previous paper.

Let X be an $S^*\times S^*$-sorted set. Let $VX : M \longrightarrow \omega F\ell_{M[X],T}$ be the restriction of the standard interpretation $\omega_{M[X]}$ to M. Let us denote by $\varepsilon_X : X \longrightarrow U(VX)$ be the $S^*\times S^*$-sorted function defined by $\varepsilon_X(x) = \omega_{M[X]}(e_M x e_M)$ for every x in X.

Let $I : M \longrightarrow Q$ be an object of $Int_\omega(M,T)$ and let $f : X \longrightarrow U(I)$ be an $S^*\times S^*$-sorted function. We notice that $\langle I,f\rangle : M[X] \longrightarrow Q$ is an interpretation of $M[X]$ in Q, therefore there exists a unique ω-continuous T-module with iterate morphism $\langle I,f\rangle^\# : \omega F\ell_{M[X],T} \longrightarrow Q$ such that $\langle I,f\rangle = \omega_{M[X]}\langle I,f\rangle^\#$. We deduce that $\langle I,f\rangle^\# \in Int_\omega(M,T)(VX,I)$ and that $\varepsilon_X U(\langle I,f\rangle^\#) = f$. If $g\in Int_\omega(M,T)(VX,I)$ and $\varepsilon_X U(g) = f$ then $\omega_{M[X]} g = \langle I,f\rangle$, therefore $g = \langle I,f\rangle^\#$. So condition (3.1) is fulfilled.

We skip the easy proofs of (3.2) and (3.3). We still have to prove (3,4).

Let Q and Q' be objects of Mod_ω. The set $Mod_\omega(Q,Q')$ is ordered by $F \leq G$ if and only if $F(\alpha) \leq G(\alpha)$ for every morphism α of Q. If $\{F_n\}_{n\in\omega}$ is an increasing sequence of morphisms from $Mod_\omega(Q,Q')$, if α is a morphism from Q and if we define $F(\alpha) = \bigvee\{F_n(\alpha) \mid n\in\omega\}$ then F is an object from $Mod_\omega(Q,Q')$. Therefore $Mod_\omega(Q,Q')$ ordered as above is an ω-complete poset and $(\bigvee_{n\in\omega} F_n)(\alpha) = \bigvee\{F_n(\alpha) \mid n\in\omega\}$ for every increasing sequence $\{F_n\}_{n\in\omega}$ from $Mod_\omega(Q,Q')$ and every morphism α of Q.

Let $Q \in |Mod_\omega|$. Let us recall that the set $Int(M,Q)$ is an ω-complete posets. For every $I \in Int(M,Q)$ let us denote by $I^{\#}$ the unique morphism from $Mod_\omega(\omega F\ell_{M,T}, Q)$ such that $\omega_M I^{\#} = I$.

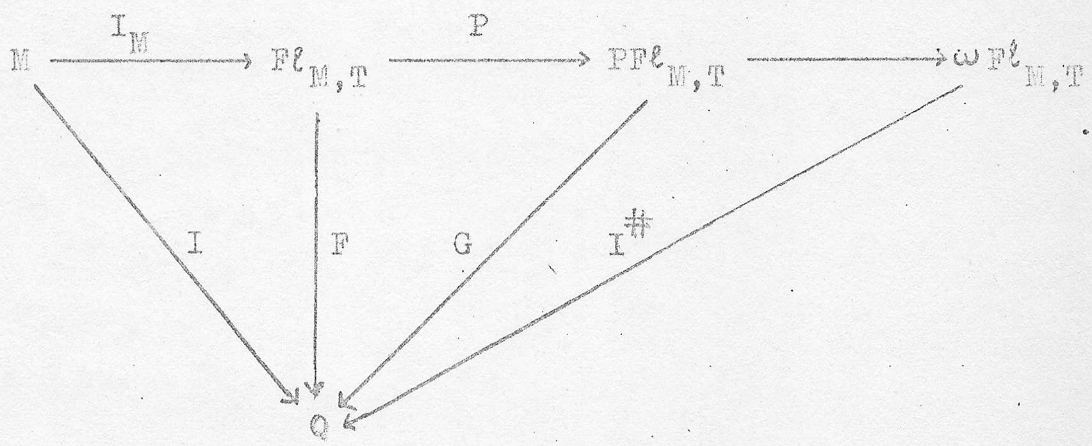
1.2. <u>Lemma</u>. If $H : T \longrightarrow Q$ is an object in Mod_ω then the function

$$\# : Int(M,Q) \longrightarrow Mod_\omega(\omega F\ell_{M,T}, Q)$$

is increasing.

<u>Proof</u>. Let $I \leq I'$ in $Int(M,Q)$.

Let $F : F\ell_{M,T} \longrightarrow Q$ ($F' : F\ell_{M,T} \longrightarrow Q$) be the unique T-module with iterate morphism such that $I_M F = I$ ($I_M F' = I'$)



We recall that $F(i,t,m) = H(i) < (I(m)H(t))^{\dagger}, 1_b >$ for $(i,t,m) \in F\ell_{M,T}(a,b)$. As $I(m) \leq I'(m)$ it follows that $F(i,t,m) \leq F'(i,t,m)$ for every $(i,t,m) \in F\ell_{M,T}(a,b)$.

Let $G : PF\ell_{M,T} \longrightarrow Q(G' : PF\ell_{M,T} \longrightarrow Q')$ be the unique morphism of ordered commutative T-modules with iterate such that $F=PG(F'=PG')$. As every morphism of $PF\ell_{M,T}$ is the image by P of a morphism form $F\ell_{M,T}$ it follows that $G(\alpha) \leq G'(\alpha)$ for every morphism α of $PF\ell_{M,T}$.

To finish the proof we note that for every $A \in \omega[PF\ell_{M,T}(a,b)]$ $I^{\#}((A]) = \bigvee\{G(\alpha) | \alpha \in A\} \leq \bigvee\{G'(\alpha) | \alpha \in A\} = I'^{\#}((A])$. ●

1.3. Proposition. Let $Q \in |Mod_{\omega}|$. The function

$$\# : Int(M,Q) \longrightarrow Mod_{\omega}(\omega F\ell_{M,T}, Q)$$

is ω-continuous.

Proof. Let $A \in \omega[Int(M,Q)]$.

As $\#$ is increasing $\{I^{\#} | I \in A\}$ is an ω-set of $Mod_{\omega}(\omega F\ell_{M,T}, Q)$; therefore there exists $\bigvee\{I^{\#} | I \in A\}$ and $(\bigvee\{I^{\#} | I \in A\})(\alpha) = \bigvee\{I^{\#}(\alpha) | I \in A\}$ for every morphism α of $\omega F\ell_{M,T}$.

If $m \in M$ then

$$(\bigvee\{I^{\#} | I \in A\})(\omega_M(m)) = \bigvee\{I^{\#}(\omega_M(m)) | I \in A\} = (\bigvee A)(m).$$

It follows that $(\bigvee A)^{\#} = \bigvee\{I^{\#} | I \in A\}$. ●

We prove (3.4). Let $\{f_n\}_{n\in\omega}$ be an increasing sequence from $Set_{S^*\times S^*}(X, U(I))$. We deduce that $\langle I, \bigvee_{n\in\omega} f_n\rangle = \bigvee_{n\in\omega}\langle I, f_n\rangle$. It follows from proposition 1.3 that $\langle I, \bigvee_{n\in\omega} f_n\rangle^{\#} = \bigvee_{n\in\omega}\langle I, f_n\rangle^{\#}$, therefore $U(\langle I, \bigvee_{n\in\omega} f_n\rangle^{\#}) = \bigvee_{n\in\omega} U(\langle I, f_n\rangle^{\#})$.

We need in the sequel some results of the previous paper "Iterative Systems of Equations". Let us recall them.

Let $s : X \longrightarrow \omega F\ell_{M[X],T}$ be a system of equations with flowcharts.

Let $I : M \longrightarrow Q$ be an object of $Int_{\omega}(M,T)$. The smallest solution s_I of s in I may be computed by $s_I = \bigvee_{n\in\omega} \eta^n(\perp_I)$ where $\perp_I$ is the smallest element of $Set_{S^*\times S^*}(X,Q)$ and the function

$$\eta : Set_{S^* \times S^*}(X,Q) \longrightarrow Set_{S^* \times S^*}(X,Q)$$

is defined for every $h \in Set_{S^* \times S^*}(X,Q)$ by

$$\eta(h) = s\langle I,h\rangle^{\#} .$$

If $K : M \longrightarrow Q'$ in another object of $Int_{\omega}(M,T)$ and F is a morphism from I to K then $s_I F = s_K$.

Let us remark that ω_M is an initial object of $Int_{\omega}(M,T)$. If s is a rational system of equations then every component of the smallest solution of s in ω_M is said to be a rational T-M-flowchart.

2. The unfoldment of the $PStr_S$-Σ-flowcharts

Let us mention that the category of ω-continuous S-sorted algebraic theories may be seen as a subcategory of the category of ω-continuous commutative $PStr_S$-modules with iterate. Indeed if T is an ω-continuous S-sorted algebraic theory then there is a unique ω – continuous theory morphism $F_T : PStr_S \longrightarrow T$ and this functor may be taken as the structural functor of the $PStr_S$-module with iterate T. Moreover, if $G : T \longrightarrow T'$ is an ω-continuous theory morphism then G is a $PStr_S$-module with iterate morphism from T to T'.

We study in this section a particular case. We work only with flowcharts over $PStr_S$, therefore we shall omit the index $PStr_S$ from our notations, e.g. we shall write $F\ell_M$ instead of $F\ell_{M,PStr_S}$. We assume that the monoid M is freely generated by the set Σ, therefore we shall replace M by Σ in our standard notations, e.g. we shall write $F\ell_{\Sigma}$, I_{Σ}, ω_{Σ} instead of $F\ell_M$, I_M, ω_M respectively.

Let $r : \Sigma^* \longrightarrow S^*$ and $r' : \Sigma^* \longrightarrow S^*$ be, as usual, the input and the output monoid morphisms.

We define the signature $p : \Sigma \longrightarrow S \times S^*$ by

$$\Sigma' = \{(\sigma,k) \mid \sigma \in \Sigma, \; k \in [\,|r(\sigma)|\,]\}$$

and $\quad p(\sigma,k) = (r(\sigma)_k, r'(\sigma))$ for every $(\sigma,k) \in \Sigma'$.

Let $CT_{\Sigma'}$ be the ω-continuous theory freely generated by Σ' and let $J : \Sigma' \longrightarrow CT_{\Sigma'}$ be its standard interpretation. Let

$$K : \Sigma \longrightarrow CT_{\Sigma'}$$

be the interpretation of Σ in the $PStr_S$-module $CT_{\Sigma'}$ defined for every $\sigma \in \Sigma$ by

$$K(\sigma) = \langle J(\sigma,1), J(\sigma,2), \ldots, J(\sigma,|r(\sigma)|) \rangle.$$

We define the unfoldment as the unique ω-continuous commutative $PStr_S$-module with iterate morphism

$$D : \omega F\ell_{\Sigma} \longrightarrow CT_{\Sigma'}$$

such that $\omega_{\Sigma} D = K$. If F is a Σ-flowchart then D(F) is the unfoldment of F.

2.1. Proposition. Let I be an interpretation of Σ in the ω-continuous S-sorted algebraic theory T. If $G : CT_{\Sigma'} \longrightarrow T$ is the unique ω-continuous theory morphism such that

$$G(J(\sigma,k)) = x_k^{r(\sigma)} I(\sigma)$$

for every $(\sigma,k) \in \Sigma'$ then $DG = I^{\#}$.

Proof. We recall that $I^{\#} : \omega F\ell_{\Sigma} \longrightarrow T$ is the unique ω-continuous communitative $PStr_S$-module with iterate such that $\omega_{\Sigma} I^{\#} = I$. As DG is an ω-continuous commutative $PStr_S$-module with iterate morphism we have only to show that $G(D(\omega_{\Sigma}(\sigma))) = I(\sigma)$ for every $\sigma \in \Sigma$. Indeed if $\sigma \in \Sigma$ then

$$G(D(\omega_{\Sigma}(\sigma))) = G(K(\sigma)) =$$

$$= G(\langle J(\sigma,1), J(\sigma,2), \ldots, J(\sigma,|r(\sigma)|) \rangle) =$$

$= \langle G(J(\sigma,1)), G(J(\sigma,2)), \ldots, G(J(\sigma,|r(\sigma)|)) \rangle =$

$= \langle x_1^{r(\sigma)} I(\sigma), x_2^{r(\sigma)} I(\sigma), \ldots, x_{r(\sigma)}^{r(\sigma)} I(\sigma) \rangle = I(\sigma).$ ∎

Proposition 2.1 shows that in order to compute the behaviour of a Σ-flowchart F under an interpretation of Σ in an ω-continuous theory it suffices to know the unfoldment of F.

Let $F = (i,t,m) \in F\ell_{\Sigma}(a,b)$. If $H : PStr_S \longrightarrow CT_{\Sigma'}$ is the unique ω-continuous theory morphism then

$$D(F) = H(i) \langle (K(m)H(t))^{\dagger}, 1_b \rangle .$$

As the components of the morphism K(m)H(t) are finite partial trees it follows that the components of $(K(m)H(t))^{\dagger}$ are rational trees. Therefore the components of the unfoldment of a finite Σ-flowchart are rational trees.

2.2. <u>Proposition.</u> The components of the unfoldment of a rational $PStr_S$- Σ -flowcharts are context-free trees.

<u>Proof.</u> Let $s : X \longrightarrow \omega F\ell_{\Sigma \cup X}$ be a rational system of equations with $PStr_S$- Σ -flowcharts. Let s_ω be the smallest solution of s in $\omega_{\Sigma} : \Sigma \longrightarrow \omega F\ell_{\Sigma}$ and let $s_\omega D = s_K$ be its unfoldment. Therefore we have to show that for every $x \in X$ and for every $i \in [|rX(x)|]$ the morphism $x_i^{rX(x)} s_K(x) \in CT_{\Sigma'}(rX(x)_i, r'X(x))$ is defined by a context-free tree.

Let us recall that $s_K = \bigvee_{n \in \omega} \eta^n(\perp_K)$ where $\perp_K$ is the smallest element of $Set_{S^* \times S^*}(X, CT_{\Sigma'})$ and the function

$$\eta : Set_{S^* \times S^*}(X, CT_{\Sigma'}) \longrightarrow Set_{S^* \times S^*}(X, CT_{\Sigma'})$$

is defined for every $h \in Set_{S^* \times S^*}(X, CT_{\Sigma'})$ by

$$\eta(h) = s \langle K, h \rangle^{\#}$$

where $\langle K,h\rangle^{\#} : \omega Fl_{\Sigma\cup X} \longrightarrow CT_{\Sigma'}$ is the unique ω-continuous commutative $PStr_S$-module with iterate morphism such that

$$\langle K,h\rangle^{\#}(\omega_{\Sigma\cup X}(\sigma)) = K(\sigma) \quad \text{for every } \sigma \in \Sigma$$

and $\langle K,h\rangle^{\#}(\omega_{\Sigma\cup X}(x)) = h(x)$ for every $x \in X$.

We are looking for another way to compute s_K. Let us first unfold the system s.

Let $X' = \{(x,i) \mid x \in X,\ i \in [\,|rX(x)|\,]\}$ and let $q : \Sigma' \cup X' \longrightarrow S \times S^*$ be the extention of p defined for every $x \in X$ and $i \in [\,|rX(x)|\,]$ by $q(x,i) = (rX(x)_i, r'X(x))$. Let

$$D' : \omega Fl_{\Sigma\cup X} \longrightarrow CT_{\Sigma'\cup X'}$$

be the unique ω-continuous commutative $PStr_S$-module with iterate morphism such that for every $\sigma \in \Sigma \cup X$.

$$D'(\omega_{\Sigma\cup X}(\sigma)) = \langle J'(\sigma,1),\dots,J'(\sigma,|\langle r,rX\rangle(\sigma)|)\rangle$$

where $J' : \Sigma' \cup X' \longrightarrow CT_{\Sigma'\cup X'}$ is the standard interpretation of $CT_{\Sigma'\cup X'}$.

We form a "context-free" system

$$s' : X' \longrightarrow CT_{\Sigma'\cup X'}$$

defined for every $(x,i) \in X'$ by

$$s'(x,i) = x_i^{rX(x)} D'(s(x)).$$

As every morphism $s'(x,i)$ is defined by a rational tree, it follows that the solution s'_J of s' in J is defined by context free trees. To finish the proof, we show that for every $(x,i) \in X'$

$$x_i^{rX(x)} s_K(x) = s'_J(x,i) \ .$$

To compute s'_J we need the function

$$\gamma : Int(X', CT_{\Sigma'}) \longrightarrow Int(X', CT_{\Sigma'})$$

defined for every interpretation g of X' in $CT_{\Sigma'}$ by

$$\gamma(g) = s'\langle J,g\rangle^{\#}.$$

It follow that $s'_J = \bigvee_{n\in\omega} \gamma^n(\perp_J)$.

Let u : $Set_{S^*\times S^*}(X,CT_{\Sigma'}) \longrightarrow Int(X',CT_{\Sigma'})$ be the function defined for every $h\in Set_{S^*\times S^*}(X,CT_{\Sigma'})$ and $(x,i)\in X'$ by

$$u(h)(x,i)=x_i^{rX(x)}h(x) .$$

Let $h\in Set_{S^*\times S^*}(X,CT_{\Sigma'})$. As $D'\langle J,u(h)\rangle^{\#}$ is an ω-continuous commutative $PStr_S$-module with iterate morphism and for every $\sigma\in\Sigma\cup X$

$(D'\langle J,u(h)\rangle^{\#})(\omega_{\Sigma\cup X}(\sigma)) =$

$= \langle J,u(h)\rangle^{\#}(\langle J'(\sigma,1),\dots,J'(\sigma,|\langle r,rX\rangle(\sigma)|)\rangle) =$

$= \langle\langle J,u(h)\rangle(\sigma,1),\dots,\langle J,u(h)\rangle(\sigma,|\langle r,rX\rangle(\sigma)|)\rangle =$

$=$ if $\sigma\in\Sigma$ then $K(\sigma)$ else $h(\sigma)$

it follows that $D'\langle J,u(h)\rangle^{\#} = \langle K,h\rangle^{\#}$.

Let us prove that $\eta u = u\gamma$. If $h\in Set_{S^*\times S^*}(X,CT_{\Sigma'})$ and $(x,i)\in X'$ then

$u(\eta(h))(x,i) = x_i^{rX(x)}\eta(h)(x) = x_i^{rX(x)}\langle K,h\rangle^{\#}(s(x)) =$

$= x_i^{rX(x)}\langle J,u(h)\rangle^{\#}(D'(s(x))) =$

$= \langle J,u(h)\rangle^{\#}(x_i^{rX(x)}D'(s(x))) =$

$= \langle J,u(h)\rangle^{\#}(s'(x,i)) = \gamma(u(h))(x,i).$

We deduce $\eta^n u = u\gamma^n$ for every nonnegative integer n. As $u(\perp_K) = \perp_J$ it follows that $\gamma^n(\perp_J) = u(\eta^n(\perp_K))$. Therefore for every $(x,i)\in X'$

$x_i^{rX(x)}s_K(x) = x_i^{rX(x)}\bigvee_{n\in\omega}\eta^n(\perp_K)(x) =$

$= \bigvee_{n\in\omega} x_i^{rX(x)}\eta^n(\perp_K)(x) = \bigvee_{n\in\omega} u(\eta^n(\perp_K))(x,i) =$

$= \bigvee \gamma^n(\perp_J)(x,i) = s'_J(x,i).$ ●

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