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Dan TIBA

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*) The National Institute for Scientific and Technical Creation, Department of Mathematics, Bd.Pacii 220, 79622 Bucharest, Romania

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INTRODUCTION. We consider the unconstrained distributed control problem:

(P) Minimize $L(y,u)$

subject to

(1.1) $y_{tt} - \Delta y + f(y, \nabla y, y_t) = u$ in Q,

(1.2) $y(0,x) = y_0(x)$, $y_t(o,x) = v_0(x)$ in Ω

(1.3) $y(t,x) = 0$ in Σ

Here Ω is a bounded domain in the Euclidean space R^n, $Q = \Omega \times]o,T[$, $\Sigma = \partial\Omega \times]0,T[$ and $u \in L^2(Q)$, $y_0 \in H_0^1(\Omega)$, $v_0 \in L^2(\Omega)$. Function $f: R^{n+2} \to R$ satisfies the growth condition

(1.4) $\forall \delta > 0\ \exists C > 0$:

$|f(y) - f(z)| \leq C(1 + |f(y)|) \cdot |y - z|$ for $|y - z| < \delta$ where $y,z \in R^{n+2}$ and $|\cdot|$ stands for the modulus or the Euclidean norm, as appropriate. Similar conditions were used by F.H.Clarke [7], V.Barbu [2], V.Komornik and D.Tiba [9] previously. They allow a large class of examples including polynomials and exponentials.

We assume that $L: L^2(Q) \times L^2(Q) \to R$ is convex, continuous and continuously differentiable with respect to u.

The main result of this paper gives first order optimality conditions for problem (P) and will be stated and proved in section 2. It extends some results of Tiba [12] §4, V.Komornik and D.Tiba [9], J.F.Bonnans [4] Ch.III. In one space dimension, by the method of characteristics M.Brokate [5] obtains necessary conditions for a more general problem.

However, our approach is different and may be applied to a large class of problems.

It is based on arguments both from the theory of optimal control for variational inequalities V.Barbu [1], D.Tiba [12] and the theory of singular control problems as developped by J.L.lions [8].

As a general remark, we don't need imbedding theorems of Sobolev type and therefore we have no conditions on the dimension of Ω. We use only the simplest existence and regularity results for the linear hyperbolic equations and this explains the many possibilities of aplication of the method.

By (1.4) f is locally Lipschitz on R^{n+2} and we express the optimality conditions for (P) by means of the Clarke [6] generalized gradient of f, denoted Df.

In the last section we collect several technical results used throughout the proof.

2. THE MAIN RESULT

As it is wellknown, the state system (1.1)-(1.3) may be not well posed for such a general nonlinear term. See Lions [8], Ch.2, § 1.4 for a thorough discussion of an example when $f(y, \nabla y, y_t) = f(y)$.

We define the pair [y,u] to be admissible for (P) if $u \in L^2(Q)$, $f(y, \nabla y, y_t) \in L^2(Q)$ and y is a solution of (1.1)-(1.3).

The existence of admissible pairs may be justified in special cases and in this respect we quote Tiba [10] §4, [12] §4, Brokate [5] §5 (well posed problems) and Lions [8] Ch.2, Bonnans [4] Ch.3, Tiba [11] (unstable systems).

We assume the existence of an optimal pair $[y^*, u^*]$ which achieves the infimum in (P).

THEOREM 2.1. <u>There is</u> $p^* \in L^2(Q)$ <u>and</u> $\gamma^* \in L^2(Q)$, $\gamma^* \in Df(y^*, \nabla y^*, y_t^*)$ <u>such that the following optimality conditions are satisfied:</u>

(2.1) $-p^* = \partial_2 L(y^*, u^*)$

(2.2) $\langle p^*, \xi_{tt} - \Delta \xi + [\gamma^*, (\xi, \nabla \xi, \xi_t)] \rangle = \langle \partial_1 L(y^*, u^*), \xi \rangle$ <u>for all</u>

$\xi \in C^2(\bar{Q})$, $\xi(0,x) = \xi_t(0,x) = 0$ in Ω and $\xi(t,x) = 0$ on Σ.

Above we denote by $\langle .,. \rangle$ the inner product in $L^2(Q)$ and by $[.,.]$ the inner product in R^{n+2}.

$\partial_1 L(y^*,u^*)$, $\partial_2 L(y^*,u^*) \in L^2(Q)$ are the two components of $\partial L(y^*,u^*)$, the subdifferential of L.

In order to prove Theorem 2.1. we consider the approximate optimization problem (P_ε), $\varepsilon > 0$:

(P_ε) Minimize

$$L_\varepsilon(y,u) + 1/2 \,|\, f^\varepsilon(y, \nabla y, y_t) - f(y^*, \nabla y^*, y_t^*)|^2_{L^2(Q)} + 1/2 \,|\, u - u^* |^2_{L^2(Q)} +$$
$$1/2 \,|\, y - y^* |^2_{H^2(Q)} + 1/2\varepsilon \,|\, y_{tt} - \Delta y + f^\varepsilon(y, \nabla y, y_t) - u \,|^2_{L^2(Q)}$$

Here L_ε and f^ε are regularizations of functions L,f given by:

$$(2.3)\ L_\varepsilon(y,u) = \inf\left\{ |(z,v) - (y,u)|^2_{L^2(Q)}/2\varepsilon + L(z,v) \right\},$$

$$(2.4)\ f^\varepsilon(z) = \int_{R^{n+2}} f(z - \varepsilon\tau)\rho(\tau)d\tau, \qquad z \in R^{n+2}$$

with $\rho \in C_0(R^{n+2})$, supp $\rho \subset S(0,1)$, $\int_{R^{n+2}} \rho(\tau)d\tau = 1$,
$\rho \geq 0$, $\rho(-\tau) = \rho(\tau)$.

This is a variant of the so called "adapted penalization" method; the idea to penalize the nonlinear term originates from the theory of singular control problems, Lions [8], while the penalization of $y - y^*$ in higher order Sobolev spaces was previously used by Barbu [2].

A pair $[y,u]$ is called ε-admissible if $u \in L^2(Q)$, $y - y^* \in H^2(Q)$, $f^\varepsilon(y, \nabla y, y_t) \in L^2(Q)$, $y(0,x) = y_0(x)$ and $y_t(0,x) = v_0(x)$ in Ω, $y(t,x) = 0$ on Σ.

PROPOSITION 2.2. The pair $[y^* + z, v]$ is ε-admissible for any $v \in L^2(Q)$, $z \in C^2(\bar{Q})$, $z(0,x) = z_t(0,x) = 0$ in Ω, $z(t,x) = 0$ on Σ.

PROOF.

We have to show that $f^\varepsilon(y^* + z, \nabla y^* + \nabla z, y_t^* + z_t) \in L^2(Q)$ for all z with the above properties:

$$|\, f^\varepsilon(y^* + z, \nabla y^* + \nabla z, y_t^* + z_t)\,|$$
$$\leq \int_{S(0,1)} |f(w - \varepsilon\tau)|\,\rho(\tau)d\tau \leq |f(y^*, \nabla y^*, y_t^*)| +$$
$$+ \int_{S(0,1)} |f(w - \varepsilon\tau) - f(y^*, \nabla y^*, y_t^*)|\,\rho(\tau)d\tau$$

where w denotes $(y^* + z, \nabla y^* + \nabla z, y_t^* + z_t)$.

By (1.4) we infer

$$|f^{\varepsilon}(y^* + z, \nabla y^* + \nabla z, y_t^* + z_t)| \le |f(y^*, \nabla y^*, y_t^*)| + C\int_{S(0,1)} (1 + |f(y^*, \nabla y^*, y_t^*)|)\,|(z, \nabla z, z_t) - \varepsilon\tau|\,\rho(\tau)\,d\tau$$

since the term $(z, \nabla z, z_t) - \varepsilon\tau$ is uniformly bounded on Q.

Finally, we get

$$|f^{\varepsilon}(y^* + z, \nabla y^* + \nabla z, y_t^* + z_t)| \le C_1(1 + |f(y^*, \nabla y^*, y_t^*)|)$$

and the proof is finished.

PROPOSITION 2.3. (P_ε) has at least one optimal pair $[y_\varepsilon, u_\varepsilon]$

PROOF.

Let $[y_k, u_k]$ be a minimizing sequence for (P_ε). Then $\{u_k\}$, $\{f^{\varepsilon}(y_k, \nabla y_k, y_{kt})\}$ are bounded in $L^2(Q)$ and $\{y_k - y^*\}$ is bounded in $H^2(Q)$. On a subsequence, we have:

$u_k \to \tilde{u}$ weakly in $L^2(Q)$,

$y_k \to \tilde{y}$ strongly in $L^2(Q)$,

$\nabla y_k \to \nabla\tilde{y}$ strongly in $L^2(Q)$,

$y_{kt} \to \tilde{y}_t$ strongly in $L^2(Q)$,

$f^{\varepsilon}(y_k, \nabla y_k, y_{kt}) \to \tilde{f}$ weakly in $L^2(Q)$.

To identify $\tilde{f}$ we reason as follows. For any $\eta > 0$, there is $Q_\eta \subset Q$ measurable, meas $(Q - Q_\eta) < \eta$ and $(y_k, \nabla y_k, y_{kt}) \to (\tilde{y}, \nabla\tilde{y}, \tilde{y}_t)$ uniformly on Q_η, by the Egorov theorem. Next (2.4) gives $f^{\varepsilon}(y_k, \nabla y_k, y_{kt}) \to f^{\varepsilon}(\tilde{y}, \nabla\tilde{y}, \tilde{y}_t)$ uniformly on Q_η, that is

$$\tilde{f} = f^{\varepsilon}(\tilde{y}, \nabla\tilde{y}, \tilde{y}_t) \text{ a.e.}\, Q.$$

Obviously $[\tilde{y}, \tilde{u}]$ is an ε - admissible pair and, by the weak lower semicontinuity of the norm, it is ε - optimal. We denote it $[y_\varepsilon, u_\varepsilon]$.

LEMMA 2.4. For $\varepsilon \to 0$ we have

(2.5) $u_\varepsilon \to u^*$ strongly in $L^2(Q)$,

(2.6) $y_\varepsilon - y^* \to 0$ strongly in $H^2(Q)$,

(2.7) $y_{\varepsilon t} \to y_t^*$ strongly in $L^2(Q)$,

(2.8) $\nabla y_\varepsilon \to \nabla y^*$ strongly in $L^2(Q)^n$, (2.9) $f^{\varepsilon}(y_{\varepsilon}, \nabla y_{\varepsilon}, y_{\varepsilon t}) \to f(y^*, \nabla y^*, y^*_t)$ strongly in $L^2(Q)$.

PROOF.

Let J_{ε} be the cost functional associated with (P_{ε}).

$$J_{\varepsilon}(y_{\varepsilon}, u_{\varepsilon}) \leq J_{\varepsilon}(y^*, u^*) = L_{\varepsilon}(y^*, u^*) + 1/2 \left| f^{\varepsilon}(y^*, \nabla y^*, y^*_t) - f(y^*, \nabla y^*, y^*_t) \right|^2_{L^2(Q)} + 1/2\varepsilon \left| y^*_{tt} - \Delta y^* + f^{\varepsilon}(y^*, \nabla y^*, y^*_t) - u^* \right|^2_{L^2(Q)} = L_{\varepsilon}(y^*, u^*) + 1/2(1 + 1/\varepsilon) \left| f^{\varepsilon}(y^*, \nabla y^*, y^*_t) - f(y^*, \nabla y^*, y^*_t) \right|^2_{L^2(Q)} \leq L_{\varepsilon}(y^*, u^*) + C_1(1 + 1/\varepsilon)\, \varepsilon^2 (1 + \left| f(y^*, \nabla y^*, y^*_t) \right|^2_{L^2(Q)})$$

by Lemma 3.1 from the Appendix.

Therefore:

(2.10) $\limsup_{\varepsilon \to 0} J_{\varepsilon}(y_{\varepsilon}, u_{\varepsilon}) \leq L(y^*, u^*)$.

We obtain the estimates:

(2.11) $\{u_{\varepsilon}\}$ bounded in $L^2(Q)$,

(2.12) $\{y_{\varepsilon} - y^*\}$ bounded in $H^2(Q)$,

(2.13) $\{f^{\varepsilon}(y_{\varepsilon}, \nabla y_{\varepsilon}, y_{\varepsilon t})\}$ bounded in $L^2(Q)$,

(2.15) $\{L_{\varepsilon}(y_{\varepsilon}, u_{\varepsilon})\}$ bounded,

(2.16) $y_{\varepsilon tt} - \Delta y_{\varepsilon} + f^{\varepsilon}(y_{\varepsilon}, \nabla y_{\varepsilon}, y_{\varepsilon t}) - u_{\varepsilon} \to 0$ strongly in $L^2(Q)$.

By the properties of L_{ε} and (2.15) we see:

$$\varepsilon/2 \left| \partial L_{\varepsilon}(y_{\varepsilon}, u_{\varepsilon}) \right|^2_{L^2(Q) \times L^2(Q)} + L((I + \varepsilon \partial L)^{-1}(y_{\varepsilon}, u_{\varepsilon})) \leq C.$$

Since L is bounded from below by an affine function and $\{y_{\varepsilon}\}$, $\{u_{\varepsilon}\}$ are bounded, it yields

$$\varepsilon^{1/2} \left| \partial L_{\varepsilon}(y_{\varepsilon}, u_{\varepsilon}) \right|_{L^2(Q) \times L^2(Q)} \leq C$$

and we conclude that

$$[y_{\varepsilon}, u_{\varepsilon}] - (I + \varepsilon \partial L)^{-1}(y_{\varepsilon}, u_{\varepsilon}) = \varepsilon \partial L_{\varepsilon}(y_{\varepsilon}, u_{\varepsilon}) \to 0$$

strongly in $L^2(Q) \times L^2(Q)$.

Denote by $[\tilde{y}, \tilde{u}]$ the limit of $[y_{\varepsilon}, u_{\varepsilon}]$. Then

(2.17) $\lim_{\varepsilon \to 0} (I + \varepsilon \partial L)^{-1}(y_{\varepsilon}, u_{\varepsilon}) = [\tilde{y}, \tilde{u}]$

in the same topology (strong - weak) of $L^2(Q)^2$.

By an argument similar to the proof of P2.3. we see that $f^{\varepsilon}(y_{\varepsilon}, \nabla y_{\varepsilon}, y_{\varepsilon t}) \to f(\tilde{y}, \nabla\tilde{y}, \tilde{y}_t)$ weakly in $L^2(Q)$. Therefore $[\tilde{y},\tilde{u}]$ is an admissible pair for (P) and by (2.17) we infer (2.18) $\liminf_{\varepsilon \to 0} J_{\varepsilon}(y_{\varepsilon}, u_{\varepsilon}) \geq L(\tilde{y},\tilde{u}) + 1/2 |\tilde{u} - u^*|^2_{L^2(Q)} + 1/2 |\tilde{y} - y^*|^2_{H^2(Q)}$.

Taking into account (2.10) and the optimality of $[y^*, u^*]$, we get $\tilde{y} = y^*$, $\tilde{u} = u^*$. From (2.10), (2.18) we remark that

$$L_{\varepsilon}(y^*, u^*) + C \cdot \varepsilon \geq J_{\varepsilon}(y_{\varepsilon}, u_{\varepsilon}) \geq L((I + \varepsilon \partial L)^{-1}(y_{\varepsilon}, u_{\varepsilon})) +$$
$$+ 1/2 |f^{\varepsilon}(y_{\varepsilon}, \nabla y_{\varepsilon}, y_{\varepsilon t}) - f(y^*, \nabla y^*, y^*_t)|^2_{L^2(Q)} +$$
$$+ 1/2 |u_{\varepsilon} - u^*|^2_{L^2(Q)} + 1/2 |y_{\varepsilon} - y^*|^2_{H^2(Q)}.$$

Then:

$$1/2 |f^{\varepsilon}(y_{\varepsilon}, \nabla y_{\varepsilon}, y_{\varepsilon t}) - f(y^*, \nabla y^*, y^*_t)|^2_{L^2(Q)} +$$
$$+ 1/2 |u_{\varepsilon} - u^*|^2_{L^2(Q)} + 1/2 |y_{\varepsilon} - y^*|^2_{H^2(Q)} \leq C \cdot \varepsilon + L_{\varepsilon}(y^*, u^*) -$$
$$- L((I + \varepsilon \partial L)^{-1}(y_{\varepsilon}, u_{\varepsilon}))$$

and the Lemma 2.4. is proved.

LEMMA 2.5. There is $p_{\varepsilon} \in L^2(Q)$ such that the approximate optimality system is satisfied:

$$(2.19)\ -p_{\varepsilon} = \partial_2 L_{\varepsilon}(y_{\varepsilon}, u_{\varepsilon}) + u_{\varepsilon} - u^*$$

$$(2.20)\ \langle p_{\varepsilon}, \xi_{tt} - \Delta \xi + [\nabla f^{\varepsilon}(y_{\varepsilon}, \nabla y_{\varepsilon}, y_{\varepsilon t}), (\xi, \nabla \xi, \xi_t)] \rangle =$$
$$= \langle f^{\varepsilon}(y_{\varepsilon}, \nabla y_{\varepsilon}, y_{\varepsilon t}) - f(y^*, \nabla y^*, y^*_t), [\nabla f^{\varepsilon}(y_{\varepsilon}, \nabla y_{\varepsilon}, y_{\varepsilon t}),$$
$$(\xi, \nabla \xi, \xi_t)] \rangle + \langle \partial_1 L_{\varepsilon}(y_{\varepsilon}, u_{\varepsilon}), \xi \rangle + \langle y_{\varepsilon} - y^*, \xi \rangle_{H^2},$$

for all $\xi \in C^2(\bar{Q})$, $\xi(0,x) = \xi_t(0,x) = 0$ on Ω and $\xi(t,x) = 0$ on Σ.

Here we use the notations of Theorem 2.1. We denote by $\langle \cdot, \cdot \rangle_{H^2}$ the inner product in $H^2(Q)$ and by ∇f^{ε} the gradient of $f^{\varepsilon}(.)$ as a function on R^{n+2}.

PROOF.

We take

$$p_{\varepsilon} = -1/\varepsilon (y_{\varepsilon tt} - \Delta y_{\varepsilon} + f^{\varepsilon}(y_{\varepsilon}, \nabla y_{\varepsilon}, y_{\varepsilon t}) - u_{\varepsilon}).$$

To obtain (2.19) one has to compute the subdifferential of J_{ε} with respect to u and to use the minimum property of u_{ε}.

By Proposition 2.2. the pairs $[y_\varepsilon + s\xi, u_\varepsilon]$ are ε-admissible for all $s \in R$ and we infer (2.20) from

$$o = \lim_{s \to 0} J_\varepsilon(y_\varepsilon + s\xi, u_\varepsilon) - J_\varepsilon(y_\varepsilon, u_\varepsilon)/s$$

PROOF OF THEOREM 2.1.

By the assumption on L and (2.19) we see that p_ε is strongly convergent in $L^2(Q)$ to p^*. On the other hand Lemma 3.3. from Appendix shows that $\{\nabla f^\varepsilon(y_\varepsilon, \nabla y_\varepsilon, y_{\varepsilon t})\}$ is bounded in $L^2(Q)^{n+2}$. Then, Lemma 3 from Barbu [3] shows that on a subsequence

$\nabla f^\varepsilon(y_\varepsilon, \nabla y_\varepsilon, y_{\varepsilon t}) \to \gamma^* \in Df(y^*, \nabla y^*, y^*_t)$ weakly in $L^2(Q)^{n+2}$. Here we use (2.6) - (2.8).

One can pass to the limit in (2.19), (2.20) to obtain (2.1), (2.2).

REMARK 2.1. When f is differentiable, we can assume only that L is continuous. Then, the argument is a combination of the above proof and of the proof of V. Komornik and D.Tiba [9].

REMARK 2.2. This method to derive the optimality conditions is, in a certain sense, independent of the form of the state system and may be applied in various situations. For instance, similar results may be obtained for unconstrained control problems governed by parabolic semilinear equations:

$$y_t - \Delta y + g(y, \nabla y) = u \quad \text{in } Q,$$
$$y(0,x) = y_0(x) \qquad \text{in } \Omega, y(t,x) = 0 \text{ on } \Sigma$$

under condition (1.4) for g. This can be compared with the work of V.Barbu [1] § 5.3.

3. APPENDIX

In this section we prove some technical lemmas on the behaviour of f^ε.

LEMMA 3.1. For all $\varepsilon \in [0,1]$ and $y \in R^{n+2}$, we have

(3.1) $|f^\varepsilon(y) - f(y)| \le C\varepsilon(1 + |f(y)|)$

with C independent of ε.

PROOF.

$$|f^\varepsilon(y) - f(y)| \le \int_{S(0,1)} |f(y - \varepsilon\tau) - f(y)| \rho(\tau)d\tau \le$$
$$\le C_1\varepsilon \int_{S(0,1)} (1 + |f(y)|)|\tau| \rho(\tau) \le C\varepsilon(1 + |f(y)|)$$

by hypothesis (1.4).

LEMMA 3.2. For ε sufficiently small, we have:

(3.2.) $|f(y)| \leq 1 + 2|f^{\varepsilon}(y)|$, $y \in R^{n+2}$.

PROOF.

$$|f(y)| \leq |f^{\varepsilon}(y)| + |f^{\varepsilon}(y) - f(y)| \leq |f^{\varepsilon}(y)| +$$
$$+ C\varepsilon(1 + |f(y)|).$$

LEMMA 3.3. For ε sufficiently small, we have:

(3.3) $|\nabla f^{\varepsilon}(y)| \leq C(1 + |f^{\varepsilon}(y)|)$

with C independent of ε.

PROOF.

It is enough to show (3.3) for a component i, $1 \leq i \leq n+2$ of the gradient ∇f^{ε}. We denote it f_i^{ε}:

$$|f_i^{\varepsilon}(y)| \leq \lim_{h \to 0} 1/|h| \int_{S(0,1)} |f(y - \varepsilon\tau) -$$
$$f(y_1 - \varepsilon\tau_1, \ldots, y_i + h - \varepsilon\tau_i, \ldots, y_{n+2} - \varepsilon\tau_{n+2})| \rho(\tau)d\tau \leq$$
$$\leq \lim_{h \to 0} C/|h| \int_{S(0,1)} (1 + |f(y - \varepsilon\tau)|)|h| \rho(\tau)d\tau \leq$$
$$\leq C \int_{S(0,1)} (1 + |f(y)| + |f(y - \varepsilon\tau) - f(y)|)\rho(\tau)d\tau \leq$$
$$\leq C(1 + |f(y)|) + C^2 \int_{S(0,1)} (1 + |f(y)|)\varepsilon|\tau| \rho(\tau)d\tau \leq$$
$$\leq C_2(1 + |f(y)|) \leq C_3(1 + |f^{\varepsilon}(y)|).$$

Here, we have used several times assumption (1.4) and (3.2).

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Dan TIBA

Dept. of Mathematics

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Bd. Pacii 220

Bucuresti 79622

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