INSTITUTUL
DE
MATEMATICA

INSTITUTUL NATIONAL
PENTRU CREATIE
STIINTIFICA SI TEHNICA

ISSN 0250 3638

AN OPERATOR THEORETIC APPROACH TO ANALYTIC FUNCTIONS INTO A GRASSMANN MANIFOLD

by

Mircea MARTIN

PREPRINT SERIES IN MATHEMATICS
No.44/1985

BUCURESTI

AN OPERATOR THEORETIC APPROACH TO ANALYTIC FUNCTIONS INTO A GRASSMANN MANIFOLD

by

Mircea MARTIN*)

July 1985

*) Department of Mathematics, National Institute for Scientific and Technical Creation, Bd.Pacii 220, 79622 Bucharest, Romania.

AN OPERATOR THEORETIC APPROACH TO ANALYTIC FUNCTIONS INTO A GRASSMANN MANIFOLD

Mircea MARTIN

INTRODUCTION

Let D be an open and connected subset of $\mathbb{C}^m$, and let H be a complex, separable, finite or infinite dimensional Hilbert space. For any positive integer n, we let $Gr(n,H)$ denote the Grassmann manifold associated with n and H, that is, the set of all n-dimensional subspaces of H. In the main body of this paper we shall be concerned with the class $A_n(D)$ of all $Gr(n,H)$-valued analytic functions defined on D.

Two functions f and $\tilde{f}$ in $A_n(D)$ are said to be congruent, if there exists a unitary operator on H which moves each subspace $f(z)$ onto $\tilde{f}(z)$, for all points z in D.

We follow Griffiths (cf., [6]) in saying that the functions f and $\tilde{f}$ have order of contact k, where k is a positive integer, if they agree in an osculation sense up to order k. Two congruent functions have order of contact k for any k.

A more or less expected converse of this remark is contained in the congruence theorem (cf., [2] , [8]), which asserts that, under a non-degeneracy condition, two functions in $A_n(D)$ are congruent, if and only if they have order of contact n.

This theorem originates from the already mentioned work of Griffits [6] . In the stated above form, the congruence theorem was proved, in the case where $m=1$, by Cowen and Douglas [2] .

The general case was discussed in [8] . Although the methods used in [8] are in essence different from that of [2] , just as in [2] the proof given in [8] has the inherent defect to be an indirect one. More precisely, the congruence theorem was obtained as a consequence of a rather deep understanding of the local equivalence of hermitian holomorphic vector bundles of rank n over D. The trouble with a such approach is that many qualitative simple properties of analytic functions into a Grassmann manifold are inevitably not used explicitely.

The aim of the present paper is to give a new and simpler proof of the congruence theorem. The proof uses certain operator theoretic techniques developed in [1] . In fact, the main results of the paper, Theorem 2.4 and Theorem 3.7, could be regarded as essentialy strengthened versions of Theorem A and, respectively, Theorem B from [1] .

Significant examples of functions in $A_n(D)$ arise, in the case where H is infinite dimensional, in connection with the class $B_n(D)$ introduced by Cowen and Douglas (cf., [2] , [3] , [4]). The elements of this class are m- tuples of commuting operators on H, and to any m- tuple T in $B_n(D)$ corresponds in an obvious fashion a function f_T from D into Gr(n,H). Using a result proved by Curto and Salinas (cf., [5] , Theorem 2.2) one obtains that f_T is an analytic function. Moreover, two m- tuples from $B_n(D)$ are simultaneously unitarily equivalent if and only if their associated functions are congruent.

This last remark constitutes a good reason for the study of congruent functions in the class $A_n(D)$.

We now give a brief outline of this paper. Section 1 contains some preliminaries on smooth and analytic functions from D into Gr(n,H). In Section 2 we associate to any function f in $A_n(D)$ and any set $\mathcal{X}$ of bounded linear operators on H, a chain of fields of finite dimensional C^*-algebras over D. The local structure of a such object is presented in Theorem 2.4. The discussion of congruent functions in $A_n(D)$ is carried out in Section 3. The main result of this section, Theorem 3.7, is a consequence of Theorem

2.4, and the congruence theorem appears as a particular case. Finally, in Section 4 we digrees in order to relate the results of Section 3 to the Cowen-Douglas class $B_n(D)$.

1. ANALYTIC FUNCTIONS INTO A GRASSSMANN MANIFOLD

Throughout the paper D will denote an open and connected subset of $\mathbb{C}^m$ and H will be a complex separable, finite or infinite dimensional Hilbert space. Given a positive integer n, we shall denote by Gr(n,H) the set of all n-dimensional subspaces of H.

1.1. If K is a subset of H, we let span K denote the closed subspace of H generated by K.

Assume that f is a function from D into Gr(n,H) and let D_0 be an open subset of D. A collection $\{h_\alpha : 1 \leq \alpha \leq n\}$ of H-valued functions on D_0 will be referred to as a frame for f over D_0 if

(1.1.1) $f(z) = \text{span}\,\{\ h_\alpha(z) : 1 \leq \alpha \leq n\ ;\ z \in D_0\}$.

The frame is called smooth, respectively analytic, if all functions h_α, $1 \leq \alpha \leq n$, are smooth, respectively analytic, on D_0.

DEFINITION. A function $f : D \to Gr(n,H)$ is said to be smooth, respectively analytic, if for any z_0 in D there exist an open neighborhood D_0 of z_0 and a smooth, respectively an analytic, frame for f over D_0. The set of all analytic functions from D into Gr(n,H) will be denoted by $A_n(D)$.

1.2. Let L(H) be the C^*-algebra of all bounded linear operator on H and let E(D, L(H)) be the space of all smooth functions from D into L(H). With pointwise sum, product and involution, the space E(D, L(H)) becomes a unital involutive algebra. Identifying each operator in L(H) with a constant function on D, one obtains a natural inclusion of L(H) into E(D, L(H)). The unit of L(H) will be denoted by $\mathbb{1}$.

For K a closed subspace of H, we let $[K]$ denote the self-adjoint projection in L(H) with the range K. Given a Gr(n, H)-valued function f on D we shall denote by $[f]$ the function defined as follows:

$$[f] : D \to L(H), \quad [f](z) = [f(z)] \ ; \quad z \in D.$$

It is plain that f is smooth if and only if $[f]$ is a self-adjoint projection in E(D, L(H)).

1.3. In order to state the next result we introduce the notations

(1.3.1) $\partial_i = \partial / \partial z_i, \quad \bar{\partial}_i = \partial / \partial \bar{z}_i; \quad 1 \leq i \leq m.$

PROPOSITION. Let f: $D \to Gr(n, H)$ be a smooth function and let us put $p = [f]$. The following conditions are equivalent:

(i) f is analytic

(ii) $(1-p)\bar{\partial}_i p = 0; \quad 1 \leq i \leq m.$

PROOF. Assume that f is analytic and let z_0 be a point in D. Let $\{h_\alpha : 1 \leq \alpha \leq n\}$ be an analytic frame for f over an open neighborhood D_0 of z_0. From (1.1.1) one obtains that there exists a smooth frame $\{g_\alpha : 1 \leq \alpha \leq n\}$ for f over D_0 such that

(1.3.2) $p(z)h = \sum_{\alpha=1}^{n} \langle h, h_\alpha(z) \rangle g_\alpha(z); \ z \in D_0, \ h \in H,$

where $\langle , \rangle$ denotes the inner product on H.

In fact the functions g_α, $1 \leq \alpha \leq n$, are real-analytic, hence p is real-analytic too.

From (1.3.2) we have

$$(\partial_i p)(z)h = \sum_{\alpha=1}^{n} \langle h, h_\alpha(z) \rangle (\partial_i g_\alpha)(z); \ 1 \leq i \leq m,$$

hence

(1.3.3) $(\partial_i p)(1-p) = 0; \ 1 \leq i \leq m.$

Since $(\partial_i p)^* = \bar{\partial}_i p$, the conditions (ii) and (1.3.3) are equivalent.

Assume now that f is smooth and p satisfies the condition (ii). Let z_0 be a point in D and let $\{h_\alpha : 1 \leq \alpha \leq n\}$ be a smooth frame for f over an open neighborhood D_0 of z_0. Then we have

(1.3.4) $\bar\partial_i h_\alpha = \bar\partial_i (p\, h_\alpha) = (\bar\partial_i p) h_\alpha + p\, \bar\partial_i h$; $1 \leq i \leq m$.

Since $\bar\partial_i p = p\, \bar\partial_i p$, we obtain that $\bar\partial_i h_\alpha(z)$ belongs to $f(z)$ for all z in D_0. If follows that

(1.3.5) $\bar\partial_i h_\alpha = \sum_{\beta=1}^{n} \xi^i_{\alpha\beta} h_\beta$; $1 \leq i \leq m$, $1 \leq \alpha \leq n$,

where $\{ \xi^i_{\alpha\beta} : 1 \leq i \leq m,\ 1 \leq \alpha, \beta \leq n \}$ is a collection of complex-valued smooth functions on D_0.

Using (1.3.5) we find

(1.3.6) $\bar\partial_j \bar\partial_i h_\alpha = \sum_{\beta=1}^{n} \left(\bar\partial_j \xi^i_{\alpha\beta} + \sum_{\gamma=1}^{n} \xi^i_{\alpha\gamma} \xi^j_{\gamma\beta} \right) h_\beta$;

$1 \leq i, j \leq m$, $1 \leq \alpha \leq n$.

Let us define the n x n matrix $\xi = (\xi_{\alpha\beta})$ of $(0,1)$-forms on D_0, as follows

(1.3.7) $\xi_{\alpha\beta} = \sum_{i=1}^{m} \xi^i_{\alpha\beta}\, d\bar z_i$; $1 \leq \alpha, \beta \leq n$.

The exterior derivative d acting on smooth forms can be decomposed to obtain

(1.3.8) $d = \partial + \bar\partial$; $\partial = \sum_{i=1}^{m} \partial_i dz_i$, $\bar\partial = \sum_{i=1}^{m} \bar\partial_i d\bar z_i$

Since $\bar\partial_j \bar\partial_i = \bar\partial_i \bar\partial_j$, a simple computation shows that, using a matrix notation, from (1.3.6) we have

(1.3.9) $\bar\partial \xi - \xi \wedge \xi = 0$.

Now, by the well-known generalization of Grothendiek's Theorem proved by Malgrange (cf., [7]), it follows that, eventually decreasing D_0, there exists a n x n matrix $\eta = (\eta_{\alpha\beta})$ of complex-valued smooth functions on D_0, such that

(1.3.10) $\bar\partial \eta + \eta \wedge \xi = 0$,

(1.3.11) $\eta(z)$ is an invertible matrix; $z \in D_0$.

Thies implies that the collection $\{ k_\alpha : 1 \leq \alpha \leq n \}$ defined by

(1.3.12) $k_\alpha : D_0 \to H$, $k_\alpha = \sum_{\beta=1}^{n} \eta_{\alpha\beta} h_\beta$; $1 \leq \alpha \leq n$, is a smooth frame for f over D_0. From (1.3.12), (1.3.5) and (1.3.10) we have succesively

$$\bar{\partial}_i k_\alpha = \sum_{\beta=1}^{n} (\bar{\partial}_i \eta_{\alpha\beta}) h_\beta + \sum_{\gamma=1}^{n} \eta_{\alpha\gamma} \bar{\partial}_i h_\gamma =$$

$$\sum_{\beta=1}^{n} (\bar{\partial}_i \eta_{\alpha\beta} + \sum_{\gamma=1}^{n} \eta_{\alpha\gamma} \xi^i_{\gamma\beta}) h_\beta = 0$$

for all $1 \leq i \leq m$ and $1 \leq \alpha \leq n$. Thus, $\{k_\alpha : 1 \leq \alpha \leq n\}$ is an analytic frame, hence f is analytic.

1.4. Our next task is to give some consequences of Proposition 1.3.

Let $(Z^+)^m$ be the set of all m- tuples $I=(i_1,\ldots, i_m)$ of nonnegative integers. We shall use the following standard notations:

(1.4.1) $D_I=(\partial_1)^{i_1}\ldots(\partial_m)^{i_m}$, $\bar{D}_I=(\bar{\partial}_1)^{i_1}\ldots(\bar{\partial}_m)^{i_m}$,

(1.4.2) $|I| = i_1+\ldots+i_m$.

For any A in E(D, L(H)) we have

(1.4.3) $(D_I\bar{D}_J A)^* = D_J \bar{D}_I A^*$; $I,J \in (Z^+)^m$.

If $I=(0,\ldots,0)$ then we put $D_I A = \bar{D}_I A = A$.

PROPOSITION. Let $p=[f]$ be the self-adjoint projection in E(D, L(H)) associated with a function f in the class $A_n(D)$. Then we have

(1.4.4) $(\bar{D}_I p)p=0$, $\bar{D}_I p = p(\bar{D}_I p)$; $|I| \geq 1$,

(1.4.5) $p(D_I p)=0$, $D_I p = (D_I p)p$; $|I| \geq 1$,

(1.4.6) $D_I \bar{D}_J p = (D_I p)(\bar{D}_J p) - (\bar{D}_J p)(D_I p)$; $|I| = |J| = 1$.

PROOF. For the first relation in (1.4.4) we shall proceed by induction. If $|I| = 1$, then we obtain

$\bar{D}_I p = \bar{D}_I (p^2) = (\bar{D}_I p)p + p(\bar{D}_I p)$.

By proposition 1.3 we have $p(\bar{D}_I p) = \bar{D}_I p$, thus $(\bar{D}_I p)p=0$.

For $|I| \geq 2$ let us put $I=J+K$ with J,K in $(Z^+)^m$ and $|J| = 1$. Assume that $(\bar{D}_K p)p=0$. Since $(\bar{D}_J p)p=0$, it follows

$0=(\bar{D}_J((\bar{D}_K p)p))p = (\bar{D}_I p)p + (\bar{D}_K p)(\bar{D}_J p)p = (\bar{D}_I p)p$.

For the second relation in (1.4.4) we proceed by induction too. If $|I| = 1$ then we already know that $\bar{D}_I p = p\,\bar{D}_I p$. For $|I| \geq 2$ we put $I=J+K$ as above and assume that $\bar{D}_K p = p\bar{D}_K p$. Since $(\bar{D}_J p)p=0$ we have

$$\bar{D}_I p = \bar{D}_J (p\bar{D}_K p) = (\bar{D}_J p)\ (\bar{D}_K p) + p\bar{D}_I p = p\bar{D}_I p,$$

and the proof of (1.4.4) is complete.

The relations (1.4.5) are obtained from (1.4.4.) using (1.4.3).

Finally, if I and J are such that $|I| = |J| = 1$, then using $\bar{D}_J p = p(\bar{D}_J p)$ and $p(D_I p)=0$ we have succesively

$$D_I\bar{D}_J p = D_I(p\bar{D}_J p) = (D_I p)(\bar{D}_J p) + p(D_I\bar{D}_J p) =$$

$$=(D_I p)(\bar{D}_J p) + \bar{D}_J(pD_I p) - (\bar{D}_J p)(D_I p) =$$

$$=(D_I p)\ (\bar{D}_J p) - (\bar{D}_J p)\ (D_I p).$$

1.5. From (1.4.4) and (1.4.5) we obtain that

$$(1.5.1) \quad (D_I p)\ (D_J p) = 0 = (\bar{D}_J p)\ (\bar{D}_I p);\ |I|, |J| \geq 1\ .$$

Then, by a repeated use of (1.4.6), cleary we have

LEMMA. For any I and J the derivative $D_I\bar{D}_J p$ can be expressed as a sum of monom i als of the following two types

(i) $\pm\ (D_{I_1} p)\ (\bar{D}_{J_1} p) \ldots \qquad (D_{I_k} p)(\bar{D}_{J_k} p)$

(ii) $\pm\ (\bar{D}_{J_1} p)\ (D_{I_1} p) \ldots \qquad (\bar{D}_{J_k} p)\ (D_{I_k} p)$

where $k \geq 1$ and $I_1+\ldots+I_k = I$, $J_1+\ldots+J_k = J$.

1.6. For the rest of this section we assume that f is a function in $A_n(D)$ and $p=[f]$. We know that p is real-analytic. Let us define

$$(1.6.1)\ E(f) = \mathrm{span}\left\{ p(z)h;\ z \in D,\ h \in H \right\}.$$

This subspace of H will be referred to as the esential space of f.

For any z_0 in D we also introduce

$$(1.6.2)\ E(f;z_0) = \mathrm{span}\left\{ D_I p(z_0)h;\ I \in (Z^+)^m,\ h \in H \right\}.$$

LEMMA. We have $E(f;z_0)=E(f)$.

PROOF. Consider the projections $E_0 = [E(f;z_0)]$ and $E = [E(f)]$.

We clearly have $E_0 = E_0 E$ and also

$$E_0 D_I p(z_0) = D_I p(z_0);\ I \in (Z^+)^m.$$

Since $\bar{D}_J p = p\bar{D}_J p$ one finds

$$E_0 \bar{D}_J p(z_0) = \bar{D}_J p(z_0)\ ;\ J \in (Z^+)^m,$$ and by Lemma 1.5.

we obtain

$$E_0 D_I \bar{D}_J p(z_0) = D_I \bar{D}_J p(z_0);\ I, J \in (Z^+)^m.$$

Since the function p is real-analytic, we conclude that there exists an open subset D_0 of D such that

$$E_0 p(z) = p(z);\ z \in D_0.$$

Now, the real-analytic function

$$D \ni z \longrightarrow (1-E_0)p(z) \in L(H)$$

vanishes on D_0, hence it vanishes identically on D, that is

$$E_0 p(z) = p(z);\ z \in D.$$

Thus $E_0 E = E$, hence $E_0 = E$.

2. THE MAIN TECHNICAL RESULT

Throughout this section p will denote the self-adjoint projection in $E(D, L(H))$ associated with a function f in $A_n(D)$ and $\mathcal{X}$ will be a fixed subset of L(H), containing the identity operator $\mathbb{1}$.

2.1. For any nonnegative integer k, let us consider the following two self-adjoint subsets of $E(D,L(H))$:

$$\mathcal{G}^k = \{(\bar{D}_J p)Y^* X(D_I p)\ :\ 0 \leq |I|, |J| \leq k\ ;\ X, Y \in \mathcal{X}\}$$

$$\mathcal{T}^k = \{(\bar{D}_J p)Y^* X(D_I p)\ :\ 0 \leq |I|, |J| \leq k+1;\ |I| + |J| \leq 2k+1;\ X, Y \in \mathcal{X}\}.$$

Given a point z in D we put

$$\mathcal{G}^k(z) = \{S(z) : S \in \mathcal{G}^k\}\ ;\ \mathcal{T}^k(z) = \{T(z) : T \in \mathcal{T}^k\}\ ;\ \mathcal{G}^\infty(z) = \bigcup_{k \geq 0} \mathcal{G}^k(z),$$

and let $\mathcal{A}^k(z)$, $\mathcal{B}^k(z)$ and $\mathcal{A}^\infty(z)$ denote the C^*-algebras generated in L(H) by $\mathcal{G}^k(z)$, $\mathcal{T}^k(z)$ and $\mathcal{G}^\infty(z)$, respectively.

By Proposition 1.4 one observes that all these C^*-algebras are finite dimensional and have the common unit p(z).

Finally, for any open subset D_0 of D let us introduce the following involutive subalgebras of $E(D_0, L(H))$:

$$\Gamma(D_0, \mathcal{A}^k) = \{A \in E(D_0, L(H)) : A(z) \in \mathcal{A}^k(z),\ z \in D_0\},$$

$$\Gamma(D_0, \mathcal{B}^k) = \{A \in E(D_0, L(H)) : A(z) \in \mathcal{B}^k(z),\ z \in D_0\},$$

$$\Gamma(D_0, \mathcal{A}^\infty) = \{A \in E(D_0, L(H)) : A(z) \in \mathcal{A}^\infty(z),\ z \in D_0\}.$$

Of course we have

$$\Gamma(D_0, \mathcal{A}^k) \subset \Gamma(D_0, \mathcal{B}^k) \subset \Gamma(D_0, \mathcal{A}^{k+1}) \subset \Gamma(D_0, \mathcal{A}^\infty).$$

The next two results are direct consequences of Proposition 1.4 and Lemma 1.5. The proofs are simple, therefore we shall omit them.

2.2. LEMMA. For any A in $\Gamma(D_0, \mathcal{A}^k)$ and $|I| = |J| = 1$ we have

(2.2.1) $p(D_I A) \in \Gamma(D_0, \mathcal{B}^k)$; $(\bar{D}_J A)p \in \Gamma(D_0, \mathcal{B}^k)$,

(2.2.2) $p(D_I \bar{D}_J A)p \in \Gamma(D_0, \mathcal{A}^{k+1})$.

2.3. LEMMA. Let D_0 and k be such that $p(D_I A)$ belongs to $\Gamma(D_0, \mathcal{A}^k)$ for any A in $\Gamma(D_0, \mathcal{A}^k)$ and $|I| = 1$. Then

(2.3.1) $\Gamma(D_0, \mathcal{A}^k) = \Gamma(D_0, \mathcal{A}^\infty)$.

Now we are ready to state the main tehnical result of the paper.

2.4. THEOREM. There exist an open nonempty subset D_0 of D and an integer $1 \leq k \leq n$, with the properties:

(i) $\Gamma(D_0, \mathcal{A}^k) = \Gamma(D_0, \mathcal{A}^\infty)$,

(ii) if $\varphi : \Gamma(D_0, \mathcal{A}^\infty) \longrightarrow E(D_0, L(H))$ is a morphism of complex algebras which satisfies

(2.4.1) $\varphi(p(D_I \bar{D}_J A)p) = \varphi(p)(D_I \bar{D}_J\, \varphi(A))\, \varphi(p)$

for all A in $\Gamma(D_0, \mathcal{A}^{k-1})$ and $0 \leq |I|, |J| \leq 1$, then

(2.4.1) $\varphi(p(D_I\bar{D}_J A)p) = \varphi(p)(D_I\bar{D}_J\,\varphi(A)\,\varphi(p)$

for all A in $\Gamma(D_0, \mathcal{A}^{\infty})$ and all I, J in $(Z^+)^m$.

2.5. This theorem is a strengthened version of Theorem A from [1]. At the present moment, using the results of Section 1, its proof is more or less similar with the proof of Theorem A given in [1]. However, for the reader's convenience, we prefer to include in what follows a complete proof.

We begin with a well-known result (see for instance [2], Lemma 3.4 and [9]).

2.6. LEMMA. Let A be a self-adjoint element of $E(D, L(H))$ such that $A = pAp$. Then there exist:

(i) an open nonempty subset D_0 of D;

(ii) a collection $\{p_\alpha : 1 \leq \alpha \leq \ell\}$ of self-adjoint orthogonal projection in $E(D_0, L(H))$;

(iii) a collection $\{\mu_\alpha : 1 \leq \alpha \leq \ell\}$ of real-valued smooth functions on D_0, with $\mu_\alpha(z) \neq \mu_\beta(z), z \in D_0, \alpha \neq \beta$, related as follows:

(2.6.1) $p(z) = \sum_{\alpha=1}^{\ell} p_\alpha(z);\ z \in D_0$

(2.6.2) $A(z) = \sum_{\alpha=1}^{\ell} \mu_\alpha(z) p_\alpha(z);\ z \in D_0$.

Moreover, from the preceding relations one obtains

$$p_\alpha(z) = \prod_{\beta \neq \alpha} (A(z) - \mu_\beta(z)p_\beta(z)) \,/\, (\mu_\alpha(z) - \mu_\beta(z))\ ;\ z \in D_0.$$

2.7. Now we return to the finite dimensional C^*-algebras $\mathcal{A}^k(z)$, $\mathcal{B}^k(z)$ and $\mathcal{A}^{\infty}(z)$ associated with p and $\mathcal{X}$. Given a finite dimensional C^*-algebra $\mathcal{A}$ we shall denote by $d(\mathcal{A})$ the cardinal of any maximal set of mutually orthogonal self-adjoint minimal projections in A, and let us put

(2.7.1) $d_z^k = d(\mathcal{A}^k(z));\ d_z^{\infty} = d(\mathcal{A}^{\infty}(z))$.

of course we have

(2.7.2) $d_z^0 \leq d_z^1 \leq \dots \leq d_z^k \leq \dots \leq d_z^\infty \leq n$

therefore we can find an open nonempty subset D_0 of D and an integer $1 \leq k \leq n$ such that

(2.7.3) $d_z^{k-1} = d_z^k$; $z \in D_0$.

Moreover, using some well-known facts about the structure of finite dimensional C^*-algebras (see for instance [10], Chap .I, §11), by a repeated use of Lemma 2.6. and eventuall decreasing D_0, we may suppose in what follows that there exists:

(i) a sequence $d_1, \dots, d_\ell$ of positive integers;

(ii) a system $Q_1, \dots, Q_\ell$ of mutually orthogonal self- adjoin central projections in $\Gamma(D_0, \mathcal{A}^{k-1})$;

(iii) a collection p_α^i : $1 \leq i \leq \ell$, $1 \leq \alpha \leq d_i$, of mutually orthogonal self-adjoint projections in $\Gamma(D_0, \mathcal{A}^{k-1})$;

(iv) a collection $U_{\alpha\beta}^i$: $1 \leq i \leq \ell$, $1 \leq \alpha, \beta \leq d_i$, of elements of $\Gamma(D_0, \mathcal{B}^{k-1})$, such that

(2.7.4) $d_1 + \dots + d_\ell = d_z^{k-1}$; $z \in D_0$

(2.7.5) $\sum_{\alpha=1}^{d_i} p_\alpha^i = Q_i$,

(2.7.6) $U_{\alpha\alpha}^i = p_\alpha^i$, $U_{\alpha\beta}^{i*} = U_{\beta\alpha}^i$, $U_{\alpha\beta}^i \; U_{\gamma\delta}^i = \Delta_{\beta\gamma} U_{\alpha\delta}^i$

where $\Delta_{\beta\gamma}$ means the Kronecher symbol.

Clearly, by (2.7.5) we obtain that all Q_i, $1 \leq i \leq \ell$, are central projections in $\Gamma(D_0, \mathcal{A}^{k-1})$. By (2.7.4) and (2.7.3) we conclude that all $p_\alpha^i(z)$, $1 \leq i \leq \ell$, $1 \leq \alpha \leq d_i$ are minimal projections in $\mathcal{A}_{(z)}^{k-1}$ and also in $\mathcal{A}_{(z)}^k$, for any z in D_0. Now, given I in $(Z^+)^m$ with $|I| = 1$ and $1 \leq i \leq \ell$, we know from Lemma 2.2 that $p(D_I Q_i)$ is in $\Gamma(D_0, \mathcal{B}^{k-1})$ and since Q_i is a central projection in $\Gamma(D_0, \mathcal{B}^{k-1})$ we have

$$p(D_I\Omega_i) = pD_I(\Omega_i\Omega_i) = p(D_I\Omega_i)\Omega_i + \Omega_i(D_I\Omega_i) = 2\Omega_i(D_I\Omega_i)\Omega_i$$

whence it follows

(2.7.7) $p(D_I\Omega_i) = 0 = (\bar{D}_I\Omega_i)p$

From this last relation it is easy to check that

(2.7.8) $\Omega_i p(D_I\bar{D}_J B)p = p(D_I\bar{D}_J B)p\Omega_i$; $1 \leq i \leq \ell$,

for all B in $\Gamma(D_0, \mathcal{B}^{k-1})$ and $0 \leq |I| + |J| \leq 1$. In particular one obtains that Ω_i, $1 \leq i \leq \ell$, are central projections in $\Gamma(D_0, \mathcal{A}^k)$.

Now, since the projections $p^i_\alpha(z)$, $1 \leq i \leq \ell$, $1 \leq \alpha \leq d_i$, are minimal in $\mathcal{A}^k_{(z)}$ for any z in D_0, by (2.7.6) and the preceding remark we have that, for any A in $\Gamma(D_0, \mathcal{A}^k)$, there exists a uniquely determined collection of complex-valued smooth functions on D_0,

$$\{\mu^i_{\alpha\beta}(A) : 1 \leq i \leq \ell, \ 1 \leq \alpha, \beta \leq d_i\}$$

such that

(2.7.9) $$A = \sum_{i=1}^{\ell} \sum_{\alpha,\beta=1}^{d_i} \mu^i_{\alpha\beta}(A)\, U^i_{\alpha\beta}$$

2.8. Let D_0 and k be as above and let $\varphi : \Gamma(D_0, \mathcal{A}^\infty) \longrightarrow E(D_0, L(H))$ be a morphism of complex algebras. In order to prove Theorem 2.4, it sufficies to show that

(2.8.1) $pD_I U^i_{\alpha\beta} \in \Gamma(D_0, \mathcal{A}^k)$

(2.8.2) $\varphi(p(D_I U^i_{\alpha\beta})p) = \varphi(p)(D_I\varphi(U^i_{\alpha\beta}))\varphi(p)$,

(2.8.3) $(p(\bar{D}_I U^i_{\alpha\beta})p) = \varphi(p)(\bar{D}_I\varphi(U^i_{\alpha\beta}))\varphi(p)$,

for all $|I| = 1$, $1 \leq i \leq \ell$, $1 \leq \alpha, \beta \leq d_i$.

Indeed, (i) of Theorem 2.4 will be a consequence of (2.8.1), (2.7.9) and Lemma 2.3, and (ii) will follow from (2.8.2), (2.8.3) and (2.7.9).

Our next task is to prove (2.8.1), (2.8.2) and (2.8.3). Let us consider the subsets of $\Gamma(D_0, \mathcal{B}^{k-1})$ defined by

$$\mathcal{Y}_i = \{p^i_\alpha A(D_K\bar{D}_L B)C\, p^i_\beta : 1 \leq \alpha, \beta \leq d_i,$$
$$A, B, C \in \Gamma(D_0, \mathcal{A}^{k-1}),\ 0 \leq |K| + |L| \leq 1\ ;\ 1 \leq i \leq \ell\}.$$

If z is a point in D_0 then each $U^i_{\alpha\beta}$ (z) is a finite product of elements belonging to $\mathcal{G}_i$, evalued in z. Therefore, eventually decreasing D_0 we may suppose that any $U^i_{\alpha\beta}$ is a finite product of $U^i_{\gamma\delta}$'s belonging to $\mathcal{G}_i$. Thus we are allowed to prove (2.8.1), (2.8.2) and (2.8.3) assuming that $U^i_{\alpha\beta}$ is an element of $\mathcal{G}_i$.

Let us first assume that $U^i_{\alpha\beta} = p^i_\alpha\ A(D_K\ B\ )C\ p^i_\beta$ where A, B, C are in $\Gamma(D_0, \mathcal{A}^{k-1})$ and $|K| = 1$.

Given I in $(Z^+)^m$ with $|I| = 1$, we derive easily that

(2.8.4) $p(D_I\ U^{i*}_{\alpha\beta}),\ (\bar{D}_I\ U^i_{\alpha\beta})p \in \Gamma(D_0, \mathcal{A}^k)$

and using (2.4.1) we also find

(2.8.5) $\varphi(p(D_I\ U^{i*}_{\alpha\beta})p) = \varphi(p)(D_I\ \varphi(U^{i*}_{\alpha\beta}))\ \varphi(p)$,

(2.8.6) $\varphi(p(\bar{D}_I\ U^i_{\alpha\beta})p) = \varphi(p)(\bar{D}_I\ \varphi(U^i_{\alpha\beta}))\ \varphi(p)$.

The rest of the proof will be based on the following simple result.

LEMMA. Let $\mathcal{A}$ be an involutive algebra and let V, W in $\mathcal{A}$ be given such that VWV=V. Then for each derivation δ on $\mathcal{A}$ we have

(2.8.7) $\delta V = V(\delta F) + \delta(E)V - V(\delta W)V$,

where F=WV and E=VW.

Proof of Lemma. Since EV = V we obtain

$V(\delta F) + \delta(E)V = V(\delta W)V + VW(\delta V) + \delta(E)V =$

$= V(\delta W)V + E(\delta V) + \delta(E)V = V(\delta W)V + \delta V.$

Now let us put in (2.8.7) $\delta = D_I$, $V = U^i_{\alpha\beta}$, $W = U^{i*}_{\alpha\beta}$.
Clearly $E = p^i_\alpha$; $F = p^i_\beta$ and we find

$$(2.8.8)\quad D_I U^i_{\alpha\beta} = U^i_{\alpha\beta}\,(D_I p^i_\beta) + (D_I p^i_\alpha)\, U^i_{\alpha\beta} - U^i_{\alpha\beta}\,(D_I U^{i*}_{\alpha\beta})\, U^i_{\alpha\beta}.$$

Since $p(D_I p^i_\alpha) \in \Gamma(D_0, \mathcal{A}^k)$, from (2.8.8) and (2.8.4) it follows that

$$(2.8.9)\quad p(D_I U^i_{\alpha\beta}) \in \Gamma(D_0, \mathcal{A}^k)$$

On the other hand, if we put in (2.8.7) $\delta = D_I$ and $V = \varphi(U^i_{\alpha\beta})$, $W = \varphi(U^{i*}_{\alpha\beta})$ then $E = \varphi(p^i_\alpha)$, $F = \varphi(p^i_\beta)$ and we obtain

$$(2.8.9)\quad D_I\,\varphi(U^i_{\alpha\beta}) = \varphi(U^i_{\alpha\beta})\,(D_I\,\varphi(p^i_\beta)) + \varphi(D_I p^i_\alpha)\,\varphi(U^i_{\alpha\beta}) - \varphi(U^i_{\alpha\beta})\,(D_I\,\varphi(U^{i*}_{\alpha\beta}))\,\varphi(U^i_{\alpha\beta}).$$

Using (2.8.5), (2.8.8) and (2.4.1), from (2.8.9) it follows

$$(2.8.10)\quad \varphi(p(D_I U^i_{\alpha\beta})p) = \varphi(p)(D_I\,\varphi(U^i_{\alpha\beta}))\,\varphi(p).$$

Thus (2.8.1), (2.8.2) and (2.8.3) are proved.
For the second case, when $U^i_{\alpha\beta} = p^i_\alpha\, A(\bar{D}_L\, \mathcal{B})C\, p^i_\beta$, we proceed analogously. The proof of Theorem 2.4 is complete.

3. THE CONGRUENCE THEOREM

Let f and $\tilde{f}$ be two functions in the class $A_n(D)$. We shall denote by p and $\tilde{p}$ the self-adjoint projections in $E(D, L(H))$ associated with f and $\tilde{f}$, respectively.

3.1. DEFINITION (cf. [6], [2]). The functions f and $\tilde{f}$ are said to be congruent, if there exists a unitary operator U in $L(H)$ such that

(3.1.1) $U p(z) = \tilde{p}(z) U$; $z \in D$.

3.2. DEFINITION (cf., [6], [2]). Let k be a nonnegative integer. The functions f and $\tilde{f}$ are said to have order of contact k, if for any point z in D there exist :

(i) an open neighborhood D_0 of z;

(ii) two analytic frames $\{h_\alpha : 1 \leq \alpha \leq n\}$ and $\{\tilde{h}_\alpha : 1 \leq \alpha \leq n\}$ for f, respectively $\tilde{f}$, over D_0;

(iii) a unitary operator U_z in $L(H)$, such that

(3.2.1) $U_z D_I h_\alpha(z) = D_I \tilde{h}_\alpha(z)$; $I \in (Z^+)^m$, $0 \leq |I| \leq k$; $1 \leq \alpha \leq n$.

It is not dificult to see that we have:

3.3. LEMMA. The functions f and $\tilde{f}$ have order of contact k, if and only if for any point z in D there exists a unitary operator U_z in $L(H)$ such that

(3.3.1) $U_z D_I p(z) = D_I \tilde{p}(z) U_z$; $I \in (Z^+)^m$, $0 \leq |I| \leq k$.

As a consequence, if f and $\tilde{f}$ are congruent then they have order of contact k for any k.

3.4. Before continuing we make another remark. Let U_z be as above and consider $V_z = U_z p(z)$.

From (3.3.1) one obtains

(3.4.1) $V_z^* V_z = p(z)$, $V_z V_z^* = \tilde{p}(z)$

hence V_z is a partial isometry in $L(H)$. Moreover, from (3.3.1) one finds

(3.4.2) $V_z \bar{D}_J p(z) D_I p(z) V_z^* = \bar{D}_J \tilde{p}(z) D_I \tilde{p}(z)$;

$I, J \in (Z^+)^m$, $0 \leq |I|, |J| \leq k$.

3.5. THE CONGRUENCE THEOREM. Let f, $\tilde{f}$ be two functions in $A_n(D)$ such that

(3.5.1) $E(f) = E(\tilde{f}) = H$

The following conditions are equivalent:

(i) f and $\tilde{f}$ are congruent;

(ii) f and $\tilde{f}$ have order of contact n;

(iii) for any z in D there exists a partial isometry V_z in $L(H)$ so that

(3.5.2) $V_z^* V_z = p(z)$, $V_z V_z^* = \tilde{p}(z)$

(3.5.3) $V_z \bar{D}_J p(z) D_I p(z) V_z^* = \bar{D}_J \tilde{p}(z)\ D_I \tilde{p}(z)$; $0 \leq |I|, |J| \leq n$.

3.6. Clearly we have to prove only that (iii) implies (i). This will follow from the next theorem, which is a generalisation of Theorem B from [1]. In order to state it we need some notation. Let f and $\tilde{f}$ be as above and let $\mathcal{X}$ be a subset of $L(H)$ containing the identity operator 1. Assume that the condition (3.5.1) is satisfied and consider a map $\psi : \mathcal{X} \to L(H)$ such that $\psi(1) = 1$.

3.6. THEOREM. The following conditions are equivalent:

(i) ψ is the restriction of an inner automorphism in $L(H)$ induced by a unitary operator U, which satisfies

$U\ p(z)\ U^* = \tilde{p}(z)$, $z \in D$.

(ii) for any z in D there exists a partial isometry V_z in $L(H)$ so that

(3.6.1) $V_z^* V_z = p(z)$; $V_z V_z^* = \tilde{p}(z)$,

(3.6.2) $V_z \bar{D}_J p(z) Y^* X D_I p(z) V_z^* = \bar{D}_J \tilde{p}(z)\ \psi(Y)^*\ \psi(X) D_I \tilde{p}(z)$,

for all X, Y in $\mathcal{X}$ and $0 \leq |I|, |J| \leq n$.

PROOF. It is clear that (i) implies (ii). The converse is based on Theorem 2.4. We associate with f and $\mathcal{X}$ the open nonempty subset D_0 of D and the integer $1 \leq k \leq n$ which appear in Theorem 2.4. Given A in $\Gamma(D_0, \mathcal{A}^\infty)$, let us define

(3.6.3) $\varphi(A)(z) = V_z A(z) V_z^*$; $z \in D_0$.

Since $\Gamma(D_0, \mathcal{A}^\infty) = \Gamma(D_0, \mathcal{A}^k)$, from (3.6.1) and (3.6.2) we obtain that φ is a well-defined morphism of complex algebras from $\Gamma(D_0, \mathcal{A}^\infty)$ into $E(D_0, L(H))$, and the conditions (2.4.1) in Theorem 2.4 are satisfied. Thus, from Theorem 2.4 we conclude by induction that

(3.6.4) $V_z \bar{D}_J p(z) Y^* X D_I p(z) V_z^* = \bar{D}_J \tilde{p}(z)\ \psi(Y)^*\ \psi(X) D_I \tilde{p}(z)$, for all z in D_o, X and Y in $\mathcal{X}$, I and J in $(Z^+)^m$.

Let z_0 be a fixed point in D_0. Since $1 \in \mathcal{X}$, Lemma 1.6 and the assumption (3.5.1) give

(3.6.5) $H = \operatorname{span}\{ XD_Ip(z_0)h : X \in \mathcal{X},\ I \in (Z^+)^m,\ h \in H \} =$

$= \operatorname{span}\{ \psi(X)D_I\tilde{p}(z_0)h;\ X \in \mathcal{X},\ I \in (Z^+)^m,\ h \in H \}.$

Let U in $L(H)$ be defined by the equations

(3.6.6) $U(XD_Ip(z_0)h) = \psi(X)D_Ip(z_0)V_0 h;\ X \in \mathcal{X},$

$I \in (Z^+)^m,\ h \in H.$

From (3.6.4) and (3.6.5) we derive that U is a unitary operator on H and also

(3.6.7) $\psi(X) = U X U^*\ ;\ X \in \mathcal{X},$

(3.6.8) $D_I\tilde{p}(z_0) = U D_Ip(z_0) U^*\ ;\ I \in (Z^+)^m.$

Since p and $\tilde{p}$ are real-analytic, using (3.6.8) and the remarks given at 1.5 we conclude that

(3.6.8) $\tilde{p}(z) = U p(z) U\ ;\ z \in D.$

The proof is complete.

4. THE COWEN-DOUGLAS CLASS $B_n(D)$.

Let us denote as above by D an open and connected subset of $\mathbb{C}^m$ and by H a separable, infinite dimensional, complex Hilbert space. Given a m- tuple $T=(T_1,\dots,T_m)$ of commuting bounded linear operators on H, and a point $z=(z_1,\dots,z_m)$ in D, we define

$K(T;z) = \{ h \in H:\ (z_1-T_1)h = \dots = (z_m-T_m)h = 0 \}.$

4.1. DEFINITION (cf., [2], [3]). The m- tuple T is said to be in the class $B_n(D)$, where n is a positive integer, if and only if

(i) $\dim K(T;z) = n\ ;\ z \in D,$

(ii) $\operatorname{span} \bigcup_{z \in D} K(T;z) = H$,

(iii) range $(z_i-T_i) = H;\ 1 \leq i \leq m,\ z \in D.$ Meu 23668

4.2. Let $T=(T_1,\ldots,T_m)$ be in $\mathcal{B}_n(D)$ and denote by f_T the function defined as follows:

$$f_T: D \longrightarrow Gr(n,H),\ f_T(z)=K(T;z).$$

Arguing as in [3] or using a result of Curto and Salinas (cf., [4], Theorem 2.2) one obtains that f_T is analytic. Moreover, we have as a direct consequence of the definitions:

LEMMA. Let $T=(T_1,\ldots,T_m)$ and $\tilde{T}=(\tilde{T}_1,\ldots,\tilde{T}_m)$ be two m-tuples in the class $\mathcal{B}_n(D)$. The following conditions are equivalent:

(i) there exists a unitary operator U on H such that

$$U\,T_i=\tilde{T}_i U\ ;\ 1 \leq i \leq m.$$

(ii) the functions f_T and $f_{\tilde{T}}$ are congruent.

4.3. We now wish to obtain an operator theoretic interpretation of the order of contact of the functions f_T and $f_{\tilde{T}}$. Before stating precisely what we are able to find out, we shall give some preliminary results.

Let $T=(T_1,\ldots T_m)$ be a fixed m-tuples in $\mathcal{B}_n(D)$ and let $p=[f_T]$. For any $z=(z_1,\ldots,z_m)$ in D we have

(4.3.1) $(z_j-T_j)p(z) = 0\ ;\ 1 \leq j \leq m$.

Let $I = (i_1,\ldots i_m)$ be in $(Z^+)^m$ and fix $1 \leq j \leq m$. If $i_j \geq 1$, then, differentiating the equation (4.3.1), one obtains

(4.3.2) $(z_j-T_j)D_I p(z) = -i_j\, D_{I(j)}p(z)$, $I(j)=(i_1,\ldots,i_j-1,$ $,\ldots,i_m)$.

If $i_j = 0$, then one finds

(4.3.3) $(z_j - T_j)D_I p(z) = 0$

Given $I = (i_1,\ldots,i_m)$ and $J = (j_1,\ldots,j_m)$ in $(Z^+)^m$, let us put $I - J = (i_1 - j_1,\ldots,i_m - j_m)$. If $I - J$ belongs to $(Z^+)^m$ we write $I \geq J$. We shall use also the notation $I! = i_1! \ldots i_m!$.

Now, for $z = (z_1,\ldots,z_m)$ in D, we let $T^J(z)$ denote the operator in $\mathcal{L}(H)$ defined by

(4.3.4) $T^J(z) = (z_1 - T_1)^{j_1} \ldots (z_m-T_m)^{j_m}$.

By a repeated use of (4.3.2) and (4.3.3) we have:

(4.3.5) $T^J(z)D_Ip(z) = (-1)^{|J|}(I!/(I-J)!)D_{I-J}p(z);\ I-J \in (Z^+)^m$,

(4.3.6) $T^J(z)\ D_Ip(z) = 0 \qquad ;\ I-J \notin (Z^+)^m$.

Let z be a fixed point in D, and let $\{h_\alpha : 1 \leq \alpha \leq n\}$ be an analytic frame for f_T over an open neighborhood D_0 of z. For any nonnegative integer k, let us introduce

(4.3.7) $E^{(k)}(T;z) = \text{span}\{D_I h_\alpha(z) : 0 \leq |I| \leq k,\ 1 \leq \alpha \leq n\}$.

We easily derive

(4.3.8) $E^{(k)}(T;\ z) = \text{span}\{D_I p(z)h : 0 \leq |I| \leq k,\ h \in H\}$.

From (4.3.5) and (4.3.6) one sees that

(4.3.9) $T^J(z)\ D_I h_\alpha(z) = (I!\ /(I-J)!)\ D_{I-J}h_\alpha(z)$;
$1 \leq \alpha \leq n,\ I - J \in (Z^+)^m$,

(4.3.10) $T^J(z)D_I h_\alpha(z) = 0 \quad ;\ 1 \leq \alpha \leq n,\ I - J \notin (Z^+)^m$.

These equations, together with

(4.3.11) $\text{span} \bigcup_{k \geq 0} E^{(k)}(T;z) = E(f_T;z) = H$

imply that :

LEMMA. (i) The vectors $\{D_I h_\alpha(z) : I \in (Z^+)^m,\ 1 \leq \alpha \leq n\}$ are independent in H, and

$$\text{span}\{D_I h_\alpha(z) : I \in (Z^+)^m,\ 1 \leq \alpha \leq n\} = H.$$

(ii) $\dim E^{(k)}(T;z) = (1+m+\ldots m^k)\ n$.

4.4. Now we introduce another collection of subspaces:

(4.4.1) $K^{(k)}(T;z) = \{h \in H : T^I(z)h=0,\ |I| = k+1\}$.

Of course $K^{(0)}(T;z) = K(T;z) = E^{(0)}(T;\ z)$.

LEMMA. For any nonnegative integer k we have

(4.4.2) $K^{(k)}(T;z) = E^{(k)}(T;z)$.

PROOF. By (4.3.9) and (4.3.10) one finds that

$E^{(k)}(T;\ z) \subset K^{(k)}(T;\ z)$.

Let h be in $K^{(k)}(T;z)$. By Lemma 4.3, there exists a unique collection of complex numbers

$\{c_{I,\alpha} : I \in (Z^+)^m,\ 1 \leq \alpha \leq n\}$ such that

$$h = \sum_{I,\alpha} c_{I,\alpha}\ D_I h_\alpha(z).$$

Let J in $(Z^+)^m$ with $|J| = k+1$. Since $T^J(z)h=0$, one obtains

$$0 = \sum_{I \geqslant J} \sum_{\alpha} c_{I,\alpha}(I!/(I-J)!)\, D_{I-J}h_\alpha(z).$$

By Lemma 4.3 again, we have

$$c_{I,\alpha} = 0;\ I \geqslant J,\quad 1 \leq \alpha \leq n.$$

Since J is an arbitrary element in $(Z^+)^m$ with $|J| = k+1$, we conclude

$$c_{I,\alpha} = 0;\ |I| \geqslant k+1,\ 1 \leq \alpha \leq n,$$

hence h belongs to $E^{(k)}(T;\ z)$.

4.5. Let $T=(T_1,\dots,T_m)$ and $\tilde{T}=(\tilde{T}_1,\dots,\tilde{T}_m)$ be two m-tuples in $\mathcal{B}_n(D)$. Our next task is to give an alternate means of the order of contact. Explicity, we have:

PROPOSITION. The following conditions are equivalent:

(i) f_T and $f_{\tilde{T}}$ have order of contact k.

(ii) for any z in D there exists a unitary operator

$U_z : K^{(k)}(T;z) \to K^{(k)}(\tilde{T};z)$ so that

(4.5.1) $\tilde{T}_i \mid K^{(k)}(\tilde{T};z) = U_z T_i U_z^* \mid K^{(k)}(\tilde{T};z);\quad 1 \leq i \leq m.$

PROOF. Let $p = [f_T]$ and $\tilde{p} = [f_{\tilde{T}}]$. Assume that f_T and $f_{\tilde{T}}$ have ordar of contact k and let z be a fixed point in D. Then there exist two analytic frames $\{h_\alpha : 1 \leq \alpha \leq n\}$ and $\{\tilde{h}_\alpha : 1 \leq \alpha \leq n\}$ for f_T, respectively $f_{\tilde{T}}$, over on open neighborhood D_0 of z, and a unitary operator U_z in $\mathcal{L}(H)$ so that:

(4.5.2) $U_z D_I h_\alpha(z) = D_I \tilde{h}_\alpha(z);\ 0 \leq |I| \leq k,\ 1 \leq \alpha \leq n.$

By Lemma 4.4. and using the equations (4.3.9), (4.3.10), it follows easily that U_z has the required properties.

In order to prove the converse, let us assume that z is a fixed point in D, and let U_z be a unitary from $K^{(k)}(T;z)$ onto $K^{(k)}(\tilde{T};z)$, satisfying the condition (4.5.1). It is enough to show that

(4.5.3) $D_I\tilde{p}(z) = U_z D_I p(z)\, U_z^*;\ 0 \leq |I| \leq k.$

We shall proceed by induction . First we remark that

$0 = (z_j - \tilde{T}_j)\tilde{p}(z) = U_z(z_j - T_j)U_z^*\tilde{p}(z),$

hence

$(z_j - T_j)\ U_z^* \tilde{p}(z)\ U_z = 0,\ 1 \leq j \leq n.$

Since dim range $U_z^* \tilde{p}(z) U_z = n$ = dim range $p(z)$ we conclude that

(4.5.4) $U_z^* \tilde{p}(z)\ U_z = p(z).$

Thus (4.5.3) is proved for $|I| = 0$.

Assume now that (4.5.3) holds for all $|I| \leq \ell$, and let $I = (i_1, \ldots i_m)$ be such that $|I| = \ell + 1 \leq k$. Let $1 \leq j \leq m$ with $i_j \geq 1$. By (4.3.2) one finds

$U_z(z_j - T_j)\ U_z^* D_I \tilde{p}(z) = (z_j - \tilde{T}_j)\ D_I \tilde{p}(z) = -i_j D_{I(j)} \tilde{p}\ (z).$

From the induction assumption one obtains

$(z_j - T_j)\ U_z^*\ D_I \tilde{p}(z) U_z = -i_j D_{I(j)} p(z)$

whence

(4.5.5) $(z_j - T_j)\ (U_z^* D_I \tilde{p}(z) U_z - D_I p(z)) = 0$

By (4.3.3) the equation (4.5.5) is also true if $i_j = 0$. Therefore, there is a complex number c such that

(4.5.6) $U_z^* D_I \tilde{p}(z)\ U_z - D_I p(z) = c\ p(z).$

It follows that

$c\ p(z) = p(z) U_z^*\ D_I \tilde{p}(z)\ U_z - p(z) D_I p(z)$

But $p(z) U_z^* = U_z \tilde{p}(z)$ and $\tilde{p}(z) D_I \tilde{p}(z) = 0 = p(z) D_I p(z)$ (cf., Proposition 1.4), hence $c = 0$. The proof is complete.

4.6. We are now ready to state the main result of this section. It could be regarded as a generalisation of Theorem 1.6 from [2] (see also [1], Theorem C).

Let $T = (T_1, \ldots, T_m)$ and $\tilde{T} = (\tilde{T}_1, \ldots, \tilde{T}_m)$ be two m-tuples in $\mathcal{B}_n(D)$. We denote by T' the commutant of $\{T_1, \ldots, T_m\}$ and assume that $\mathcal{X}$ is a subset of T' containing the identity operator 1 and all operators $T_1, \ldots, T_m$.

Let $\psi : \mathcal{X} \to L(H)$ be a map such that

(4.6.1) $\psi(1) = 1,\ \psi(T_j) = \tilde{T}_j\ ;\ 1 \leq j \leq m,$

(4.6.2) $\psi(X) \in \tilde{T}';\ X \in \mathcal{X}.$

Then we have:

THEOREM. The following conditions are equivalent:

(i) ψ is the restriction to $\mathcal{X}$ of an inner automorphism in $\mathcal{L}(H)$;

(ii) for any z in D there exists a unitary operator $U_z : K^{(n)}(T;z) \to K^{(n)}(\tilde{T};z)$ so that

(4.6.3) $\psi(X) \mid K^{(n)}(\tilde{T};z) = U_z X U_z^* \mid K^{(n)}(\tilde{T};z)$; $X \in \mathcal{X}$.

PROOF. It sufficies to prove that, under our assumptions, the present condition (ii) implies the condition (ii) in Theorem 3.6.

Let $p=[f_T]$ and $\tilde{p}=[f_{\tilde{T}}]$. From Proposition 4.5. we obtain

(4.6.4) $U_z D_I p(z) U_z^* = D_I \tilde{p}(z)$; $z \in D$, $0 \leq |I| \leq n$.

Now let us put $V_z = U_z p(z)$. By (4.6.3) and (4.6.4) we derive easily the desired relations (3.6.1) and (3.6.2). This concludes the proof.

NATIONAL INSTITUTE FOR SCIENTIFIC AND
TECHNICAL CREATION - INCREST
Department of Mathematics
Bd. Păcii 220,
79622 BUCHAREST
R O M A N I A

- R E F E R E N C E S -

1. APOSTOL, C., MARTIN, M.: A C^*-algebra approach to the Cowen-Douglas theory. Operator Theory: Advances and Applications No.2, Basel-Boston-Stuttgard: Birkhauser 1981, 45 - 51.

2. COWEN, M.J., DOUGLAS, R.G.: Complex geometry and operator theory. Acta Math. 141(1978), 187 - 261.

3. COWEN, M.J., DOUGLAS, R.G.: Operator possessing an open set of eigenvalues. Colloquia Mathematica Societatis Janos Bolyai, 35.Functions,Series,Operators.North-Holland 1980, 323-341.

4. CURTO, R.E.; SALINAS, N.: Generalized Bergman kernels and the Cowen-Douglas theory. Amer. J. Math. 106:2(1984), 447 - 488.

5. DOUGLAS, R.G.: Banach Algebra Techniques in Operator Theory. New York: Academic Press 1972.

6. GRIFFITHS, P.A.: On Cartan's method of Lie groups and moving frames as applied to uniqueness and existence questions in differential geometry. Duke Math. J. 41(1974), 775 - 814.

7. MALGRANGE, R.: Lectures on the Theory of Functions of Several Complex Variables. Tata Institute of Fundamental Research, Bombay, 1958.

8. MARTIN, M., Hermitian geometry and involutive algebras. Math. Z. 188(1985), 359 - 382.

9. NOMIZU, K.: Characteristic roots and vectors of a differential family of symmetric matrices. Linear and multilinear Algebra 1 (1973), 159 - 162.

10. TAKESAKI, M.: Theory of Operator Algebras. I.Berlin-Heidelberg - New York: Springer 1979.