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1. Introduction

Let $P_0(D)$ be an elliptic operator, $V_S(x,D)$ a short range and $V_L(x,D)$ a long range perturbation. The purpose of this paper is to prove the existence and the asymptotic completeness of the wave operators $W_\pm(P,P_1) = \text{s-}\lim_{t\to\pm\infty} e^{itP} e^{-itP} P_{ac}(P_1)$, where $P = P_0 + V_S + V_L$ and $P_1 = P_0 + V_L$.

These operators satisfy the following assumptions:

(A.1) $P_0(D)$ is an elliptic pseudodifferential operator on R^n, of order $m>0$, formally self-adjoint (i.e. $P_0(D) \in L^m_{\varrho,(0,\infty)}(R^n)$, there are two constants $R, c>0$ such that $|P_0(\xi)| \geq c|\xi|^m$, for $|\xi| \geq R$, and $(P_0u,v)_{L^2} = (u,P_0v)_{L^2}$, $u,v \in \mathcal{S}(R^n)$) such that $\lim_{|\xi|\to\infty} P_0(\xi) = \infty$.

(A.2) If $\Lambda_c(P_0) = \{P_0(\xi);\quad R^n,\ P_0'(\xi) = 0\}$ is the set of the critical values of P_0, then $\Lambda_c(P_0)$ is at most a countable set.

(B) $V_L(x,D) \in L^m_{\varrho,(\mu,1+\mu)}(R^n)$, $\varrho > 1/2$, $\mu > 0$, is formally self-adjoint and it has an extension to a compact operator $V_L : H^m(R^n) \longrightarrow L^2(R^n)$.

(C) $V_S(x,D) \in L^m_{\varrho,(\theta,\sigma)}(R^n)$, $\varrho > 1/2$, $\theta > 1$, is formally self-adjoint and it has an extension to a compact operator $V_S: H^m(R^n) \longrightarrow L^2(R^n)$.

The classes of operators $L^m_{\rho,(\theta,\sigma)}(R^n)$ will be defined in the next section.

Now, we shall make some remarks on these assumptions.

- The assumption (A.1) implies that $P_0(D): \mathcal{S}(R^n) \longrightarrow \mathcal{S}(R^n)$ has a unique self-adjoint extension on $L^2(R^n)$ ($D(P_0) = H^m(R^n)$). Moreover, since $\lim_{|\xi|\to\infty} P_0(\xi) = \infty$, this operator is bounded from below and $\sigma(P_0) = \sigma_{ess}(P_0) = [c, \infty)$, where $c = \{\inf P_0(\xi); \xi\in R^n\}$ Let us remark that the condition $\lim_{|\xi|\to\infty} P_0(\xi) = \infty$ is not a restrictive one, because the ellipticity and the symmetry of P_0 assure that $\lim_{|\xi|\to\infty} P_0(\xi) = \infty$ or $\lim_{|\xi|\to\infty} P_0(\xi) = -\infty$.
- The last condition of the assumption (A.1) implies that $\Lambda_c(P_0)$ is closed and Sard's theorem shows that $\Lambda_c(P_0)$ has Lebesgue measure zero. It is known that if P_0 is a differential operator with constant coefficients then $\Lambda_c(P_0)$ is a finite set.
- From the assumptions (B) and (C) it follows that $P_1 = P_0 + V_L$ and $P = P_1 + V_S$ are self-adjoint operators in $L^2(R^n)$, with $D(P) = D(P_1) = D(P_0) = H^m(R^n)$, bounded from below and that $\sigma_{ess}(P) = \sigma_{ess}(P_1) = \sigma_{ess}(P_0) = [c, \infty)$.

The wave operators are given by $W_\pm(P,P_1) = \text{s-}\lim_{t\to\pm\infty} e^{itP}e^{-itP} P_{ac}(P_1)$, if the limits exist.

<u>Definition 1.1.</u> Suppose that $W_\pm(P,P_1)$ exist. We say that they are complete if $\text{Ran } W_+ = \text{Ran } W_- = \mathcal{H}_{ac}(P)$.

If, in addition, $\sigma_{sing}(P) = \Phi$, we say that the wave operators are asymptotically complete.

The main result of this paper is

Theorem 1.2. (a) If P_0, V_L, V_S satisfy the assumptions (A.1), (B), (C) and if $P_1 = P_0 + V_L$, $P = P_0 + V_L + V_S$, then the wave operators $W_\pm(P,P_1)$ exist and they are complete.

(b) If, in addition, P_0 satisfies (A.2), then the wave operators are asymptotically complete.

The proof of this theorem is based on the estimates contained in the principle of limiting absorption and on some results about smooth operators.

If $V_L = 0$, Theorem 1.2. can be proved with time dependent methods (see [1], [2]).

Results similar to those contained in Theorem 1.2. were proved by Lavine in [3] for $P_0(D) = -\Delta$ and $V_S(x,D) = V_S(x)$, $V_L(x,D) = V_L(x)$.

In the last section we shall apply Theorem 1.2. to the perturbed Hamiltonian of a relativistic particle with spin-zero.

2. Preliminaries

In this section we list some results which we shall use in the proof of Theorem 1.2..

First of all we give some results concerning the pseudodifferential operators used in this paper. The proofs are standard in the theory of global pseudodifferential operators (see [4], [5] and [7]).

Definition 2.1. Let $m, \theta, \sigma \in R$, $\theta \leq \sigma$, $0 < \rho \leq 1$. We say that A is in $S^m_{\rho,(\theta,\sigma)}(R^n)$ if $A \in C^\infty(R^n x R^n)$ and for every $\alpha, \beta \in N^n$ there exists $C = C(\alpha,\beta) > 0$ such that

$$|D_x^\alpha D_\xi^\beta A(x,\xi)| \leqslant C(1+|x|)^{-\mu(|\alpha|)}(1+|\xi|)^{m-\varrho|\beta|}, \; \forall x,\xi \in R^n,$$

where $\mu(0) = \theta$, $\mu(k) = \sigma$, $k \in N$, $k \geqslant 1$.

We denote by $S^{-\infty}_{\varrho,(\theta,\sigma)}(R^n) = \bigcap_{m\in R} S^m_{\varrho,(\theta,\sigma)}(R^n)$.

Proposition 2.2. If $A_j \in S^{m_j}_{\varrho,(\theta,\sigma)}(R^n)$, $m_j \searrow -\infty$, there exists $A \in S^{m_0}_{\varrho,(\theta,\sigma)}(R^n)$ such that, for all $k \geqslant 1$,

$$A - \sum_{j=0}^{k-1} A_j \in S^{m_k}_{\varrho,(\theta,\sigma)}(R^n).$$

Moreover, if B is another symbol with this property then $A - B \;\; S^{-\infty}_{\varrho,(\theta,\sigma)}(R^n)$.

We say that A is the asymptotic sum of A_j, and write

$$A \sim \sum_{j\geqslant 0} A_j$$

We define $L^m_{\varrho,(\theta,\sigma)}(R^n)$ as the linear space of the continuous linear maps $A(x,D): \mathcal{S}(R^n) \longrightarrow \mathcal{S}(R^n)$ given by

$$A(x,D)u(x) = (2\pi)^{-n} \int e^{i(x,\xi)} A(x,\xi)\hat{u}(\xi)d\xi \;, \; u \in \mathcal{S}(R^n),$$

where $A \in S^m_{\varrho,(\theta,\sigma)}(R^n)$.

Proposition 2.3. (a) If $A_j(x,D) \in L^{m_j}_{\varrho,(\theta_j,\sigma_j)}(R^n)$, $j = 1, 2$, then $A(x,D) = A_1(x,D)A_2(x,D) \in L^m_{\varrho,(\theta,\sigma)}(R^n)$, where $m = m_1 + m_2$, $\theta = \theta_1 + \theta_2$, $\sigma = \inf(\theta_1 + \sigma_2, \theta_2 + \sigma_1)$.

Moreover

$$A - \sum_{|\alpha|<k} (1/\alpha!)\partial_\xi^\alpha A_1 D_x^\alpha A_2 \in S^{m-\varrho k}_{\varrho,(\theta_1+\sigma_2,\theta_1+\sigma_2)}(R^n).$$

(b) If $A = A(x,D) \in L^m_{\varrho,(\theta,\sigma)}(R^n)$, then his formal adjoint A^* belongs to the same space of operators and if $A^* = A^*(x,D)$, then

$$A^*(x,\xi) - \sum_{|\alpha|<k} (1/\alpha!)\partial_\xi^\alpha D_x^\alpha \overline{A(x,\xi)} \in S^{m-\varrho k}_{\varrho,(\sigma,\sigma)}(R^n).$$

The most important facts we use in the sequel are the continuity properties of these operators. In order to state them it is convenient to introduce the following weighted Sobolev spaces:

$$H^s_\alpha(R^n) = \{ u \in \mathcal{S}'(R^n);\ (1+|x|^2)^{\alpha/2} u \in H^s(R^n) \},$$

which are Hilbert spaces under the norms:

$$\|u\|_{s,\alpha} = \|\varrho^\alpha u\|_s = \left(\int |(1+|\xi|^2)^{s/2} \widehat{\varrho^\alpha u}(\xi)|^2 d\xi \right)^{1/2},$$

where $\varrho : R^n \longrightarrow R$, $\varrho(x) = (1+|x|^2)^{1/2}$.

Proposition 2.4. (a) If $s_1 \geq s_2$, $\alpha_1 \geq \alpha_2$, then $H^{s_1}_{\alpha_1}(R^n)$ is continuously embedded in $H^{s_2}_{\alpha_2}(R^n)$.

(b) If $s_1 > s_2$, $\alpha_1 > \alpha_2$, then this embedding is compact.

Proposition 2.5. If $A = A(x,D) \in L^m_{\varrho,(\theta,\sigma)}(R^n)$ and $s, \alpha \in R$, then A has a unique extension to a bounded operator $A : H^s_\alpha(R^n) \longrightarrow H^{s-m}_{\alpha+\theta}(R^n)$.

A class of compact operators is given by the following:

Proposition 2.6. (a) Let $A(x,D) \in L^0_{\varrho,(0,0)}(R^n)$ such that

$$(2.1) \qquad \lim_{|x|+|\xi|\to\infty} A(x,\xi) = 0$$

Then $A(x,D)$ induces a compact linear map : $L^2(R^n) \longrightarrow L^2(R^n)$.

(b) $A(x,D) \in L^m_{\varrho,(\theta,\sigma)}(R^n)$ has a compact extension to an operator in $\mathcal{L}(H^s_\alpha(R^n), H^{s-m}_{\alpha+\theta}(R^n))$ if and only if $\varrho^{s-m}(D)\varrho^{\alpha+\theta}(x)A(x,D)\varrho^{-\alpha}(x)\varrho^{-s}(D)$ has a compact extension on $L^2(R^n)$.

(c) Let $A(x,D) \in L^m_{\varrho,(\theta,\sigma)}(R^n)$, $\theta < \sigma$, such that

$$\lim_{|x|+|\xi|\to\infty} \varrho^\theta(x)\varrho^{-m}(\xi)\, A(x,\xi) = 0$$

Then $A(x,D)$ induces a compact linear map : $H^s(R^n) \longrightarrow H^{s-m}_{\alpha+\theta}(R^n)$.

Proof. (a) If supp A(,) is compact, then A $L^{-\delta}_{\varrho,(\delta,\delta)}(R^n)$

for $\delta > 0$. In this case our statement follows from Proposition 2.4. and Proposition 2.5..

If supp $A(\, , \,)$ is not a compact set, we choose $\varphi \in C_0^\infty(R^{2n})$ such that $\varphi(x,\xi) = 0$ for $|x| + |\xi| \geq 2$ and $\varphi(x,\xi) = 1$ for $|x| + |\xi| \leq 1$. If we set $A_\varepsilon(x,\xi) = \varphi(\varepsilon x, \varepsilon \xi) A(x,\xi)$, it follows that $A_\varepsilon(x,D) : L^2(R^n) \longrightarrow L^2(R^n)$ is a compact operator for every $\varepsilon > 0$. So it suffices to prove that $A_\varepsilon(x,D) \longrightarrow A(x,D)$ when $\varepsilon \longrightarrow 0$ in the uniform operator topology of $\mathcal{L}(L^2(R^n))$.

In order to prove this, we assume for a moment that $A(\, , \,)$ satisfies:

$$(2.1)' \qquad \lim_{|x|+|\xi|\to\infty} |D_x^\alpha D_\xi^\beta A(x,\xi)| = 0 \, , \; |\alpha|, |\beta| \leq [n/2] + 1.$$

Since

$$\| A_\varepsilon(x,D) - A(x,D) \|_{\mathcal{L}(L^2(R^n))} \leq C \sup_{\substack{x,\xi \in R^n \\ |\alpha|,|\beta| \leq [n/2]+1}} |D_x^\alpha D_\xi^\beta (A - A_\varepsilon)(x,\xi)| \, ,$$

(Calderon-Vaillancourt's Theorem [5]), we need only to verify that:

$$\lim_{\varepsilon\to 0} \sup_{x,\xi\in R^n} |D_x^\alpha D_\xi^\beta (A - A_\varepsilon)(x,\xi)| = 0 \, , \; |\alpha|, |\beta| \leq [n/2] + 1 \, .$$

But this follows from

$$| D_x^\alpha D_\xi^\beta (A - A_\varepsilon)(x,\xi)| \leq C_{\alpha\beta} \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \beta_1+\beta_2=\beta}} |D_x^{\alpha_1} D_\xi^{\beta_1} (1 - \varphi(\varepsilon x, \varepsilon\xi))| \, |D_x^{\alpha_2} D_\xi^{\beta_2} A(x,\xi)| \, ,$$

$$D_x^{\alpha_1} D_\xi^{\beta_1} (1 - \varphi(\varepsilon x, \varepsilon \xi)) = 0 \, , \qquad |x| + |\xi| \leq 1/\varepsilon \, ,$$

$$\{ D_x^{\alpha_1} D_\xi^{\beta_1} (1 - \varphi(\varepsilon \cdot, \varepsilon \cdot)) \}_{0<\varepsilon\leq 1} \quad \text{is bounded and}$$

$$\lim_{|x|+|\xi|\to\infty} | D_x^{\alpha_2} D_\xi^{\beta_2} A(x,\xi)| = 0 \, .$$

To conclude the proof of the proposition we note that (2.1)' follows from (2.1) and the fact that for $u \in C_b^2(R^n)$ we have

$$\| D_j u \|_\infty^2 \leq 4 \|u\|_\infty \|D_j^2 u\|_\infty \, ,$$

where $C_b^2(R^n) = \{ u \in C^2(R^n) \; ; \; D^\alpha u \in L^\infty(R^n), |\alpha| \leq 2 \}$.

(b) is a consequence of Proposition 2.3. (a), and (c) follows from (a), (b), Proposition 2.3., 2.4. and 2.5..

In order to apply the results of [7] we need

Proposition 2.7. If $A(x,D) = P_0(D) + V(x,D)$, where $P_0(D)$ satisfies (A.1) and $V(x,D) \in L^m_{\rho,(\theta,\sigma)}(R^n)$, $\theta > 0$, is a compact operator from $H^m(R^n)$ to $L^2(R^n)$, then there exists $B(x,D) \in L^{-m}_{\rho,(0,\sigma)}(R$ such that

$$B(x,D)A(x,D) = I + R_1(x,D) ,$$

$$A(x,D)B(x,D) = I + R_2(x,D) ,$$

where $R_j(x,D) \in L^{-\infty}_{\rho,(0,\sigma)}(R^n)$, $j = 1, 2$.

Proof. As it is well known from the theory of pseudodifferential operators it is enough to prove that there are $R > 0$, $c > 0$ such that

$$(2.2) \qquad |P_0(\xi) + V(x,\xi)| \geq c|\xi|^m \; , |\xi| \geq R, \; x \in R^n .$$

Because $P_0(D)$ is an elliptic operator of order m, there are $R_1 > 0$, $c_1 > 0$ such that

$$|P_0(\xi)| \geq c_1 |\xi|^m \; , |\xi| \geq R_1 .$$

So (2.2) follows if we show that

$$\lim_{|\xi| \to \infty} \sup_{x \in \mathbb{R}^n} |V(x,\xi)| (1 + |\xi|)^{-m} = 0 .$$

Since $V(x,D) \in L^m_{\rho,(\theta,\sigma)}(R^n)$, there is a constant $C > 0$ such that

$$|V(x,\xi)| \leq C(1 + |x|)^{-\theta}(1 + |\xi|)^m .$$

But for every $\varepsilon > 0$ we can find a compact set $K_\varepsilon \subset R^n$ such that

$$C(1 + |x|)^{-\theta} < \varepsilon, \; x \in R^n \setminus K_\varepsilon .$$

On the other hand $V: H^m(R^n) \longrightarrow L^2(R^n)$ is compact, hence

$V(1+|D|^2)^{-m/2}: L^2(R^n) \longrightarrow L^2(R^n)$ is compact too. Theorem 1.1. chap.II from [6] implies that

$$\lim_{|\xi|\to\infty} \sup_{x\in K_\varepsilon} |V(x,\xi)|\,(1+|\xi|)^{-m} = 0 ,$$

which completes the proof.

This proposition permits to get the limiting absorption principle in the same way as in [7].

Theorem 2.8. If $H = P_0 + V$, where P_0 satisfies (A.1) and V satisfies (B), then

(a) Any eigenvalue of H, which does not belong to $\Lambda_c(P_0)$, has finite multiplicity. The only possible limit points of the point spectrum of H are in $\Lambda_c(P_0) \cup \{+\infty\}$.

(b) For every compact set $K \subset [c, \infty) \setminus (\sigma_{pp}(H) \cup \Lambda_c(P_0))$ and for every $\alpha > 1/2$, there exists a constant $C > 0$ such that

$$\|R(\lambda \pm ik)f\|_{m,-\alpha} \leqslant C\|f\|_{0,\alpha} , \quad f \in L^2_\alpha(R^n), \ \lambda \in K, \ k \in (0, 1] ,$$

where $R(\lambda \pm ik) = (H - (\lambda \pm ik))^{-1}$.

At the end of this section we state some results about smooth operators and one concerning the absence of the singular continuous spectrum. These results can be found in [8].

There are alot of equivalent definitions of H-smootheness. We give the following one:

Definition 2.9. Let A be a closed operator and $H = H^*$ in a separable Hilbert space .

(a) We say that A is H-smooth if $D(H) \subset D(A)$ and

$$\sup_{\mu\notin\mathbb{R},\ \|\varphi\|=1} \|AR(\mu)\varphi\|^2\, |\mathrm{Im}\,\mu| < \infty$$

(b) We say that A is H-smooth on Ω if AP_Ω is H-smooth, where

P_Ω is the spectral projection of H corresponding to the borelian set Ω.

Proposition 2.10. Let $H = H^*$ and $\Omega \subset R$ a borelian set. If A is H-bounded and

$$\sup_{0<k<1, \lambda\in\Omega} \| A(H - \lambda - ik)^{-1} A^* \| < \infty$$

then A is H-smooth on Ω.

Proposition 2.11. Let $H = H^*$, $H_0 = H_0^*$, B a H_0-bounded operator and A a H-bounded operator such that

$$(H\varphi, \psi) - (\varphi, H_0\psi) = (A\varphi, B\psi) \ , \quad \varphi \in D(H) \ , \quad \psi \in D(H_0) .$$

Let $S \subset R$ with $S = \bigcup_{i=1}^{\infty} \Omega_i$ and each Ω_i a bounded open interval. Suppose that:

(i) A is H-smooth on each Ω_i and B is H_0-smooth on each Ω_i.

(ii) Both $\sigma(H) \setminus S$ and $\sigma(H_0) \setminus S$ have Lebesgue measure zero.

Then the wave operators exist and they are complete.

Proposition 2.12. Let H be a self-adjoint operator and (a, b) a bounded interval. Suppose that there is a dense set D in $\mathcal{H}$ so that, for each $\varphi \in D$

$$\sup_{0<\varepsilon<1} \ \sup_{a<x<b} |(\varphi, R(x + i\varepsilon)\varphi)| < \infty$$

Then $\operatorname{Ran} P_{(a, b)} \subset \mathcal{H}_{ac}$.

3. The proof of Theorem 1.2.

We shall prove this theorem in several steps. Let $V(x,D)$ be a pseudodifferential operator that satisfies (B). It follows that the operator $P(x,D) = P_0(D) + V(x,D)$ belongs to $L^m_{\varrho,(0,\sigma)}(R^n)$, $\sigma > 1$. So it has extensions in $\mathcal{L}(H^s_\alpha(R^n), H^{s-m}_\alpha(R^n))$, α, $s \in R$. All these extensions are obtained from the operator $P(x,D) : \mathcal{S}'(R^n) \longrightarrow \mathcal{S}'(R^n)$.

In order to avoid any ambiguity we denote by $P_{s,\alpha}$ the extension of $P(x,D)$ in $\mathcal{L}(H^s_\alpha(R^n), H^{s-m}_\alpha(R^n))$. We can now state and prove the following lemma:

Lemma 3.1. For each $\alpha \in [-1, 1]$ the operators $P_{m/2,\alpha} \pm i$ are invertible.

Proof. Consider first the case $\alpha \in [0, 1]$.

We prove that $P_{m/2,\alpha} - i$ is one-to-one, $\mathrm{Ran}(P_{m/2,\alpha} - i)$ is closed and $\mathrm{Ran}(P_{m/2,\alpha} - i)$ is dense in $H^{-m/2}_\alpha(R^n)$.

Let $u \in H^{m/2}_\alpha(R^n)$ and $f \in H^{-m/2}_\alpha(R^n)$ be such that

$$(3.1) \qquad P(x,D)u - iu = f .$$

Let $B(x,D) \in L^{-m}_{\varrho,(0,\sigma)}(R^n)$ be the parametrix of $P(x,D) - i$ given by Proposition 2.7.. Applying $B(x,D)$ to (3.1) we get

$$(3.2) \qquad u = B(x,D)f - R(x,D)u \, , \quad R(x,D) \in L^{-\infty}_{\varrho,(0,\sigma)}(R^n) .$$

From the continuity properties of the pseudodifferential operators (Proposition 2.5.) it follows that

$$(3.3) \qquad \| u \|_{m/2,0} \leq C(\|u\|_{0,0} + \|f\|_{-m/2,0})$$

On the other hand, by identifying $H^{-m/2}(R^n)$ with the dual space of $H^{m/2}(R^n)$, we have

$$(Pu, u) - i(u, u) = (f, u) .$$

Since $P(x,D)$ is a formally self-adjoint operator we have that $(Pu, u) \in R$ for $u \in H^{m/2}(R^n)$.

Hence

$$\|u\|_{0,0}^2 \leqslant \|f\|_{-m/2,0} \|u\|_{m/2,0} \leqslant \varepsilon \|u\|_{m/2,0}^2 + (1/4\varepsilon) \|f\|_{-m/2,0}^2 .$$

From this and from (3.3) we conclude that

$$(3.4) \qquad \|u\|_{m/2,0} \leqslant C \|f\|_{-m/2,0} .$$

Let $v = \varrho^{\alpha} u$; then $v \in H^{m/2}(R^n)$ and verifies

$$(P(x,D) - i)v = \varrho^{\alpha} f + [P(x,D), \varrho^{\alpha}]u .$$

Because $[P(x,D), \varrho^{\alpha}] \in L^{m-1}_{\varrho,(1-\alpha,1-\alpha)}(R^n)$, it has a bounded extension defined on $H^{m/2}(R^n)$ with values in $H^{-m/2}(R^n)$. Hence

$$g = \varrho^{\alpha} f + [P(x,D), \varrho^{\alpha}]u \in H^{-m/2}(R^n)$$

and

$$\|g\|_{-m/2,0} \leqslant \|f\|_{-m/2,\alpha} + C\|u\|_{-m/2,0} .$$

From this estimate and from (3.4) we deduce that

$$\|u\|_{m/2,\alpha} = \|v\|_{m/2,0} \leqslant C\|g\|_{-m/2,0} \leqslant C_1 (\|f\|_{-m/2,\alpha} + \|u\|_{-m/2,0}) ,$$

which implies that

$$(3.5) \qquad \|u\|_{m/2,\alpha} \leqslant C\|f\|_{-m/2,\alpha} = C\|(P(x,D) - i)u\|_{-m/2,\alpha}$$

for $u \in H^{m/2}_{\alpha}(R^n)$.

This last estimate assures the fact that $P_{m/2,\alpha} - i$ is a one-to-one map and that it has closed range in $H^{-m/2}_{\alpha}(R^n)$.

Now we prove that $\mathrm{Ran}(P_{m/2,\alpha} - i)$ is dense in $H^{-m/2}_{\alpha}(R^n)$. We know that $P = P_0 + V$ is a self-adjoint operator on $L^2(R^n)$ with $D(P) = H^m(R^n)$. Hence $(P(x,D) - i)H^m(R^n) = L^2(R^n)$. Consequently for every $f \in L^2_{\alpha}(R^n)$ there exists $u \in H^m(R^n)$ so that

$(P(x,D) - i)u = f$. The proof is complete if we show that $u \in H^m_\alpha(R^n)$.

As before it follows that

$$(3.6) \qquad \|u\|_{m,0} \leq C\|f\|_{0,0} .$$

Setting again $v = \varrho^\alpha u$, we prove that $v \in H^m(R^n)$. We have

$$(P(x,D) - i)v = g , \quad g = [P, \varrho^\alpha]u + f .$$

We know that $g \in L^2(R^n)$ and that

$$\|g\|_{0,0} \leq C\|f\|_{0,\alpha} .$$

Let $v_\varepsilon = \psi_\varepsilon v$, where $\psi_\varepsilon(x) = \psi(\varepsilon x)$, $\psi \in C_0^\infty(R^n)$, $\psi(x) = 1$ for $|x| \leq 1$. Since $v \in H^m_{-\alpha}(R^n) \subset H^m_{loc}(R^n)$, it follows that $v_\varepsilon \in H^m(R^n)$ for every $\varepsilon > 0$. Also v_ε satisfies

$$(P(x,D) - i)v_\varepsilon = h_\varepsilon , \quad h_\varepsilon = \psi_\varepsilon g + [P, \psi_\varepsilon]v .$$

Because $\psi \in L^0_{\varrho,(0,1)}(R^n)$ uniformly with respect to ε , we have that $[P, \psi_\varepsilon] \in L^{m-1}_{\varrho,(1,1)}(R^n)$ uniformly with respect to ε . Hence

$$\|h_\varepsilon\|_{0,0} \leq C(\|v\|_{m,-1} + \|g\|_{0,0}) \leq C_1(\|f\|_{0,\alpha} + \|u\|_{m,0}) \leq C_2\|f\|_{0,\alpha} .$$

The estimate (3.6), with v_ε insted of u, gives

$$\|v_\varepsilon\|_{m,0} \leq C\|f\|_{0,\alpha} .$$

Then $\{v_\varepsilon\}_{0<\varepsilon\leq 1}$ is a bounded set in $H^m(R^n)$. Therefore it contains a sequence $\{v_{\varepsilon_j}\}_j$, $\lim_{j\to\infty} \varepsilon_j = 0$, which is weakly convergent to a function $w \in H^m(R^n)$. But $\lim_{\varepsilon\to 0} v_\varepsilon = v$ in $\mathcal{D}'(R^n)$, hence $v = w \in H^m(R^n)$, and the proof is complete in the case $\alpha \in [0, 1]$.

The case $\alpha \in [-1, 0)$ follows by duality.

<u>Remark 3.2.</u> If $f \in L^2(R^n) \cap H^{-m/2}_\alpha(R^n)$, then

$$(P_{m,0} - i)^{-1}f = (P_{m/2,\alpha} - i)^{-1}f$$

We prove the existence and completeness of the wave operators $W_{\pm}(P,P_1)$ by means of Proposition 2.11.. Let $P = P_0 + V_S + V_L$, $P_1 = P_0 + V_L$, P_0, V_L and V_S be as in Theorem 1.2. (a) . We can write

$$(Pu, v) - (u, P_1 v) = (V_S u, v) , \quad u, v \in H^m(R^n) .$$

We introduce the following operators: $B = B(D) =$ $= (1 + |D|^2)^{-m/4}$ and $A = A(x) = \rho(x)^{\theta/2}$. Then we have

$$(V_S u, v) = (BAV_S u, B^{-1}A^{-1} v) , \quad u, v \in H^m(R^n) .$$

Since $BAV_S \in L^{m/2}_{\rho,(\theta/2,\sigma_1)}(R^n)$, $\sigma_1 = \inf(\theta/2 + 1, \sigma - \theta/2)$ and $B^{-1}A^{-1} \in L^{m/2}_{\rho,(\theta/2,\theta/2 + 1)}(R^n)$ we have that BAV_S (resp. $B^{-1}A^{-1}$) is P (resp. P_1) bounded.

Let $[a, b] \subset [c, \infty) \setminus (\sigma_{pp}(P) \cup \sigma_{pp}(P_1) \cup \Lambda_c(P_0))$ be a compact interval. Now we can prove

<u>Lemma 3.3.</u> The operator BAV_S (resp. $B^{-1}A^{-1}$) is P (resp. P_1) smooth on $[a, b]$.

<u>Proof.</u> We prove first that $B^{-1}A^{-1}$ is P_1-smooth on $[a, b]$. For this purpose we use the criterion given by Proposition 2.10.. We must show that

$$\sup_{\lambda \in [a,b]} \sup_{k \in (0,1]} \| B^{-1}A^{-1}(P_1 - \lambda - ik)^{-1}(B^{-1}A^{-1})^* \| < \infty .$$

Because $A^{-1}B^{-1} \subset (B^{-1}A^{-1})^*$ it is enough to prove that

$$\sup_{\lambda \in [a,b]} \sup_{k \in (0,1]} \| B^{-1}A^{-1}(P_1 - \lambda - ik)^{-1}A^{-1}B^{-1} \| < \infty .$$

We use the identity

$$(P_1 - \lambda - ik)^{-1} = (P_1 - i)^{-1} + (\lambda - ik - i)(P_1 - i)^{-2} + $$
$$+ (\lambda - ik - i)^2 (P_1 - i)^{-1}(P_1 - \lambda - ik)^{-1} \cdot$$
$$\cdot (P_1 - i)^{-1}$$

The definition domain of the operator $B^{-1}A^{-1}(P_1 - i)^{-1}A^{-1}B^{-1}$ is $H^{m/2}(R^n)$ and on this set this operator is equal to $B^{-1}A^{-1}(P_{1\ m/2,0} - i)^{-1}A^{-1}B^{-1}$. It follows that $B^{-1}A^{-1}(P_1 - i)^{-1}A^{-1}B^{-1}$ is a bounded operator. In the same manner it can be proved that $B^{-1}A^{-1}(P_1 - i)^{-2}A^{-1}B^{-1}$ is a bounded operator.

Also $B^{-1}A^{-1}(P_1 - i)^{-1}(P_1 - \lambda - ik)^{-1}(P_1 - i)^{-1}A^{-1}B^{-1}$ is equal to $B^{-1}A^{-1}(P_{1\ m/2,-\theta/2} - i)^{-1}(P_1 - \lambda - ik)^{-1}(P_{1\ m/2,\theta/2} - i)^{-1}A^{-1}B^{-1}$ on $H^{m/2}(R^n)$.

We have the diagram

$$L^2 \xrightarrow{B^{-1}} H^{-m/2} \xrightarrow{A^{-1}} H^{-m/2}_{\theta/2} \xrightarrow{(P_1-i)^{-1}} L^2_{\theta/2} \xrightarrow[\text{l.a.p.}]{(P_1 - -ik)^{-1}} H^m_{-\theta/2} \hookrightarrow$$

$$\hookrightarrow H^{-m/2}_{-\theta/2} \xrightarrow{(P_1-i)^{-1}} H^{m/2}_{-\theta/2} \xrightarrow{A^{-1}} H^{m/2} \xrightarrow{B^{-1}} L^2 .$$

where all the operators are bounded and $(P_1 - \lambda - ik)^{-1}: L^2_{\theta/2} \longrightarrow H^m_{-\theta/2}$ is uniformly bounded with respect to $\lambda \in [a, b]$ and $k \in (0, 1]$ (Theorem 2.8.).

Therefore $\{B^{-1}A^{-1}(P_1 - i)^{-1}(P_1 - \lambda - ik)^{-1}(P_1 - i)^{-1}A^{-1}B^{-1}\}_{\lambda,k}$ is a family of uniformly bounded operators with respect to $\lambda \in [a, b]$ and $k \in (0, 1]$.

In order to prove that BAV_S is P-smooth on $[a, b]$, we remark that $BAV_S = BAV_S ABB^{-1}A^{-1} = V_1 B^{-1}A^{-1}$, where $V_1 = BAV_S AB$ is a bounded operator. From this it follows that

$$\|BAV_S(P - \lambda - ik)^{-1}(BAV_S)^*\| \leq \|V_1\|^2 \|B^{-1}A^{-1}(P - \lambda - ik)^{-1}A^{-1}B^{-1}\|$$

which, according to the first part of the proof accomplishes our proof.

The proof of Theorem 1.2.

(a) Let $N = \sigma_{pp}(P) \cup \sigma_{pp}(P_1) \cup \Lambda_c(P_0)$ which is a closed set and which has Lebesgue measure zero. We take $S = [c, \infty) \setminus N = \bigcup_{j=1}^{\infty} \Omega_j$, where Ω_j are bounded open intervals with the closure also contained in S. Lemma 3.3. ensures that (i) from Proposition 2.11. is satisfied. Since $\sigma_{ess}(P) = \sigma_{ess}(P_1) = [c, \infty)$, it follows that the sets $\sigma(P) \setminus S$ and $\sigma(P_1) \setminus S$ have Lebesgue measure zero. Hence all the hypothesis of Proposition 2.11. are satisfied.

(b) It is enough to prove that $\sigma_{sing}(P) = \emptyset$. Theorem 2.8. and Proposition 2.12. imply that $\sigma_{sing}(P) \subset \sigma_{pp}(P) \cup \Lambda_c(P_0)$. But from the assumption (A.2) this is at most a countable set, so that $\sigma_{sing}(P) = \emptyset$.

4. An example

We are going to apply the results obtained in the previous sections to the perturbed Hamiltonian of a relativistic particle with spin-zero.

In this case the operators are given by the following symbols

$$P_0(\xi) = (m^2 + |\xi|^2)^{1/2}, \quad m > 0,$$

$$P_1(x,\xi) = (m^2 + |\xi - A(x)|^2 + V(x))^{1/2},$$

$$P(x,\xi) = P_1(x,\xi) + W(x),$$

where A, V and W satisfy :

(i) $A \in C^{\infty}(R^n; R^n)$, $V \in C^{\infty}(R^n; R)$ and there exist constants θ, C_0, $C_\alpha > 0$ such that:

$$|A(x)|,\ |V(x)| \leq C_0(1+|x|)^{-\theta}\ ,\ \forall x \in R^n,$$

$$|D^\alpha A(x)|,\ |D^\alpha V(x)| \leq C_\alpha(1+|x|)^{-(1+\theta)}\ ,\ \forall x \in R^n,\ |\alpha| \neq 0\ .$$

(ii) $W \in C^\infty(R^n;R)$ and there exist constants $\sigma > 1$, $C_\alpha > 0$ such that

$$|D^\alpha W(x)| \leq C_\alpha(1+|x|)^{-\sigma}, \forall x \in R^n, \forall \alpha \in N^n.$$

(iii) $m^2 + |\xi - A(x)|^2 + V(x) > 0\ ,\ \forall\ x \in R^n, \forall \xi \in R^n.$

Remark 4.1. 1) Under the hypothesis (i) and (iii) we have

(iii)′ $m^2 + |\xi - A(x)|^2 + V(x) \geq C(1+|\xi|^2)^{1/2}, \forall x, \xi \in R^n$

for a positive constant C.

2) Using (i) and (iii)′ it follows that

$$(m^2 + |\xi - A(x)|^2 + V(x))^{1/2} \in S^1_{1,(0,1+\theta)}(R^n)$$

and that

$$(P_1(x,\xi) + P_0'(\xi))^{-1} \in S^{-1}_{1,(0,1+\theta)}(R^n).$$

We have now that

$$V_L(x,\xi) = P_1(x,\xi) - P_0(\xi) =$$

$$= (-2\langle A(x),\xi\rangle + |A(x)|^2 + V(x))/(P_0(\xi) + P_1(x,\xi))$$

is in $S^0_{1,(\theta,1+\theta)}(R^n)$.

It is easy now to see that all the assumptions in the introduction are fullfiled if we take $V_S(x,\xi) = W(x)$. (That V_L and V_S are compact operators from $H^1(R^n)$ to $L^2(R^n)$ results from Proposition 2.5.) So we can apply Theorem 1.2. to these operators.

Remark 4.2. 1) These results are valid also when there exists a constant $A \in R^n$ such that $A(x) - A$ verifies (i).

2) We can proceed in a similar manner if:

$$P_0(\xi) = (m^2 + |\xi|^2)^{1/2}\ ,\ m > 0\ ,$$

$$P_1(x,\xi) = (m^2 + |\xi - A(x)|^2)^{1/2} + W(x) ,$$

$$P(x,\xi) = (m^2 + |\xi - A(x)|^2 + V(x))^{1/2} + W(x) .$$

Here $A(x)$ and $W(x)$ satisfy (i), $V(x)$ satisfies (ii) and

$$m^2 + |\xi - A(x)|^2 + V(x) > 0 , \quad \forall x, \xi \in R^n .$$

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