INSTITUTUL
DE
MATEMATICA

INSTITUTUL NATIONAL
PENTRU CREATIE
STIINTIFICA SI TEHNICA

ISSN 0250 3638

ABSOLUTELY CONTINUOUS POTENTIAL KERNELS ON HOMOGENEOUS SPACES

by

Lucian BEZNEA

PREPRINT SERIES IN MATHEMATICS
No.43/1985

BUCURESTI

ABSOLUTELY CONTINUOUS POTENTIAL KERNELS ON HOMOGENEOUS SPACES

by

Lucian BEZNEA*)

June 1985

*) *Department of Mathematics, The National Institute for Scientific and Technical Creation, Bd.Pacii 220, 79622 Bucharest, Romania.*

ABSOLUTELY CONTINUOUS POTENTIAL KERNELS ON HOMOGENEOUS SPACES

by

Lucian Beznea

Introduction. The aim of this paper is to present facts of potential theory (in H-cones language; see [7]) associated to a semigroup of continuous invariant (under the action of G) kernels on the homogeneous space X=G/H , where G is a locally compact group with countable base and H is a compact subgroup of G. We start from the works [5], [11] and [12]. Generally speaking, the program is the same as in [5]. Following this way, we make the connection with the so called "arithmetics of convolution semigroups" exposed in [11] (see §2). More clearly: when transported on the group G, the convolution semigroups on G/H (for various compact subgroups H of G), which generate our theory are (following [12]) exactly the convolution semigroups on G which are only continuous (not $\{e\}$-continuous). Thus, by the semigroup point of view we are still on the group G. "Nice" properties for the associated cone of excessive functions are obtained only if we think it on the associated homogeneous space (see §4, §5).

We also show (see §6) that the dual potential structre (which can always be constructed in the H-cones framework) is generated, on X=G/H, in the same way as the initial one.

§1. Preliminaries and Notations.

The basic facts concerning integration theory and topology on groups are taken from [8], [10] and [11].

Let G be a locally compact topological group with countable base (with the group operation written multiplicatively and with the neutral element denoted by e), K a compact subgroup of G and Y the homogeneous space G/K of left cosets aK, $a\in G$. The group itself will often be considered as the homogeneous space corresponding to $K=\{e\}$. We denote by X the homogeneous space G/H , where H is a compact subgroup of G. Y=G/K will be one of the homogeneous spaces X (K=H) or G ($K=\{e\}$). X is endowed with the quotient topology. We denote by $\mathcal{T}_Y$ the topology on Y. Let Π be the natural projection $a \longmapsto aK$ of G onto G/K . The element $\Pi(a)\in Y$ will be denoted by $\bar{a}$, for all $a\in G$. Π is a continuous, proper, open maping . We denote by ay the action of an element $a\in G$ on the element $y\in Y$: $ay=\overline{ab}$, with $y=\bar{b}$, $b\in G$. Let $\mathcal{B}(Y)$, $C(Y)$, $C_0(Y)$, $K(Y)$ denote respectively the Borel measurable , continuous , continuous vanishing at infinity , continuous with compact support real-valued functions on Y . The σ-algebra of Borel measurable sets on Y is denoted by $\mathcal{B}_Y$. For $\mathcal{H}\subset\mathcal{B}(Y)$, $\mathcal{H}^+$ and $\mathcal{H}_b$ denote respectively the positive, bounded elements of $\mathcal{H}$. We denote by $\mathcal{F}(Y)$ the set of all Borel measurable positive numerical functions on Y and by $M(Y)$ (resp. $M_b(Y)$) the set of all positive (resp. bounded positive) Radon measures on Y.

Let ω be a fixed <u>right invariant</u> Haar measure on G and Δ the modular function of G , hence :

(1.1) $$d\omega(ba)=\frac{1}{\Delta(b)}d\omega(a) \qquad a , b\in G .$$

If g is a numerical function on G , $a\in G$ we denote by g_a , ${}_ag$, $\check{g}$, g^* the functions on G defined by : $g_a(b)=g(ba)$, ${}_ag(b)=g(ab)$, $g(b)=g(b^{-1})$, $g^*(b)=f(b^{-1})\cdot\Delta(b)$, for every $b\in G$.

We denote by ω_H the normed Haar measure of the compact subgroup H of G and we have : $\omega_H\in M_b(G)$, $\omega_H(G)=1$. If $\mathcal{H}$ is a set of numerical functions on G , we define :

$$_H\mathcal{H}^\natural=\left\{g\in\mathcal{H} \;/\; g_h=g \text{ , for all } h\in H\right\} ,$$

$$_H^\natural\mathcal{H}=\left\{g\in\mathcal{H} \;/\; {}_hg=g \text{ , for all } h\in H\right\} ,$$

and :

$$_H\mathcal{H}={}_H\mathcal{H}^\natural\cap{}_H^\natural\mathcal{H} .$$

For $g\in{}_H\mathcal{H}^\natural$, we define the numerical function $\bar{g}$ on X by :

(1.2) $\bar{g}(\bar{a})=g(a)$, for $a\in G$.

If $g\in\mathcal{B}(G)$ is bounded on aH , for every $a\in G$, $g^\natural$ denotes the function on G defined by

$$g^\natural(a)=\int g(ah)d\omega_H(h) \quad , \quad \text{for every } a\in G .$$

<u>1.1 Remark.</u> a) If $\mathcal{H}$ is one of the vector lattices $\mathcal{B}_b(G)$, $C(G)$, $C_0(G)$, $K(G)$, the above correspondence $g \mapsto g^\natural$ defines a positive linear operator from $\mathcal{H}$ to ${}_H\mathcal{H}^\natural$.

b) Extending the above operation on $\mathcal{F}(G)$, we obtain an additive positive homogeneous operator from $\mathcal{F}(G)$ to ${}_H\mathcal{F}^\natural(G)$. In fact , this is a continuous kernel on G (see § 2) . If $g \in \mathcal{F}(G)$ is lower semicontinuous , then $g^\natural$ has the same property .

c) If $g \in \mathcal{B}_b(G) \cup C(G) \cup \mathcal{F}(G)$ then :

$$g \in (\mathcal{B}_b(G) \cup C(G) \cup \mathcal{F}(G))^\natural \Longleftrightarrow g^\natural = g ,$$

hence we can also denote by $\bar{\bar{g}}$ (see (1.2)) the function on X defined by :

$$\bar{\bar{g}}(\bar{\bar{a}}) = \overline{g^\natural}(\bar{\bar{a}}) \quad , \quad \text{for every } a \in G .$$

d) If $g \in \mathcal{B}_b(G)$ (resp. $g \in \mathcal{F}(G)$, $g \in C(G)$, $g \in C_0(G)$, $g \in K(G)$) then $\bar{\bar{g}} \in \mathcal{B}_b(X)$ (resp. $\bar{\bar{g}} \in \mathcal{F}(X)$, $\bar{\bar{g}} \in C(X)$, $\bar{\bar{g}} \in C_0(X)$, $\bar{\bar{g}} \in K(X)$) .

<u>1.2 Lemma.</u> The map

$$f \longmapsto f' = f \circ \pi$$

is a positive linear bijection between $\mathcal{B}_b(X)$ (resp. $C(X)$, $C_0(X)$, $K(X)$) and ${}_H\mathcal{B}_b^\natural(G)$ (resp. ${}_HC^\natural(G)$, ${}_HC_0^\natural(G)$, ${}_HK^\natural(G)$) , whose inverse is given by (1.2) . Analogously , $\mathcal{F}(X)$ is in bijection with ${}_H\mathcal{F}^\natural(G)$.

For $\nu \in M(G)$, $\check{\nu}$ will be the measure on G defined by

$$\check{\nu}(g) = \nu(\check{g}) \quad , \quad \text{for every } g \in K(G) .$$

We have : $\check{\omega} = \Delta \cdot \omega$. Then

$$(1.3) \qquad \widecheck{g.\omega} = g^*.\omega \quad ,$$

where $g.\omega$ denotes the Radon measure with density g with respect to ω.

For ν_1 , $\nu_2 \in M(G)$ we define (if it exists) the measure $\nu_1 * \nu_2$ on G by

$$\nu_1 * \nu_2(g) = \iint g(ab)\,d\,\nu_1(a)\,d\,\nu_2(b) \quad , \quad \text{for } g \in K(G)$$

and we observe that :

(1.4) $$\widecheck{\nu_1 * \nu_2} = \check{\nu}_2 * \check{\nu}_1 \quad .$$

If $\nu \in M(G)$ and $g \in \mathcal{F}(G)$, we define the function $\nu * g$ on G by

$$\nu * g(a) = \int g(b^{-1}a)\,d\nu(b) \quad , \quad \text{for all } a \in G \, .$$

If g_1, $g_2 \in \mathcal{F}(G)$, we denote by $g_1 * g_2$ the function :

$$g_1 * g_2\,(a) = \int g_2(b^{-1}a) g_1(b) \Delta(b) d\omega(b) \quad , \text{for all } a \in G.$$

If g_1, $g_2 \in L^1_{loc}(\omega)$ (i.e g_1, g_2 are ω-locally integrable), g_1, $g_2 \geq 0$ we deduce by (1.1) and (1.4) :

(1.5) $$g_1 \cdot \omega * g_2 \cdot \omega = (g_1 * g_2) \cdot \omega \quad ,$$

(1.6) $$(g_1 * g_2)^* = g_2^* * g_1^* \quad .$$

On $M_b(Y)$ we can consider the induced vague topology (determined by the duality with $K(Y)$) and the weak topology (determined by the duality with $C_b(Y)$). The set $M_b(G)$ is a topological semigroup (with respect to convolution) with the weak topology (see [11, Theorem 1.2.2]) .

If f is a numerical function on Y and $a \in G$, we define the function ${}_a f$ on Y by

$${}_a f(y) = f(ay) \quad , \text{for every } y \in Y.$$

Let

$${}_K^{\natural}M(Y) = \{\mu \in M(Y) / \mu({}_k f) = \mu(f) \ , \text{for all } k \in K, f \in K(Y)\} \, ,$$

$${}_H M^{\natural}(G) = \{\nu \in M(G) / \nu(g_h) = \nu(g) \ , \text{for all } h \in H, g \in K(G)\}$$

and

$${}_H M(G) = {}_H^{\natural}M(G) \cap {}_H M^{\natural}(G) \, .$$

<u>1.3 Lemma.</u> If $\nu \in M(G)$,the following statements are equivalente :

a) $\omega_H * \nu = \nu$ (resp. $\nu * \omega_H = \nu$, $\omega_H * \nu * \omega_H = \nu$).

b) $\nu \in {}_H^{\natural}M(G)$ (resp. $\nu \in {}_H M^{\natural}(G)$, $\nu \in {}_H M(G)$).

For $\nu \in M(G)$, the measure $\bar{\nu}$ M(X) is defined by :

(1.7) $$\bar{\nu}(f) = \nu(f \circ \pi) \quad , \text{for all } f \in K(X).$$

If $\mu \in M(X)$, let us define $\mu' \in {}_H M(G)$ by :

(1.8) $$\mu'(g) = \mu(\bar{g}) \quad , \quad \text{for all } g \in K(G) \; .$$

1.4 Lemma. ([8, ch.VII, §2]). There is an additive positive homogeneous homeomorfism in the vague topology between $M(X)$ (resp. $M_b(X)$, ${}^{\natural}_H M(X)$) and ${}_H M^{\natural}(G)$ (resp. ${}_H M_b^{\natural}(G)$, ${}_H M(G)$) . The correspondence is given by (1.7) and (1.8) .

Proof. This is a consequence of Fubini's theorem and of the fact that $\omega_H(G)=1$. □

Applying (1.1) for a function $g \in {}^{\natural}_H K^+(G)$, $g \neq 0$, it follows that

$$\Delta(h)=1 \quad , \text{ for every } h \in H \; ,$$

and now again by (1.1) we have : $\omega \in {}_H M(G)$.

If $\mu_1, \mu_2 \in M(X)$, we define the measure $\mu_1 * \mu_2 \in M(X)$ by :

(1.9) $$\mu_1 * \mu_2 = \overline{\mu_1' * \mu_2'} \; ,$$

when the convolution $\mu_1' * \mu_2'$ exists .

By Lemma 1.3 we deduce that (if the convolution exists) :

(1.10) $$\nu_1 \in M(G) \; , \; \nu_2 \in {}_H M^{\natural}(G) \implies \nu_1 * \nu_2 \in {}_H M^{\natural}(G) \; .$$

From (1.9) , (1.10) and the bijection between $M(X)$ and ${}_H M^{\natural}(G)$ (see Lemma 1.4) we deduce that , if $\mu_1, \mu_2 \in M(X)$, then :

(1.11) $$\mu_1' * \mu_2' = (\mu_1 * \mu_2)' \; .$$

Analogously , if $\nu_1, \nu_2 \in {}_H M^{\natural}(G)$, then

(1.12) $$\nu_1 * \nu_2 = (\bar{\nu}_1 * \bar{\nu}_2)' \; .$$

If $\mu \in M(X)$, we define the measure $\check{\mu} \in M(X)$ by :

$$\check{\mu} = \overline{\check{\mu'}} \; .$$

and we notice that , if $\nu \in {}_H M(G)$, then $\check{\nu} \in {}_H M(G)$, hence :

(1.13) $$\check{\mu}' = \check{\mu'} \quad , \text{for } \mu \in {}^{\natural}_H M(X) \; .$$

If $\mu_1, \mu_2 \in {}^{\natural}_H M(X)$, from (1.4) and (1.13) there results :

(1.14) $$\widecheck{\mu_1 * \mu_2} = \check{\mu}_2 * \check{\mu}_1 \; .$$

We recall now some results about absolute continuity and quasi-invariant measures. A measure $\mu \in M(Y)$ is called quasi-invariant if for $A \in \mathcal{B}_Y$:

$$\mu(A)=0 \Rightarrow \mu(aA)=0 \quad \text{, for every } a \in G.$$

1.5 Lemma. a)([8, ch. VII, §2, Corollaire 1]) Let $\mu_1, \mu_2 \in M(X)$. Then: $\mu_1 \ll \mu_2$ (μ_1 is absolutely continuous with respect to μ_2) if and only if $\mu_1' \ll \mu_2'$.

b)([8, ch. VII, §2, no. 5]) A measure $\mu \in M(Y)$ is quasi-invariant if and only if μ is equivalent with $\bar{\omega}$ (i.e. $\mu \ll \bar{\omega}$ and $\bar{\omega} \ll \mu$), where $\bar{\omega} = \omega$ when Y=G.

c)([8, ch. VII, §2, Proposition 6]). Let $f \in \mathcal{F}(X)$. Then:

$$f \in L^1_{loc}(\bar{\omega}) \Longleftrightarrow f' \in L^1_{loc}(\omega).$$

If $\mu = f.\bar{\omega}$ with $f \in L^1_{loc}(\bar{\omega})$ then $\mu' = f'.\omega$.

1.6 Remark. $\bar{\omega}$ and $\check{\bar{\omega}}$ are quasi-invariant measures on Y.

§2. Convolution Semigroups on X and Translation to the Group Space.

Most of the results and the terminology in this section are adapted from [11].

A family $(\mu_t)_{t>0} \subset {}^{\natural}_K M(G/K)$ with the properties

(2.1) $$\mu_t(Y) \leq 1 \text{ ,}$$

(2.2) $$\mu_t * \mu_s = \mu_{t+s} \quad \text{, for every } t,s>0$$

is called convolution semigroup on Y=G/K. In the sequel, the considered convolution semigroups will be non trivial ($\mu_t \neq 0$, for all $t>0$). $\mu_t(Y)$ being the bounded (different from zero) solution of the functional equation $u(t+s)=u(t)u(s)$, it must have the form:

(2.3) $$\mu_t(Y)=e^{-ct} \text{ , } c \geq 0.$$

A convolution semigroup $S=(\mu_t)_{t>0}$ on Y is called (see [11, 1.5.6]) continuous (resp. 0-continuous) if the map $t \mapsto \mu_t$ is continuous from $(0,\infty)$ into $M_b(Y)$ (resp. if the limit $\mu_0 = \lim_{t \to 0} \mu_t$ exists in $M_b(Y)$).

On $M_b(Y)$ we consider the weak topology .

2.1 Remark. a) If Y= G , by the weak continuity of the convolution on G , the limit measure μ_0 (from the definition above) is an idempotent of $M_b(G)$ hence , by Theorem 1.2.7 in [11] , it is of the form $\mu_0=\omega_K$ for some compact subgroup K of G (see also [12,10.2] and [13,Theorem 3.1]) . In this case such a 0-continuous convolution semigroup is called K-continuous .

b) From (2.3) and Theorem 1.1.9 in [11] , for example , in the definitios of continuous and K-continuous convolution semigroup on G , we can also consider the vague topology .

c) If $S=(\nu_t)_{t>0}$ is a convolution semigroup on G , by (2.3) the family $S_1=(e^{ct}\cdot\nu_t)_{t>0}$ is a convolution semigroup of probability measures on G . S and S_1 satisfy simultaneously the additional continuity properties , hence we can apply the results of [11] .

2.2 Theorem. ([11,Theorem 1.5.8]) Let $S=(\nu_t)_{t>0}$ be a convolution semigroup on G . The following statements are equivalent :

a) S is 0-cotinuous .

b) S is continuous .

In either case $\nu_0=\lim_{t\to 0}\nu_t$ satisfies

(2.4) $$\nu_t=\nu_0*\nu_t=\nu_t*\nu_0 \quad , \quad \text{for all } t>0 .$$

A convolution semigroup $(\mu_t)_{t>0}$ on Y is called $\{\bar{e}\}$-continuous if :

(2.5) $$\mu_t \xrightarrow[t\to 0]{} \varepsilon_{\bar{e}} \quad \text{in the vague topology .}$$

2.3 Remark. a) By Lemma 1.4,(1.11) and(1.12) we deduce that the family $(\mu_t)_{t>0}$ (resp. $(\nu_t)_{t>0}$) is a convolution semigroup on X (resp. on G with $\nu_t \in {}_H M(G)$, for all $t>0$) if and only if $(\mu_t')_{t>0}$ (resp. $(\bar{\nu}_t)_{t>0}$) is a convolution semigroup on G (resp. on X) .

b) Using (1.14) (resp.(1.4)) and observing that $\check{\varepsilon}_{\bar{e}} = \varepsilon_{\bar{e}}$ (resp. $\check{\omega}_H = \omega_H$) , it follows :

the family $(\mu_t)_{t>0} \subset {}_H^{\natural}M(X)$ (resp. $(\nu_t)_{t>0} \subset {}_H M(G)$) is an $\{\bar{e}\}$-continuous (resp. H-continuous) convolution semigroup on X (resp. on G) if and only if $(\check{\mu}_t)_{t>0}$ (resp. $(\check{\nu}_t)_{t>0}$) is an $\{\bar{e}\}$-continuous (resp. H-continuous) convolution semigroup on X (resp. on G) .

The following result is the variant of Theorem 2.2 which appears in the earlier paper [12].

<u>2.4 Theorem.</u> ([12 , §10]) a) Let $(\mu_t)_{t>0}$ be an $\{\bar{e}\}$-continuous convolution semigroup on the homogeneous space X=G/H . Then $(\mu_t')_{t>0}$ is an H-continuous convolution semigroup on G.

b) Conversly , if $(\nu_t)_{t>0}$ is a continuous convolution semigroup on G , then there exists a compact subgroup H of G such that $\nu_t \in {}_H M(G)$ for all $t>0$ and $(\bar{\nu}_t)_{t>0}$ is an $\{\bar{e}\}$-continuous convolution semigroup on X= G/H .

<u>Proof.</u> Observe that if $\nu_0 = \omega_H$ with H a compact subgroup of G, then (2.4) means that $\nu_t \in {}_H M(G)$ for all $t>0$. Because $\varepsilon_{\bar{e}}' = \omega_H$ and $\bar{\omega}_H = \varepsilon_{\bar{e}}$, the statements are a direct consequence of Theorem 2.2 together with Lemma 1.4 and Remark 2.3 .□

For every $\nu \in M(G)$ we define , following [11, 1.2.3] , the <u>invariance subgroup of ν</u> by :

$$I(\nu) = \{a \in G \ / \ \nu = \varepsilon_a * \nu = \nu * \varepsilon_a\} .$$

If $\nu \in M_b(G)$, then $I(\nu)$ is a compact subgroup of G (see [11,Theorem 1.2.4.]) . If $S = (\nu_t)_{t>0}$ is a convolution semigroup on G , we denote by I(S) the compact subgroup of G :

$$I(S) = \bigcap_{t>0} I(\nu_t)$$

<u>2.5 Corollary.</u> Let $S=(\nu_t)_{t>0}$ be an H-continuous convolution semigroup on G . Then $H=I(S)$.

<u>Proof.</u> Because $\nu_t \in {}_H M(G)$ for all $t>0$, we have $H \subset I(S)$. But this implies (Lemma 1.3) that :

$$(2.6) \qquad \omega_H * \omega_{I(S)} = \omega_{I(S)} * \omega_H = \omega_{I(S)} \ .$$

For every $t>0$ $I(S) \subset I(\nu_t)$, and therefore , by Lemma 1.3,

$$(2.7) \qquad \nu_t * \omega_{I(S)} = \nu_t \ .$$

By hypothesis $\lim_{t \searrow 0} \nu_t = \omega_H$, hence , from (2.7) and the continuity of the convolution , it results that :

$$(2.8) \qquad \omega_H * \omega_{I(S)} = \omega_H \ .$$

From (2.6) and(2.8) it follows $\omega_H = \omega_{I(S)}$, so we finish the proof. □

The next corollary describes the continuity of the convolution semigroup S in terms of the invariance subgroup I(S) of S (see [6, Corollary 8.17] for the abelian group case) .

<u>2.6 Corollary.</u> Let $S=(\nu_t)_{t>0}$ be a convolution semigroup on G . Then :

a) S is continuous $\Longleftrightarrow \lim_{t \searrow 0} \nu_t = \omega_{I(S)}$.

b) S is $\{e\}$ — continuous $\Longleftrightarrow I(S)=\{e\}$ and S is a continuous semigroup.

<u>Proof.</u> This is a consequence of Theorem 2.4 and Corollary 2.5 .□

We complete this section with the correspondence between convolution semigroups and semigroups of continuous invariant kernels.

We recall that a <u>continuous kernel</u> on Y is a linear and positive map $T:K(Y) \longrightarrow C(Y)$. For every $y \in Y$, extending to $\mathcal{F}(Y)$ the Radon measure $T_y \in M(Y)$ defined by : $T_y(f)=T(f)(y)$, for all $f \in K(Y)$, we obtain a kernel

$$T: \mathcal{F}(Y) \longrightarrow \mathcal{F}(Y)$$

on the measurable space $(Y, \mathcal{B}_Y)$ (see [7,§1.1]).

A continuous kernel T on Y is called <u>invariant</u> if:

$$T({}_a f) = {}_a T(f) \quad , \text{ for all } f \in K(Y) \; , \; a \in G \; .$$

Let N(Y) (resp. $N_i(Y)$) denote the set of all continuous (resp. continuous invariant) kernels on Y and :

$${}_H N(G) = \left\{ T \in N(G) \; / \; T(g)_h = T(g_h) = T(g) \; , \text{ for all } g \in K(G), \; h \in H \right\},$$

$${}_H N_i(G) = {}_H N(G) \cap N_i(G) \; .$$

<u>2.8 Remark.</u> For a kernel $T \in N(G)$, the following statements are equivalent :

a) $T \in {}_H N(G)$.

b) $T(g^{\natural}) = T(g)^{\natural} = T(g)$, for every $g \in K(G)$.

If $T \in N(G)$, we define the continuous kernel $\overline{T} \in N(X)$ by :

(2.9) $\quad \overline{T}(f) = \overline{T(f')}$, for every $f \in K(X)$.

If $T \in N(X)$, we denote by T' the continuous kernel on G defined by :

(2.10) $\quad T'(g) = T(\overline{g})'$, for every $g \in K(G)$,

and we observe that $T' \in {}_H N(G)$.

<u>2.9 Lemma.</u> (2.9)and (2.10) establish a bijection between N(X) (resp. $N_i(X)$) and ${}_H N(G)$ (resp. ${}_H N_i(G)$) .

For $\mu \in {}_K^{\natural} M(G/K)$, we define (see [12 , 8.1]) the <u>convolution operator T_μ of μ</u> by

(2.11) $\quad T_\mu f(\overline{a}) = \int_{G/K} f(ay) d\mu(y)$, for every $a \in G$, $f \in K(G/K)$.

<u>2.10 Remark.</u> a) ([3, Lemma 0.1]). If $\nu \in M(G)$, then (2.10) defines a continuous kernel $T_\nu \in N(G)$.

<u>2.11 Lemma.</u> a) If $\nu \in M(G)$, then :

$$\nu \in {}_H M(G) \Longleftrightarrow T_\nu \in {}_H N(G)$$

b) If $\mu, \eta \in {}_H^{\natural} M(X)$, then (2.10) defines a continuous kernel $T_\mu \in N(X)$ and :

(2.12) $\quad T_\mu = \overline{T_{\mu'}}$, (2.13) $T'_{\mu} = T_{\mu'}$,

(2.14) $T_{\mu * \eta} = T_{\mu} \circ T_{\eta}$,

if the convolution exists.

c) ([11, 1.5.4]). If $\mu \in {}^{b}_{H}M_b(X)$, then : $T_{\mu}(C_0(X)) \subseteq C_0(X)$, $T_{\mu}(C_b(X)) \subseteq C_b(X)$ and T_{μ} is a bounded linear operator on $C_b(X)$ with $\|T_{\mu}\| = \mu(X)$.

<u>Proof.</u> Let us observe that assertion a) is a consequence of the definitions .

b) Let $\mu \in {}^{b}_{H}M(X)$ and $f \in K(X)$. We shall show that :

(2.15) $T_{\mu}(f) = \overline{T_{\mu'}(f')}$

Indeed , if $a \in G$, from assertion a) , Remark 2.8 , (2.11) and (1.8) it follows : $\overline{T_{\mu'}(f')}(\overline{a}) = T_{\mu'}(f')(a) = \mu'({}_a f') = \mu({}_a f) = T_{\mu}(f)(\overline{a})$. By (2.15) and Lemma 1.2 results : $T_{\mu} \in N(X)$. From (2.15) results (2.12), using (2.9) . Because $\mu' \in {}_{H}M(G)$, by assertion a) and Lemma 2.9 , (2.13) is equivalent with (2,12) . (2.14) has a direct proof.

Assertion c) is a consequence of 1.5.4 in [11] , using (2.12) and Lemma 1.2 . □

<u>2.12 Lemma.</u> ([12, § 2],[14, Proposition I 3.2]). Let $T \in N(X)$ (resp. $T' \in N(G)$) . The following statements are equivalent :

a) $T \in N_i(X)$ (resp. $T' \in {}_{H}N_i(G)$) .

b) There exist $\mu \in {}^{b}_{H}M(X)$ (resp. $\mu' \in {}_{H}M(G)$) such that $T = T_{\mu}$ (resp. $T' = T_{\mu'}$) .

In addition μ (resp. μ') is unique with this property .

<u>Proof.</u> From Lemma 2.11 a) and b) it suffices to consider the case $T \in N(X)$. If $T \in N_i(X)$, we define the measure $\mu \in M(X)$ by :

(2.16) $\mu = T_{\overline{e}}$

One can easily verify that $\mu \in {}^{b}_{H}M(X)$ and $T = T_{\mu}$. □

A family $(S_t)_{t>0} \subseteq N(Y)$ is called semigroup of continuous kernel on Y if :

(2.17) $\qquad S_t(1) \leq 1$

(2.18) $\qquad S_t \circ S_s = S_{t+s}$, for every $s,t>0$.

If $S=(\mu_t)_{t>0}$ is a convolution semigroup on Y , we define the associated semigroup of continuous kernels on Y by :

$$S_t = T_{\mu_t} \quad , \quad \text{for all } t>0 \quad .$$

From (2.1) and (2.2) we deduce (2.17) and (2.18) for this family .

2.13 Remark. a) Let $S=(\nu_t)_{t>0}$ be a convolution semigroup on G . By Theorem 2.2 , Lemma 2.11 a) and Remark 2.8 we obtain that: S is H-continuous if and only if

(2.19) $\qquad S_t g(a) \xrightarrow[t\to 0]{} T_{\omega_H} g(a) = g^{\natural}(a)$, for $a \in G$, $g \in K(G)$

b) Let $S=(\mu_t)_{t>0}$ be a convolution semigroup on X . Then , from (2.5), it follows : S is $\{\bar{e}\}$-continuous if and only if

(2.20) $\qquad S_t f(x) \xrightarrow[t\to 0]{} f(x)$, for $x \in X$ and $f \in K(X)$.

In this case , by Lemma 2.11 c) , the family $(S_t)_{t>0}$ induces an invariant Feller semigroup on X .

2.14 Lemma. ([12, §9]). The corespondence $\mu_t \longmapsto S_t$ between the $\{\bar{e}\}$-continuous (resp. H-continuous) convolution semigroups on X (resp. on G) and the semigroups of continuous invariant kernels on X (resp. on G) which satisfy (2.20) (resp. (2.19)) is one-to-one . (See also [11, Lemma 4.1.1])

Proof. The assertions follow from Lemma 2.12 and Remark 2.13 .□

§3. Absolutely Continuous Potential Kernels on X .

Let $(\mu_t)_{t>0}$ be a continuous convolution semigroup on $Y=G/K$. For every $\alpha>0$ we define the measure $k_\alpha \in {}^{\natural}_{K}M(G/K)$ by :

$$k_\alpha(f)=\int_0^\infty e^{-\alpha t}\mu_t(f)dt \quad , \quad \text{for } f\in K(Y) \quad .$$

We have :

$$k_\alpha(Y)\leq\frac{1}{\alpha} \quad , \quad \text{hence } k_\alpha\in M_b(Y) \quad ,$$

(3.1) $$k_\alpha - k_\beta=(\beta-\alpha)k_\alpha * k_\beta \quad , \quad \text{for } \alpha,\beta>0 .$$

The family $(k_\alpha)_{\alpha>0}$ is called the resolvent of measures associated with the convolution semigroup $(\mu_t)_{t>0}$. For a fixed $f\in K^+(Y)$, let :

$$k(f)=\sup_{\alpha>0} k_\alpha(f)=\int_0^\infty \mu_t\ (f)dt \leq +\infty .$$

The semigroup $(\mu_t)_{t>0}$ is called transient if $k(f)<+\infty$ for every $f\in K^+(Y)$, and the measure k is called the potential kernel associated with $(\mu_t)_{t>0}$. We remark that for every $\alpha>0$:

$$k-k_\alpha=\alpha\cdot(k*k_\alpha) \ ,$$

(3.2) $$k*k_\alpha=k_\alpha*k \quad .$$

A measure $k\in {}^{\natural}_{K}M(G/K)$ is called potential kernel if there exists a transient convolution semigroup $(\mu_t)_{t>0}$ such that k should be its potential kernel . We write :

$$k=\int_0^\infty \mu_t dt$$

Let $k=\int_0^\infty \mu_t dt$ be a potential kernel on Y. We define the family $\mathcal{R}=(R_\alpha)_{\alpha\geq 0}$ by : $R_0=T_k$, $R_\alpha=T_{k_\alpha}$, for all $\alpha>0$ and we call $\mathcal{R}$ the resolvent of kernels associated with the potential kernel k . Remark that:

$$R_\alpha(1)\leq\frac{1}{\alpha} \quad , \quad \text{for all } \alpha>0$$

$$R_\alpha - R_\beta=(\beta-\alpha)R_\alpha\circ R_\beta \quad , \quad \text{for } \alpha,\beta\geq 0$$

$$R_\alpha \circ R_\beta = R_\beta \circ R_\alpha \quad , \quad \text{for } \alpha,\beta \geq 0$$

$$R_0(f) = \sup_{\alpha>0} R_\alpha(f) \quad , \quad \text{for every } f \in \mathcal{F}(Y) \ .$$

Let us denote :

$$\mathcal{S}(\mathcal{R}) = \{ s \in \mathcal{F}(Y) \ / \ \alpha R_\alpha s \leq s \ , \text{ for every } \alpha > 0 \} ,$$

$$\mathcal{S}'(\mathcal{R}) = \{ s \in \mathcal{S}(\mathcal{R}) \ / \ \lim_{\alpha\to\infty} \alpha R_\alpha s = s \} ,$$

$$\mathcal{E}(\mathcal{R}) = \{ s \in \mathcal{S}'(\mathcal{R}) \ / \ [s=\infty] \text{ is } \mathcal{R}\text{-negligible} \} .$$

We recall that $A \in \mathcal{B}_Y$ is <u>$\mathcal{R}$-negligible</u> if $R_\alpha(1_A)=0$ for all $\alpha > 0$.

<u>3.1 Remark</u>. Let $S=(\mu_t)_{t>0}$ be an $\{\bar{e}\}$-continuous convolution semigroup on X (therefore $S'=(\mu'_t)_{t>0}$ is an H-continuous convolution semigroup on G , by Theorem 2.4) .

a) If $(k_\alpha)_{\alpha>0}$ is the resolvent of measures associated with S , then $(k'_\alpha)_{\alpha>0}$ is the associated resolvent of the semigroup S' on G .

b) S is transient if and only if S' is transient and :

$$k=\int_0^\infty \mu_t \, dt \iff k'=\int_0^\infty \mu'_t \, dt \quad . \tag{3.3}$$

c) If $\mathcal{R}=(R_\alpha)_{\alpha\geq 0}$ is the reslvent of kernels associated with k , then by Lemma 2.9 , (2.13) and (3.3) , the family $\mathcal{R}'=(R'_\alpha)_{\alpha\geq 0}$ is the resolvent of kernels associated with k' .

d) Let $\alpha>0$ be fixed . Then k_α (resp. k'_α) is a potential kernel on X (resp. on G) with : $k_\alpha=\int_0^\infty e^{-\alpha t}\mu_t \, dt$ (resp. $k'_\alpha=\int_0^\infty e^{-\alpha t}\mu'_t \, dt$) .

The associated resolvent of kernels is denoted by : $\mathcal{R}_\alpha=(R_{\alpha+\beta})_{\beta\geq 0}$ (resp. $\mathcal{R}'_\alpha=(R'_{\alpha+\beta})_{\beta\geq 0}$) .

e) From (2.20) (i.e. S is $\{\bar{e}\}$-continuous) it follows:

$$\alpha R_\alpha f(x) \xrightarrow[\alpha\to\infty]{} f(x) \quad , \text{for every } f \in K(X) \ , \ x \in X$$

and from (2.19) (i.e. S' is H-continuous) it results that :

$$\alpha R'_\alpha g(a) \xrightarrow[\alpha\to\infty]{} g^\natural(a) \quad , \text{for every } g \in K(G) \ , \ a \in G \ .$$

In the sequel , the potential kernel $k=\int_0^\infty \mu_t \, dt$ (resp.

$k' = \int_0^\infty \mu'_t dt$) on X (resp. on G) is associated with the $\{\bar{e}\}$-continuous (resp. H-continuous) convolution semigroup $(\mu_t)_{t>0}$ (resp. $(\mu'_t)_{t>0}$) on X (resp. on G) .

We recall that a continuous kernel $T \in N(Y)$ is <u>absolutely continuous</u> (with respect to the measure $\eta \in M(Y)$) if : $T(1_A)=0$,for every $A \in \mathcal{B}_Y$ with $\eta(A)=0$.(i.e. $T_y \ll \eta$ for every $y \in Y$) . The measure η can be chosen to be finite .

<u>3.2 Lemma.</u> Let $T=T_\mu \in N_i(Y)$ and $\eta \in M(Y)$ be quasi-invariant . Then the following statements are equivalent :

a) T is absolutely continuous with respect to η .

b) $\mu \ll \eta$.

<u>Proof.</u> The implication "a)$\Rightarrow$b)" is obvious by (2.16) .

"b)$\Rightarrow$a)" . Let $A \in \mathcal{B}_Y$, such that $\eta(A)=0$ and $a \in G$. Then : $\eta(a^{-1}A)=0$, hence $\mu(a^{-1}A)=0$ by hypothesis b) and thus $T_{\bar{a}}(A)= (T_\mu)_{\bar{a}}(A)=\mu({}_a(1_A))=0$. □

<u>3.3 Lemma.</u> Let $T \in N_i(Y)$, $T \neq 0$ and $f \in \mathcal{F}(Y)$, Then :

$$T(f)=0 \ , \bar{\omega}\text{- a.e.(almost everywhere)} \Rightarrow \bar{\omega}(f)=0 \ .$$

<u>Proof.</u> Let $\mu \in {}^{\natural}_{K}M(G/K)$, with $T=T_\mu$. Then $\bar{\omega}(T(f))=0$ implies $\omega(T(f)')=0$. From (2.13) , we have $T(f)'=T'(f')=T_{\mu'}(f')$ (obviously if Y=G , then $f'=f$, $T'=T$, $\mu'=\mu$). So $0=\omega(T_{\mu'}(f'))= \iint f'(ab)d\mu'(b)d\omega(a)=\mu'(\omega(f'))$. Since $\mu' \neq 0$,we deduce that $\omega(f')=0$, hence $\bar{\omega}(f)=0$. □

<u>3.4 Lemma.</u> Let $k=\int_0^\infty \mu_t dt$ (resp. $k'=\int_0^\infty \mu'_t dt$) be a potential kernel on X (resp. on G) and $A \in \mathcal{B}_X$ (resp. $A \in \mathcal{B}_G$) .

a) If $\alpha > 0$ then :

$$R_0(1_A)=0 \text{ (resp. } R'_0(1_A)=0) \Leftrightarrow R_\alpha(1_A)=0 \text{ (resp. } R'_\alpha(1_A)=0) \ .$$

b) If R_0 (resp. R_0') is absolutely continuous with respect to $\eta \in M(X)$ (resp. $\eta' \in M(G)$) we have :

$R_0(1_A)=0$ (resp. $R_0'(1_A)=0$) , η-a.e. (resp. η'-a.e.) $\Rightarrow R_0(1_A)=0$ (resp. $R_0'(1_A)=0$) .

<u>Proof</u>. Assertion a) is a consequence of the resolvent equation . For b) it suffices to notice that : $\alpha R_\alpha R_0(1_A)=0$ and $R_0(1_A) \in \mathcal{S}'(\mathcal{R})$. □

<u>3.5 Theorem</u>. ([3 , Proposition 3.5]). Let $k=\int_0^\infty \mu_t dt$ (resp. $k'=\int_0^\infty \mu_t' dt$) be a potential kernel on X (resp. on G) such that R_0 (resp. R_0') is absolutely continuous with respect to $\eta \in {}_H^b M_b(X)$ (resp. $\eta' \in M_b(G)$) . Then :

a) $k \ll \overline{\omega}$ (resp. $k' \ll \omega$) .

b) R_0 (resp. R_0') is absolutely continuous with respect to $\overline{\omega}$ (resp. ω) . Clearly , if $A \in \mathcal{B}_X$ (resp. $A \in \mathcal{B}_G$) , then :

(3.4) $\quad R_0(1_A)=0$ (resp. $R_0'(1_A)=0$) $\Longleftrightarrow \overline{\omega}(A)=0$ (resp. $\omega(A)=0$) .

<u>Proof</u>. It suffices to prove the first assertion of b) (i.e. the implication "$\Longleftarrow$" of (3.4)) . Indeed , the assertion a) holds by Lemma 3.2 and the other implication of (3.4) results from Lemma 3.3 .

By Lemma 3.4 a) we can replace the potential kernel k with k_α , for some $\alpha > 0$. The measures η and k_α being finite , we may without loss of generality , assume that there exists the convolution $\eta * k$ (resp. $\eta' * k'$) denoted by η_0 (resp. η_0') . By (2.14) it follows : $T_{\eta_0} = T_{\eta * k} = T_\eta \circ T_k$, hence

$$\eta_0(A)=\eta(R_0(1_A)) \quad , \quad \text{for every } A \in \mathcal{B}_X \quad . \tag{3.5}$$

Let $a \in G$ and $A \in \mathcal{B}_X$ with $\eta_0(A)=0$. From (3.5) we deduce $\eta(R_0(1_A))=0$ and by Lemma 3.4 b) it follows that :

$$R_0(1_A)=0 \tag{3.6}$$

From (3.5) , the invariance of R_0 and (3.6) , it results that :

$$\eta_0(aA)=\eta(R_0({}_{a^{-1}}1_A))=\eta({}_{a^{-1}}R_0(1_A))=0 .$$

We have just come to prove that R_0 is absolutely continuous with respect to η_0 and that η_0 is quasi-invariant . We now have by Lemma 1.5 b) : R_0 is absolutely continuous with respect to ω . □

From now on $k=\int_0^\infty \mu_t dt$ will be a potential kernel on X with

(*) $$k \ll \overline{\omega} .$$

§4. H-Cones of Functions on X .

For the H-cones terminology and results see [7] .

4.1 Lemma. a) If $t\in \mathcal{S}(\mathcal{R})$, then $t^{\natural}\in \mathcal{S}(\mathcal{R}')$ and $\widehat{t^{\natural}}=\hat{t}$, where $\hat{t}=\sup_{\alpha>0}\alpha R'_{\alpha} t$.

b) $\mathcal{S}'(\mathcal{R}')\subseteq {}_H\mathcal{F}^{\natural}(G)$.

c) The correspondence between $\mathcal{F}(X)$ and ${}_H\mathcal{F}^{\natural}(G)$ induces a bijection between $\mathcal{S}(\mathcal{R})$ (resp. $\mathcal{S}'(\mathcal{R})$, $\mathcal{E}(\mathcal{R})$) and ${}_H\mathcal{S}(\mathcal{R}')^{\natural}$ (resp. $\mathcal{S}'(\mathcal{R}')$, $\mathcal{E}(\mathcal{R}')$) . Hence , if $s\in \mathcal{F}(X)$, then :

(4.1) $s\in \mathcal{S}'(\mathcal{R})$ (resp. $s\in\mathcal{E}(\mathcal{R})$) $\Longleftrightarrow s'\in \mathcal{S}'(\mathcal{R}')$ (resp. $s'\in \mathcal{E}(\mathcal{R}')$) .

Proof. a) Because $R'_{\alpha}\in {}_H N(G)$, for every $\alpha>0$, by Remark 2.8 we have :

(4.2) $$\alpha R'_{\alpha}(t^{\natural})=\alpha R'_{\alpha}(t)\leq t .$$

and again by Remark 2.8 : $\alpha R'(t^{\natural})=(\alpha R'_{\alpha}(t^{\natural}))^{\natural}\leq t^{\natural}$ so $t^{\natural}\in \mathcal{S}(\mathcal{R}')$. By (4.2) we also deduce $\widehat{t^{\natural}}=\hat{t}$.

Assertion b) is true because $\alpha R'_{\alpha} t\in {}_H\mathcal{F}^{\natural}(G)$ and $t=\sup_{\alpha>0}\alpha R'_{\alpha} t$, if $t\in\mathcal{S}'(\mathcal{R}')$.

c) By (2.15) we have

$$\alpha R_{\alpha}(s)=\overline{\alpha R'_{\alpha}(s')}$$

hence

(4.3) $\alpha R_\alpha(s) \leq s \iff \alpha R'_\alpha(s') \leq s'$

and

(4.4) $\alpha R_\alpha s \nearrow s \iff \alpha R'_\alpha(s') \nearrow s'$.

The bijection between $\mathcal{S}(\mathcal{R})$ (resp. $\mathcal{S}'(\mathcal{R})$) and ${}_H\mathcal{S}(\mathcal{R}')^{\natural}$ (resp. $\mathcal{S}'(\mathcal{R}')$) results now by (4.3) (resp. (4.4) , the assertion b)) and Remark 1.1 . The proof is complete observing that :

$$R_\alpha([s=\infty])=0 \iff R'([s'=\infty])=0 \text{ , for } s\in\mathcal{F}(X) \text{ and } \alpha>0 . \square$$

The follwing theorem is a consequence of the hypothesis (*) . (The kernel R_0 results proper) .

<u>4.2 Theorem.</u> ([7, Theorem 4.4.6]). $\mathcal{E}(\mathcal{R})$ and $\mathcal{E}(\mathcal{R}')$ are standard H-cones .

<u>4.3 Remark.</u> By Remark 1.1 b) and because

$$s_1 \leq s_2 \iff s'_1 \leq s_2' \text{ , for } s_1 \text{ , } s_2 \in \mathcal{E}(\mathcal{R}) \text{ ,}$$

the correspondence established above between $\mathcal{E}(\mathcal{R})$ and $\mathcal{E}(\mathcal{R}')$ is an H-cone properties preserver .

<u>4.4 Lemma.</u> a) If $U\in\mathcal{T}_X$ (resp. $U\in\mathcal{T}_G$) then :

$$R_0(1_U)>0 \text{ on } U \text{ (resp. } R'_0(1_{UH})>0 \text{ on } UH).$$

b) ([4, Lemma 1.8]). If $s\in\mathcal{S}(\mathcal{R})$ (resp. $t\in\mathcal{S}(\mathcal{R}')$) is lower semicontinuous ,then $s\in\mathcal{S}'(\mathcal{R})$ (resp. $t^{\natural}\in\mathcal{S}'(\mathcal{R}')$).

<u>Proof.</u> The assertions (in the case of the space X) follow by standard arguments of Feller resolvent, using (2.20). If $U\in\mathcal{T}_G$, then: $R'_0(1_{UH})=$ $=R_0(\overline{1}_{UH})'=R_0(1_{\Pi(UH)})\circ\Pi$,hence:

$$R'_0(1_{UH})>0 \text{ on } \Pi^{-1}(\Pi(UH))=UH.$$

If $t\in\mathcal{S}(\mathcal{R}')$, by Lemma 4.1 a) results $t^{\natural}\in{}_H\mathcal{S}(\mathcal{R}')^{\natural}$ and from Lemma 4.1 c) we have $\overline{t^{\natural}}\in\mathcal{S}(\mathcal{R})$. The function $\overline{t^{\natural}}$ being lower semicontinuous on X, it follows that: $\overline{t^{\natural}}\in\mathcal{S}'(\mathcal{R})$. By (4.1) it now results: $t^{\natural}=\overline{t^{\natural}}'\in\mathcal{S}'(\mathcal{R}')$. $\square$

The assertion b) of the lemma above implies that :

(4.5) $\mathcal{E}(\mathcal{R})$ and $\mathcal{E}(\mathcal{R}')$ contain the positive constant functions .

The assertions of the following lemma are adapted from [5] . We can do that because their proofs do not require the $\{e\}$-continuity of the convolution semigroup on G . Then , by (4.1) and Remark 4.3 we can transport them on X .

4.5 Lemma. a) If $s \in \mathcal{E}(\mathcal{R})$ (resp. $s' \in \mathcal{E}(\mathcal{R}')$) , then :

s (resp. s') is a weak unit in $\mathcal{E}(\mathcal{R})$ (resp. $\mathcal{E}(\mathcal{R}')$) $\Longleftrightarrow s > 0$ on X (resp. $s' > 0$ on G) .

b) If $s \in \mathcal{S}'(\mathcal{R})$ (resp. $s' \in \mathcal{S}'(\mathcal{R}')$) , then s (resp. s') is lower semicontinuous on X (resp. on G) .

c) If we denote by $\mathcal{A}(Y)$ the elements of $\mathcal{F}(Y)$ wich are bounded and have compact support ; then : $1 \in \mathcal{A}(X)$ (resp. $1' \in \mathcal{A}(G)$) $\Rightarrow R_0(1) \in C_b(X) \cap \mathcal{E}(\mathcal{R})_0$ (resp. $R'_0(1') \in C_b(G) \cap \mathcal{E}(\mathcal{R}')_0$) , where $\mathcal{E}(\mathcal{R})_0$ denotes the universally continuous elements of the H-cone $\mathcal{E}(\mathcal{R})$.

d) If $s \in \mathcal{E}(\mathcal{R})$ (resp. $s' \in \mathcal{E}(\mathcal{R}')$) ,there exist $(1_n)_{n \in N} \subseteq \mathcal{A}(X)$ (resp. $(1'_n)_{n \in N} \subseteq \mathcal{A}(G)$) such that

$$R_0(1_n) \nearrow s \quad (\text{resp. } R'_0(1'_n) \nearrow s') \quad , \text{ when } n \to \infty .$$

e) Every $s \in \mathcal{E}(\mathcal{R})_0$ (resp. $s' \in \mathcal{E}(\mathcal{R}')_0$) is continuous and bounded on X (resp. on G) .

f) If $a \in G$ then :

(4.6) $s \in \mathcal{E}(\mathcal{R})$ (resp. $s' \in \mathcal{E}(\mathcal{R}')$) $\Longrightarrow {}_a s \in \mathcal{E}(\mathcal{R})$ (resp. ${}_a s' \in \mathcal{E}(\mathcal{R}')$)

(4.7) $s \in \mathcal{E}(\mathcal{R})_0$ (resp. $s' \in \mathcal{E}(\mathcal{R}')_0$) $\Longrightarrow {}_a s \in \mathcal{E}(\mathcal{R})_0$ (resp. ${}_a s' \in \mathcal{E}(\mathcal{R}')_0$).

4.6 Theorem. ([5,Theorem 2.1.11]). a) $\mathcal{E}(\mathcal{R})$ is a standard H-cone of functions on X .

b) $\mathcal{E}(\mathcal{R}')$ has all the properties of standard H-cones except the points separation one .

Proof. The assertion a) follows by the same arguments as in [5] , using Theorem 4.2 , (4.5) and Lemma 4.5 . Notice that the points separation property on X for $\mathcal{E}(\mathcal{R})$ is a direct consequence of the $\{\bar{e}\}$-continuity of the convolution semigroup .

To prove b) , observe that the correspondence between $\mathcal{E}(\mathcal{R})$ and $\mathcal{E}(\mathcal{R}')$ preserves all the properties of standard H-cone of functions, except the points separation one (see Remark 4.3) .□

4.7 Remark. a) From Remark 2.3 b) , it follows that $\check{k}=\int_0^\infty \check{\mu}_t dt$ (resp. $\check{k}'=\int_0^\infty \check{\mu}'_t dt$) is a potential kernel on X (resp. on G) . Let $\check{\mathcal{R}}$ (resp. $\check{\mathcal{R}}'$) denote the resolvent of kernels associated with $\check{k}$ (resp. $\check{k}'$) . By Remark 1.6 and Lemma 1.5 b) from $k \ll \bar{\omega}$ (resp. $k' \ll \omega$), it results that $\check{k} \ll \bar{\omega}$ (resp. $\check{k}' \ll \omega$) . Hence , by Theorem 4.6 , we conclude that :

$\mathcal{E}(\check{\mathcal{R}})$ is a standard H-cone of functions on X .

b) For $\mathcal{E}(\mathcal{R}')$ we can only prove a slighter separation property on G . More precisely , for a,b ∈G we have:

(4.8) $b^{-1}a \notin H \Longleftrightarrow$ there exists $s' \in \mathcal{E}(\mathcal{R}')$ with $s'(a) \neq s'(b)$.

Proof. of b) . The condition $b^{-1}a \notin H$ means that $\bar{a} \neq \bar{b}$ and we apply the points separation property of $\mathcal{E}(\mathcal{R})$ on X together with (4.1) .□

The following corollary is a direct consequence of (4.8) .

4.8 Corollary. Let $\mathcal{H}$ denote one of the sets $\mathcal{E}(\mathcal{R}')$, $\mathcal{E}(\mathcal{R}')_0$, $R'_0(\mathcal{A}(G))$.

a) Then :

H is a normal subgroup of G $\Longleftrightarrow \mathcal{H} \subseteq_H {}^h\mathcal{F}(G) \Longleftrightarrow \mathcal{H} \subseteq_H \mathcal{F}(G)$.

b) If H={e} , we have :

G is abelian $\Longleftrightarrow s_a = {}_a s$, for every $s \in \mathcal{H}$ and $a \in G$.

Proof. It suffices to make the proof only in the case $\mathcal{H} = \mathcal{E}(\mathcal{R}')$, because in the other cases $\mathcal{H}$ is increasingly dense in $\mathcal{E}(\mathcal{R}')$.

By (4.8) and Remark 4.1 b) it follows that :

H is normal $\Longleftrightarrow$ ($\overline{ha}=\bar{a}$ for every $a\in G$, $h\in H$) $\Longleftrightarrow$ ($s(ha)=s(a)$ for every $a\in G$, $h\in H$, $s\in\mathcal{H}$) $\Longleftrightarrow$ $\mathcal{H}\subseteq_H^h\mathcal{F}(G) \Longleftrightarrow \mathcal{H}\subseteq_H\mathcal{F}(G)$.□

§5. The Natural and the Fine Topologies .

Because hypothesis (*) is satisfied , it results , by Theorem 4.6 and Remark 4.7 a) , that $\mathcal{E}(\mathcal{R})$ and $\mathcal{E}(\check{\mathcal{R}})$ are standard H-cones of functions on X . We can define on X (see [7, §4.3]) the natural topology with respect to $\mathcal{E}(\mathcal{R})$ (resp. to $\mathcal{E}(\check{\mathcal{R}})$) as being the coarsest topology on X which makes continuous all the universally continuous elements of $\mathcal{E}(\mathcal{R})$ (resp. $\mathcal{E}(\check{\mathcal{R}})$) and it will be denoted by $\mathcal{T}_n$ (resp. $\check{\mathcal{T}}_n$) .

The coarsest topology on X which makes continuous all the elements of $\mathcal{E}(\mathcal{R})$ (resp. $\mathcal{E}(\check{\mathcal{R}})$) is called the fine topology on X with respect to $\mathcal{E}(\mathcal{R})$ (resp. $\mathcal{E}(\check{\mathcal{R}})$) and will be denoted by $\mathcal{T}_f$ (resp. $\check{\mathcal{T}}_f$) .

If $\mathcal{T}$ is a topology on Y , we denote by $\mathcal{V}_y(\mathcal{T})$ the neighbourhoods system of the point $y\in Y$ and by $\mathcal{T}'$ the topology on G :

$$\mathcal{T}' = \pi^{-1}(\mathcal{T}) .$$

Obviously $\mathcal{T}_n\subseteq\mathcal{T}_f$, $\check{\mathcal{T}}_n\subseteq\check{\mathcal{T}}_f$ and by Lemma 4.5 e) it follows that : $\mathcal{T}_n$, $\check{\mathcal{T}}_n\subseteq\mathcal{T}_X$.

5.1 Lemma. ([4, Lemma 2.1]). Let $a\in G$.

a) If $V\in\mathcal{T}_f$ (resp. $V\in\mathcal{T}'_f$) then $aV\in\mathcal{T}_f$ (resp. $aV\in\mathcal{T}'_f$) .

b) If $V\in\mathcal{T}_n$ (resp. $V\in\mathcal{T}'_n$) then $aV\in\mathcal{T}_n$ (resp. $aV\in\mathcal{T}'_n$) .

The assertions hold if we replace $\mathcal{T}_f$, $\mathcal{T}_n$ (resp. $\mathcal{T}'_f$, $\mathcal{T}'_n$) by $\check{\mathcal{T}}_f$, $\check{\mathcal{T}}_n$ (resp. $\check{\mathcal{T}}'_f$, $\check{\mathcal{T}}'_n$) .

Proof. If we consider the map $\sigma_a : X\to X$ defined by

$$\sigma_a(x)=ax \quad , \quad \text{for all } x\in X \quad ,$$

we obtain ${}_a s=s\circ\sigma_a$, hence by (4.6) σ_a results $\mathcal{T}_f$-homeomorfism on X . This proves a) . Analogously , from (4.7) , assertion b) holds . □

5.2 Lemma. ([4, Lemma 2.2]) . If $V \in \mathcal{T}_X$ (resp. $V \in \mathcal{T}'_X$) , $V \neq \phi$ there exists $x \in V$ (resp. $a \in V$) such that $V \in \mathcal{V}_x(\mathcal{T}_n)$ (resp. $V \in \mathcal{V}_a(\mathcal{T}'_n)$) .

5.3 Theorem. ([4, Thorem 2.3]). We have :

$$\mathcal{T}_n = \check{\mathcal{T}}_n = \mathcal{T}_X \quad .$$

Let us first expose the following preliminary result of group topology .

5.4 Lemma. ([10 , Theorem 4.9]). Let G be a topological group , $U \in \mathcal{V}_e(\mathcal{T}_G)$ and let F be any compact subset of G . Then there exist $V \in \mathcal{V}_e(\mathcal{T}_G)$ such that $aVa^{-1} \subseteq U$ for all $a \in F$.

5.5 Corollary. Let G be a topological group , H a compact subgroup of G and $U \in \mathcal{V}_e(\mathcal{T}_G)$. Then , there exists $V \in \mathcal{V}_e(\mathcal{T}_G)$, $V \subseteq U$, $V=V^{-1}$ and $HVH \subseteq UH$.

Proof. By Lemma 5.4 there exists $V \in \mathcal{V}_e(\mathcal{T}_G)$ such that $hVh^{-1} \subseteq U$ for all $h \in H$. Hence , if $h,h' \in H$, then :

$$hVh' = (hVh^{-1})hh' \subseteq (hVh^{-1})H \subseteq UH \quad . \ \square$$

Proof of Theorem 5.3 . Let $\mathcal{T}^4_G = \mathcal{T}'_X$ and notice that :

$$\mathcal{T}^4_G = \{ V \in \mathcal{T}_G \ / \ V=VH \} \quad .$$

Because $\mathcal{T}_n \subset \mathcal{T}_X$, it suffices to prove that :

$$\mathcal{T}^4_G \subseteq \mathcal{T}'_n$$

or equivalently :

(5.1) $\quad V \in \mathcal{T}^4_G \ , \ a \in V \Longrightarrow V \in \mathcal{V}_a(\mathcal{T}'_n)$.

Let $V \in \mathcal{T}^4_G$ and $a \in V$. First we shall show that there is $W \in \mathcal{V}_e(\mathcal{T}_G)$ such that :

(5.2) $\quad aWW \subseteq V \ , \ W=W^{-1} \ , \ W=WH$.

Indeed, by the continuity of the multiplication in $(a,e)\in G\times G$, there exists $U\in\mathcal{V}_e(\mathcal{T}_G)$ such that

(5.3) $$aUU\subseteq V \quad .$$

From Corollary 5.5 , there exists $W_1\in\mathcal{V}_e(\mathcal{T}_G)$ with

(5.4) $$HW_1H\subseteq UH \quad\text{and}\quad W_1\subseteq U \quad .$$

By (5.3) and (5.4) it results that : $aW_1HW_1H\subseteq aW_1UH\subseteq aUUH\subseteq VH=V$, therefore :

(5.5) $$aW_1HW_1H\subseteq V \quad .$$

Again by Corollary 5.5 , applied for $W_1\in\mathcal{V}_e(\mathcal{T}_G)$, there exists $W_2\in\mathcal{V}_e(\mathcal{T}_G)$ such that :

$$HW_2H\subseteq W_1H \quad , \quad W_2\subseteq W_1 \quad , \quad W_2=W_2^{-1} \quad .$$

From (5.5) we now deduce : $aHW_2HW_2H\subseteq V$ and if we denote $W=HW_2H$ we obtain (5.2) .

We have : $aW\in\mathcal{V}_a(\mathcal{T}_G)$ hence by Lemma 5.2 there exists $b\in aW$ such that $aW\in\mathcal{V}_b(\mathcal{T}'_n)$. By Lemma 5.1 b) it results that :

(5.6) $$ab^{-1}(aW)\in\mathcal{V}_a(\mathcal{T}'_n) \quad .$$

Then $b\in aW$ implies $b^{-1}a\in W^{-1}=W$ and by (5.2) we have :

(5.7) $$a\in a(b^{-1}a)W\subseteq aWW\subseteq V \quad .$$

By (5.6) and (5.7) it results that $V\in\mathcal{V}_a(\mathcal{T}'_n)$, so that (5.1) holds . □

§ 6. Resolvents in Duality on X .

Hypothesis (*) implies that there exists $f\in L^1_{loc}(\overline{\omega})$ and $f_\alpha\in L^1(\overline{\omega})$ (i.e. f_α is $\overline{\omega}$-integrable) , for every $\alpha>0$, such that : $k=f.\overline{\omega}$, $k_\alpha=f_\alpha\overline{\omega}$, for all $\alpha>0$. By Lemma 1.5 c) it results that:

$$k'=f'.\omega$$

$$k'_\alpha=f'_\alpha.\omega \quad , \quad \text{for all } \alpha>0$$

and $f'\in L^1_{loc}(\omega)$, $f'_\alpha\in L^1(\omega)$.

6.1 Lemma. Let $g\in\mathcal{F}(G)$ such that :

(6.1) $$g*f'_\alpha=f'_\alpha*g \quad , \quad \text{for all } \alpha>0 \quad .$$

Then :

(6.2) $$g\in \mathcal{E}(\check{\mathcal{R}}') \Longleftrightarrow g^{*}\in \mathcal{E}(\mathcal{R}') .$$

Proof. One can easily verify that :

(6.3) $$T_{f'\omega}(g) = g*f'^{*} .$$

We now deduce from (1.3) , (1.6) and (6.3) :

$$\check{R}'_{\alpha}(g) = T_{f'^{*}_{\alpha}\omega}(g) = g*f'_{\alpha} , \quad \text{for all } \alpha>0 ,$$

$$R'_{\alpha}(g^{*})^{*} = f'_{\alpha}*g$$

and by (6.1) it follows $R'_{\alpha}(g^{*})^{*} = \check{R}'_{\alpha}g$, so that

$$\alpha R'_{\alpha}g^{*} \nearrow g^{*} \Longleftrightarrow \alpha\check{R}'_{\alpha}g \nearrow g .$$

The proof is complete noticing that :

$$g^{*}<\infty \quad \omega\text{-a.e.} \Longleftrightarrow g<\infty \quad \omega\text{-a.e.} \quad .\square$$

6.2 Lemma. ([3, Theorem 3.10]). a) We can assume that

(6.4) $$f\in \mathcal{E}(\check{\mathcal{R}})(\text{resp. } f'\in \mathcal{E}(\check{\mathcal{R}}')),$$

(6.5) $$f_{\alpha}\in \mathcal{E}(\check{\mathcal{R}}_{\alpha})(\text{resp. } f'_{\alpha}\in \mathcal{E}(\check{\mathcal{R}}'_{\alpha})) \quad , \text{ for all } \alpha>0.$$

b) f and f_{α}(resp. f' and f'_{α}), $\alpha>0$, are lower semicontinuous functions on X (resp. on G).

Proof. By (4.1), Lemma 4.1 b) and Lemma 1.5 c) it suffices to prove the assertions for f' and f'_{α} , $\alpha>0$. As in [3], we can replace f^{*} and f^{*}_{α} such that

(6.6) $$f'^{*}\in \mathcal{E}(\mathcal{R}') \text{ and } f'^{*}_{\alpha}\in \mathcal{E}(\mathcal{R}'_{\alpha}) , \text{ for all } \alpha>0$$

and f'^{*}, f'^{*}_{α} are lower semicontinuous. By (3.1) and (3.2) we deduce , using also (1.5): $f'_{\alpha}*f'_{\beta}=f'_{\beta}*f'_{\alpha}$, $f'*f'_{\alpha}=f'_{\alpha}*f'$ for all $\alpha,\beta>0$. Hence we can apply (6.2) and from (6.6) the assertions hold.$\square$

Let us define the functions:

$$G(a,b)=f'(a^{-1}b)\Delta(a),$$

$$G_{\alpha}(a,b)=f'_{\alpha}(a^{-1}b)\Delta(a),$$

for all $a,b\in G$, $\alpha>0$.

6.3 Lemma. a) If $g\in K(G)$, $a\in G$ and $\alpha\geq 0$ we have:

(6.7) $$R'_\alpha g(a) = \int g(b)G_\alpha(a,b)d\omega(b)$$

(6.8) $$\check{R}'_\alpha g(a) = \int g(b)G_\alpha(b,a)d\omega(b)$$

where $G_0(\cdot,\cdot)=G(\cdot,\cdot)$.

b) The resolvents $\mathcal{R}'=(R'_\alpha)_{\alpha\geq 0}$ and $\check{\mathcal{R}}'=(\check{R}'_\alpha)_{\alpha\geq 0}$ are in duality with respect to ω , i.e. for $g_1,g_2\in K(G)$ we have :

$$\int g_1(a)R'_\alpha g_2(a)d\omega(a) = \int \check{R}'_\alpha g_1(a)g_2(a)d\omega(a) \quad .$$

c) If we denote by G^a_α (resp. $G_{\alpha a}$) , $a\in G$ the functions on G $b\mapsto G_\alpha(a,b)$ (resp. $b\mapsto G_\alpha(b,a)$) , then for all $\alpha\geq 0$:

$$G^a_\alpha \in \mathcal{E}(\check{\mathcal{R}}'_\alpha) ,$$

$$G_{\alpha a}\in \mathcal{E}(\mathcal{R}'_\alpha) \qquad , \text{ where } \mathcal{R}_0=\mathcal{R}.$$

<u>Proof</u>. a) Let us verify (6.7) and (6.8). Using (1.1) we have :

$R'_\alpha g(a)=\int g(ab)f_\alpha(b)d\omega(b)=\int g(c)f_\alpha(a^{-1}c)d\omega(a^{-1}c)=$

$=\int g(c)\Delta(a)f_\alpha(a^{-1}c)d\omega(c)=\int g(c)G_\alpha(a,c)d\omega(c)$ and $\check{R}'_\alpha g(a)=$

$\int g(ab)f^*_\alpha(b)d\omega(b)=\int g(ab)f_\alpha(b^{-1})\Delta(b)d\omega(b)=$

$=\int g(c)f_\alpha(c^{-1}a)\Delta(c)\Delta(a^{-1})d\omega(a^{-1}c)= \int g(c)f_\alpha(c^{-1}a)\Delta(c)d\omega(c)=$

$=\int g(c)G_\alpha(c,a)d\omega(c)$.

The assertion b) is a direct consequence of (6.7) and (6.8) .

c) We have $G^a_\alpha(b)=f_\alpha(a^{-1}b)\Delta(a)=\Delta(a)\cdot{}_{a^{-1}}f(b)$. By (6.4) , (6.5) and (4.6) we deduce that : $G^a_\alpha\in\mathcal{E}(\check{\mathcal{R}}'_\alpha)$, for all $\alpha\geq 0$. Analogously , $G_{\alpha a}=\Delta(a)\cdot{}_{a^{-1}}(f^*_\alpha)$, hence by (6.6) and (4.6) $G_{\alpha a}\in\mathcal{E}(\mathcal{R}'_\alpha)$, for all $\alpha\geq 0$. □

Because $G^a_\alpha, G_{\alpha a}\in{}_H\mathcal{F}^b(G)$, $\alpha\geq 0$, for all $a\in G$, we can define the function :

$$\overline{G}_\alpha : X\times X \longrightarrow \overline{R}_+ \quad \text{by}$$

$$\overline{G}_\alpha(\overline{a},\overline{b}) = G_\alpha(a,b) \quad , \quad \text{for all } a,b\in G \quad .$$

<u>6.4 Theorem</u>. a) If $g\in K(X)$, then :

(6.9) $$R_\alpha g(x)=\int g(y)\overline{G}_\alpha(x,y)d\overline{\omega}(y) \quad ,$$

(6.10) $$\check{R}_\alpha g(x)=\int g(y)G_\alpha(y,x)d\overline{\omega}(y)\ ,$$

for all $x\in X$.

b) The resolvents $\mathcal{R}$ and $\check{\mathcal{R}}$ are in duality with respect to $\overline{\omega}$.

c) For all $\alpha\geqslant 0$ and $x\in X$ we have :

$$\overline{G}_\alpha^x\in\mathcal{E}(\check{\mathcal{R}}_\alpha)\ ,$$

$$\overline{G}_{\alpha_x}\in\mathcal{E}(\mathcal{R}_\alpha)\ ,$$

where $\overline{G}_\alpha^x(\cdot)=\overline{G}_\alpha(x,\cdot)$ and $\overline{G}_{\alpha_x}(\cdot)=\overline{G}_\alpha(\cdot,x)$.

Proof. a) Obviously we have :

(6.11) $$\overline{G}_\alpha(\overline{a},\cdot)=\overline{G_\alpha^a}(\cdot)$$

and

(6.12) $$\overline{G}_\alpha(\cdot,\overline{a})=\overline{G_{\alpha_a}}(\cdot)\ .$$

Let $g\in K(X)$ and $x=\overline{a}$. From (6.11) it follows that :

$$\int g(y)\overline{G}_\alpha(x,y)d\overline{\omega}(y)=\int g(y)\overline{G_\alpha^a}(y)d\overline{\omega}(y)=\int g'(b)G_\alpha^a(b)d\omega(b)=R'_\alpha(g')(a)=$$
$$=R_\alpha(g)(\overline{a})\ .$$

This proves (6.9) . Analogously , we deduce (6.10) from (6.12) . Assertion b) is a consequence of the assertion b) of Lemma 6.3 . Assertion c) results from Lemma 6.3 c) , (6.11) , (6.12) and (4.1) .□

From [7,Theorem 1.2.2] we deduce the following :

6.5 Corollary. $\mathcal{E}(\check{\mathcal{R}})$ (resp. $\mathcal{E}(\check{\mathcal{R}}')$) is the dual of the standard H-cone of functions on X $\mathcal{E}(\mathcal{R})$ (resp. of the standard H-cone $\mathcal{E}(\mathcal{R}')$) .

Final remark. Let H be a compact subgroup of a Lie group G and let D be a second order (strictly)elliptic differential operator on X=G/H , invariant by left translations , with elements of G . Then , by [12,§8] there exists exactly one $\{\overline{e}\}$-continuous convolution semigroup on X with the "infinitesimal operator" D .

We want to point out that (in the transient case) the potential kernel k associated with such a convolution semigroup satisfies the absolute continuity condition (i.e. $k \ll \overline{\omega}$) . Indeed , following [1] and [2] , the semigroup itself is absolutely continuous with respect to the volume element (see [9, ch X, §1]) associated to the coefficients of D . Hence we can apply Theorem 3.5 .

References

[1] R.Azencott : Diffusions sur les variétés différentiables. C.R. Acad.Sci. Paris,274,sérieA,651-654(1972).

[2] R.Azencott : Behavior of diffusion semi-groups at infinity. Bull.Soc.math. France,102,193-240(1974).

[3] M.Banulescu : Absolutely continuous potential kernels on locally compact groups. Bull.Math. de la Soc.Sci.Math. de la R.S.Roum,25(73),no.3(1981).

[4] M.Banulescu : H-cones of functions on topological groups. Bull.Math. de la Soc.Sci.Math. de la R.S.Roum,26(74),no.4(1982).

[5] M.Banulescu : Potentials on locally compact non-abelian groups. In:Complex Analysis-Fifth Romanian-Finish Seminar.Part2,Proc. Bucharest 1981,129-154.,Lectures Notes in Math.vol.1014,Berlin-Heidelberg-New York:Springer 1983.

[6] Ch.Berg, G.Forst : Potential theory on locally compact abelian groups. Berlin-Heidelberg-New York; Springer1975.

[7] N.Boboc,Gh.Bucur,A.Cornea : Order and convexity in potential theory:H-cones. Lectures Notes in Math.vol.853,Berlin-Heidelberg New York:Springer 1981.

[8] N.Bourbaki : Elements de Mathematique XXIX:Livre VI:Integration. Chap.7-8.Paris Hermann 1963.

[9] S.Helgason : Differential geometry and symmetric spaces.New York and London:Academic Press 1962.

[10] E.Hewitt,K.A.Ross : Abstract harmonic analysis I.Berlin-Göttingen -Heidelberg:Springer 1963.

[11] H.Heyer : Probability measures on locally compact groups.Berlin-Heidelberg-New York:Springer 1977.

[12] G.A.Hunt : Semi-groups of measures on Lie groups.Trans.Amer. Math.Soc.81,264-293(1956).

[13] K.R.Parthasarathy : Probability measures on metric spaces. New York-London:Academic Press 1967.

[14] J.P.Roth : Operateurs dissipatifs... .Ann.Inst.Fourier 26, fasc.4,1-97(1976).