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by

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Ivan Singer

§0. Introduction

This paper is a continuation of our papers [20]-[22], and it is parallel to [13]. In [20], among other results, we have given some new examples of infimal generators (in the sense of [12]) of some complete lattices, occurring in the study of generalized convexity and functional hulls, with applications to the theory of optimization problems; in [21], we have started an axiomatic study of the theory of Fenchel-Moreau conjugations $f\in\overline{R}^X\rightarrow f^{c(\varphi)}\in\overline{R}^W$ (where X and W are two sets and $\varphi:X\times W\rightarrow\overline{R}=[-\infty,+\infty]$ is a coupling functional in [22], we have shown some relations between dualities $\Delta:2^X\rightarrow 2^W$ (where X and W are two sets), in the sense of Evers and van Maaren [6], polarities $\pi(\rho):2^X\rightarrow 2^W$ (where $\rho\subseteq X\times W$ is a binary relation), in the sense of Birkhoff [1], Ch.IV, §5 (see also [11]), coupling functionals and conjugations; finally, in [13], there are given some applications of infimal generators to characterizations and representations of dualities $\Delta:E\rightarrow F$, where E and F are two complete lattices and, in particular, of dualities $\Delta:A^X\rightarrow B^W$, where $(A,\leqslant)$, $(B,\leqslant)\subseteq(\overline{R},\leqslant)$ are such that A^X and B^W are complete lattices.

The aim of the present paper is to give some applications of infimal generators to the study of dualities between complete lattices, both in the general and in some particular cases. Also, we shall introduce and study the quasi-complement of an element of a complete lattice, with respect to a family of infimal generators.

The main result of §1 of the present paper is a theorem on the extension of an antitone mapping $\Delta_0:Y\rightarrow F$, where Y is a family of infimal generators of a complete lattice E and F is a complete lattice, to a duality $\Delta:E\rightarrow F$.

In §2 we shall use infimal generators to give some formulae for $\Delta^*(z)$, $\Delta^*\Delta(x)$ and Fix $(\Delta^*\Delta)$, where $\Delta^*:F\rightarrow E$ is the dual of the duality $\Delta:E\rightarrow F$, and $\Delta^*\Delta:E\rightarrow E$ is the associated hull operator, with invariancy class Fix $(\Delta^*\Delta)$. Also, we shall show that, for the elements x of a complete lattice E, the theories of $\Delta^*\Delta$-hulls and $\mathcal{M}$-convex hulls (in the sense of [12]), where $\mathcal{M}\subseteq E$, are equivalent.

In §3 we shall consider the set D=D(E,F) of all dualities $\Delta:E\rightarrow F$, endowed with the natural partial order, and we shall use infimal generators to study the lattice operations generated by this partial order. Also, we shall introduce, for any element x of a complete lattice E, the concept of the quasi-complement $\bar{x}=\bar{x}(Y)$ of x, with respect to a family Y of infimal generators of E, which will permit us to define, for any duality $\Delta:E\rightarrow F$, the quasi-complement $\bar{\Delta}=\bar{\Delta}(Y,T)$ of Δ, with respect to two families of infimal generators $Y\subseteq E$ and $T\subseteq F$; the quasi-complements $\bar{x}=\bar{x}(Y)$ may also have interest for other applications.

In §4, considering the particular case when $E=(2^X,\supseteq)$, $F=(2^W,\supseteq)$ and $Y\subseteq E$, $T\subseteq F$ are the families of all singletons in X and W respectively, we shall show that, even in this particular case, the results of the preceding sections yield some new results. Also, we shall obtain some lattice-theoretic properties of the relations of [22] between dualities $\Delta:(2^X,\supseteq)\rightarrow(2^W,\supseteq)$, binary relations $\rho\subseteq X\times W$, polarities ([1],[11]) and coupling functionals $\varphi:X\times W\rightarrow\bar{R}$.

In §5, we shall consider the particular case when $E=\bar{R}^X$, $F=\bar{R}^W$ (where X and W are two sets), Y is the family of infimal generators of $E=\bar{R}^X$, given in [21], [13], consisting of all functionals of the form $\chi_{\{x\}}\dot{+}d$, where $\dot{+}$ denotes the "upper addition", in the sense of Moreau ([15],[16]), $\chi_{\{x\}}$ denotes the indicator functional of the singleton $\{x\}$ $(x\in X)$, and $d\in\bar{R}$, identified with the constant functional $h_d(x)\equiv d$ $(x\in X)$, and $T\subseteq F=\bar{R}^W$ is defined similarly. We shall show that, even in this particular case, the results of the preceding sections yield some new results on dualities $\Delta:\bar{R}^X\rightarrow\bar{R}^W$ and conjugations $c:\bar{R}^X\rightarrow\bar{R}^W$ and, again, we shall give some related lattice-theoretic properties.

Finally, in §6 (Appendix), returning to the general case, we shall give some relations between dualities $\Delta:E\rightarrow F$ and coupling functionals $\varphi:Y\times T\rightarrow\bar{R}$, where $Y\subseteq E$ and $T\subseteq F$ are families of infimal generators of the complete lattices E and F, respectively.

The notions and notations which we shall use, will be explained in the subsequent sections.

§1. Dualities between complete lattices. Families of infimal and supremal generators.

We recall that if $E=(E,\leqslant)$ and $F=(F,\leqslant)$ are two complete lattices, a mapping $\Delta:E\rightarrow F$ is called a duality ([6], [13]), or, a polarity ([1], [17], [18]) if for every index set I (including the empty set $I=\emptyset$), we have

$$\Delta(\inf_{i\in I} x_i)=\sup_{i\in I}\Delta(x_i) \qquad (\{x_i\}_{i\in I}\subseteq E); \qquad (1.1)$$

we shall use here the term "duality" (reserving "polarity" for (4.54) below).

In any complete lattice, we shall denote by $+\infty(-\infty)$ the greatest (smallest) element and we shall adopt the usual conventions

$$\inf \emptyset=+\infty , \sup \emptyset=-\infty ; \tag{1.2}$$

thus [13], a duality $\Delta:E\to F$ is nothing else than a complete inf-anti-homomorphism satisfying (by (1.1) for $I=\emptyset$)

$$\Delta(+\infty)=-\infty . \tag{1.3}$$

It is well-known that (by (1.1) applied to x_1,x_2), each inf-anti-homomorphism $\Delta:E\to F$ is <u>antitone</u>, i.e.,

$$x_1,x_2\in E, \quad x_1\leq x_2 \Rightarrow \Delta(x_1)\geq\Delta(x_2), \tag{1.4}$$

whence, for $x_2=+\infty$, we obtain

$$\Delta(+\infty)=\min\{\Delta(x)| x \in E\}; \tag{1.5}$$

hence [13], a complete inf-anti-homomorphism Δ of E <u>onto</u> F satisfies (1.3), and thus it is a duality.

Let us give now some examples, which will be considered in the sequel.

<u>Example 1.1.</u> For any sets X and W, let $E=(2^X, \supseteq)$, the lattice of all subsets of X, ordered by containment (i.e., $G_1\leq G_2$ if and only if $G_1 \supseteq G_2$; hence, $\sup=\bigcap$, $\inf=\bigcup$, $+\infty=\emptyset$, $-\infty=X$ and, by (1.2), $\bigcup\emptyset=\emptyset$, $\bigcap \emptyset=X$), and let $F=(2^W,\supseteq)$. Then, a mapping $\Delta:E\to F$ is a duality if and only if it is a "duality between the sets X and W", in the sense of [6], i.e., for every index set I we have

$$\Delta(\bigcup_{i\in I} G_i)=\bigcap_{i\in I}\Delta(G_i) \qquad (\{G_i\}_{i\in I} \subseteq 2^X). \tag{1.6}$$

<u>Example 1.2.</u> For any sets X and W, let $E=(\overline{R}^X, \leq)$, $F=(\overline{R}^W,\leq)$, with the usual pointwise order (where $\overline{R}=[-\infty,+\infty]$). We recall that a mapping $c:\overline{R}^X\to\overline{R}^W$ is called [21] a "conjugation", if for every index set I we have (denoting $c(f)$ by f^c)

$$(\inf_{i\in I} f_i)^c=\sup_{i\in I} f_i^c \qquad (\{f_i\}_{i\in I} \subseteq\overline{R}^X), \tag{1.7}$$

$$(f\dot{+}d)^c=f^c\dot{-}d \qquad (f\in\overline{R}^X, \ d\in\overline{R}), \tag{1.8}$$

where $\dot{+}$ and $\dot{-}$ denote the "upper" and "lower" addition on $\overline{R}$, respectively, i.e. ([15], [16]), $+\infty\dot{+}-\infty=+\infty$, $+\infty\dot{-}-\infty=-\infty$. By (1.7), every conjugation $c:E\to F$ is a duality (but, clearly, the converse is not true). Thus, as shown by examples 1.1 and 1.2, the dualities between complete lattices yield a unified point of view, both for the usual dualities $\Delta: (2^X,\supseteq)\to(2^W,\supseteq)$ and for the dualities $\Delta:\overline{R}^X\to\overline{R}^W$ and conjugations $c:\overline{R}^X\to \overline{R}^W$. Moreover, they also encompass the cases involving two complete lattices E and F "of different types", such as in

<u>Example 1.3.</u> a) Let E be a complete lattice, $\mathcal{M}\subseteq$ E and $F=(2^{\mathcal{M}}, \supseteq)$, and define a mapping $\Delta_{\mathcal{M}}:E\to F=(2^{\mathcal{M}},\supseteq)$ by

$$\Delta_{\mathcal{M}}(x)=\{m\in\mathcal{M}\mid m\leq x\} \qquad (x\in L). \qquad (1.9)$$

Then $\Delta_{\mathcal{M}}$ is a duality, in the sense (1.1), since for any $\{x_i\}_{i\in I}\subseteq E$ we have

$$\Delta_{\mathcal{M}}(\inf_{i\in I} x_i)=\{m\in\mathcal{M}\mid m\leq\inf_{i\in I} x_i\}=\bigcap_{i\in I}\{m\in\mathcal{M}\mid m\leq x_i\}=\sup_{i\in I}\Delta_{\mathcal{M}}(x_i);$$

in [12], $\Delta_{\mathcal{M}}|_{E_o}:E_o\to F$, where $E_o=\{x\in E\mid x=\sup\Delta_{\mathcal{M}}(x)\}$ (the complete lattice of all "$\mathcal{M}$-convex" elements of E), is called "the Minkowski duality" (however, in [12] the word "duality" is not used in the sense (1.1), but merely in the sense of "simultaneous study of a pair of objects").

b) In the particular case when $E=(2^X,\supseteq)$ and $\mathcal{M}\subseteq 2^X$, (1.9) becomes the duality

$$\Delta(G)=\{M\in\mathcal{M}\mid G\subseteq M\}, \qquad (1.10)$$

called [6] "the duality associated to the family of sets $\mathcal{M}$".

c) In the particular case when $E=(\overline{R}^X,\leq)$ and $\mathcal{M}\subseteq\overline{R}^X$, (1.9) becomes the duality

$$\Delta_{\mathcal{M}}(f)=\{m\in\mathcal{M}\mid m\leq f\} \qquad (f\in\overline{R}^X); \qquad (1.11)$$

in [6], a "duality" $\Delta:\overline{R}^X\to(2^{\mathcal{M}},\supseteq)$ similar to (1.11), is also considered, via (1.6) and identification of functionals $f\in\overline{R}^X$ with their hypographs in $2^{X\times R}$ (see [6], §8).

We recall that if $Y\subseteq E_o\subseteq E$, where E is a complete lattice, Y is called [12] a <u>family of infimal generators of</u> E_o, if for each $x\in E_o$ there exists $Y_x\subseteq Y$ such that

$$x=\inf Y_x . \qquad (1.12)$$

A <u>family of supremal generators of</u> E_o is defined [12] similarly, with inf replaced by sup.

<u>Example 1.4.</u> If E is a coatomic (respectively, an atomic) lattice, i.e. (see e.g. [1], [9]), each $x\in E$ is an infimum of coatoms of E (respectively, a supremum of atoms of E), then <u>the family of all coatoms (atoms) of</u> E is a family of infimal (supremal) generators of E.

<u>Example 1.5.</u> For the complete lattice $E=(2^X,\supseteq)$, where X is a set, $\mathcal{M}\subseteq\mathcal{C}(\subseteq 2^X)$ is a family of infimal (supremal) generators of $\mathcal{C}$ if and only if it is a unional (intersectional) basis of $\mathcal{C}$. The coatoms (atoms) of the lattice $E=(2^X,\supseteq)$ are the singletons $\{x\}$ (respectively, the sets $X\setminus\{x\}$), where $x\in X$, so they form a family Y of infimal (supremal) generators of E (see example 1.4).

<u>Example 1.6.</u> a) For any set X, if $E=(\overline{R}^X,\leq)$, then, by [21], lemma 3.1 and remark 3.1 b), we have

$$f=\inf_{x\in X}\{\chi_{\{x\}}\dot{+}f(x)\} \qquad (f\in\bar{R}^X), \qquad (1.13)$$

$$f=\sup_{x\in X}\{-\chi_{\{x\}}\dot{+}f(x)\} \qquad (f\in\bar{R}^X), \qquad (1.14)$$

where, for any $G\subseteq X$, χ_G denotes the "indicator functional" of G, i.e.,

$$\chi_G(x)=0 \text{ if } x\in G, \quad =+\infty \text{ if } x\notin G; \qquad (1.15)$$

hence, the families

$$Y_1=\{\chi_{\{x\}}\dot{+}d \mid x\in X,\ d\in\bar{R}\}\subset\bar{R}^X, \qquad (1.16)$$

$$Y_2=\{-\chi_{\{x\}}\dot{+}d \mid x\in X,\ d\in\bar{R}\}\subset\bar{R}^X, \qquad (1.17)$$

are families of infimal, respectively, supremal generators of $E=(\bar{R}^X,\leqslant)$. Note that $E=(\bar{R}^X,\leqslant)$ has no coatoms and no atoms.

b) Similarly, using [21], formula (3.7), it follows that

$$Y_1'=Y_1\setminus\{-\infty,+\infty\}=\{\chi_{\{x\}}\dot{+}d \mid x\in X,\ d\in R\}\subset(R\cup\{+\infty\})^X \qquad (1.18)$$

(where $R=(-\infty,+\infty)$) is a family of infimal generators of $(\bar{R}^X,\leqslant)$.

For further examples of infimal (supremal) generators, see [12] and [20]; see also the "first proof" and "second proof", given at the end of §2 below.

<u>Proposition 1.1.</u> Y <u>is a family of infimal generators of</u> $E_o\subseteq E$ <u>if and only if</u>

$$x = \inf\{y\in Y \mid x\leqslant y\} \qquad (x\in E_o). \qquad (1.19)$$

<u>Proof.</u> For $E_o=E$, this has been proved in [13], proposition 1.1. In the general case, the proof is the same.

<u>Remark 1.1.</u> a) <u>If</u> Y <u>is a family of infimal generators of a complete lattice</u> E, <u>then so is</u> $Y\setminus\{+\infty\}$, since by (1.19) and (1.2) we have

$$x = \inf\{y\in Y \mid x\leqslant y<+\infty\} \qquad (x\in E); \qquad (1.20)$$

similarly, if Y is a family of supremal generators, then so is $Y\setminus\{-\infty\}$.

b) If Y is a family of infimal generators of E, then, by (1.19), (1.20),

$$-\infty = \inf Y = \inf(Y\setminus\{+\infty\}). \qquad (1.21)$$

<u>Corollary 1.1.</u> <u>If</u> Y <u>is a family of infimal generators of a complete lattice</u> E, <u>then</u>

$$\{y\in Y \mid x\leqslant y<+\infty\}\neq\emptyset \Leftrightarrow x<+\infty \qquad (x\in E). \qquad (1.22)$$

<u>Remark 1.2.</u> If, in addition, $+\infty\notin Y$, then

$$\{y\in Y \mid x\leqslant y\}\neq\emptyset \Leftrightarrow x<+\infty \qquad (x\in E). \qquad (1.23)$$

Proposition 1.1 suggests to give

Definition 1.1. Let E be a complete lattice and let Y be a family of infimal generators of E. We shall say that $M\subseteq Y$ is an upper conical subset of Y, if there exists $x\in E$ such that

$$M=\{y\in Y \mid x\leqslant y\}. \qquad (1.24)$$

Remark 1.3. By (1.19) (with $E_0=E$), the element x in definition 1.1 is uniquely determined by M, namely, $x=\inf M$.

Proposition 1.2. For $M\subseteq Y$, the following statements are equivalent:

1^0. M is an upper conical subset of Y.

2^0. We have the implication

$$y\in Y,\ \inf M\leqslant y \Rightarrow y\in M. \qquad (1.25)$$

3^0. We have

$$M=\{y\in Y \mid \inf M\leqslant y\}. \qquad (1.26)$$

Proof. $1^0\Rightarrow 3^0$. If 1^0 holds, then, by (1.24) and remark 1.3, we have (1.26).

The equivalence $2^0\Leftrightarrow 3^0$ and the implication $3^0\Rightarrow 1^0$ are obvious.

Corollary 1.2. Let E be a complete lattice and let Y be a family of infimal generators of E. Then

$$\omega : x\to\{y\in Y \mid x\leqslant y\} \qquad (1.27)$$

is a one-to-one mapping of E onto the family

$$\mathcal{U}(Y)=\{\{y\in Y \mid x\leqslant y\} \mid x\in E\}\subseteq 2^Y \qquad (1.28)$$

of all upper conical subsets of Y.

Remark 1.4. a) If $E=(2^X,\supseteq)$ (where X is a set) and Y is the family of all singletons $\{x\}$, where $x\in X$, and if we identify each collection of singletons $M\subseteq Y$ with $\bigcup_{\{x\}\in M}\{x\}\subseteq E$, then (1.27) is the identical mapping of E onto itself.

b) If $E=(\overline{R}^X,\leqslant)$ (where X is a set) and $Y=Y_1'$ of (1.18), then, since $\chi_{\{x\}}\dot{+}d\to(x,d)$ is a one-to-one mapping of Y onto $X\times R$, we can identify Y with $X\times R$; with this identification, (1.27) is nothing else than the mapping $f\to \mathrm{Epi}\, f=\{(x,d)\in X\times R \mid f(x)\leqslant d\}$ of $\overline{R}^X$ onto the family $\mathcal{E}_0$ of all "epigraphic subsets" ([16],[12]) of $X\times R$ (by (1.35) below).

c) The mapping (1.27) is a lattice isomorphism and a complete sup-homomorphism of $(E,\leqslant)$ onto $(\mathcal{U}(Y),\supseteq)$.

Let us recall now the following result of [13]:

Theorem 1.1 ([13], theorem 1.1). Let E, F be two complete lattices, and Y a family of infimal generators of E. For a mapping $\Delta: E \to F$, the following statements are equivalent:

1°. Δ is a duality.

2°. For every index set I (including $I=\emptyset$) we have

$$\Delta(\inf_{i\in I} y_i) = \sup_{i\in I} \Delta(y_i) \qquad (\{y_i\}_{i\in I} \subseteq Y). \tag{1.29}$$

These statements imply

3°. We have

$$\Delta(x) = \sup \{\Delta(y) \mid y\in Y,\ x\leqslant y\} \qquad (x\in E). \tag{1.30}$$

Of course, the implication $1^\circ \Rightarrow 3^\circ$ follows immediately from (1.19) and (1.1). As has been observed in [13], remark 1.2 a), any constant mapping $\Delta(x) \equiv y_0 \in F\setminus\{-\infty\}$ $(x\in E)$ satisfies 3°, but not 1°; further examples that $3^\circ \not\Rightarrow 1^\circ$ follow from proposition 1.3 below.

Corollary 1.3. Let E, F be two complete lattices and $Y\subseteq E$ a family of infimal generators of E. If $\Delta_1, \Delta_2: E \to F$ are two dualities such that $\Delta_1|_Y = \Delta_2|_Y$, then $\Delta_1 = \Delta_2$.

Remark 1.5. a) If $\Delta: E\to F$ is a duality, then, by (1.1) and remark 1.1 a), we have

$$\Delta(x) = \sup \{\Delta(y) \mid y\in Y,\ x\leqslant y<+\infty\} \qquad (x\in E). \tag{1.31}$$

b) For $E=(2^X, \supseteq)$, with Y = the family of all singletons $\{x\}$, where $x\in X$ (see examples 1.5 and 1.1) and for any complete lattice F, formula (1.30) becomes

$$\Delta(G) = \sup_{x\in G} \Delta(\{x\}) \qquad (G\subseteq X), \tag{1.32}$$

and the implication $3^\circ \Rightarrow 2^\circ$ in theorem 1.1 is also valid; indeed, if $\Delta: (2^X, \supseteq) \to F$ satisfies (1.32), then for any $\{G_i\}_{i\in I} \subseteq 2^X$ we have

$$\Delta(\bigcup_{i\in I} G_i) = \sup_{x\in \bigcup_{i\in I} G_i} \Delta(\{x\}) = \sup_{i\in I} \sup_{x\in G_i} \Delta(\{x\}) = \sup_{i\in I} \Delta(G_i).$$

c) For $E=(\overline{R}^X, \leqslant)$, with $Y=Y_1$ of (1.16), and for any complete lattice F, theorem 1.1 yields that a mapping $\Delta: \overline{R}^X \to F$ is a duality if and only if for all $\{x_i\}_{i\in I} \subseteq X$ and $\{d_i\}_{i\in I} \subseteq \overline{R}$ we have

$$[\inf_{i\in I} (\chi_{\{x_i\}} \dotplus d_i)]^\Delta = \sup_{i\in I} (\chi_{\{x_i\}} \dotplus d_i)^\Delta, \tag{1.33}$$

and, if this holds, then

$$f^\Delta = \sup_{x\in X} (\chi_{\{x\}} \dotplus f(x))^\Delta \qquad (f\in \overline{R}^X). \tag{1.34}$$

Indeed, if Δ is a duality, then, by (1.30), (1.16) and the obvious equivalence

$$f\leqslant \chi_{\{x\}} \dotplus d \Leftrightarrow f(x)\leqslant d \qquad (f\in\overline{R}^X,\ x\in X,\ d\in\overline{R}), \tag{1.35}$$

and by (1.4), we obtain

$$f^{\Delta}=\sup_{\substack{x\in X,\, d\in\overline{R}\\ f\leq\chi_{\{x\}}\dot{+}d}}(\chi_{\{x\}}\dot{+}d)^{\Delta}=\sup_{\substack{x\in X,\, d\in\overline{R}\\ f(x)\leq d}}(\chi_{\{x\}}\dot{+}d)^{\Delta}=$$

$$=\sup_{x\in X}(\chi_{\{x\}}\dot{+}f(x))^{\Delta}\qquad(f\in\overline{R}^X).$$

Formulae (1.34), (1.35) have been observed in [13]; moreover, by [13], theorem 2.1, $\Delta:\overline{R}^X\to F$ <u>is a duality if and only if for all</u> $x\in X$ <u>and</u> $\{d_i\}_{i\in I}\subseteq\overline{R}$ <u>we have</u> (1.33) <u>with</u> $x_i=x$ $(i\in I)$, <u>i.e.</u>,

$$(\chi_{\{x\}}\dot{+}\inf_{i\in I}d_i)^{\Delta}=\sup_{i\in I}(\chi_{\{x\}}\dot{+}d_i)^{\Delta}\qquad(x\in X,\{d_i\}_{i\in I}\subseteq\overline{R}).\qquad(1.36)$$

We recall that if X and W are two sets, then any functional $\varphi:X\times W\to\overline{R}$ is called ([15],[16]) a "coupling functional". For any $\varphi:X\times W\to\overline{R}$, the mapping $c(\varphi):\overline{R}^X\to\overline{R}^W$, defined ([15],[16]) by

$$f^{c(\varphi)}(w)=\sup_{x\in X}\{\varphi(x,w)\dot{+}-f(x)\}\qquad(f\in\overline{R}^X,\ w\in W),\qquad(1.37)$$

is a conjugation (see example 1.2), called "the Fenchel-Moreau conjugation with respect to the coupling functional φ"; note that, in particular,

$$(\chi_{\{x\}})^{c(\varphi)}(w)=\varphi(x,w)\qquad(x\in X,\ w\in W).\qquad(1.38)$$

Conversely, by [21], theorem 3.1, for every conjugation $c:\overline{R}^X\to\overline{R}^W$, there exists a unique coupling functional $\varphi_c:X\times W\to\overline{R}$ such that $c=c(\varphi_c)$, namely,

$$\varphi_c(x,w)=(\chi_{\{x\}})^c(w)\qquad(x\in X,\ w\in W);\qquad(1.39)$$

φ_c is called [21] "the coupling functional associated to c". For extensions of [21], theorem 3.1, to arbitrary dualities $\Delta:\overline{R}^X\to\overline{R}^W$, see [13], §3.

Remark 1.5 c) above suggests to give the following characterizations of conjugations $c:\overline{R}^X\to\overline{R}^W$:

<u>Proposition 1.3. For a mapping</u> $c:\overline{R}^X\to\overline{R}^W$, <u>the following statements are equivalent</u>:

1°. c <u>is a conjugation</u>.

2°. <u>We have</u> (1.34) (<u>with</u> $\Delta=c$) <u>and</u>

$$(\chi_{\{x\}}\dot{+}d)^c=(\chi_{\{x\}})^c\dot{+}-d\qquad(x\in X,\ d\in\overline{R}).\qquad(1.40)$$

<u>Proof</u>. 1°⇒2°. If 1° holds, then, by (1.13) and (1.7), we have (1.34). Furthermore, by (1.8), we have (1.40).

2°⇒1°. If 2° holds, then

$$f^c=\sup_{x\in X}(\chi_{\{x\}}\dot{+}f(x))^c=\sup_{x\in X}((\chi_{\{x\}})^c\dot{+}-f(x))\qquad(f\in\overline{R}^X),$$

whence $f^c=f^{c(\varphi)}$ $(f\in\overline{R}^X)$, with $\varphi=\varphi_c:X\times W\to\overline{R}$ of (1.39).

The following "extension theorem" will be useful in the sequel.

Theorem 1.2. Let E,F be two complete lattices, and Y a family of infimal generators for E. For a mapping $\Delta_o:Y\to F$, the following statements are equivalent:

1^o. There exists a duality $\Delta:E\to F$ such that $\Delta|_Y=\Delta_o$.

2^o. Δ_o is antitone and for each family $\{y_i\}_{i\in I}\subseteq Y$ we have

$$\sup\{\Delta_o(y)|\,y\in Y,\ \inf_{i\in I} y_i\leqslant y\}=\sup_{i\in I}\Delta_o(y_i). \tag{1.41}$$

3^o. Δ_o is antitone and the mapping $\Delta:E\to F$ defined by

$$\Delta(x)=\sup\{\Delta_o(y)|\,y\in Y,\ x\leqslant y\}\qquad (x\in E), \tag{1.42}$$

is a duality.

In this case, the duality Δ of 1^o is uniquely determined, namely, it is given by (1.42).

Proof. $1^o\Rightarrow 2^o$. If 1^o holds, then Δ is antitone, whence so is $\Delta_o=\Delta|_Y$. Furthermore, since Δ is a duality, by theorem 1.1 we have (1.30), whence, using also (1.29), we obtain

$$\sup\{\Delta_o(y)|\,y\in Y,\ \inf_{i\in I} y_i\leqslant y\}=\sup\{\Delta(y)|\,y\in Y,\inf_{i\in I} y_i\leqslant y\}=$$
$$=\Delta(\inf_{i\in I} y_i)=\sup_{i\in I}\Delta(y_i)=\sup_{i\in I}\Delta_o(y_i).$$

$2^o\Rightarrow 3^o$. Observe that if $\Delta_o:Y\to F$ is antitone, then for the mapping $\Delta:E\to F$ defined by (1.42) we have (even when Δ is not a duality)

$$\Delta(y')=\sup\{\Delta_o(y)|\,y\in Y,\ y'\leqslant y\}=\Delta_o(y')\qquad (y'\in Y),$$

that is, $\Delta|_Y=\Delta_o$. Hence, by (1.41), for any $\{y_i\}_{i\in I}\subseteq Y$ we get

$$\Delta(\inf_{i\in I} y_i)=\sup\{\Delta(y)|\,y\in Y,\ \inf_{i\in I} y_i\leqslant y\}=$$
$$=\sup\{\Delta_o(y)|\,y\in Y,\ \inf_{i\in I} y_i\leqslant y\}=\sup_{i\in I}\Delta_o(y_i)=\sup_{i\in I}\Delta(y_i),$$

and therefore, by theorem 1.1, Δ is a duality.

$3^o\Rightarrow 1^o$. By the observation made in the above proof of the implication $2^o\Rightarrow 3^o$, the duality Δ of 3^o satisfies $\Delta|_Y=\Delta_o$.

Finally, if Δ is as in 1^o, then, by (1.30) and $\Delta|_Y=\Delta_o$, we obtain (1.42).

Remark 1.6. a) If $+\infty\notin Y$, then condition (1.41) is satisfied for $I=\emptyset$ (by (1.2)).

b) If $+\infty\in Y$, then condition (1.41) is satisfied for $I=\emptyset$ if and only if

$$\Delta_o(+\infty)=-\infty. \qquad (1.43)$$

c) For $I\neq\emptyset$ and for every mapping $\Delta_o:Y\to F$, we have the inequality $\geq$ in (1.41); indeed, since $\inf_{i\in I} y_i\leq y_k$ $(k\in I)$, we have

$$\Delta_o(y_k)\in\{\Delta_o(y)\mid y\in Y,\ \inf_{i\in I} y_i\leq y\} \qquad (k\in I),$$

whence the assertion follows.

Corollary 1.4. Let $E=(2^X,\supseteq)$, where X is a set, let Y be the family of all singletons $\{x\}$ $(x\in X)$, and let F be a complete lattice. Then every mapping $\Delta_o:Y\to F$ is antitone and can be extended to a (unique) duality $\Delta:E\to F$, namely,

$$\Delta(G)=\sup_{x\in G}\Delta_o(\{x\}) \qquad (G\subseteq X). \quad (1.44)$$

Proof. Obviously, every $\Delta_o:Y\to F$ is antitone (since distinct singletons are not comparable). Also, $+\infty\notin Y$. Finally, let $\{\{x_i\}\}_{i\in I}\subseteq Y$, $I\neq\emptyset$, and let $x\in X$ be such that $\inf_{i\in I}\{x_i\}\leq\{x\}$, i.e., $x\in\{x_i\}_{i\in I}$. Then $\Delta_o(\{x\})\leq\sup_{i\in I}\Delta_o(\{x_i\})$, whence we obtain the inequality $\leq$ in (1.41), which, together with remark 1.6 c), a), yields (1.41). Hence, by theorem 1.2, the conclusion follows.

For $E=(\overline{R}^X,\leq)$ one obtains, with the method of [13] mentioned in remark 1.5 c) above (or, using directly [13], theorem 2.1), the following result:

Proposition 1.4. Let $E=(\overline{R}^X,\leq)$, where X is a set, let $Y=Y_1$ be the family (1.16), and let F be a complete lattice. For a mapping $\Delta_o:Y\to F$, the following statements are equivalent:

1°. There exists a duality $\Delta:\overline{R}^X\to F$ such that $\Delta|_Y=\Delta_o$.

2°. For any index set I we have

$$(\chi_{\{x\}}\dot{+}\inf_{i\in I} d_i)^{\Delta_o}=\sup_{i\in I}(\chi_{\{x\}}\dot{+}d_i)^{\Delta_o} \qquad (x\in X,\ \{d_i\}_{i\in I}\subseteq\overline{R}). \quad (1.45)$$

In this case, the duality Δ of 1° is uniquely determined, namely, it is given by (1.34).

§2. Duals of dualities. Hulls. Generalized convexity

We recall that if E and F are two complete lattices, the "dual" $\Delta^*:F\to E$ of any mapping $\Delta:E\to F$ is defined (see e.g. [18]) by

$$\Delta^*(z)=\inf\{x\in E\mid\Delta(x)\leq z\} \qquad (z\in F); \qquad (2.1)$$

note that, by (1.3), we have $+\infty\in\{x\in E\mid\Delta(x)\leq z\}\neq\emptyset$ $(z\in F)$.

In the sequel we shall assume that Δ is a duality. In this case, Δ^* is a duality, too, and we have the equivalence (see e.g. [18])

$$\Delta(x)\leq z \Leftrightarrow \Delta^{*}(z)\leq x \qquad (x\in E,\ z\in F), \qquad (2.2)$$

whence $\Delta^{**}=\Delta$. Thus, each result on dualities can be "dualized", interchanging the roles of E, F and Δ,Δ^{*}. For example, the "dualization" of (2.1) is

$$\Delta(x)=\inf\{z\in F\mid \Delta^{*}(z)\leq x\} \qquad (x\in E); \qquad (2.3)$$

the right hand side of (2.3) is nothing else than $\Delta^{**}(x)$.

Clearly, $\Delta^{*}\Delta: E\to E$ is a "hull operator" ("from below"), i.e.,

$$x_1\leq x_2 \Rightarrow \Delta^{*}\Delta(x_1)\leq \Delta^{*}\Delta(x_2) \qquad (x_1,x_2\in E), \qquad (2.4)$$

$$\Delta^{*}\Delta(x)\leq x \qquad (x\in E), \qquad (2.5)$$

$$\Delta^{*}\Delta(x)=\Delta^{*}\Delta\Delta^{*}\Delta(x) \qquad (x\in E). \qquad (2.6)$$

and hence $\Delta^{*}\Delta$ is also a "hull operator" in the sense of [14], i.e.,

$$\Delta^{*}\Delta(x)=\max\{x'\in \mathrm{Fix}\ (\Delta^{*}\Delta)\mid x'\leq x\} \qquad (x\in E), \qquad (2.7)$$

where, by definition (and by (2.6)),

$$\mathrm{Fix}\ (\Delta^{*}\Delta)=\{x\in E\mid \Delta^{*}\Delta(x)=x\}=\{\Delta^{*}\Delta(x)\mid x\in E\}; \qquad (2.8)$$

indeed, by (2.8) and (2.5), we have $\Delta^{*}\Delta(x)\in\{x'\in \mathrm{Fix}\ (\Delta^{*}\Delta)\mid x'\leq x\}$, whence the inequality $\leq$ in (2.7) and, on the other hand, by (2.4), for each $x'\in \mathrm{Fix}\ (\Delta^{*}\Delta)$ with $x'\leq x$ we have $x'=\Delta^{*}\Delta(x')\leq\Delta^{*}\Delta(x)$.

We shall now give some formulae for $\Delta^{*}(z)$, $\Delta^{*}\Delta(x)$ and Fix $(\Delta^{*}\Delta)$ using infimal generators. _In the sequel, we shall denote by_ Y _and_ T _two families of infimal generators, of_ E _and_ F _respectively_.

Note first that, dualizing (1.30), we obtain

$$\Delta^{*}(z)=\sup\{\Delta^{*}(t)\mid t\in T,\ z\leq t\} \qquad (z\in F). \qquad (2.9)$$

Furthermore, let us recall

Proposition 2.1 ([13], proposition 1.2). _Let_ Y _be a family of infimal generators of_ E _and let_ $\Delta: E\to F$ _be a duality. Then_

$$\Delta^{*}(z)=\inf\{y\in Y\mid \Delta(y)\leq z\} \qquad (z\in F). \qquad (2.10)$$

Proof [13]. By (1.19) and (2.2), we have

$$\Delta^{*}(z)=\inf\{y\in Y\mid \Delta^{*}(z)\leq y\}=\inf\{y\in Y\mid \Delta(y)\leq z\} \qquad (z\in F).$$

Remark 2.1. The dualization of (2.10) is

$$\Delta(x)=\inf\{t\in T\mid \Delta^{*}(t)\leq x\} \qquad (x\in E), \qquad (2.11)$$

where T is a family of infimal generators of F.

Applying (2.1), (2.10) to $z=\Delta(x)$ and dualizing, we obtain

Proposition 2.2. We have

$$\Delta^{*}\Delta(x)=\inf\{x'\in E\mid \Delta(x')\leq\Delta(x)\}=\inf\{y\in Y\mid \Delta(y)\leq\Delta(x)\} \qquad (x\in E), \qquad (2.12)$$

$$\Delta\Delta^{*}(z)=\inf\{z'\in F\mid \Delta^{*}(z')\leq\Delta^{*}(z)\}=\inf\{t\in T\mid \Delta^{*}(t)\leq\Delta^{*}(z)\} \qquad (z\in F). \qquad (2.13)$$

Remark 2.2. a) The expressions in (2.12) contain explicitly only E, Y and Δ, but not F.

b) We shall not state separately the obvious dualizations of the subsequent results, but we shall use such dualizations freely, whenever necessary.

Corollary 2.1. We have

$$\mathrm{Fix}\,(\Delta^*\Delta)=\{x\in E \mid x\leq y \quad (y\in Y,\ \Delta(y)\leq\Delta(x))\}= \\ =\{x\in E \mid \Delta(x)\not\geq\Delta(y) \quad (y\in Y,\ y\not\geq x)\}. \tag{2.14}$$

Proof. We have $x\leq y$ for all $y\in Y$ with $\Delta(y)\leq\Delta(x)$ if and only if $x\leq\inf\{y\in Y \mid \Delta(y)\leq\Delta(x)\}=\Delta^*\Delta(x)$ (by (2.12)), so it remains to apply (2.5).

In order to express $\Delta^*\Delta(x)$ and $\mathrm{Fix}\,(\Delta^*\Delta)$ in terms of separation, let us first give the following useful property of infimal generators T of a complete lattice F (which also follows from remark 1.4 c)):

Lemma 2.1. We have the equivalence

$$z\leq z' \Leftrightarrow \{t\in T \mid z'\leq t\}\subseteq\{t\in T \mid z\leq t\} \qquad (z,z'\in F). \tag{2.15}$$

Proof. If $z\leq z'$, $t\in T$ and $z'\leq t$, then $z\leq z'\leq t$. Conversely, if $\{t\in T \mid z'\leq t\}\subseteq\{t\in T \mid z\leq t\}$, then, by (1.19), we obtain $z\leq z'$.

Theorem 2.1. We have

$$\Delta^*\Delta(x)=\inf\{y\in Y \mid \nexists t\in T,\ x\geq\Delta^*(t),\ y\not\geq\Delta^*(t)\}= \\ =\inf\{y\in Y \mid y\geq\Delta^*(t) \quad (t\in T,\ x\geq\Delta^*(t))\} \qquad (x\in E), \tag{2.16}$$

$$\mathrm{Fix}\,(\Delta^*\Delta)=\{x\in E \mid \forall y\in Y,\ y\not\geq x, \exists t\in T, x\geq\Delta^*(t), y\not\geq\Delta^*(t)\}. \tag{2.17}$$

Proof. Applying (2.12), (2.15) (with $z=\Delta(y)$, $z'=\Delta(x)$) and (2.2), we obtain (2.16). Furthermore, by (2.15), (2.2), for $x\in E$ and $y\in Y$ there exists $t\in T$ with $x\geq\Delta^*(t)$, $y\not\geq\Delta^*(t)$ if and only if $\Delta(y)\not\leq\Delta(x)$. Hence, (2.17) follows from the right hand side of (2.14).

Remark 2.3. a) If $x\geq\Delta^*(t)$ and $y\not\geq\Delta^*(t)$, then one can say that the set $\{x'\in E \mid x'\geq\Delta^*(t)\}$, or, simply, the element $\Delta^*(t)$, "separates" x from y.

b) One can also express $\Delta^*\Delta(x)$ as a supremum. Indeed, by (2.9) for $z=\Delta(x)$ and (2.2), we have

$$\Delta^*\Delta(x)=\sup\{\Delta^*(t) \mid t\in T,\ \Delta(x)\leq t\}= \\ =\sup\{\Delta^*(t) \mid t\in T, \Delta^*(t)\leq x\} \qquad (x\in E). \tag{2.18}$$

We recall (see [12], [3]) that for any $\mathcal{M}\subseteq E$, the "$\mathcal{M}$-convex hull" of an element $x\in E$ is, by definition, the element

$$\mathcal{C}(\mathcal{M})(x)=\sup\{m\in\mathcal{M} \mid m\leq x\}\in E, \tag{2.19}$$

and x is said to be "$\mathcal{M}$-convex" if $x=\mathcal{C}(\mathcal{M})(x)$; clearly, $\mathcal{C}(\mathcal{M}):E\to E$ is a "hull operator", in the sense that (2.4)-(2.7) hold for $\mathcal{C}(\mathcal{M})$ instead of $\Delta^*\Delta$, where

$$\text{Fix } \mathcal{C}(\mathcal{M})=\{x\in E \mid \mathcal{C}(\mathcal{M})(x)=x\}=\{\mathcal{C}(\mathcal{M})(x) \mid x\in E\}. \tag{2.20}$$

Remark 2.3 b) shows that if $\Delta:E\to F$ is a duality, then for

$$\mathcal{M}=\{\Delta^*(t) \mid t\in T\}\subseteq E \tag{2.21}$$

we have

$$\Delta^*\Delta(x)=\mathcal{C}(\mathcal{M})(x) \qquad (x\in E). \tag{2.22}$$

Moreover, let us show now that <u>every result on $\Delta^*\Delta$, where $\Delta:E\to F$ is a duality, is equivalent to a result on $\mathcal{C}(\mathcal{M})$, where $\mathcal{M}\subseteq E$ and, in particular, theorem</u> 2.1 <u>is equivalent to</u>

<u>Theorem 2.2. For any $\mathcal{M}\subseteq E$, we have</u>

$$\mathcal{C}(\mathcal{M})(x)=\inf\{y\in Y \mid \nexists m\in\mathcal{M},\ x\geqslant m,\ y\ngeqslant m\} \qquad (x\in E), \tag{2.23}$$

$$\text{Fix } \mathcal{C}(\mathcal{M})=\{x\in E \mid \forall y\in Y,\ y\ngeqslant x,\ \exists\, m\in\mathcal{M},\ x\geqslant m,\ y\ngeqslant m\}. \tag{2.24}$$

Indeed, we have shown above that <u>for any duality</u> $\Delta:E\to F$ there exists $\mathcal{M}\subseteq E$ such that (2.22) holds, namely, $\mathcal{M}$ of (2.21); in particular, for $\mathcal{M}$ of (2.21), theorem 2.2 yields theorem 2.1. Let us prove now the converse, i.e., that <u>for each $\mathcal{M}\subseteq E$ there exist a complete lattice</u> F <u>and a duality</u> $\Delta:E\to F$, <u>such that</u> (2.22) <u>holds</u>. We shall give two different proofs, since both are revealing.

<u>First proof</u>. Given $\mathcal{M}\subseteq E$, let $F=(2^{\mathcal{M}},\supseteq)$, let $\Delta_{\mathcal{M}}:E\to F$ be the duality (1.9) of example 1.3, and let $T\subseteq F$ be the family of infimal generators of F, consisting of all singletons $\{m\}$, where $m\in\mathcal{M}$. Then, by (2.10), (1.9) and (1.19), we have

$$\Delta^*_{\mathcal{M}}(\{m\})=\inf\{y\in Y \mid \Delta_{\mathcal{M}}(y)\ni m\}=\inf\{y\in Y \mid m\leqslant y\}=m \qquad (m\in\mathcal{M}), \tag{2.25}$$

whence, by (2.9), (1.9) and (2.19), we obtain

$$\Delta^*_{\mathcal{M}}(\Delta_{\mathcal{M}}(x))=\sup\{\Delta^*_{\mathcal{M}}(\{m\}) \mid m\in\mathcal{M}, \Delta_{\mathcal{M}}(x)\ni m\}=$$
$$=\sup\{m\in\mathcal{M} \mid m\leqslant x\}=\mathcal{C}(\mathcal{M})(x) \qquad (x\in E). \tag{2.26}$$

Note also that, applying theorem 2.1 to F, T and $\Delta_{\mathcal{M}}$ above, and taking into account (2.25), we obtain theorem 2.2.

<u>Second proof</u>. Given $\mathcal{M}\subseteq E$, for any $\{x_i\}_{i\in I}\subseteq E$ we have, by (2.19),

$$\mathcal{C}(\mathcal{M})(\inf_{i\in I} x_i)=\sup\{m\in\mathcal{M} \mid m\leqslant \inf_{i\in I} x_i\}=\sup\{m\in\mathcal{M} \mid m\leqslant x_i \quad (i\in I)\}=$$
$$=\inf_{i\in I} \sup\{m\in\mathcal{M} \mid m\leqslant x_i\}=\inf_{i\in I} \mathcal{C}(\mathcal{M})(x_i), \tag{2.27}$$

whence, if $\mathcal{C}(\mathcal{M})(x_i)=x_i \quad (i\in I)$, then

$$\mathcal{C}(\mathcal{M})(\inf_{i\in I} x_i)=\inf_{i\in I} \mathcal{C}(\mathcal{M})(x_i)=\inf_{i\in I} x_i. \tag{2.28}$$

Now, let $F=(\mathrm{Fix}\,\mathcal{C}(\mathcal{M}))^-$, i.e., the set Fix $\mathcal{C}(\mathcal{M})$ of (2.20) endowed with the reverse order $\geq_E$ (that is, $x_1 \leq_F x_2$ if and only if $x_1 \geq_E x_2$). Then, by (2.28), F is a complete semi-lattice, and hence a complete lattice (see e.g.[1], Ch.IV, theorem 2). Furthermore, by (2.27), $\mathcal{C}(\mathcal{M}):E\to F$ is a duality, whence, by (2.12), we obtain

$$\mathcal{C}(\mathcal{M})^*(\mathcal{C}(\mathcal{M})(x))=\inf\{x'\in E\,|\,\mathcal{C}(\mathcal{M})(x')\geq_E \mathcal{C}(\mathcal{M})(x)\}=\mathcal{C}(\mathcal{M})(x) \quad (x\in E). \quad (2.29)$$

Indeed, the last equality in (2.29) holds, since, on the one hand, by $\mathcal{C}(\mathcal{M})(\mathcal{C}(\mathcal{M})(x))=\mathcal{C}(\mathcal{M})(x)$ we have $\mathcal{C}(\mathcal{M})(x)\in\{x'\in E\,|\,\mathcal{C}(\mathcal{M})(x')\geq_E \geq_E \mathcal{C}(\mathcal{M})(x)\}$, whence the inequality $\leq_E$ in (2.29) and, on the other hand, for each $x'\in E$ with $\mathcal{C}(\mathcal{M})(x')\geq_E \mathcal{C}(\mathcal{M})(x)$ we have $x'\geq_E \mathcal{C}(\mathcal{M})(x')\geq_E \geq_E \mathcal{C}(\mathcal{M})(x)$, whence the inequality $\geq_E$ in (2.29); for some more general results, see [4], §1. Finally, by (2.19), (2.20), $T=\mathcal{M}$ is a family of infimal generators of $F=(\mathrm{Fix}\,\mathcal{C}(\mathcal{M}))^-$; hence, applying theorem 2.1 to F, T and the duality $\mathcal{C}(\mathcal{M})$ above, and taking into account that $\mathcal{C}(\mathcal{M})^*(m)=m$ for all $m\in\mathcal{M}$ (by (2.29) for x=m and (2.19)), we obtain again theorem 2.2.

<u>Remark 2.4.</u> a) One can also give a simple direct proof of theorem 2.2.

b) From the above observation that $T=\mathcal{M}$ is a family of infimal generators of $F=(\mathrm{Fix}\,\mathcal{C}(\mathcal{M}))^-\subseteq E^-$ and from (1.12), it follows that

$$\mathrm{Fix}\,\mathcal{C}(\mathcal{M})=\{x\in E\,|\,\exists\mathcal{M}_x\subseteq\mathcal{M},\ x=\sup_E \mathcal{M}_x\}. \quad (2.30)$$

§3. <u>Partial order and lattice operations for dualities.</u>

Let E and F be two complete lattices, with families of infimal generators Y and T respectively.

We recall that the natural order on F^E, the family of all mappings $\Delta:E\to F$, is defined "pointwise", i.e., for $\Delta_1,\Delta_2:E\to F$ we write $\Delta_1\leq\Delta_2$, if

$$\Delta_1(x)\leq\Delta_2(x) \quad (x\in E). \quad (3.1)$$

Now we shall consider the set $D=D(E,F)$ of all dualities $\Delta:E\to F$, endowed with the partial order induced by the above (i.e., $D=(D,\leq)$), and we shall study the lattice operations generated by this partial order.

<u>Remark 3.1.</u> D has a smallest element Θ, and a greatest element Ω, i.e., we have

$$\Theta\leq\Delta\leq\Omega \quad (\Delta\in D), \quad (3.2)$$

where $\Theta\in D$ and $\Omega\in D$ are the dualities defined by

$$\Theta(x)=-\infty \quad (x\in E), \quad (3.3)$$

$$\Omega(x)=+\infty, \text{ if } x<+\infty, \quad =-\infty, \text{ if } x=+\infty. \quad (3.4)$$

<u>Proposition 3.1. We have</u> $\Delta_1 \leq \Delta_2$ <u>if and only if</u>

$$\Delta_1(y) \leq \Delta_2(y) \qquad (y \in Y). \tag{3.5}$$

<u>Proof.</u> If (3.5) holds, then, by (1.30),

$$\Delta_1(x)=\sup\{\Delta_1(y) | y\in Y,\ x\leq y\} \leq \sup\{\Delta_2(y) | y\in Y, x\leq y\}=\Delta_2(x) \quad (x\in E).$$

For the supremum, respectively, the infimum, of a family $\{\Delta_j\}_{j\in J} \subseteq D$, we shall use the notations $\bigvee_{j\in J} \Delta_j$, $\bigwedge_{j\in J} \Delta_j$, instead of $\sup_{j\in J} \Delta_j$ and $\inf_{j\in J} \Delta_j$, in order to avoid writing $\sup_{j\in J} (\Delta_j(x))$ and $\inf_{j\in J} (\Delta_j(x))$, in the right hand sides of (3.8), (3.9), etc. below. We recall that, by definition,

$$\bigvee_{j\in J}\Delta_j=\min\{\Delta\in D | \Delta_j\leq\Delta \ (j\in J)\} \qquad (\{\Delta_j\}_{j\in J} \subseteq D), \tag{3.6}$$

$$\bigwedge_{j\in J}\Delta_j=\max\{\Delta\in D | \Delta\leq\Delta_j \ (j\in J)\} \qquad (\{\Delta_j\}_{j\in J} \subseteq D), \tag{3.7}$$

provided that they exist in D.

<u>Theorem 3.1.</u> a) $D=(D,\leq)$ <u>is a complete lattice, and for any</u> $\Delta_j \in D$ $(j\in J)$ <u>we have</u>

$$(\bigvee_{j\in J}\Delta_j)(x)=\sup_{j\in J} \Delta_j(x) \qquad (x\in E), \tag{3.8}$$

$$(\bigwedge_{j\in J}\Delta_j)(x)\leq \sup\{\inf_{j\in J} \Delta_j(y) | y\in Y,\ x\leq y\}\leq \inf_{j\in J} \Delta_j(x) \qquad (x\in E). \tag{3.9}$$

b) <u>For</u> $\Delta_j\in D$ $(j\in J)$, <u>the following statements are equivalent:</u>

1^o. <u>We have</u>

$$(\bigwedge_{j\in J}\Delta_j)(y)=\inf_{j\in J} \Delta_j(y) \qquad (y\in Y). \tag{3.10}$$

2^o. <u>There exists a duality</u> $\Delta\in D$ <u>such that</u>

$$\Delta(y)=\inf_{j\in J} \Delta_j(y) \qquad (y\in Y). \tag{3.11}$$

3^o. <u>For each family</u> $\{y_i\}_{i\in I} \subseteq Y$ <u>we have</u>

$$\sup\{\inf_{j\in J} \Delta_j(y) | y\in Y, \inf_{i\in I} y_i\leq y\}=\sup_{i\in I} \inf_{j\in J} \Delta_j(y_i). \tag{3.12}$$

4^o. <u>The mapping</u> $\Delta:E\rightarrow F$ <u>defined by</u>

$$\Delta(x)=\sup\{\inf_{j\in J} \Delta_j(y) | y\in Y,\ x\leq y\} \qquad (x\in E), \tag{3.13}$$

<u>is a duality.</u>

5^o. <u>We have</u>

$$(\bigwedge_{j\in J}\Delta_j)(x)=\sup\{\inf_{j\in J} \Delta_j(y) | y\in Y,\ x\leq y\} \qquad (x\in E). \tag{3.14}$$

<u>Proof.</u> a) Let $\Delta_j \in D$ $(j\in J)$. Since F is a complete lattice, we

can define $\Delta':E\to F$ by

$$\Delta'(x)=\sup_{j\in J}\Delta_j(x) \qquad (x\in E). \qquad (3.15)$$

Then $\Delta'\in D$, since for any $\{x_i\}_{i\in I}\subseteq E$ we have, by $\Delta_j\in D$ $(j\in J)$,

$$\sup_{j\in J}\Delta_j(\inf_{i\in I} x_i)=\sup_{j\in J}\sup_{i\in I}\Delta_j(x_i)=\sup_{i\in I}\sup_{j\in J}\Delta_j(x_i).$$

Furthermore, $\Delta_j(x)\leqslant\Delta'(x)$ $(x\in E,\ j\in J)$, so $\Delta_j\leqslant\Delta'$ $(j\in J)$. Finally, if $\Delta\in D, \Delta_j\leqslant\Delta$ $(j\in J)$, then $\Delta'(x)=\sup_{j\in J}\Delta_j(x)\leqslant\Delta(x)$ $(x\in E)$, so $\Delta'\leqslant\Delta$. Thus, $\bigvee_{j\in J}\Delta_j=\min\{\Delta\in D|\ \Delta_j\leqslant\Delta\ (j\in J)\}=\Delta'\in D$, and (3.8) holds.

By the above, $(D,\leqslant)$ is a complete semi-lattice for $\bigvee$, and hence a complete lattice (see e.g. [1], Ch.IV, theorem 2).

Now, since F is a complete lattice, we can define $\Delta'':E\to F$ by

$$\Delta''(x)=\inf_{j\in J}\Delta_j(x) \qquad (x\in E). \qquad (3.16)$$

Then, for each $\Delta\in D$ such that $\Delta(x)\leqslant\Delta_j(x)$ $(x\in E,\ j\in J)$, we have $\Delta\leqslant\Delta''$, whence by (3.7), $\bigwedge_{j\in J}\Delta_j\leqslant\Delta''$ (in F^E). Hence, by (1.30) (applied to $\bigwedge_{j\in J}\Delta_j\in D$), we obtain, for each $x\in E$,

$$(\bigwedge_{j\in J}\Delta_j)(x)=\sup\{\inf(\bigwedge_{j\in J}\Delta_j)(y)|\ y\in Y,\ x\leqslant y\}\leqslant\sup\{\Delta''(y)|\ y\in Y,\ x\leqslant y\},$$

i.e., the first inequality in (3.9). Finally, since each Δ_j is antitone, we have

$$\inf_{j\in J}\Delta_j(x)\geqslant\inf_{j\in J}\Delta_j(y) \qquad (y\in Y,\ x\leqslant y),$$

whence we obtain the second inequality in (3.9).

b) The equivalences $2^o\Leftrightarrow3^o\Leftrightarrow4^o$ follow from theorem 1.2, applied to the antitone mapping $\Delta_o=\Delta''|_Y:Y\to F$, i.e., to

$$\Delta_o(y)=\inf_{j\in J}\Delta_j(y) \qquad (y\in Y). \qquad (3.17)$$

The implication $1^o\Rightarrow2^o$ is obvious (with $\Delta=\bigwedge_{j\in J}\Delta_j$).

$2^o\Rightarrow1^o$. If $\Delta\in D$ is as in 2^o, then $\Delta(y)\leqslant\Delta_j(y)$ $(y\in Y,\ j\in J)$, whence, by proposition 3.1, $\Delta\leqslant\Delta_j$ $(j\in J)$ and hence, by (3.7), $\Delta\leqslant\bigwedge_{j\in J}\Delta_j$. Thus, by (3.11),

$$\inf_{j\in J}\Delta_j(y)\leqslant(\bigwedge_{j\in J}\Delta_j)(y) \qquad (y\in Y), \qquad (3.18)$$

whence, by (3.9), we obtain (3.10).

$1^o\Rightarrow5^o$. If 1^o holds, then by the last statement of theorem

1.2, applied to $\bigwedge_{j\in J}\Delta_j:E\to F$ and to $\Delta_o:Y\to F$ of (3.17), we have (3.14).

$5^o\Rightarrow 1^o$. If 5^o holds, then, since Δ_o of (3.17) is antitone, by the observation made in the proof of theorem 1.2, implication $2^o\Rightarrow 3^o$ (with $\Delta=\bigwedge_{j\in J}\Delta_j$), we have (3.10).

Remark 3.2. a) Let $E=(2^X,\supseteq)$, $F=(2^W,\supseteq)$. If we take Y=E, then, in general, the first inequality in (3.9) is strict, even when J is finite; if we take Y=the family of all singletons $\{x\}$, where $x\in X$ (see example 1.5), then, in general, the second inequality in (3.9) is strict, even when J is finite.

b) If $E=(2^X,\supseteq)$, Y= the family of all singletons $\{x\}$ ($x\in X$) and F is any complete lattice, then we have (3.10), (3.14); indeed, condition 2^o of theorem 3.1 b) is satisfied (by corollary 1.4).

Theorem 3.2. The mapping $\Delta\to\Delta^*$ is a complete lattice isomorphism of $D=D(E,F)$ onto $D^*=\{\Delta^*|\Delta\in D\}=D(F,E)$, the complete lattice of all dualities from F into E.

Proof. Clearly, $D^*\subseteq D(F,E)$. Conversely, if $\Gamma\in D(F,E)$, then for $\Delta=\Gamma^*\in D$ we have $\Delta^*=\Gamma^{**}=\Gamma$, so $D^*=D(F,E)$ and $\Delta\to\Delta^*$ maps D onto D^*. Also, $\Delta\to\Delta^*$ is one-to-one, since $\Delta_1^*=\Delta_2^*$ implies $\Delta_1=\Delta_1^{**}=\Delta_2^{**}=\Delta_2$.

Thus, it remains to show that, for any family $\{\Delta_j\}_{j\in J}\subseteq D$, we have

$$(\bigvee_{j\in J}\Delta_j)^*=\bigvee_{j\in J}\Delta_j^*, \tag{3.19}$$

$$(\bigwedge_{j\in J}\Delta_j)^*=\bigwedge_{j\in J}\Delta_j^*. \tag{3.20}$$

By (2.1), (3.8) and (2.2), we obtain

$$(\bigvee_{j\in J}\Delta_j)^*(z)=\inf\{x\in E|(\bigvee_{j\in J}\Delta_j)(x)\leq z\}=$$
$$=\inf\{x\in E|\Delta_j(x)\leq z\ (j\in J)\}=$$
$$=\inf\{x\in E|x\geq\Delta_j^*(z)\ (j\in J)\}=\sup_{j\in J}\Delta_j^*(z)=(\bigvee_{j\in J}\Delta_j^*)(z)\quad (z\in F),$$

i.e., (3.19). Hence (see e.g. [1], Ch.II, §5),

$$\Delta_1\leq\Delta_2\Leftrightarrow\Delta_1^*\leq\Delta_2^*\qquad(\Delta_1,\Delta_2\in D). \tag{3.21}$$

Finally, by (2.2), (3.7), (3.21) and $\Delta^{**}=\Delta$, we have

$$(\bigwedge_{j\in J}\Delta_j)^*(z)\leq x\Leftrightarrow(\bigwedge_{j\in J}\Delta_j)(x)\leq z\Leftrightarrow\Delta(x)\leq z\quad(\Delta\in D,\ \Delta\leq\Delta_j\ (j\in J))\Leftrightarrow$$
$$\Leftrightarrow\Delta^*(z)\leq x\ \ (\Delta^*\in D^*,\ \Delta^*\leq\Delta_j^*\ (j\in J))\Leftrightarrow(\bigwedge_{j\in J}\Delta_j^*)(z)\leq x\qquad(x\in E,\ z\in F),$$

which implies (3.20).

Proposition 3.2. For $\Delta_j \in D$ $(j \in J)$, we have

$$(\bigvee_{j\in J}\Delta_j)^*(\bigvee_{j\in J}\Delta_j)(x)=\inf\{y\in Y \mid \sup_{j\in J}\Delta_j(y)\leq \sup_{j\in J}\Delta_j(x)\}=$$

$$=\sup_{j\in J}\Delta_j^*(\sup_{k\in J}\Delta_k(x)) \qquad (x\in E). \qquad (3.22)$$

Proof. By (2.12) and (3.8), we have the first equality in (3.22). Also, by (3.8) and (3.19),

$$(\bigvee_{j\in J}\Delta_j)^*(\bigvee_{j\in J}\Delta_j)(x)=(\bigvee_{j\in J}\Delta_j)^*(\sup_{k\in J}\Delta_k(x))=\sup_{j\in J}\Delta_j^*(\sup_{k\in J}\Delta_k(x)) \quad (x\in E).$$

Definition 3.1. For each $x\in E$, we define the quasi-complement of x in E (with respect to Y), by

$$\bar{x}=\bar{x}(Y)=\inf\{y\in Y \mid x\not\leq y\} \qquad (x\in E). \qquad (3.23)$$

Remark 3.3. a) The quasi-complement $\bar{x}$ depends on the family of infimal generators Y. Indeed, for example, if $E=(2^X,\supseteq)$ and Y' is the family of all singletons $\{x\}$, where $x\in X$ (see example 1.5), then formula (3.23) becomes

$$\tilde{G}=\tilde{G}(Y')=\bigcup_{x'\notin G}\{x'\}=X\setminus G \qquad (G\subseteq X), \qquad (3.24)$$

where we use the notation $\tilde{G}$ instead of $\bar{G}$, in order to avoid confusion with the closure of G (when X is a topological space). On the other hand, if we take (for the same E) $Y''=2^X$ or $Y''=2^X\setminus\{\emptyset\}$, then formula (3.23) yields

$$\tilde{G}=\tilde{G}(Y'')=\bigcup_{\substack{G'\subseteq X\\ G'\setminus G\neq\emptyset}} G'=X \quad \text{if } G\neq X,$$
$$=\bigcup\emptyset=\emptyset \qquad \text{if } G=X. \qquad (3.25)$$

Note that $(2^X,\supseteq,G\to\tilde{G}(Y'))$ is a complete Boolean algebra algebra, while $(2^X,\supseteq,G\to\tilde{G}(Y''))$ is not even a complemented lattice, in the sense of (3.37) below (since $G\cap\tilde{G}(Y'')=G\neq\emptyset$ for $\emptyset\neq G\neq X$).

In the sequel we shall write $\bar{x}$ instead of $\bar{x}(Y)$, and $(E,x\to\bar{x})$ instead of $(E,\leq,x\to\bar{x}(Y))$, whenever this will lead to no confusion.

b) By (1.21) and (1.2), we have

$$\overline{+\infty}=\inf\{y\in Y \mid +\infty\not\leq y\}=\inf(Y\setminus\{+\infty\})=-\infty, \qquad (3.26)$$

$$\overline{-\infty}=\inf\{y\in Y \mid -\infty\not\leq y\}=\inf\emptyset=+\infty. \qquad (3.27)$$

Proposition 3.3. We have the implications

$$x_1,x_2\in E, \quad x_1\leq x_2\Rightarrow \bar{x}_1\geq\bar{x}_2, \qquad (3.28)$$

$$x,x'\in E, \quad \inf(x,\bar{x})\geq x'\Rightarrow \overline{x'}\geq x', \qquad (3.29)$$

$$x,x''\in E, \quad \sup(x,\bar{x})\leq x''\Rightarrow x''\geq\overline{x''}. \qquad (3.30)$$

Proof. If $x_1\leq x_2$, then $\{y\in Y \mid x_1\not\leq y\}\subseteq\{y\in Y \mid x_2\not\leq y\}$, whence (3.28) follows. Furthermore, if $\sup(x,\bar{x})\leq x''$, then $\bar{x}\leq x''$ and $x\leq x''$, whence, by (3.28), $x''\geq\bar{x}\geq\overline{x''}$. The proof of (3.29) is similar.

Proposition 3.4. For $x\in E$, the following statements are equivalent:

1°. $x\leq\bar{x}$.

2°. $\{y\in Y \mid x\not\leq y\}=\emptyset$.

3°. $\{y\in Y \mid x\leq y\}=Y$.

4°. $x=-\infty$

5°. $\bar{x}=+\infty$.

Proof. $1^\circ\Rightarrow 2^\circ$. If 2° does not hold, say $y_0\in Y$, $x\not\leq y_0$, then 1° cannot hold, since otherwise $x\leq\bar{x}=\inf\{y\in Y \mid x\not\leq y\}\leq y_0$.

The equivalence $2^\circ\Leftrightarrow 3^\circ$ is obvious.

$3^\circ\Rightarrow 4^\circ$. If 3° holds, then, by (1.19) and (1.21), we obtain $x=\inf Y=-\infty$.

The implications $4^\circ\Rightarrow 1^\circ$ and $5^\circ\Rightarrow 1^\circ$ are obvious.

Finally, if 2° holds, then $\bar{x}=\inf\emptyset=+\infty$. Thus, $2^\circ\Rightarrow 5^\circ$.

Theorem 3.3. We have

$$\inf(x,\bar{x})=-\infty \qquad (x\in E). \tag{3.31}$$

Proof. If $x\in E$, $\inf(x,\bar{x})=x'$, then, by (3.29), we have $\overline{x'}\geq x'$, whence, by proposition 3.4, implication $1^\circ\Rightarrow 4^\circ$, we obtain $x'=-\infty$.

Remark 3.4. By (3.31), one can say that every pseudo-complement is a "semi-complement for inf".

In order to consider $\sup(x,\bar{x})$, let us give

Proposition 3.5. For $x\in E$, the following statements are equivalent:

1°. $\bar{x}\leq x$.

2°. $\{y\in Y \mid \bar{x}\leq y\}=Y$.

3°. $\{y\in Y \mid \bar{x}\not\leq y\}=\emptyset$.

4°. $\bar{x}=-\infty$.

Proof. $1^\circ\Rightarrow 2^\circ$. If $\bar{x}\leq x=\inf\{y\in Y \mid x\leq y\}$, then $x\leq y$ for all $y\in Y$ such that $x\leq y$. On the other hand, by (3.23), $\bar{x}\leq y$ for all $y\in Y$ such that $x\not\leq y$.

The equivalence $2^\circ\Leftrightarrow 3^\circ$ and the implication $4^\circ\Rightarrow 1^\circ$ are obvious.

$2^\circ\Rightarrow 4^\circ$. If 2° holds, then, by (1.19) and (1.21), we obtain $\bar{x}=\inf Y=-\infty$.

Theorem 3.4. The following statements are equivalent:

1°. We have

$$\sup(x,\bar{x})=+\infty \qquad (x\in E). \tag{3.32}$$

2°. We have

$$\bar{x}\not\leq x \qquad (x\in E,\ x<+\infty). \tag{3.33}$$

3°. We have

$$\{y\in Y \mid \overline{x}\leq y\}\neq Y \qquad (x\in E,\ x<+\infty). \qquad (3.34)$$

4°. We have

$$\{y\in Y \mid \overline{x}\nleq y\}\neq\emptyset \qquad (x\in E,\ x<+\infty). \qquad (3.35)$$

5°. We have

$$\overline{x}>-\infty \qquad (x\in E,\ x<+\infty). \qquad (3.36)$$

Proof. 1° ⇒ 2°. If 2° does not hold, say $\overline{x}\leq x<+\infty$, then $\sup(x,\overline{x})=x<+\infty$.

2° ⇒ 1°. If 1° does not hold, say $x\in E$, $\sup(x,\overline{x})=x''<+\infty$, then, by (3.30), we have $x''\geq\overline{x''}$.

Finally, the equivalences 2° ⇔ ... ⇔ 5° follow from proposition 3.5.

We recall that $(E, x\to C(x))$ is called a "complemented lattice", if $C:E\to E$ is such that

$$\sup(x,C(x))=+\infty, \quad \inf(x,C(x))=-\infty \qquad (x\in E). \qquad (3.37)$$

From theorems 3.3 and 3.4 we obtain

Corollary 3.1. $(E, x\to\overline{x})$, where x is defined by (3.23), is a complemented lattice if and only if we have 2°–5° of theorem 3.4.

Proposition 3.6. We have

$$\{y\in Y \mid \overline{x}\nleq y\}\subseteq\{y\in Y \mid x\leq y<+\infty\} \qquad (x\in E). \qquad (3.38)$$

b) If $(E, x\to\overline{x})$ is a complemented lattice, then

$$\{y\in Y \mid \overline{x}\nleq y\}=\{y\in Y \mid x\leq y<+\infty\} \qquad (x\in E). \qquad (3.39)$$

Proof. a) Clearly, $+\infty\notin\{y\in Y \mid \overline{x}\nleq y\}$ $(x\in E)$. If $y\in Y\setminus\{+\infty\}$ and $\overline{x}\nleq y$, then $x\leq y$ (since otherwise $\overline{x}=\inf\{y'\in Y \mid x\nleq y'\}\leq y$).

b) If $(E, x\to\overline{x})$ is a complemented lattice and $x\leq y<+\infty$, then $\overline{x}\nleq y$ (since otherwise $\sup(x,\overline{x})\leq y<+\infty$).

Let us define the second quasi-complement of $x\in E$ by

$$\overline{\overline{x}}=\overline{(\overline{x})}=\inf\{y\in Y \mid \overline{x}\nleq y\}. \qquad (3.40)$$

Remark 3.5. From proposition 3.4 (applied to $\overline{x}$ instead of x) it follows that for an element $x\in E$ we have $\overline{x}=-\infty$ if and only if $\overline{\overline{x}}=+\infty$. Hence, condition (3.36) of theorem 3.4 is equivalent to

$$\overline{\overline{x}}<+\infty \qquad (x\in E,\ x<+\infty). \qquad (3.41)$$

Proposition 3.7. a) We have

$$x\leq\overline{\overline{x}} \qquad (x\in E). \qquad (3.42)$$

b) If $(E, x\to\overline{x})$ is a complemented lattice, then

$$x=\overline{\overline{x}} \qquad (x\in E), \qquad (3.43)$$

$$\overline{x}_1\leq x_2\Leftrightarrow x_1\geq\overline{x}_2 \qquad (x\in E). \qquad (3.44)$$

Proof. a) By (1.20), (3.38) and (3.40), we have

$$x=\inf\{y\in Y \mid x\leq y<+\infty\}\leq\inf\{y\in Y \mid \overline{x}\nleq y\}=\overline{\overline{x}} \qquad (x\in E).$$

b) The proof of (3.43) is similar, using (3.39). Finally, (3.44) follows from (3.28) and (3.43).

Remark 3.6. From (3.28) it follows that

$$x_1, x_2 \in E, \quad x_1 \leq x_2 \Rightarrow \bar{\bar{x}}_1 \leq \bar{\bar{x}}_2. \tag{3.45}$$

Furthermore, by (3.42) and (3.28) for $x_1 = x$, $x_2 = \bar{\bar{x}}$, we have $\bar{x} \geq \bar{\bar{\bar{x}}}$ $(x \in E)$, whence, by (3.42) for $\bar{x}$ instead of x, we obtain

$$\bar{x} = \bar{\bar{\bar{x}}} \qquad (x \in E); \tag{3.46}$$

thus, by (3.46) (for $\bar{x}$ instead of x), (3.42) and (3.45), the mapping $x \rightarrow \bar{\bar{x}}$ is a hull operator on $E^{-} = (E, \geq)$.

Proposition 3.8. a) We have

$$\overline{\sup_{i \in I} x_i} = \inf_{i \in I} \bar{x}_i \qquad (\{x_i\}_{i \in I} \subseteq E). \tag{3.47}$$

b) If $(E, x \rightarrow \bar{x})$ is a complemented lattice, then

$$\sup_{i \in I} \bar{x}_i = \overline{\inf_{i \in I} x_i} \qquad (\{x_i\}_{i \in I} \subseteq E). \tag{3.48}$$

Proof. a) If $\{x_i\}_{i \in I} \subseteq E$, then, by (3.23),

$$\overline{\sup_{i \in I} x_i} = \inf \{y \in Y \mid \sup_{i \in I} x_i \not\leq y\} = \inf \bigcup_{i \in I} \{y \in Y \mid x_i \not\leq y\} =$$

$$= \inf_{i \in I} \inf \{y \in Y \mid x_i \not\leq y\} = \inf_{i \in I} \bar{x}_i .$$

b) If $(E, x \rightarrow \bar{x})$ is a complemented lattice, then, by (3.43) and (3.47) (for $\bar{x}_i$ instead of x_i), we obtain

$$\sup_{i \in I} \bar{x}_i = \overline{\overline{\sup_{i \in I} \bar{x}_i}} = \overline{\inf_{i \in I} \bar{\bar{x}}_i} = \overline{\inf_{i \in I} x_i} .$$

For the proof of the next theorem, we shall also need

Lemma 3.1. If $(E, x \rightarrow \bar{x}(Y))$ is a complemented lattice, then for any $I \neq \emptyset$, $\{x_i\}_{i \in I} \subseteq E$ and $y \in Y$ with $\inf_{i \in I} x_i \leq y$ there exists $i(y) \in I$ such that $x_{i(y)} \leq y$.

Proof. If $y = +\infty$, this is obviously true. If $y < +\infty$, then, by (3.39), $\overline{\inf_{i \in I} x_i} \not\leq y$, and hence, by (3.48), $\sup_{i \in I} \bar{x}_i \not\leq y$. Therefore, $\overline{x_{i(y)}} \not\leq y$ for some $i(y) \in I$, whence, by (3.38), $x_{i(y)} \leq y$.

Lemma 3.2. If $(E, x \rightarrow \bar{x}(Y))$ is a complemented lattice, then

$$\{y \in Y \mid y_o \leq y < +\infty\} = \{y_o\} \qquad (y_o \in Y \setminus \{+\infty\}). \tag{3.49}$$

Proof. If there exist $y_o, y \in Y \setminus \{+\infty\}$ such that $y_o < y$, then $y \not\leq y_o$, whence, by (3.23),

$$\bar{y} = \inf \{y' \in Y \mid y \not\leq y'\} \leq y_o < y < +\infty , \tag{3.50}$$

so $\sup (y, \bar{y}) = y < +\infty$, in contradiction with the assumption (3.32).

Theorem 3.5. Let E be a complete lattice and let Y be a family of infimal generators of E, with $+\infty \notin Y$ and such that $(E, x \rightarrow \bar{x}(Y))$ is a complemented lattice. Then the mapping ω defined by (1.27) is a complete lattice isomorphism of E onto $(2^Y, \supseteq)$, satisfying

$$\omega(y) = \{y\} \qquad (y \in Y). \qquad (3.51)$$

Proof. By corollary 1.2, ω is a one-to-one mapping of E onto the family $\mathcal{U}(Y) \subseteq 2^Y$ of (1.28). Furthermore, by $+\infty \notin Y$ and lemma 3.2, we have (3.51).

Assume now that $\omega(E) = \mathcal{U}(Y) \neq 2^Y$, say $M \in 2^Y \setminus \mathcal{U}(Y)$, and let $x_o = \inf M$. Then we have the strict inclusion

$$M \subset \{y \in Y \mid x_o \leq y\}, \qquad (3.52)$$

i.e., there exists $y_o \in Y \setminus M$ such that $x_o \leq y_o$. We claim that

$$x_o < y_o; \qquad (3.53)$$

indeed, if $x_o = y_o \in Y$, then, by $+\infty \notin Y$, (3.49) and the strict inclusion (3.52), we would obtain $M = \emptyset$, in contradiction with $M \notin \mathcal{U}(Y)$ (since by $+\infty \notin Y$ we have $\emptyset = \{y \in Y \mid +\infty \leq y\} \in \mathcal{U}(Y)$).

Now, by $M \subset Y$, $y_o \notin M$, $+\infty \notin Y$ and (3.49), we have

$$M \subset Y \setminus \{y_o\} = \{y \in Y \mid y_o \nleq y\}, \qquad (3.54)$$

whence, by (3.53) and (3.23), we obtain

$$+\infty > y_o > x_o = \inf M \geqslant \inf\{y \in Y \mid y_o \nleq y\} = \overline{y_o},$$

so $\sup(y_o, \overline{y_o}) = y_o < +\infty$, in contradiction with the assumption (3.32). Thus, $\mathcal{U}(Y) = 2^Y$, and hence ω maps E onto 2^Y.

Finally, by (3.51) and lemma 3.1, for any $\{x_i\}_{i \in I} \subseteq E$ with $I \neq \emptyset$, there holds

$$\omega(\inf_{i \in I} x_i) = \{y \in Y \mid \inf_{i \in I} x_i \leq y\} = \{y \in Y \mid \exists i(y) \in Y,\ x_{i(y)} \leq y\} =$$
$$= \bigcup_{i \in I} \{y \in Y \mid x_i \leq y\} = \bigcup_{i \in I} \omega(x_i),$$

and for $I = \emptyset$ we have, by (1.2) and $+\infty \notin Y$,

$$\omega(\inf \emptyset) = \omega(+\infty) = \{y \in Y \mid +\infty \leq y\} = \emptyset = \bigcup \emptyset,$$

which, together with remark 1.4 c), proves that ω is a complete lattice isomorphism.

Remark 3.7. a) The assumption $+\infty \notin Y$ is not an essential restriction of the generality. Indeed, if $+\infty \in Y$, then the above results remain valid for $Y \setminus \{+\infty\}$ instead of Y.

b) Theorem 3.5 shows that, essentially (up to a complete lattice isomorphism), the only E and $Y \subseteq E$ such that $(E, x \to \overline{x}(Y))$ is a complemented lattice, are those of the form $E = (2^X, \supseteq)$ (where X is a set), and Y= the family of all singletons $\{x\}$, where $x \in X$, for which $\overline{x}(Y)$ is nothing else than the usual set complement (3.24). For any family of infimal generators on an arbitrary complete lattice E, the quasi-complements $\overline{x}(Y)$ $(x \in E)$ are well defined by (3.23), and the cardinality of the set $\{x \in E \mid \overline{x}(Y) = -\infty\}$ may serve as a measure of the "deviation" of (E, Y) from the above particular case (by corollary 3.1 and theorem 3.5).

c) From theorem 3.5 it follows that our results on $E = (2^X, \supseteq)$ and Y = the family of all singletons $\{x\}$, where $x \in X$ (e.g., corollary 1.4, proposition 3.12 below, etc.) remain valid for any pair (E, Y) such that $(E, x \to \overline{x}(Y))$ is a complemented lattice. In §§4,5 we shall consider several such pairs (E, Y) (see e.g. (4.98), etc.).

In order to define the "quasi-complement" of a duality Δ, we shall need

Lemma 3.3. *Let E, F be two complete lattices, with families of infimal generators $Y \subseteq E$, $T \subseteq F$, and let $\Delta: E \to F$ be a duality. The mapping $\Delta_o: Y \to F$, defined by*

$$\Delta_o(y) = \overline{\Delta(y)} \qquad (y \in Y) \tag{3.55}$$

(where $\overline{\Delta(y)} = \overline{\Delta(y)}(T)$), is antitone if and only if

$$\overline{\Delta(y_1)} = \overline{\Delta(y_2)} \qquad (y_1, y_2 \in Y,\ y_1 \leq y_2). \tag{3.56}$$

Proof. Assume that Δ_o is antitone and let $y_1, y_2 \in Y$, $y_1 \leq y_2$. Then, since Δ_o is antitone, $\overline{\Delta(y_1)} = \Delta_o(y_1) \geq \Delta_o(y_2) = \overline{\Delta(y_2)}$. On the other hand, since Δ is a duality, we have $\Delta(y_1) \geq \Delta(y_2)$, whence, by (3.28) (in F), we obtain $\overline{\Delta(y_1)} \leq \overline{\Delta(y_2)}$; thus, (3.56) holds.

Conversely, if (3.56) holds, then Δ_o is obviously antitone.

Definition 3.2. Let E, F be two complete lattices, with families of infimal generators $Y \subseteq E$, $T \subseteq F$, and assume that the mapping $\Delta_o: Y \to F$ defined by (3.55) is antitone. If there exists a duality $\overline{\Delta} = \overline{\Delta}(Y, T): E \to F$ such that $\overline{\Delta}|_Y = \Delta_o$, i.e., that

$$\overline{\Delta}(y) = \overline{\Delta(y)} \qquad (y \in Y) \tag{3.57}$$

(where $\overline{\Delta(y)} = \overline{\Delta(y)}(T)$), then, by theorem 1.2, it is (unique and) given by

$$\overline{\Delta}(x) = \sup\{\overline{\Delta(y)} \mid y \in Y,\ x \leq y\} \qquad (x \in E), \tag{3.58}$$

and we shall call it *the quasi-complement of the duality* Δ (with respect to Y and T). We shall write $\overline{\Delta}$ instead of $\overline{\Delta}(Y, T)$, whenever this will lead to no confusion; also, instead of writing "$\overline{\Delta}$ exists", we shall simply write: $\overline{\Delta} \in D$.

Remark 3.8. *If $E = (2^X, \supseteq)$, Y = the family of all singletons $\{x\}$ $(x \in X)$ and F is any complete lattice, with a family of infimal generators $T \subseteq F$, then we have*

$$\overline{\Delta} \in D \qquad (\Delta \in D), \tag{3.59}$$

$$\overline{\Delta}(G) = \sup_{x \in G} \overline{\Delta(\{x\})} \qquad (\Delta \in D,\ G \subseteq X); \qquad (3.60)$$

indeed, this follows from corollary 1.4 applied to Δ_o of (3.55).

Proposition 3.9. If there exists $\Delta \in D$ such that $\overline{\Delta} \in D$, then $+\infty \notin Y$ (excluding the trivial case when F is a singleton).

Proof. If $+\infty \in Y$, then, by lemma 3.3, (1.3) and (3.27),

$$\overline{\Delta(y)} = \overline{\Delta(+\infty)} = \overline{-\infty} = +\infty \qquad (y \in Y),$$

and, on the other hand, by lemma 3.3, (3.57) and (1.3) (for $\overline{\Delta} \in D$),

$$\overline{\Delta(y)} = \overline{\Delta(+\infty)} = \overline{\Delta}(+\infty) = -\infty \qquad (y \in Y),$$

whence $+\infty = -\infty$, so F is a singleton.

Proposition 3.10. If $+\infty \notin Y$, then for $\Theta \in D$ and $\Omega \in D$ of (3.3), (3.4), we have

$$\overline{\Theta} = \Omega \in D,\ \overline{\Omega} = \Theta \in D. \qquad (3.61)$$

Proof. By (1.30), (3.57), $+\infty \notin Y$, (1.22), (3.3), (3.27), (3.4) and (3.26),

$$\overline{\Theta}(x) = \sup \{\overline{\Theta(y)} \mid y \in Y,\ x \leqslant y\} = \overline{-\infty} = +\infty = \Omega(x) \text{ if } x < +\infty,$$
$$= \sup \emptyset = -\infty = \Omega(x) \text{ if } x = +\infty,$$

$$\overline{\Omega}(x) = \sup \{\overline{\Omega(y)} \mid y \in Y,\ x \leqslant y\} = \overline{+\infty} = -\infty = \Theta(x) \text{ if } x < +\infty,$$
$$= \sup \emptyset = -\infty = \Theta(x) \text{ if } x = +\infty.$$

Proposition 3.11. Assume that (3.59) holds. Then

a) We have

$$\Delta \wedge \overline{\Delta} = \Theta \qquad (\Delta \in D). \qquad (3.62)$$

b) If $F = (2^W, \supseteq)$ and T = the family of all singletons $\{w\}$ $(w \in W)$, then

$$\Delta \vee \overline{\Delta} = \Omega \qquad (\Delta \in D), \qquad (3.63)$$

so in this case $(D, \Delta \to \overline{\Delta})$ is a complemented lattice.

Proof. a) By (3.9), (3.57), (3.31) and (3.3), we have

$$(\Delta \wedge \overline{\Delta})(y) \leqslant \inf (\Delta(y), \overline{\Delta}(y)) = \inf (\Delta(y), \overline{\Delta(y)}) = -\infty = \Theta(y) \qquad (y \in Y),$$

whence, by corollary 1.3, we obtain (3.62).

The proof of b) is similar, using (3.8), (3.57) and that $\Delta(y) \cap \widetilde{\Delta(y)} = \emptyset = \Omega(y)$ for all $y \in Y$ (since $+\infty \notin Y$, by proposition 3.9).

If (3.59) holds, we shall denote $(\overline{\Delta})$ by $\overline{\overline{\Delta}}$.

Proposition 3.12. If (3.59) holds and $F = (2^W, \supseteq)$, T = the family of all singletons $\{w\}$ $(w \in W)$, then

$$\overline{\overline{\Delta}} = \Delta \qquad (\Delta \in D). \qquad (3.64)$$

Proof. By (3.57) and (3.43) (in F), we have

$$\overline{\overline{\Delta}}(y) = \overline{\overline{\Delta}(y)} = \overline{\overline{\Delta(y)}} = \Delta(y) \qquad (y \in Y), \qquad (3.65)$$

whence, by corollary 1.3, we obtain (3.64).

§4. Dualities $\Delta:(2^X,\supseteq)\rightarrow(2^W,\supseteq)$.

Let us consider now the particular case when $E=(2^X,\supseteq)$, $F=(2^W,\supseteq)$ (where X and W are two sets) and $Y\subset E$, $T\subset F$ are the families of all singletons in X and W respectively.

By example 1.1, $\Delta:E\rightarrow F$ is a duality if and only if it satisfies (1.6). The definition (2.1) of $\Delta^*:(2^W,\supseteq)\rightarrow(2^X,\supseteq)$ becomes now

$$\Delta^*(\Omega)=\bigcup_{\substack{G\subseteq X\\ \Omega\subseteq\Delta(G)}} G \qquad (\Omega\subseteq W), \tag{4.1}$$

i.e., the one of [6]. The equivalence (2.2) becomes now

$$\Omega\subseteq\Delta(G)\Leftrightarrow G\subseteq\Delta^*(\Omega) \qquad (G\subseteq X,\ \Omega\subseteq W), \tag{4.2}$$

which, together with $\Delta^{**}=\Delta$ and the principle of dualization, have been observed in [6], propositions 1.12 (vii) and 1.13.

Formulae (2.5) and (2.7) become, in this case, the known formulae (see [6])

$$G\subseteq\Delta^*\Delta(G) \qquad (G\subseteq X), \tag{4.3}$$

$$\Delta^*\Delta(G)=\bigcap_{\substack{G'\in \mathrm{Fix}\,(\Delta^*\Delta)\\ G\subseteq G'}} G' \qquad (G\subseteq X); \tag{4.4}$$

the fact that $\Delta^*\Delta$ is a hull operator on $(2^X,\supseteq)$, has been noted in [6], remark 4.1. Formulae (2.10), (2.11) become now

$$\Delta^*(\Omega)=\{x'\in X\mid \Omega\subseteq\Delta(\{x'\})\} \qquad (\Omega\subseteq W), \tag{4.5}$$

$$\Delta(G)=\{w'\in W\mid G\subseteq\Delta^*(\{w'\})\} \qquad (G\subseteq X). \tag{4.6}$$

The results of §2, expressing $\Delta^*\Delta(x)$ and Fix $(\Delta^*\Delta)$ with the aid of infimal generators, yield, even in our particular case, some new results. Thus, formulae (2.12) and (2.13) yield now

$$\Delta^*\Delta(G)=\bigcup_{\substack{G'\subseteq X\\ \Delta(G)\subseteq\Delta(G')}} G'=\{x'\in X\mid \Delta(G)\subseteq\Delta(\{x'\})\} \qquad (G\subseteq X), \tag{4.7}$$

$$\Delta\Delta^*(\Omega)=\bigcup_{\substack{\Omega'\subseteq W\\ \Delta^*(\Omega)\subseteq\Delta^*(\Omega')}} \Omega'=\{w'\in W\mid \Delta^*(\Omega)\subseteq\Delta^*(\{w'\})\} \qquad (\Omega\subseteq W); \tag{4.8}$$

in particular, for the hulls of singletons, they become

$$\Delta^*\Delta(\{x\})=\{x'\in X\mid \Delta(\{x\})\subseteq\Delta(\{x'\})\} \qquad (x\in X), \tag{4.9}$$

$$\Delta\Delta^*(\{w\})=\{w'\in W\mid \Delta^*(\{w\})\subseteq\Delta^*(\{w'\})\} \qquad (w\in W). \tag{4.10}$$

Since $A\not\subseteq B$ is equivalent to $A\setminus B\neq\emptyset$, formula (2.14) yields

$$\mathrm{Fix}\,(\Delta^*\Delta)=\{G\subseteq X\mid \Delta(G)\setminus\Delta(\{x'\})\neq\emptyset \quad (x'\in X\setminus G)\}. \tag{4.11}$$

Hence, in particular, the equalities

$$\Delta^*\Delta(\{x\})=\{x\} \qquad (x\in X) \qquad (4.12)$$

<u>hold if and only if for each pair</u> x, $x'\in X$, $x\neq x'$, <u>we have</u>

$$\Delta(\{x\})\setminus\Delta(\{x'\})\neq\emptyset. \qquad (4.13)$$

Theorem 2.1 says now that

$$\Delta^*\Delta(G)=\{x'\in X \mid \nexists w\in W,\ G\subseteq\Delta^*(\{w\}),\ x'\in X\setminus\Delta^*(\{w\})\}=$$

$$=\{x'\in X \mid x'\in\Delta^*(\{w\}) \quad (w\in W,\ G\subseteq\Delta^*(\{w\}))\} \quad (G\subseteq X), \qquad (4.14)$$

$$\text{Fix } (\Delta^*\Delta)=\{G\subseteq X \mid \forall x'\notin G,\ \exists w\in W,\ G\subseteq\Delta^*(\{w\}), x'\in X\setminus\Delta^*(\{w\})\}, \qquad (4.15)$$

where, according to remark 2.3 a), the conditions $G\subseteq\Delta^*(\{w\})$, $x'\in X\setminus\Delta^*(\{w\})$, can be expressed by saying that $\Delta^*(\{w\})$ separates G from x'; the part $\subseteq$ of (4.15) has been given in [6], proposition 1.12(v), where its "separation" aspect has been also noted. From (4.15) it follows, in particular, that <u>we have</u> (4.12) <u>if and only if for each pair</u> $x,x'\in X$, $x\neq x'$, <u>there exists</u> $w\in W$ <u>such that</u> $\Delta^*(\{w\})$ <u>separates</u> x <u>from</u> x'.

Remark 2.3 b) says now that

$$\Delta^*\Delta(G)=\bigcap_{w\in\Delta(G)}\Delta^*(\{w\})=\bigcap_{\substack{w\in W\\ G\subseteq\Delta^*(\{w\})}}\Delta^*(\{w\}) \qquad (G\subseteq X); \qquad (4.16)$$

clearly, the last term of (4.16) coincides with the last term of (4.14).

Formulae (2.19), (2.20) become now, for any $\mathcal{M}\subseteq 2^X$,

$$\mathcal{C}(\mathcal{M})(G)=\bigcap_{\substack{M\in\mathcal{M}\\ G\subseteq M}} M\subseteq 2^X \qquad (G\subseteq X), \qquad (4.17)$$

$$\text{Fix } \mathcal{C}(\mathcal{M})=\{G\subseteq X \mid \mathcal{C}(\mathcal{M})(G)=G\}=\{\mathcal{C}(\mathcal{M})(G) \mid G\subseteq X\}; \qquad (4.18)$$

in [20], $\mathcal{C}(\mathcal{M})(G)$ has been called "the $\mathcal{M}$-convex hull of the set G" and Fix $\mathcal{C}(\mathcal{M})$ has been denoted by $\mathcal{C}(\mathcal{M})$. The theory of "$\mathcal{M}$-convex sets" (i.e. of the sets in Fix $\mathcal{C}(\mathcal{M})$), originates in [2].

Theorem 2.2 and formula (2.30) mean now that, for any $\mathcal{M}\subseteq 2^X$,

$$\mathcal{C}(\mathcal{M})(G)=\{x'\in X \mid \nexists M\in\mathcal{M},\ G\subseteq M,\ x'\in X\setminus M\} \qquad (G\subseteq X), \qquad (4.19)$$

$$\text{Fix } \mathcal{C}(\mathcal{M})=\{G\subseteq X \mid \forall x'\notin G,\ \exists M\in\mathcal{M},\ G\subseteq M,\ x'\in X\setminus M\}, \qquad (4.20)$$

$$\text{Fix } \mathcal{C}(\mathcal{M})=\{G\subseteq X \mid \exists \mathcal{M}_G\subseteq\mathcal{M},\quad G=\bigcap_{M\in\mathcal{M}_G} M\}. \qquad (4.21)$$

In [20], an equivalent way has been followed: Fix $\mathcal{C}(\mathcal{M})$ has been defined by (4.20), then $\mathcal{C}(\mathcal{M})(G)$ by (4.4) (with $\Delta^*\Delta$ replaced by $\mathcal{C}(\mathcal{M})$) and then (4.21), (4.18), (4.17) and (4.19) have been deduced from these definitions ([20], propositions 1.3 - 1.6).

The observation made before theorem 2.2, says now that <u>the theories of</u> $\Delta^*\Delta$-<u>hulls</u>, <u>where</u> $\Delta:(2^X,\supseteq)\to(2^W,\supseteq)$ <u>is a duality</u>, <u>and</u> $\mathcal{M}$-<u>convex</u>

hulls, where $\mathcal{M}\subseteq 2^X$, are equivalent, via (2.21), which becomes now

$$\mathcal{M}=\{\Delta^*(\{w\})\mid w\in W\}\subseteq 2^X, \tag{4.22}$$

respectively, via the dualities (1.10) or $\mathcal{C}(\mathcal{M})$. Hence, since by [20], §1, the theories of $\mathcal{M}$-convex hulls, where $\mathcal{M}\subseteq 2^X$, and "W-convex hulls" (of sets $G\subseteq X$), where $W\subseteq \overline{R}^X$ ([7], [20]), are equivalent, it follows that so are the theories of $\Delta^*\Delta$-hulls and W-convex hulls (of sets $G\subseteq X$), where $W\subseteq\overline{R}^X$; we recall that the W-convex hull of $G\subseteq X$ is defined by $\mathcal{C}(W)(G)=\mathcal{C}(\mathcal{M}_1)(G)$, with

$$\mathcal{M}_1=\{S_d(w)\mid (w,d)\in W\times R\}\subseteq 2^X, \tag{4.23}$$

where

$$S_d(w)=\{x\in X\mid w(x)\leqslant d\}\qquad (w\in\overline{R}^X,\ d\in R). \tag{4.24}$$

Indeed, given any duality $\Delta:(2^X,\supseteq)\to(2^W,\supseteq)$, for $V\subseteq\overline{R}^X$ defined by

$$V=\{-\chi_{X\setminus\Delta^*(\{w\})}\mid w\in W\} \tag{4.25}$$

and for $\mathcal{M}$ of (4.22) we have, by (2.22) and and [20], theorem 1.1,

$$\Delta^*\Delta(G)=\mathcal{C}(\mathcal{M})(G)=\mathcal{C}(V)(G)\qquad (G\subseteq X). \tag{4.26}$$

Conversely, given any family $W\subseteq\overline{R}^X$, for the duality $\Delta:(2^X,\supseteq)\to\to(2^{W\times R},\supseteq)$ defined by

$$\Delta(G)=\{(w,d)\in W\times R\mid \sup w(G)\leqslant d\}\qquad (G\subseteq X) \tag{4.27}$$

we have, by (4.5) and (4.24),

$$\Delta^*(\{w,d\})=\{x'\in X\mid w(x')\leqslant d\}=S_d(w)\qquad (w\in W, d\in R), \tag{4.28}$$

whence, by (4.16) and [20], formula (1.52), we obtain

$$\Delta^*\Delta(G)=\bigcap_{(w,d)\in\Delta(G)} S_d(w)=\mathcal{C}(W)(G)\qquad (G\subseteq X); \tag{4.29}$$

or, alternatively, for $\mathcal{M},\mathcal{M}_1$ of (4.22), (4.23), we have $\mathcal{M}=\mathcal{M}_1$ (by (4.28)), whence, by (2.22), $\Delta^*\Delta(G)=\mathcal{C}(\mathcal{M}_1)(G)=\mathcal{C}(W)(G)\quad (G\subseteq X)$.

Similarly, the results of §3 yield, even in our particular case, some new results. Formula (3.1), defining the partial order $\Delta_1\leqslant\Delta_2$, becomes

$$\Delta_1(G)\supseteq\Delta_2(G)\qquad (G\subseteq X), \tag{4.30}$$

and the dualities Θ,Ω of (3.3), (3.4) are now

$$\Theta(G)=W\qquad (G\subseteq X), \tag{4.31}$$

$$\Omega(G)=\emptyset,\ \text{ if } G\neq\emptyset,\qquad =W,\ \text{ if } G=\emptyset. \tag{4.32}$$

Proposition 3.1 means now that

$$\Delta_1 \leq \Delta_2 \Leftrightarrow \Delta_1(\{x\}) \supseteq \Delta_2(\{x\}) \qquad (x \in X). \qquad (4.33)$$

For any dualities $\Delta_1, \Delta_2, \Delta_j : (2^X, \supseteq) \to (2^W, \supseteq)$ $(j \in J)$, one can denote the inequality $\Delta_1 \leq \Delta_2$ by $\Delta_1 \supseteq \Delta_2$ and call the dualities of (3.6) and (3.7), the "intersection" and the "union" of the Δ_j's, respectively; in symbols,

$$\bigcap_{j \in J} \Delta_j = \bigvee_{j \in J} \Delta_j, \quad \bigcup_{j \in J} \Delta_j = \bigwedge_{j \in J} \Delta_j. \qquad (4.34)$$

Then, formulae (3.8), (3.10) and (3.14) become

$$(\bigvee_{j \in J} \Delta_j)(G) = (\bigcap_{j \in J} \Delta_j)(G) = \bigcap_{j \in J} \Delta_j(G) \qquad (G \subseteq X), \qquad (4.35)$$

$$(\bigwedge_{j \in J} \Delta_j)(\{x\}) = (\bigcup_{j \in J} \Delta_j)(\{x\}) = \bigcup_{j \in J} \Delta_j(\{x\}) \qquad (x \in X), \qquad (4.36)$$

$$(\bigwedge_{j \in J} \Delta_j)(G) = (\bigcup_{j \in J} \Delta_j)(G) = \bigcap_{x \in G} \bigcup_{j \in J} \Delta_j(\{x\}) \qquad (G \subseteq X); \qquad (4.37)$$

note that (4.36) and (4.37) always hold, by remark 3.2 b).

Formula (3.23) means now (3.24) of remark 3.3 a), and the subsequent results of §3, on $\bar{x} \in E$, reduce now to well-known properties of complementary sets (3.24). Note that $(2^X, \supseteq, G \to \tilde{G})$ is a complete Boolean algebra.

By remark 3.8, we have now (3.59), and formulae (3.57) and (3.60) become now, respectively,

$$\bar{\Delta}(\{x\}) = W \setminus \Delta(\{x\}) \qquad (x \in X), \qquad (4.38)$$

$$\bar{\Delta}(G) = \bigcap_{x \in G} (W \setminus \Delta(\{x\})) \qquad (G \subseteq X). \qquad (4.39)$$

Also, by proposition 3.11 b), $(D, \Delta \to \bar{\Delta})$ is now a complemented complete lattice. Moreover, we have

<u>Theorem 4.1. The mapping $\eta : \Delta \to \bar{\Delta}$ is a complete lattice anti-automorphism of $D = D((2^X, \supseteq), (2^W, \supseteq))$.</u>

<u>Proof.</u> By (3.64), η is a one-to-one mapping of D onto itself. Furthermore, by (3.57), (4.36), (3.48), and (4.35),

$$\overline{(\bigwedge_{j \in J} \Delta_j)}(\{x\}) = \overline{(\bigwedge_{j \in J} \Delta_j)(\{x\})} = \overline{\bigcup_{j \in J} \Delta_j(\{x\})} = \bigcap_{j \in J} \overline{\Delta_j(\{x\})} =$$

$$= \bigcap_{j \in J} \bar{\Delta}_j(\{x\}) = (\bigvee_{j \in J} \bar{\Delta}_j)(\{x\}) \qquad (x \in X),$$

whence, by corollary 1.3, we obtain

$$\overline{\bigwedge_{j \in J} \Delta_j} = \bigvee_{j \in J} \bar{\Delta}_j. \qquad (4.40)$$

Finally, by (3.64) and (4.40) (for $\bar{\Delta}_j$ instead of Δ_j),

$$\overline{\bigvee_{j \in J} \Delta_j} = \overline{\bigvee_{j \in J} \bar{\bar{\Delta}}_j} = \overline{\overline{\bigwedge_{j \in J} \bar{\Delta}_j}} = \bigwedge_{j \in J} \bar{\Delta}_j. \qquad (4.41)$$

Remark 4.1. From theorem 4.1 it follows (e.g., by [1], Ch.II, §5), that

$$\Delta_1 \leqslant \Delta_2 \Leftrightarrow \overline{\Delta_1} \geqslant \overline{\Delta_2}. \qquad (4.42)$$

We shall denote $(\overline{\Delta})^*$ by $\overline{\Delta}^*$ (recall that we have now (3.59)).

Proposition 4.1. We have

$$\overline{\Delta^*} = \overline{\Delta}^* \in D^* \qquad (\Delta^* \in D^*). \qquad (4.43)$$

Proof. By (4.5), (4.38), (4.2) and (3.24), we have

$$\overline{\Delta}^*(\{w\}) = \{x \in X \mid w \in \overline{\Delta}(\{x\})\} = \{x \in X \mid w \in W \setminus \Delta(\{x\})\} =$$

$$= X \setminus \Delta^*(\{w\}) = \widetilde{\Delta^*(\{w\})} \qquad (w \in W). \qquad (4.44)$$

Thus, for each $\Delta^* \in D^*$ there exists a duality from $(2^W, \supseteq)$ into $(2^X, \supseteq)$, namely, $\overline{\Delta}^*$, satisfying (4.44), whence, by definition 3.2, we obtain (4.43).

Combining theorems 3.2, 4.1 and proposition 4.1, we obtain

Corollary 4.1. The mapping $\Delta \to \overline{\Delta}^* = \overline{\Delta^*}$ is a complete lattice anti-isomorphism of $D = D((2^X, \supseteq), (2^W, \supseteq))$ onto $D^* = \{\Delta^* \mid \Delta \in D\} = D((2^W, \supseteq), (2^X, \supseteq))$. Hence, for any $\Delta_j \in D$ $(j \in J)$ we have

$$(\overline{\bigvee_{j \in J} \Delta_j})^* = \overline{\bigvee_{j \in J} \Delta_j^*} = \bigwedge_{j \in J} \overline{\Delta}_j^*, \qquad (4.45)$$

$$(\overline{\bigwedge_{j \in J} \Delta_j})^* = \overline{\bigwedge_{j \in J} \Delta_j^*} = \bigvee_{j \in J} \overline{\Delta}_j^*. \qquad (4.46)$$

Proposition 4.2. We have

$$\overline{\Delta}(G) = \{w \in W \mid G \cap \Delta^*(\{w\}) = \emptyset\} \qquad (G \subseteq X), \qquad (4.47)$$

$$\overline{\Delta}^* \overline{\Delta}(G) = \bigcap_{\substack{w \in W \\ G \subseteq X \setminus \Delta^*(\{w\})}} (X \setminus \Delta^*(\{w\})) =$$

$$= \{x' \in X \mid \Delta(\{x'\}) \subseteq \bigcup_{x \in G} \Delta(\{x\})\} \qquad (G \subseteq X). \qquad (4.48)$$

Proof. By (4.6) and (4.44), we have

$$\overline{\Delta}(G) = \{w \in W \mid G \subseteq \overline{\Delta}^*(\{w\})\} = \{w \in W \mid G \cap (X \setminus \overline{\Delta}^*(\{w\}) = \emptyset\} =$$

$$= \{w \in W \mid G \cap \Delta^*(\{w\}) = \emptyset\} \qquad (G \subseteq X).$$

Furthermore, by (4.16) and (4.44),

$$\overline{\Delta}^* \overline{\Delta}(G) = \bigcap_{\substack{w \in W \\ G \subseteq \overline{\Delta}^*(\{w\})}} \overline{\Delta}^*(\{w\}) = \bigcap_{\substack{w \in W \\ G \subseteq X \setminus \Delta^*(\{w\})}} (X \setminus \Delta^*(\{w\}) \qquad (G \subseteq X).$$

Finally, by (4.7) and (4.39), we obtain

$$\overline{\Delta}^*\overline{\Delta}(G)=\{x'\in X|\bigcap_{x\in G}(W\setminus\Delta(\{x\}))\subseteq W\setminus\Delta(\{x'\})\}=$$

$$=\{x'\in X|\Delta(\{x'\})\subseteq\bigcup_{x\in G}\Delta(\{x\})\}\qquad(G\subseteq X).$$

Remark 4.1. By (4.48) we have, in particular,

$$\overline{\Delta}^*\overline{\Delta}(\{x\})=\{x'\in X|\Delta(\{x'\})\subseteq\Delta(\{x\})\}\qquad(x\in X),\qquad(4.49)$$

whence, by (4.9),

$$\Delta^*\Delta(\{x\})\cap\overline{\Delta}^*\overline{\Delta}(\{x\})=\{x'\in X|\Delta(\{x'\})=\Delta(\{x\})\}\qquad(x\in X).\qquad(4.50)$$

Let us give now some examples of complementary dualities $\overline{\Delta}$ and "complementary hull operators" $\overline{\Delta}^*\overline{\Delta}$.

Example 4.1. Let X be a set and $W\subseteq\overline{R}^X$. Then for the duality $\Delta:(2^X,\supseteq)\to(2^{W\times R},\supseteq)$ of (4.27) we have, by (4.47), (4.28), (4.44) and (4.48),

$$\overline{\Delta}(G)=\{(w',d')\in W\times R|w'(x)>d'\ (x\in G)\},\qquad(4.51)$$

$$\overline{\Delta}^*(\{w,d\})=\{x'\in X|w(x')>d\}\qquad(w\in W,\ d\in R),\qquad(4.52)$$

$$\overline{\Delta}^*\overline{\Delta}(G)=\bigcap_{\substack{w\in W,\ d\in R\\ w(x)>d\ (x\in G)}}\{x'\in X|w(x')>d\}\qquad(G\subseteq X).\qquad(4.53)$$

In the particular case when X is a locally convex space and $W=X^*$, the family of all continuous linear functionals on X, by (4.29) we have $\Delta^*\Delta(G)=\overline{co}\ G$, the closed convex hull of G, and, by (4.53), $\overline{\Delta}^*\overline{\Delta}(G)=$ $=$eco G, the "evenly convex hull" [8] of G, i.e., the intersection of all open half-spaces containing G (see e.g. [20]). Thus, the complementary hull operator to the closed convex hull is the evenly convex hull.

Example 4.2. By interchanging Δ and $\overline{\Delta}$ in example 4.1 and using (3.64), it follows that the complementary hull operator to the evenly convex hull is the closed convex hull.

We recall that if $\varrho\subseteq X\times W$ is a binary relation (where $(x,w)\in\varrho$ is also denoted by $x\varrho w$), then, following Birkhoff [1], Ch.IV, §5 (see also [11]), the "ϱ-polar" of any set $G\subseteq X$ is defined by

$$G^{\pi(\varrho)}=\{w\in W|(x,w)\in\varrho\quad(x\in G)\}\qquad(4.54)$$

and the mapping $\pi(\varrho):2^X\to 2^W$ is called the "polarity between subsets of X and subsets of W defined by ϱ". In [22], theorem 1.1 and its proof, we have shown that the theories of binary relations, polarities and dualities are equivalent, since a) every polarity $\pi(\varrho)$ determines uniquely the binary relation ϱ, namely,

$$\varrho=\{(x,w)\in X\times W|w\in\{x\}^{\pi(\varrho)}\};\qquad(4.55)$$

b) every polarity $\pi(\varrho):2^X\to 2^W$ is a duality and, conversely, for every duality $\Delta:(2^X,\supseteq)\to(2^W,\supseteq)$ there exists a unique set $\rho_\Delta\subseteq X\times W$ such that

$$\Delta=\pi(\rho_\Delta), \tag{4.56}$$

namely,

$$\rho_\Delta=\{(x,w)\in X\times W \mid w\in\Delta(\{x\})\}. \tag{4.57}$$

Let us give now a slight generalization of a known lemma (see e.g. [19], Ch.II, §18, formulae (6), (6') and §20), which will be used repeatedly in the sequel:

Lemma 4.1. Let A be a complete Boolean algebra, B a lattice and u an order isomorphism of A onto B. Then B is a complete Boolean algebra and u is a complete Boolean algebra isomorphism.

Proof. Let $u(x_i)\in B$, where $x_i\in A$ $(i\in I\neq\emptyset)$. Then $x_k\leqslant\sup_{i\in I} x_i$ $(k\in I)$, whence $u(x_k)\leqslant u(\sup_{i\in I} x_i)$ $(k\in I)$. On the other hand, if $u(x)\in B$, where $x\in A$, and if $u(x_i)\leqslant u(x)$ $(i\in I)$, then $x_i\leqslant x$ $(i\in I)$, whence $\sup_{i\in I} x_i\leqslant x$, and hence $u(\sup_{i\in I} x_i)\leqslant u(x)$. This proves that $\sup_{i\in I} u(x_i)= =u(\sup_{i\in I} x_i)\in B$, and the proof of the corresponding fact for $\inf_{i\in I}$ is similar. Hence, if $\sup(x,\bar{x})=+\infty$, $\inf(x,\bar{x})=-\infty$, where $x\in A$, then $\sup(u(x),u(\bar{x}))= =u(\sup(x,\bar{x}))=u(+\infty)=+\infty$ and, similarly, $\inf(u(x),u(\bar{x}))=-\infty$. Finally, since u is a lattice isomorphism, it preserves the distributivity of A.

Theorem 4.2. $D=(D,\leqslant,\Delta\to\bar{\Delta})$ is a complete Boolean algebra, and the mapping $\rho_\cdot:\Delta\to\rho_\Delta$ is a complete Boolean algebra isomorphism of D onto $(2^{X\times W},\supseteq,\rho\to\tilde{\rho})$, with inverse $(\rho_\cdot)^{-1}=\pi:\rho\to\pi(\rho)$.

Proof. By the above, $\rho_\cdot:\Delta\to\rho_\Delta$ is a one-to-one mapping of D onto $2^{X\times W}$. Moreover, by (4.33), (4.57) and example 1.1,

$$\Delta_1\leqslant\Delta_2\Leftrightarrow\rho_{\Delta_1}\supseteq\rho_{\Delta_2}\Leftrightarrow\rho_{\Delta_1}\leqslant\rho_{\Delta_2}. \tag{4.58}$$

Hence, since $(2^{X\times W},\supseteq,\rho\to\tilde{\rho})$ is a complete Boolean algebra, from lemma 4.1 (with $u=(\rho_\cdot)^{-1}$) it follows that D is a complete Boolean algebra and $\rho_\cdot$ is a complete Boolean algebra isomorphism. Finally, by (4.56), we have $(\rho_\cdot)^{-1}=\pi$.

Remark 4.2. a) By the usual definition of "inclusion" for binary relations (see e.g. [17]), we have "$\rho_1\subseteq\rho_2$" if and only if $(x,w)\in\rho_1$ implies $(x,w)\in\rho_2$, or, equivalently, $\rho_1\subseteq\rho_2$ as subsets of $X\times W$, i.e., $\rho_2\leqslant\rho_1$ in $(2^{X\times W},\supseteq)$. Hence, the usual operations (see e.g. [17]) $\bigcap_{j\in J}\rho_j$, $\bigcup_{j\in J}\rho_j$ and $\tilde{\rho}$ are, respectively, the operations $\bigvee_{j\in J}\rho_j$, $\bigwedge_{j\in J}\rho_j$ and $\tilde{\rho}$ in $(2^{X\times W},\supseteq)$. Furthermore, by (4.54) (for ρ^{-1}), we have

$$\pi(\rho^{-1})=\pi(\rho)^* \qquad (\rho\subseteq X\times W), \tag{4.59}$$

where $\rho^{-1}\subseteq W\times X$ is the "inverse" binary relation, defined (see e.g. [17]) by

$$\rho^{-1}=\{(w,x)\in W\times X \mid (x,w)\in\rho\} \tag{4.60}$$

and $\pi(\varrho)^*: 2^W \to 2^X$ is the polarity defined ([11], formula (2.2)) by

$$Q^{\pi(\varrho)^*} = \{x \in X \mid (x,w) \in \varrho \quad (w \in Q)\} \qquad (Q \subseteq W); \qquad (4.61)$$

note also that $\pi(\varrho)^*$ of (4.61) coincides with Δ^* of (4.5), for $\Delta = \pi(\varrho)$ of (4.54).

b) From the above, we obtain the following simple proof of theorem 3.2 for this case: The mapping $\Delta \to \Delta^*$ coincides with the composition of the complete lattice isomorphisms "onto"

$$\Delta \to \varrho_\Delta \to (\varrho_\Delta)^{-1} \to \pi((\varrho_\Delta)^{-1}) = \Delta^*, \qquad (4.62)$$

where the last equality follows from (4.59) and (4.56).

Using the relations between coupling functionals $\varphi: X \times W \to \overline{R}$ and dualities $\Delta: (2^X, \supseteq) \to (2^W, \supseteq)$, given in [22], we shall show now some relations between the natural partial order and lattice operations for coupling functionals $\varphi \in (\overline{R}^{X \times W}, \leq)$ and the partial order and lattice operations for dualities $\Delta: (2^X, \supseteq) \to (2^W, \supseteq)$.

We recall that for any coupling functional $\varphi: X \times W \to \overline{R}$, "the duality $\Delta_\varphi: (2^X, \supseteq) \to (2^W, \supseteq)$ associated to φ" is defined [22] by

$$\Delta_\varphi(G) = \{w' \in W \mid \varphi(x,w') \geq -1 \quad (x \in G)\} \qquad (G \subseteq X). \qquad (4.63)$$

In particular, for $\varphi \equiv -\infty$ and $\varphi \equiv +\infty$, we obtain

$$\Delta_{-\infty} = \Omega, \Delta_{+\infty} = \Theta, \qquad (4.64)$$

where Ω and Θ are the dualities (4.32), (4.31). Furthermore, we recall [22] that a coupling functional $\varphi: X \times W \to \overline{R}$ is said to be "<u>of type</u> $\{0, -\infty\}$", if $\varphi(X \times W) \subseteq \{0, -\infty\}$, i.e., if φ can assume only the values 0 and $-\infty$. Clearly, for any coupling functional $\varphi: X \times W \to \overline{R}$ of type $\{0, -\infty\}$, we have

$$\Delta_\varphi(G) = \{w' \in W \mid \varphi(x,w') = 0 \quad (x \in G)\} \qquad (G \subseteq X), \qquad (4.65)$$

$$W \setminus \Delta_\varphi(G) = \{w' \in W \mid \exists x \in G, \ \varphi(x,w') = -\infty\} \qquad (G \subseteq X). \qquad (4.66)$$

According to [22], theorem 2.1, <u>for each duality</u> $\Delta: (2^X, \supseteq) \to (2^W, \supseteq)$ <u>there exists a unique coupling functional</u> $\varphi = \varphi_\Delta$ <u>of type</u> $\{0, -\infty\}$, <u>such that</u> $\Delta = \Delta_\varphi$, <u>namely</u>,

$$\varphi_\Delta(x,w) = -\chi_{\Delta(\{x\})}(w) = -\chi_{\Delta^*(\{w\})}(x) \qquad (x \in X, \ w \in W); \qquad (4.67)$$

this φ_Δ is called [22] "<u>the coupling functional associated to the duality</u> Δ".

<u>Theorem 4.3.</u> <u>The mapping</u> $\Delta_\cdot: \varphi \to \Delta_\varphi$ <u>is a lattice anti-homomorphism and a complete inf-anti-homomorphism of</u> $\overline{R}^{X \times W}$ <u>onto</u> $D = D((2^X, \supseteq), (2^W, \supseteq))$, <u>with kernel</u>

$$\mathrm{Ker}\ \Delta_\cdot = \{\varphi \in \overline{R}^{X \times W} \mid \varphi(x,w) \geq -1 \quad ((x,w) \in X \times W)\}. \qquad (4.68)$$

<u>Proof.</u> By the above mentioned result of [22] on φ_Δ, the mapping $\Delta_\cdot$ maps $\overline{R}^{X \times W}$ <u>onto</u> D. Furthermore, it is a lattice anti-homomorphism and a complete inf-antihomomorphism, i.e.,

$$\Delta_{\max(\varphi_1,\varphi_2)}=\Delta_{\varphi_1}\wedge\Delta_{\varphi_2} \qquad (\varphi_1,\varphi_2\in\overline{R}^{X\times W}), \tag{4.69}$$

$$\Delta_{\inf_{j\in J}\varphi_j}=\bigvee_{j\in J}\Delta_{\varphi_j} \qquad (\{\varphi_j\}_{j\in J}\subseteq\overline{R}^{X\times W}); \tag{4.70}$$

indeed, for any $\varphi_1,\varphi_2\in\overline{R}^{X\times W}$, $\{\varphi_j\}_{j\in J}\subseteq\overline{R}^{X\times W}$ and $G\subseteq X$ we have

$$\{w'\in W \mid \max(\varphi_1,\varphi_2)(x,w')\geqslant -1 \quad (x\in G)\}=$$

$$=\bigcap_{x\in G}(\{w'\in W\mid \varphi_1(x,w')\geqslant -1\}\cup\{w'\in W\mid \varphi_2(x,w')\geqslant -1\}), \tag{4.71}$$

$$\{w'\in W\mid \inf_{j\in J}\varphi_j(x,w')\geqslant -1 \quad (x\in G)\}=\bigcap_{x\in G,\ j\in J}\{w'\in W\mid \varphi_j(x,w')\geqslant -1\}, \tag{4.72}$$

whence, by (4.63), (4.37) and (4.35), we obtain (4.69), (4.70).

Finally, by (4.63), we have $\varphi(x,w)\geqslant -1$ $((x,w)\in X\times W)$ if and only if

$$\Delta_\varphi(G)=W \qquad (G\subseteq X), \tag{4.73}$$

i.e., $\Delta_\varphi=\textcircled{\scriptsize{W}}$ of (4.29); hence, by remark 3.1, we obtain (4.68).

<u>Remark 4.3.</u> a) From theorem 4.3 (or, directly from the definition (4.63) of Δ_φ) it follows that $\Delta_\circ:\overline{R}^{X\times W}\to D$ is antitone, and hence

$$\Delta_{\sup_{j\in J}\varphi_j}\leqslant\bigwedge_{j\in J}\Delta_{\varphi_j} \qquad (\{\varphi_j\}_{j\in J}\subseteq\overline{R}^{X\times W}). \tag{4.74}$$

b) The mapping $\Delta_\circ$ of theorem 4.3 is not a complete lattice anti-homomorphism, since (4.71) does not extend to infinite families $\{\varphi_j\}_{j\in J}\subseteq\overline{R}^{X\times W}$ (with max replaced by sup). However, (4.71) extends to infinite families $\{\varphi_j\}_{j\in J}\subseteq\{0,-\infty\}^{X\times W}$, and hence $\Delta_\circ^r=\Delta_\circ|_{\{0,-\infty\}^{X\times W}}$ is a complete lattice anti-isomorphism of the complete sublattice $\{0,-\infty\}^{X\times W}$ of $\overline{R}^{X\times W}$, onto D, with inverse $\Delta\to\varphi_\Delta$ (of (4.67)); for a sharpening, see theorem 4.4 below. Thus, we have

$$\varphi_{\bigvee_{j\in J}\Delta_j}=\inf_{j\in J}\varphi_{\Delta_j} \qquad (\{\Delta_j\}_{j\in J}\subseteq D), \tag{4.75}$$

$$\varphi_{\bigwedge_{j\in J}\Delta_j}=\sup_{j\in J}\varphi_{\Delta_j} \qquad (\{\Delta_j\}_{j\in J}\subseteq D). \tag{4.76}$$

c) By theorem 4.3 and e.g. [9], Ch. I, §3, theorem 11, one can define <u>a congruence</u> τ <u>on</u> $\overline{R}^{X\times W}$, by

$$\varphi_1\equiv\varphi_2(\tau)\Longleftrightarrow\Delta_{\varphi_1}=\Delta_{\varphi_2}, \tag{4.77}$$

and the (well defined) mapping

$$\beta:[\varphi]=\{\varphi'\in\overline{R}^{X\times W}\mid \varphi'\equiv\varphi(\tau)\}\to\Delta_\varphi \qquad (\varphi'\in[\varphi],\ \varphi\in\overline{R}^{X\times W}), \tag{4.78}$$

<u>is a lattice anti-isomorphism of the quotient lattice</u> $\overline{R}^{X\times W}/\tau$ <u>onto</u> D, with

$$\beta^{-1}(\Delta)=[\varphi_\Delta] \qquad (\Delta\in D) \tag{4.79}$$

(see (4.67)); for a sharpening, see theorem 4.4 below. The equivalence relation $\varphi_1 \equiv \varphi_2(\tau)$ of (4.77) (not regarded as a lattice congruence) has been introduced in [22], where it has been also observed that (4.78) is a (well defined) one-to-one mapping of $\overline{R}^{X\times W}/\tau$ onto D, satisfying (4.79). Furthermore, in [22] it has been also shown that

$$\varphi_1 \equiv \varphi_2(\tau) \Leftrightarrow \{(x,w)\in X\times W \mid \varphi_1(x,w)\geq -1\} = \{(x,w)\in X\times W \mid \varphi_2(x,w)\geq -1\}, \quad (4.80)$$

and hence <u>each equivalence class</u> $[\varphi]$ <u>(where</u> $\varphi\in\overline{R}^{X\times W}$) <u>contains an unique</u> φ^o <u>of type</u> $\{0,-\infty\}$, <u>namely,</u>

$$\varphi^o = -\chi_{\{(x,w)\in X\times W \mid \varphi(x,w)\geq -1\}}. \quad (4.81)$$

Moreover, let us observe that the mapping

$$\sigma : [\varphi] \to \varphi^o \qquad (\varphi\in\overline{R}^{X\times W}) \quad (4.82)$$

<u>is a lattice isomorphism of</u> $\overline{R}^{X\times W}/\tau$ <u>onto</u> $\{0,-\infty\}^{X\times W}$; for a sharpening, see theorem 4.4 below. Note also that, by (4.68), the definition (4.78) of $[\varphi]$, (4.80) and the definition [9] of Ker τ, we have

$$\text{Ker}\ \Delta_\circ = [+\infty], \quad \text{Ker}\ \tau = -\infty. \quad (4.83)$$

<u>Definition 4.1.</u> a) Given a coupling functional $\varphi\in\{0,-\infty\}^{X\times W}$, we define the <u>complementary coupling functional</u> $\overline{\varphi}\in\{0,-\infty\}^{X\times W}$ by

$$\overline{\varphi} = -\chi_{\{(x,w)\in X\times W \mid \varphi(x,w)=-\infty\}}. \quad (4.84)$$

b) Given an equivalence class $[\varphi]\in\overline{R}^{X\times W}/\tau$, where τ is the congruence (4.77), we define the <u>complementary equivalence class</u> $\overline{[\varphi]}\in\overline{R}^{X\times W}/\tau$ by

$$\overline{[\varphi]} = [\overline{\varphi^o}],$$

where $\varphi^o\in\{0,-\infty\}^{X\times W}$ is that of (4.81). (4.85)

<u>Remark 4.4.</u> a) By the obvious formula

$$\varphi = -\chi_{\{(x,w)\in X\times W \mid \varphi(x,w)=0\}} \qquad (\varphi\in\{0,-\infty\}^{X\times W}), \quad (4.86)$$

we have

$$\min(\varphi,\overline{\varphi}) = -\infty, \quad \max(\varphi,\overline{\varphi}) = 0 \qquad (\varphi\in\{0,-\infty\}^{X\times W}), \quad (4.87)$$

so $(\{0,-\infty\}^{X\times W}, \leq, \varphi\to\overline{\varphi})$ is a complemented lattice (since 0 is its greatest element).

b) By $[\varphi]=[\varphi^o]$ and (4.85), (4.87), we have

$$\min([\varphi],\overline{[\varphi]}) = \min([\varphi^o],[\overline{\varphi^o}]) = [\min(\varphi^o,\overline{\varphi^o})] = -\infty \qquad (\varphi\in\overline{R}^{X\times W}), \quad (4.88)$$

$$\max([\varphi],\overline{[\varphi]}) = [\max(\varphi^o,\overline{\varphi^o})] = [0] = [+\infty] \qquad (\varphi\in\overline{R}^{X\times W}), \quad (4.89)$$

so $(\overline{R}^{X\times W}/\tau, \leq, [\varphi]\to\overline{[\varphi]})$ is a complemented lattice.

<u>Theorem 4.4.</u> $(\{0,-\infty\}^{X\times W}, \leq, \varphi\to\overline{\varphi})$ <u>and</u> $(\overline{R}^{X\times W}/\tau, \leq, [\varphi]\to\overline{[\varphi]})$ <u>are complete Boolean algebras, and the diagram</u>

$$\begin{array}{ccccc} \overline{R}^{X\times W} & \xrightarrow{\ q\ } & \overline{R}^{X\times W}/\tau & \xrightarrow{\ \sigma\ } & \{0,-\infty\}^{X\times W} \\ & \searrow^{\Delta_\bullet} & \downarrow\beta & \swarrow^{\Delta^r_\bullet} & \downarrow\xi \\ & & D((2^X,\supseteq),(2^W,\supseteq)) & \xrightarrow{\ \varrho_\bullet\ } & (2^{X\times W},\supseteq) \end{array} \tag{4.90}$$

is commutative, where σ (of remark 4.3 c)) and $\varrho_\bullet$ (of theorem 4.2) ar complete Boolean algebra isomorphisms onto, $\beta, \Delta^r_\bullet$ (of remark 4.3 c), l and

$$\xi : \varphi \rightarrow \{(x,w)\in X\times W \mid \varphi(x,w)=0\} \qquad (\varphi\in\{0,-\infty\}^{X\times W}) \tag{4.91}$$

are complete Boolean algebra anti-isomorphisms onto, with

$$\xi^{-1}(\varrho)=-\chi_\varrho \qquad (\varrho\subseteq X\times W), \tag{4.92}$$

$\Delta_\bullet$ is as in theorem 4.3, and the quotient mapping

$$q : \varphi \rightarrow [\varphi] \qquad (\varphi\in\overline{R}^{X\times W}) \tag{4.93}$$

is a lattice homomorphism and a complete inf-homomorphism onto.

Proof. The statements on $\{0,-\infty\}^{X\times W}$ and ξ are immediate , while those on $\overline{R}^{X\times W}/\tau$, σ and β follow similarly to theorem 4.2, using theorem 4.2 and lemma 4.1. Furthermore, by (4.57), (4.65), (4.91), (4.82), (4.81), (4.80), (4.78) and (4.93), we have

$$\varrho_{\Delta^r_\varphi} = \{(x,w)\in X\times W \mid w\in\Delta_\varphi(\{x\})\}=\xi(\varphi) \qquad (\varphi\in\{0,-\infty\}^{X\times W}),$$

$$\Delta^r_{\sigma([\varphi])} = \Delta_{\varphi^o} = \Delta_\varphi = \beta([\varphi]) \qquad (\varphi\in\overline{R}^{X\times W}),$$

$$\beta q(\varphi)=\beta([\varphi])=\Delta_\varphi \qquad (\varphi\in\overline{R}^{X\times W}),$$

so the diagram (4.90) is commutative. Finally, the statement on q fo lows from $q=\beta^{-1}\Delta_\bullet$ and theorem 4.3.

Remark 4.5. a) By $\varphi\equiv\varphi^o(\tau)$ $(\varphi\in\overline{R}^{X\times W})$, theorem 4.4, (4.39) (for $\Delta=\Delta_\varphi$) and (4.63), we have

$$\overline{\Delta_\varphi}(G)=\overline{\Delta_{\varphi^o}}(G)=\Delta_{\overline{\varphi^o}}(G)=\bigcap_{x\in G}(W\setminus\Delta_\varphi(\{x\})) =$$

$$=\bigcap_{x\in G}\{w'\in W \mid \varphi(x,w')<-1\} \qquad (\varphi\in\overline{R}^{X\times W},\ G\subseteq X); \tag{4.94}$$

on the other hand,using also (4.67) and (4.2), we obtain

$$\overline{\varphi_\Delta}(x,w)=\varphi_{\overline{\Delta}}(x,w)=-\chi_{W\setminus\Delta(\{x\})}(w)=-\chi_{X\setminus\Delta^*(\{w\})}(x) \quad (\Delta\in D, x\in X, w\in W). \tag{4.95}$$

b) Since the families

$$Y=\{\{(x,w)\} \mid x\in X,\ w\in W\}\subset 2^{X\times W}, \tag{4.96}$$

$$Y'=\{X\times W\setminus\{(x,w)\} \mid x\in X, w\in W\}\subset 2^{X\times W}, \tag{4.97}$$

are infimal, respectively, supremal generators of $(2^{X\times W},\supseteq)$ (see example 1.5), from theorem 4.4 we obtain the following families of i fimal generators of the other complete Boolean algebras of (4.90):

$$(\rho_\circ)^{-1}(Y)=\pi(Y)=\{\pi(\{x,w\})\mid x\in X,\ w\in W\}\ \text{in}\ D((2^X,\supseteq),(2^W,\supseteq)), \qquad (4.98)$$

$$\zeta^{-1}(Y')=\{-\chi_{X\times W\setminus\{(x,w)\}}\mid x\in X, w\in W\}\ \text{in}\ \{0,-\infty\}^{X\times W}, \qquad (4.99)$$

$$\sigma^{-1}\zeta^{-1}(Y')=\{[-\chi_{X\times W\setminus\{(x,w)\}}]\mid x\in X, w\in W\}\ \text{in}\ \overline{R}^{X\times W}/\tau, \qquad (4.100)$$

where, by (4.54),

$$G^{\pi(\{x,w\})}=\{w\} \quad \text{if } G=\{x\}, \qquad (4.101)$$
$$= \emptyset \quad \text{if } G\neq\{x\}.$$

Also, the complements in (4.39) and definition 4.1 a), b) coincide with the quasi-complements with respect to these families of infimal generators, in the sense of definition 3.1 (by the uniqueness of the complement in a Boolean algebra). Thus, we obtain

$$\overline{\Delta} = \bigwedge_{\substack{x\in X, w\in W\\ \Delta\not\leq\pi(\{x,w\})}} \pi(\{x,w\}) = \bigwedge_{\substack{x\in X, w\in W\\ w\in W\setminus\Delta(\{x\})}} \pi(\{x,w\}) \quad (\Delta\in D), \qquad (4.102)$$

$$\overline{\varphi} = \inf_{\substack{x\in X, w\in W\\ \varphi(x,w)=0}} (-\chi_{X\times W\setminus\{(x,w)\}}) \quad (\varphi\in\{0,-\infty\}^{X\times W}). \qquad (4.103)$$

One can also give similar formulae for the other complements occurring in this paper, but we omit them.

c) By theorems 4.3, 4.4, and the remark made after (1.5), $\Delta_\circ$ and $\Delta_\circ^r$ are dualities, whence, by (1.30) applied to Y_1 of (1.16) in $\overline{R}^{X\times W}$ and to $\zeta^{-1}(Y')$ of (4.99) in $\{0,-\infty\}^{X\times W}$, and by (1.35), we obtain, respectively,

$$\Delta_\varphi = \bigvee_{\substack{x\in X, w\in W, d\in\overline{R}\\ \varphi(x,w)\leq d}} \Delta_{\chi_{\{(x,w)\}}\dot{+}d} \quad (\varphi\in\overline{R}^{X\times W}), \qquad (4.104)$$

$$\Delta_\varphi = \bigvee_{\substack{x\in X, w\in W\\ \varphi(x,w)=-\infty}} \Delta_{-\chi_{X\times W\setminus\{(x,w)\}}} \quad (\varphi\in\{0,-\infty\}^{X\times W}). \qquad (4.105)$$

One can also give similar formulae for the other complete inf-anti-homomorphisms onto and complete anti-isomorphisms onto occurring in this paper, but we omit them.

d) In general, q and $\sigma q:\varphi\to\varphi^{\circ}$ are not complete sup-homomorphisms.

Let us consider now the mapping $\overline{\Delta}_\circ:\varphi\to\overline{\Delta_\varphi}$, given by (4.94).

<u>Corollary 4.2.</u> a) <u>The mapping $\overline{\Delta}_\circ:\varphi\to\overline{\Delta_\varphi}$ is a lattice homomorphism and a complete inf-homomorphism of $\overline{R}^{X\times W}$ onto D, with kernel</u>

$$\operatorname{Ker}\bar{\Delta}_{\circ}=\{\varphi\in\bar{R}^{X\times W}\mid\varphi(x,w)<-1\quad((x,w)\in X\times W)\}. \tag{4.106}$$

b) The mapping $\bar{\Delta}_{\circ}^{r}=\bar{\Delta}_{\circ}|_{\{0,-\infty\}^{X\times W}}$ is a complete Boolean algebra isomorphism onto D.

Proof. a) The first statement follows from theorems 4.3 and 4.1, since $\bar{\Delta}_{\circ}=\eta\Delta_{\circ}$. Note that here, for each $\Delta\in D$, we have

$$\Delta=\Delta_{\varphi_{\Delta}}=\overline{\Delta_{\varphi_{\Delta}}}=\Delta_{\overline{\varphi_{\Delta}}}. \tag{4.107}$$

Finally, by (3.64) and (3.61), we have $\overline{\Delta_{\varphi}}=\Theta$ if and only if $\Delta_{\varphi}=\Omega$, i.e., by (4.32),

$$\begin{aligned}\Delta_{\varphi}(G)&=\emptyset\quad\text{if } G\neq\emptyset,\\ &=W\quad\text{if } G=\emptyset;\end{aligned} \tag{4.108}$$

but, by (4.63), this happens if and only if $\varphi(x,w)<-1$ $((x,w)\in X\times W)$. Thus, there holds (4.106).

The proof of b) is similar, using theorem 4.4.

Remark 4.6. a) By corollary 4.2 a), we have

$$\overline{\Delta_{\max(\varphi_1,\varphi_2)}}=\overline{\Delta_{\varphi_1}}\vee\overline{\Delta_{\varphi_2}}\qquad(\varphi_1,\varphi_2\in\bar{R}^{X\times W}), \tag{4.109}$$

$$\overline{\Delta_{\inf_{j\in J}\varphi_j}}=\bigwedge_{j\in J}\overline{\Delta_{\varphi_j}}\qquad(\{\varphi_j\}_{j\in J}\subseteq\bar{R}^{X\times W}), \tag{4.110}$$

$$\overline{\Delta_{\sup_{j\in J}\varphi_j}}\geqslant\bigvee_{j\in J}\overline{\Delta_{\varphi_j}}\qquad(\{\varphi_j\}_{j\in J}\subseteq\bar{R}^{X\times W}). \tag{4.111}$$

b) By corollary 4.2 b), we have the equality sign in (4.111) for all $\{\varphi_j\}_{j\in J}\subseteq\{0,-\infty\}^{X\times W}$, and

$$\varphi_{\overline{\bigvee_{j\in J}\Delta_j}}=\sup_{j\in J}\varphi_{\overline{\Delta_j}}\qquad(\{\Delta_j\}_{j\in J}\subseteq D), \tag{4.112}$$

$$\varphi_{\overline{\bigwedge_{j\in J}\Delta_j}}=\inf_{j\in J}\varphi_{\overline{\Delta_j}}\qquad(\{\Delta_j\}_{j\in J}\subseteq D). \tag{4.113}$$

§5. Dualities $\Delta:\bar{R}^X\to\bar{R}^W$ and conjugations $c:\bar{R}^X\to\bar{R}^W$.

Let us consider now the case when $E=\bar{R}^X$, $F=\bar{R}^W$ (where X and W are two sets) and $Y=\{\chi_{\{x\}}\dot{+}d\mid x\in X, d\in\bar{R}\}$, $T=\{\chi_{\{w\}}\dot{+}d\mid w\in W, d\in\bar{R}\}$. For $\Delta:\bar{R}^X\to\bar{R}^W$, $\mathcal{M}\subseteq\bar{R}^X$, $f\in\bar{R}^X$, we shall write f^{Δ}, $f^{\Delta\Delta^*}$, $f^{\mathcal{C}(\mathcal{M})}$, instead of $\Delta(f)$, $\Delta^*\Delta(f)$, $\mathcal{C}(\mathcal{M})(f)$.

Let us first note the following complements to the equivalence (1.35), which are immediate: For any $f\in\bar{R}^X$ and $\chi_{\{x\}}\dot{+}d\in Y$, we have

$$f\leqslant\chi_{\{x\}}\dot{+}d\iff f(x)>d, \tag{5.1}$$

$$\begin{aligned}f\geqslant\chi_{\{x\}}\dot{+}d&\iff f(x)\geqslant d\text{ and } f(x')=+\infty\quad(x'\in X\setminus\{x\})\\ &\iff f=\chi_{\{x\}}\dot{+}d'\in Y\text{ for some } d'\geqslant d.\end{aligned} \tag{5.2}$$

The definition (2.1) of $\Delta^{*}:\overline{R}^{W}\to\overline{R}^{X}$ becomes now

$$g^{\Delta^{*}} = \inf \{h\in\overline{R}^{X} \mid h^{\Delta}\leq g\}, \qquad (5.3)$$

i.e., the one given in [21], formula (4.1), for any mapping $\Delta:\overline{R}^{X}\to\overline{R}^{W}$ (not necessarily a duality). The equivalence (2.2) becomes now

$$f^{\Delta}\leq g \Leftrightarrow g^{\Delta^{*}}\leq f \qquad (f\in\overline{R}^{X},\ g\in\overline{R}^{W}), \qquad (5.4)$$

which, for a conjugation $\Delta=c:\overline{R}^{X}\to\overline{R}^{W}$, has been observed in [21], proposition 4.1; also, by [21], theorem 4.1, if $\Delta=c:\overline{R}^{X}\to\overline{R}^{W}$ is a conjugation, then so is $c^{*}:\overline{R}^{W}\to\overline{R}^{X}$ and we have $c^{**}=c$, whence a principle of dualization for conjugations.

Formula (2.3) becomes now

$$f^{\Delta}= \inf \{g\in\overline{R}^{W} \mid g^{\Delta^{*}}\leq f\} \qquad (f\in\overline{R}^{X}), \qquad (5.5)$$

and formulae (2.4)-(2.8) remain unchanged. Formulae (2.9)-(2.13) yield now, by (1.35) and (1.15),

$$g^{\Delta^{*}}=\sup \{(\chi_{\{w\}} \dot{+} e)^{\Delta^{*}} \mid w\in W,\ e\in\overline{R},\ g(w)\leq e\} \qquad (g\in\overline{R}^{W}), \qquad (5.6)$$

$$g^{\Delta^{*}}(x)=\inf \{d\in\overline{R} \mid (\chi_{\{x\}}\dot{+}d)^{\Delta}\leq g\} \qquad (g\in\overline{R}^{W},\ x\in X), \qquad (5.7)$$

$$f^{\Delta}(w)=\inf \{d\in\overline{R} \mid (\chi_{\{w\}}\dot{+}d)^{\Delta^{*}}\leq f\} \qquad (f\in\overline{R}^{X},\ w\in W), \qquad (5.8)$$

$$f^{\Delta\Delta^{*}}(x)=\inf \{h(x) \mid h\in\overline{R}^{X},\ h^{\Delta}\leq f^{\Delta}\}=$$
$$=\inf \{d\in\overline{R} \mid (\chi_{\{x\}}\dot{+}d)^{\Delta}\leq f^{\Delta}\} \qquad (f\in\overline{R}^{X}, x\in X), \qquad (5.9)$$

$$g^{\Delta^{*}\Delta}(w)=\inf \{s(w) \mid s\in\overline{R}^{W},\ s^{\Delta^{*}}\leq g^{\Delta^{*}}\}=$$
$$=\inf \{d\in\overline{R} \mid (\chi_{\{w\}}\dot{+}d)^{\Delta^{*}}\leq g^{\Delta^{*}}\} \qquad (g\in\overline{R}^{W},\ w\in W). \qquad (5.10)$$

In particular, if $\Delta=c:\overline{R}^{X}\to\overline{R}^{W}$ is a conjugation, then, using (1.8) and [16], proposition 3 c), from (5.7) above we obtain

$$g^{c^{*}}(x)=\inf \{d\in\overline{R} \mid (\chi_{\{x\}})^{c} \dot{+} -d\leq g\}=$$
$$=\inf \{d\in\overline{R} \mid (\chi_{\{x\}})^{c}\dot{+}-g\leq d\}=$$
$$=\sup ((\chi_{\{x\}})^{c}\dot{+}-g)(W) \qquad (g\in\overline{R}^{W},\ x\in X), \qquad (5.11)$$

which has been given in [21], formula (4.22); similarly, (5.8)-(5.10) yield the remaining formulae of [21], corollary 4.4.

Formula (2.14) yields now, using (1.35),

$$f=f^{\Delta\Delta^{*}} \Leftrightarrow f(x)\leq d \qquad (x\in X,\ d\in\overline{R},\ (\chi_{\{x\}}\dot{+}d)^{\Delta}\leq f^{\Delta}); \qquad (5.12)$$

hence, if $\Delta=c:\overline{R}^{X}\to\overline{R}^{W}$ is a conjugation, then, by (1.8) and [16], proposition 3 c), we obtain

$$f=f^{cc^{*}} \Leftrightarrow f(x)\leq d \qquad (x\in X, d\in\overline{R}, (\chi_{\{x\}})^{c}\dot{+}-f^{c}\leq d). \qquad (5.13)$$

Theorem 2.1 yields now, using (5.1), that

$$f^{\Delta\Delta^*}=\inf\{\chi_{\{x\}}\dot{+}d\,|\,x\in X,\ d\in\overline{R},\ \nexists\, w\in W,\ e\in\overline{R},$$

$$f\geq(\chi_{\{w\}}\dot{+}e)^{\Delta^*},\ (\chi_{\{w\}}\dot{+}e)^{\Delta^*}(x)>d\}\quad(f\in\overline{R}^X),\qquad(5.14)$$

$$f=f^{\Delta\Delta^*}\Leftrightarrow\forall x\in X,\forall d\in\overline{R},\ f(x)>d,\ \exists w\in W,\ \exists e\in\overline{R},$$

$$f\geq(\chi_{\{w\}}\dot{+}e)^{\Delta^*},\ (\chi_{\{w\}}\dot{+}e)^{\Delta^*}(x)>d,\qquad(5.15)$$

where, according to remark 2.3 a), the conditions

$$f\geq(\chi_{\{w\}}\dot{+}e)^{\Delta^*},\ (\chi_{\{w\}}\dot{+}e)^{\Delta^*}(x)>d\qquad(5.16)$$

can be expressed by saying that $(\chi_{\{w\}}\dot{+}e)^{\Delta^*}$ separates f from $\chi_{\{x\}}\dot{+}d$. In particular, if $\Delta=c:\overline{R}^X\rightarrow\overline{R}^W$ is a conjugation, then, using (1.8), the equivalence $a\dot{+}-e>d\Leftrightarrow a\dot{+}-d>e$ $(a,d,e\in\overline{R})$ and [16], proposition 3 c), it follows that (5.16) is equivalent to

$$(\chi_{\{w\}})^{c^*}(x)\dot{+}-d>e\geq(\chi_{\{w\}})^{c^*}\dot{+}-f.\qquad(5.17)$$

Remark 2.3 b) yields now, using (1.35),

$$f^{\Delta\Delta^*}=\sup\{(\chi_{\{w\}}\dot{+}d)^{\Delta^*}\,|\,w\in W,\ d\in\overline{R},\ f^{\Delta}(w)\leq d\}=$$

$$=\sup\{(\chi_{\{w\}}\dot{+}d)^{\Delta^*}\,|\,w\in W,\ d\in\overline{R},\ (\chi_{\{w\}}\dot{+}d)^{\Delta^*}\leq f\}\quad(f\in\overline{R}^X).\qquad(5.18)$$

In particular, if $\Delta=c:\overline{R}^X\rightarrow\overline{R}^W$ is a conjugation, then, by (1.8), formula (5.18) becomes

$$f^{cc^*}=\sup\{(\chi_{\{w\}})^{c^*}\dot{+}-d\,|\,w\in W,\ d\in\overline{R},\ f^{c}(w)\leq d\}=$$

$$=\sup\{(\chi_{\{w\}})^{c^*}\dot{+}-d\,|\,w\in W,\ d\in\overline{R},(\chi_{\{w\}})^{c^*}\dot{+}-d\leq f\}\quad(f\in\overline{R}^X).\qquad(5.19)$$

Formula (2.19) becomes now, for any $\mathcal{M}\subseteq\overline{R}^X$,

$$f^{\mathcal{C}(\mathcal{M})}=\sup\{m\in\mathcal{M}\,|\,m\leq f\}\qquad(f\in\overline{R}^X),\qquad(5.20)$$

i.e., the "$\mathcal{M}$-convex hull" of f, in the sense of [5].

Theorem 2.2 and formula (2.30) mean now, by (5.1), that

$$f^{\mathcal{C}(\mathcal{M})}=\inf\{\chi_{\{x\}}\dot{+}d\,|\,x\in X,\ d\in\overline{R},\ \nexists\, m\in\mathcal{M},\ f\geq m,\ m(x)>d\}\quad(f\in\overline{R}^X),\quad(5.21)$$

$$\mathrm{Fix}\ \mathcal{C}(\mathcal{M})=\{f\in\overline{R}^X\,|\,\forall x\in X,\forall d\in\overline{R},f(x)>d,\exists m\in\mathcal{M},f\geq m,m(x)>d\},\qquad(5.22)$$

$$\mathrm{Fix}\ \mathcal{C}(\mathcal{M})=\{f\in\overline{R}^X\,|\,\exists\mathcal{M}_f\subseteq\mathcal{M},\ f=\sup_{h\in\mathcal{M}_f}h\}.\qquad(5.23)$$

Again, in [20] an equivalent way has been followed (see the remark made after (4.19)-(4.21) above); let us mention that, for $\mathcal{M}\subseteq\overline{R}^X$, (5.22) has been given in [5], proposition 1.6 (i).

The observation made before theorem 2.2 says now that <u>the theories of the hulls $f\rightarrow f^{\Delta\Delta^*}$, where $\Delta:\overline{R}^X\rightarrow\overline{R}^W$ is a duality, and $\mathcal{M}$-convex hulls, where $\mathcal{M}\subseteq\overline{R}^X$, are equivalent</u>, via (2.21), which becomes now

$$\mathcal{M}=\{(\chi_{\{w\}}\dot{+}d)^{\Delta^*}|w\in W, d\in\overline{R}\}\subseteq\overline{R}^X, \quad (5.24)$$

respectively, via the dualities (1.11) or $\mathcal{C}(\mathcal{M})$. In particular, if $\Delta=c:\overline{R}^X\to\overline{R}^W$ is a conjugation, then, by (1.8), formula (5.24) becomes

$$\mathcal{M}=\{(\chi_{\{w\}})^{c^*}\dot{+}-d|w\in W, d\in\overline{R}\}\subset\overline{R}^X; \quad (5.25)$$

since $(\chi_{\{w\}})^{c^*}=\varphi_c(\cdot,w)$ $(w\in W)$, where φ_c is the coupling functional (1.39) (see [21], theorem 4.2), this $\mathcal{M}$ is nothing else than

$$\mathcal{M}=\{\varphi_c(\cdot,w)\dot{+}d|w\in W, d\in\overline{R}\}, \quad (5.26)$$

i.e., <u>the family of all "elementary functionals" (associated to φ_c), in the sense of</u> [16].

Formula (3.1), defining the partial order $\Delta_1\le\Delta_2$, becomes now

$$f^{\Delta_1}\le f^{\Delta_2} \qquad (f\in\overline{R}^X), \quad (5.27)$$

and the dualities Θ,Ω of (3.3), (3.4) are now

$$f^{\Theta}\equiv-\infty \qquad (f\in\overline{R}^X), \quad (5.28)$$

$$\begin{aligned} f^{\Omega}&\equiv+\infty \quad \text{if } f\not\equiv+\infty,\\ &\equiv-\infty \quad \text{if } f\equiv+\infty. \end{aligned} \quad (5.29)$$

Proposition 3.1 means now that, for $\Delta_1,\Delta_2\in D=D(\overline{R}^X,\overline{R}^W)$,

$$\Delta_1\le\Delta_2\Leftrightarrow(\chi_{\{x\}}\dot{+}d)^{\Delta_1}\le(\chi_{\{x\}}\dot{+}d)^{\Delta_2} \qquad (x\in X, d\in\overline{R}). \quad (5.30)$$

We shall denote by $C=C(\overline{R}^X,\overline{R}^W)$ the set of all conjugations $c:\overline{R}^X\to\overline{R}^W$. Since $C\subset D=D(\overline{R}^X,\overline{R}^W)$, we have a natural partial order on C, induced by the one of D, i.e., by (5.27).

<u>Theorem 5.1. For any</u> $\varphi_j\in\overline{R}^{X\times W}$ $(j\in J)$ <u>we have (where</u> $\bigvee_{j\in J}c(\varphi_j)$, $\bigwedge_{j\in J}c(\varphi_j)$ <u>are taken in</u> D)

$$f^{\bigvee_{j\in J}c(\varphi_j)}=\sup_{j\in J}f^{c(\varphi_j)}=f^{c(\sup_{j\in J}\varphi_j)} \qquad (f\in\overline{R}^X), \quad (5.31)$$

$$(\chi_{\{x\}}\dot{+}d)^{\widehat{\bigwedge}_{j\in J}c(\varphi_j)}=\inf_{j\in J}(\chi_{\{x\}}\dot{+}d)^{c(\varphi_j)}=(\chi_{\{x\}}\dot{+}d)^{c(\inf_{j\in J}\varphi_j)} \qquad (x\in X, d\in R\cup\{+\infty\}), \quad (5.32)$$

$$f^{\widehat{\bigwedge}_{j\in J}c(\varphi_j)}=\sup\{\inf_{j\in J}(\chi_{\{x\}}\dot{+}d)^{c(\varphi_j)}|x\in X, d\in R\cup\{+\infty\}, f(x)\le d\}=$$
$$=f^{c(\inf_{j\in J}\varphi_j)} \qquad (f\in\overline{R}^X). \quad (5.33)$$

<u>Hence, C is a complete sublattice of</u> $D=D(\overline{R}^X,\overline{R}^W)$ (<u>and thus</u> $\bigvee^C_{j\in J}=\bigvee^D_{j\in J}$, $\bigwedge^C_{j\in J}=\bigwedge^D_{j\in J}$), <u>and the one-to-one mapping</u> $c(\cdot):\varphi\to c(\varphi)$, <u>defined by</u> (1.37) (<u>with inverse</u> $c\to\varphi_c$, <u>of</u> (1.39)) <u>is a complete lattice isomorphism of</u> $\overline{R}^{X\times W}$ <u>onto</u> C.

<u>Proof</u>. Let us recall that if Z is any set and $h:Z\to\overline{R}$, then, by

[16], formula (4.8) and [23], lemma 2.1, we have

$$\sup_{z\in Z}\{h(z)\dot{+}a\}=\sup_{z\in Z} h(z)\dot{+}a \qquad (a\in\overline{R}), \tag{5.34}$$

$$\inf_{z\in Z}\{h(z)\dot{+}a\}=\inf_{z\in Z} h(z)\dot{+}a \qquad (a\in R\cup\{-\infty\}). \tag{5.35}$$

From (3.8), (1.37) and (5.34), we obtain

$$f^{\bigvee_{j\in J}c(\varphi_j)}(w)=\sup_{j\in J} f^{c(\varphi_j)}(w)=\sup_{j\in J}\sup_{x\in X}\{\varphi_j(x,w)\dot{+}-f(x)\}=$$

$$=\sup_{x\in X}\{\sup_{j\in J}\varphi_j(x,w)\dot{+}-f(x)\}=f^{c(\sup_{j\in J}\varphi_j)} \qquad (f\in\overline{R}^X,\ w\in W).$$

Furthermore, by (1.8), (1.38) and (5.35) (with a=-d), we have

$$(\chi_{\{x\}}\dot{+}d)^{c(\inf_{j\in J}\varphi_j)}=(\chi_{\{x\}})^{c(\inf_{j\in J}\varphi_j)}\dot{+}-d=$$

$$=\inf_{j\in J}\{\varphi_j(x,\cdot)\dot{+}-d\}=\inf_{j\in J}(\chi_{\{x\}}\dot{+}d)^{c(\varphi_j)} \qquad (x\in X, d\in R\cup\{+\infty\}).$$

Thus, since Y_1' of (1.18) is a family of infimal generators of $\overline{R}$ condition 2° of theorem 3.1 b) for $\Delta_j=c(\varphi_j)$ $(j\in J)$ is satisfied (with $\Delta=c(\inf_{j\in J}\varphi_j)$). Hence, by theorem 3.1 b) and corollary 1.3, we ha (5.32) and (5.33). Thus, by (5.31) and (5.33),

$$\bigvee_{j\in J} c(\varphi_j)=c(\sup_{j\in J}\varphi_j)\in C \qquad (\{\varphi_j\}_{j\in J}\subseteq\overline{R}^{X\times W}), \tag{5.36}$$

$$\bigwedge_{j\in J} c(\varphi_j)=c(\inf_{j\in J}\varphi_j)\in C \qquad (\{\varphi_j\}_{j\in J}\subseteq\overline{R}^{X\times W}), \tag{5.37}$$

whence the other statements (of theorem 5.1) follow.

Remark 5.1. a) By theorem 5.1 (or, by (1.8) and (1.39)), we hav

$$c_1\le c_2\Leftrightarrow\varphi_{c_1}\le\varphi_{c_2} \qquad (c_1,c_2\in C). \tag{5.38}$$

b) By theorem 5.1 (or, by (1.37)), the smallest (greatest) ele- ments in C and D coincide, namely,

$$c(-\infty)=\Theta,\quad c(+\infty)=\Omega. \tag{5.39}$$

c) Since

$$Y_1=\{\chi_{\{(x,w)\}}\dot{+}d\mid(x,w)\in X\times W,\ d\in\overline{R}\} \tag{5.40}$$

is a family of infimal generators of $\overline{R}^{X\times W}$ (see example 1.6), from theorem 5.1 it follows that

$$c(Y_1)=\{c(\chi_{\{(x,w)\}}\dot{+}d)\mid (x,w)\in X\times W, d\in\overline{R}\} \tag{5.41}$$

is a family of infimal generators of $C(\overline{R}^X,\overline{R}^W)$, where, by (1.37) and (1.15), for each $f\in\overline{R}^X$ we have

$$f^{c(\chi_{\{(x,w)\}}\dot{+}d)}(w')=\sup_{x'\in X}\{(\chi_{\{(x,w)\}}(x',w')\dot{+}d)\dot{+}-f(x')\}=$$
$$=d\dot{+}-f(x) \quad \text{if } w'=w \text{ and } f\in\chi_{\{x\}}\dot{+}\overline{R},$$
$$=+\infty \quad \text{otherwise.} \tag{5.42}$$

Theorem 5.2. The restriction of the complete lattice isomorphism $\Delta\in D(\overline{R}^X,\overline{R}^W)\to\Delta^*\in D^*=D(\overline{R}^W,\overline{R}^X)$ (see theorem 3.2) to $C=C(\overline{R}^X,\overline{R}^W)$, i.e., the mapping $c\to c^*$ (where c^* is defined by (5.3)), is a complete lattice isomorphism of $C=C(\overline{R}^X,\overline{R}^W)$ onto $C^*=\{c^*|c\in C\}=C(\overline{R}^W,\overline{R}^X)$, the complete lattice of all conjugations from $\overline{R}^W$ into $\overline{R}^X$.

Proof. By the remarks made after formula (5.4), we have $C^*\subseteq C(\overline{R}^W,\overline{R}^X)$. Conversely, if $c'\in C(\overline{R}^W,\overline{R}^X)$, then for $c=(c')^*$ we have $c^*=(c')^{**}=c'$, so $C^*=C(\overline{R}^W,\overline{R}^X)$ and $c\to c^*$ maps C onto C^*. Hence, by theorems 3.2 and 5.1, the conclusion follows.

Remark 5.2. One can also give the following proof of theorem 5.2, similar to the argument of remark 4.2 b): Since the mapping $\varphi\to\varphi^-$ $(\varphi\in\overline{R}^{X\times W})$, defined by

$$\varphi^-(w,x)=\varphi(x,w) \qquad (x\in X, w\in W), \tag{5.43}$$

is a complete lattice isomorphism of $\overline{R}^{X\times W}$ onto $\overline{R}^{W\times X}$, the mapping $c\to c^*$ is the composition of the complete lattice isomorphisms "onto"

$$c\to\varphi_c\to(\varphi_c)^-\to c((\varphi_c)^-)=c^*; \tag{5.44}$$

indeed, by (5.11), (1.39), (5.43) and (1.37) (applied to $(\varphi_c)^-:W\times X\to\overline{R}$), we have

$$g^{c^*}(x)=\sup((\chi_{\{x\}})^c\dot{+}-g)(W)=\sup_{w\in W}\{(\varphi_c)^-(w,x)\dot{+}-g(w)\}=$$
$$=g^{c((\varphi_c)^-)}(x) \qquad (x\in X,\ g\in\overline{R}^W). \tag{5.45}$$

Let us recall that, for each coupling functional $\varphi:X\times W\to\overline{R}$, the "conjugation of type Lau $L(\varphi):\overline{R}^X\to\overline{R}^W$ associated to φ" is defined ([22], definition 3.2) by

$$f^{L(\varphi)}(w)=-\inf_{\substack{x\in X\\ \varphi(x,w)\geq -1}} f(x) \qquad (f\in\overline{R}^X,\ w\in W); \tag{5.46}$$

we shall denote by $CL=CL(\overline{R}^X,\overline{R}^W)$ the set of all conjugations of type Lau from $\overline{R}^X$ into $\overline{R}^W$.

We recall that, by [22], corollary 3.2, for each $\varphi\in\overline{R}^{X\times W}$ there exists a unique $\varphi^o\in\{0,-\infty\}^{X\times W}$, namely, φ^o of (4.81), such that

$$L(\varphi)=c(\varphi^o); \tag{5.47}$$

hence, in particular ([22], corollary 3.1),

$$L(\varphi)=c(\varphi) \qquad (\varphi\in\{0,-\infty\}^{X\times W}). \tag{5.48}$$

From (5.47) and [21], theorem 3.1, it follows that $c:\overline{R}^X \to \overline{R}^W$ is a conjugation of type Lau if and only if it satisfies (1.7), (1.8) an

$$(\chi_{\{x\}})^c \in \{0,-\infty\}^W \qquad (x\in X), \tag{5.49}$$

or, equivalently (where φ_c is the functional (1.39)),

$$\varphi_c \in \{0,-\infty\}^{X\times W}; \tag{5.50}$$

however, if c satisfies these conditions, then there are many functionals $\varphi\in\overline{R}^{X\times W}$ satisfying $L(\varphi)=c$ (see (5.60) below).

Theorem 5.3. $CL=CL(\overline{R}^X,\overline{R}^W)$ is a complete sublattice of $C=C(\overline{R}^X, \overline{R}$ and the mapping $L:\varphi\to L(\varphi)$ is a lattice homomorphism and a complete i -homomorphism of $\overline{R}^{X\times W}$ onto CL, with kernel

$$\text{Ker } L=\{\varphi\in\overline{R}^{X\times W} \mid \varphi(x,w)<-1 \quad ((x,w)\in X\times W)\}. \tag{5.51}$$

Proof. By the definition of CL, the mapping L maps $\overline{R}^{X\times W}$ onto CL. Furthermore, by (5.47), (5.36) and (5.37), for any $\{\varphi_j\}_{j\in J}\subseteq\overline{R}^{X\times}$ we have (where $\bigvee_{j\in J} L(\varphi_j)$, $\bigwedge_{j\in J} L(\varphi_j)$ are taken in C, and $\varphi_j^o=(\varphi_j)^o$),

$$\bigvee_{j\in J} L(\varphi_j)=\bigvee_{j\in J} c(\varphi_j^o)=c(\sup_{j\in J} \varphi_j^o)=L(\sup_{j\in J} \varphi_j^o)\in CL, \tag{5.52}$$

$$\bigwedge_{j\in J} L(\varphi_j)=\bigwedge_{j\in J} c(\varphi_j^o)=c(\inf_{j\in J} \varphi_j^o)=L(\inf_{j\in J} \varphi_j^o)\in CL, \tag{5.53}$$

so CL is a complete sublattice of C (and thus $\bigvee_{j\in J}^{CL}=\bigvee_{j\in J}^{C}$, $\bigwedge_{j\in J}^{CL}=\bigwedge_{j\in J}^{C}$). Also, by (5.47), (5.36), (5.37) and using that $\sigma q:\varphi\to\varphi^o$ ($\varphi\in\overline{R}^{X\times W}$) is a lattice homomorphism and a complete inf-homomorphism (see (4.90)), we obtain

$$L(\max(\varphi_1,\varphi_2))=c((\max(\varphi_1,\varphi_2))^o)=c(\max(\varphi_1^o,\varphi_2^o))=c(\varphi_1^o)\vee c(\varphi_2^o)=L(\varphi_1)\vee L(\varphi_2), \tag{5.54}$$

$$L(\inf_{j\in J} \varphi_j)=c((\inf_{j\in J} \varphi_j)^o)=c(\inf_{j\in J} \varphi_j^o)=\bigwedge_{j\in J} c(\varphi_j^o)=\bigwedge_{j\in J} L(\varphi_j), \tag{5.55}$$

and thus L is a lattice homomorphism and a complete inf-homomorphis

Finally, if $\varphi(x,w)<-1$ $((x,w)\in X\times W)$, then, by (5.46) and (5.28),

$$f^{L(\varphi)}(w)=-\inf \emptyset=-\infty=f^{\ominus}(w) \qquad (f\in\overline{R}^X,\ w\in W);$$

conversely, if $f^{L(\varphi)}=f^{\ominus}$ $(f\in\overline{R}^X)$, then, by (5.46) and (5.28), we obtai

$$-\inf_{\substack{x\in X\\ \varphi(x,w)\geq -1}} \chi_{\{x\in X \mid \varphi(x,w)\geq -1\}}=-\infty \qquad (w\in W),$$

whence $\varphi(x,w)<-1$ $(x\in X,\ w\in W)$.

Remark 5.3. a) By theorem 5.3, L is isotone, and hence

$$L(\sup_{j\in J} \varphi_j)\geq\bigvee_{j\in J} L(\varphi_j) \qquad (\{\varphi_j\}_{j\in J}\subseteq\overline{R}^{X\times W}). \tag{5.56}$$

b) In general, the mapping L of theorem 5.3 is not a complete su homomorphism. However, by (5.48), $L^r=L|_{\{0,-\infty\}^{X\times W}}=c|_{\{0,-\infty\}^{X\times W}}$ is a complete lattice isomorphism of the complete sublattice $\{0,-\infty\}^{X\times W}$ of $\overline{R}^{X\times W}$, onto CL, with

$$(L^r)^{-1}(L(\varphi))=\varphi^o \qquad (\varphi\in\overline{R}^{X\times W}) \tag{5.57}$$

(see (5.63) below); for a sharpening, see theorem 5.4 below. Hence, since $-\infty\leq\varphi\leq 0$ $(\varphi\in\{0,-\infty\}^{X\times W})$, the smallest and greatest elements of CL are, respectively,

$$L(-\infty)=c(-\infty)=\Theta, \quad L(0)=c(0), \tag{5.58}$$

where

$$f^{L(0)}(w)=f^{c(0)}(w)=-\inf f(X) \qquad (f\in\overline{R}^X, w\in W). \tag{5.59}$$

c) By theorem 5.3 and e.g. [9], Ch.I, §3, theorem 11, one can define <u>a congruence τ on</u> $\overline{R}^{X\times W}$, by

$$\varphi_1\equiv\varphi_2(\tau)\Longleftrightarrow L(\varphi_1)=L(\varphi_2), \tag{5.60}$$

and the (well defined) mapping

$$\hat{L}:[\varphi]=\{\varphi'\in\overline{R}^{X\times W}\mid \varphi'\equiv\varphi(\tau)\}\longrightarrow L(\varphi') \quad (\varphi'\in[\varphi], \varphi\in\overline{R}^{X\times W}) \tag{5.61}$$

<u>is a lattice isomorphism of the quotient lattice</u> $\overline{R}^{X\times W}/\tau$ <u>onto</u> CL, with

$$\hat{L}^{-1}(L(\varphi))=[\varphi] \qquad (\varphi\in\overline{R}^{X\times W}) \tag{5.62}$$

(the fact that L is well defined, one-to-one, satisfying (5.62), has been observed in [22], remark 3.3 c)); for a sharpening, see theorem 5.4 below. Moreover, by [22], corollay 3.3, for $\varphi_1,\varphi_2\in\overline{R}^{X\times W}$ we have $L(\varphi_1)=L(\varphi_2)$ if and only if $\Delta_{\varphi_1}=\Delta_{\varphi_2}$; thus, <u>the congruences τ of</u> (5.60) <u>and</u> (4.77) <u>coincide</u>. In particular, for φ^o of (4.81) we obtain

$$L(\varphi)=L(\varphi^o) \qquad (\varphi\in\overline{R}^{X\times W}) \tag{5.63}$$

(this also follows from (5.47) and (5.48)). Note also that, by (5.51), (5.61) and (4.80), we have

$$\mathrm{Ker}\ L=[-\infty]=\mathrm{Ker}\ \tau. \tag{5.64}$$

<u>Definition 5.1.</u> a) For each $\varphi\in\overline{R}^{X\times W}$, we define the <u>complementary conjugation of type Lau</u> $\overline{L(\varphi)}:\overline{R}^X\longrightarrow\overline{R}^W$ <u>to</u> $L(\varphi)$, by

$$f^{\overline{L(\varphi)}}(w)=-\inf_{\substack{x\in X\\ \varphi(x,w)<-1}} f(x)=f^{L(\overline{\varphi^o})}(w) \quad (f\in\overline{R}^X, w\in W), \tag{5.65}$$

with φ^o of (4.81) and $\overline{\varphi^o}$ of definition 4.1 a).

b) We define a congruence μ on $C(\overline{R}^X,\overline{R}^W)$, by

$$c(\varphi_1)\equiv c(\varphi_2)(\mu)\Longleftrightarrow \varphi_1\equiv\varphi_2(\tau), \tag{5.66}$$

with $c(\varphi_i)$ of (1.37) and τ of (5.60)=(4.77). Furthermore, for each $[c(\varphi)]$ in the quotient lattice $C(\overline{R}^X,\overline{R}^W)/\mu$, we define the <u>complementary</u>

class $\overline{[c(\varphi)]}$ to $[c(\varphi)]$, by

$$\overline{[c(\varphi)]}=[c(\overline{\varphi^{o}})]=\{c'\in C(\overline{R}^{X},\overline{R}^{W}) \mid c'\equiv c(\overline{\varphi^{o}})(\mu)\}. \qquad (5.67)$$

Remark 5.4. a) By (5.63), (5.65), (5.55), (5.54), (4.87) and (5.58), we have

$$f^{L(\varphi)\wedge\overline{L(\varphi)}}=f^{L(\varphi^{o})\wedge L(\overline{\varphi^{o}})}=f^{L(\min(\varphi^{o},\overline{\varphi^{o}}))}=f^{L(-\infty)}=f^{\Theta} \quad (f\in\overline{R}^{X},\varphi\in\overline{R}^{X\times W}), \qquad (5.68)$$

$$f^{L(\varphi)\vee\overline{L(\varphi)}}=f^{L(\varphi^{o})\vee L(\overline{\varphi^{o}})}=f^{L(\max(\varphi^{o},\overline{\varphi^{o}}))}=f^{L(0)}=f^{c(0)} \quad (f\in\overline{R}^{X},\varphi\in\overline{R}^{X\times W}), \qquad (5.69)$$

so $(CL,\leq,L(\varphi)\rightarrow\overline{L(\varphi)})$ is a complemented lattice.

b) Since the quotient mapping

$$s : c(\varphi)\longrightarrow[c(\varphi)] \qquad (\varphi\in\overline{R}^{X\times W}) \qquad (5.70)$$

is a lattice homomorphism of $C(\overline{R}^{X},\overline{R}^{W})$ onto $C(\overline{R}^{X},\overline{R}^{W})/\mu$ (see e.g. [9]), we obtain, by $\varphi\equiv\varphi^{o}(\tau)$, (5.66), (5.67), (5.48) and (5.68),

$$\inf([c(\varphi)],\overline{[c(\varphi)]})=\inf([c(\varphi^{o})],[c(\overline{\varphi^{o}})])=[c(\varphi^{o})\wedge c(\overline{\varphi^{o}})]=$$

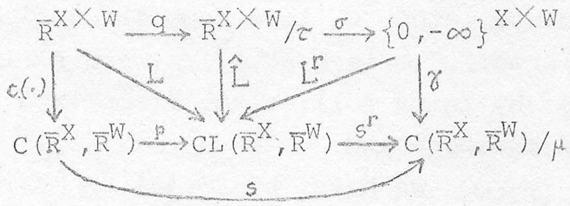
$$=[L(\varphi^{o})\wedge\overline{L(\varphi^{o})}]=[\Theta] \qquad (\varphi\in\overline{R}^{X\times W}), \qquad (5.71)$$

and, similarly,

$$\sup([c(\varphi)],\overline{[c(\varphi)]})=[c(0)] \qquad (\varphi\in\overline{R}^{X\times W}), \qquad (5.72)$$

so $(C(\overline{R}^{X},\overline{R}^{W})/\mu,\leq,[c(\varphi)]\rightarrow\overline{[c(\varphi)]})$ is a complemented lattice.

Theorem 5.4. $CL=(CL,\leq,L(\varphi)\rightarrow\overline{L(\varphi)})$ and $(C(\overline{R}^{X},\overline{R}^{W})/\mu,\leq,[c(\varphi)]\rightarrow$ $\rightarrow\overline{[c(\varphi)]})$ are complete Boolean algebras and the diagram



(5.73)

is commutative, where τ and μ are the congruences (5.60)=(4.77) and (5.66) respectively, q and σ are as in theorem 4.4, $c(\cdot)$ is the complete lattice isomorphism onto, of theorem 5.1, L is as in theorem 5.3, the mapping

$$p : c(\varphi)\longrightarrow L(\varphi) \qquad (\varphi\in\overline{R}^{X\times W}) \qquad (5.74)$$

and the quotient mapping s of (5.70) are lattice homomorphisms and complete inf-homomorphisms onto, and $\hat{L}$, L^{r} (of remark 5.3 c), b)), $s^{r}=s|_{CL}$ and $\gamma=sc(\cdot)|_{\{0,-\infty\}^{X\times W}}$ are Boolean algebra isomorphisms onto, with

$$(s^{r})^{-1}([c(\varphi)])=L(\varphi)=L(\varphi^{o}) \qquad (\varphi\in\overline{R}^{X\times W}); \qquad (5.75)$$

$$\gamma^{-1}([c(\varphi)])=\varphi^{o} \qquad (\varphi\in\overline{R}^{X\times W}), \qquad (5.76)$$

where φ^{o} is defined by (4.81).

Proof. The proofs of the statements on CL and $\hat{L}$, L^{r}, are similar to the proof of theorem 4.2, using that $\overline{R}^{X\times W}/\tau$ and $\{0,-\infty\}^{X\times W}$ are complete Boolean algebras (theorem 4.4) and using remark 5.3 c), b) and lemma 4.1. Furthermore, by (5.74) we have $pc(\cdot)=L$, whence $p=L(c(\cdot))^{-1}$ is a lattice homomorphism and a complete inf-homomorphism onto (by theorems 5.1 and 5.3). Hence, e.g. by [9], Ch. I, §3, theorem 11, one can define a congruence μ_p on $C(\overline{R}^X,\overline{R}^W)$, by

$$c_1\equiv c_2(\mu_p)\Longleftrightarrow p(c_1)=p(c_2), \tag{5.77}$$

which induces a lattice isomorphism of $C(\overline{R}^X,\overline{R}^W)/\mu_p$ onto CL. But, by $s^r=s|_{CL}$, (5.70), (5.47), (5.66), (5.60) and (5.63), we have

$$s^r(L(\varphi_1))=s^r(L(\varphi_2))\Longleftrightarrow[L(\varphi_1)]=[L(\varphi_2)]\Longleftrightarrow[c(\varphi_1^o)]=[c(\varphi_2^o)]\Longleftrightarrow$$

$$\Longleftrightarrow\varphi_1^o\equiv\varphi_2^o(\tau)\Longleftrightarrow L(\varphi_1)=L(\varphi_2) \qquad (\varphi_1,\varphi_2\in\overline{R}^{X\times W}), \tag{5.78}$$

so s^r is one-to-one. Also, by (5.70), $\varphi\equiv\varphi^o(\tau)$, (5.66), (5.47) and (5.74),

$$s(c(\varphi))=[c(\varphi)]=[c(\varphi^o)]=[L(\varphi)]=s^r(L(\varphi))=s^r(p(c(\varphi)) \qquad (\varphi\in\overline{R}^{X\times W}), \tag{5.79}$$

so $s=s^r p$, and hence s^r maps CL <u>onto</u> $C(\overline{R}^X,\overline{R}^W)/\mu$ and there holds (5.75). Moreover, by (5.77), (5.74), (5.60) and (5.66), we have

$$c(\varphi_1)\equiv c(\varphi_2)(\mu_p)\Longleftrightarrow p(c(\varphi_1))=p(c(\varphi_2))\Longleftrightarrow L(\varphi_1)=L(\varphi_2)$$

$$\Longleftrightarrow\varphi_1\equiv\varphi_2(\tau)\Longleftrightarrow c(\varphi_1)\equiv c(\varphi_2)(\mu) \qquad (\varphi_1,\varphi_2\in\overline{R}^{X\times W}),$$

whence $\mu_p=\mu$. Thus, by (5.77), (5.74) and (5.75), the lattice isomorphism of $C(\overline{R}^X,\overline{R}^W)/\mu_p=C(\overline{R}^X,\overline{R}^W)/\mu$ onto CL, induced by $\mu_p=\mu$, is the mapping

$$[c(\varphi)]\longrightarrow p(c(\varphi))=L(\varphi)=(s^r)^{-1}([c(\varphi)]) \qquad (\varphi\in\overline{R}^{X\times W}), \tag{5.80}$$

i.e., the mapping $(s^r)^{-1}$. Hence, since CL is a complete Boolean algebra (see the beginning of this proof), so is $C(\overline{R}^X,\overline{R}^W)/\mu$, and s^r is a complete Boolean algebra isomorphism (by lemma 4.1); also, $s=s^r p$ is a lattice homomorphism and a complete inf-homomorphism.

Now, for $\gamma=sc(\cdot)|_{\{0,-\infty\}^{X\times W}}$ we have, by (5.48),

$$\gamma(\varphi)=sc(\varphi)=s^r(L^r(\varphi)) \qquad (\varphi\in\{0,-\infty\}^{X\times W}), \tag{5.81}$$

and hence γ is a complete Boolean algebra isomorphism onto (since so are L^r and s^r), satisfying, by (5.75) and (5.57),

$$\gamma^{-1}([c(\varphi)])=(L^r)^{-1}(s^r)^{-1}([c(\varphi)])=(L^r)^{-1}(L(\varphi))=\varphi^o \quad (\varphi\in\overline{R}^{X\times W}),$$

i.e., (5.76). Finally, by (4.82), $\varphi^o\equiv\varphi(\tau)$, (5.61) and (4.93), we have

$$L^r\sigma([\varphi])=L(\varphi^o)=\hat{L}([\varphi]) \qquad (\varphi\in\overline{R}^{X\times W}),$$

$$\hat{L}q(\varphi)=\hat{L}([\varphi])=L(\varphi) \qquad (\varphi\in\overline{R}^{X\times W}),$$

so the diagram (5.73) is commutative.

<u>Remark 5.5.</u> a) The equivalence relation (5.66) (not regarded as a lattice congruence) has been introduced in [22], together with the mapping $\hat{c}(\cdot):\overline{R}^{X\times W}/\tau\rightarrow C(\overline{R}^X,\overline{R}^W)/\mu$ defined by

$$\hat{c}([\varphi])=[c(\varphi)] \qquad (\varphi\in\overline{R}^{X\times W}), \tag{5.82}$$

where it has been observed that $\hat{c}(\cdot)$ is one-to-one and onto. From the above it follows that $\hat{c}(\cdot)$ is a complete Boolean algebra isomorphism onto, which can be inserted in the diagram (5.73), so that the new diagram remains commutative. The fact that s^r is one-to-one and onto, satisfying (5.75), has been observed, essentially, in [22], remark 3.3 d); for (5.76), some parts of (5.78), (5.79), and some related properties, see [22].

b) From (4.99), (5.81), (5.70), (5.48) and theorem 5.4, we obtain the following families of infimal generators of $C(\overline{R}^X,\overline{R}^W)/\mu$ and CL respectively:

$$\gamma(\zeta^{-1}(Y'))=\{[c(-\chi_{X\times W\setminus\{(x,w)\}})]\mid x\in X,\ w\in W\}\text{ in } C(\overline{R}^X,\overline{R}^W)/\mu, \tag{5.83}$$

$$L^r(\zeta^{-1}(Y'))=\{c(-\chi_{X\times W\setminus\{(x,w)\}})\mid x\in X,\ w\in W\}\text{ in CL}, \tag{5.84}$$

where, by (1.37),

$$f^{c(-\chi_{X\times W\setminus\{(x,w)\}})}(w')=\begin{cases}-\inf f(X) & \text{if } w'\neq w,\\ -\inf f(X\setminus\{x\}) & \text{if } w'=w.\end{cases} \tag{5.85}$$

Also, the complements (5.67), (5.65) coincide with the quasi-complements with respect to (5.83), (5.84), in the sense of definition 3.1.

Corollary 5.1. a) The mapping $\overline{L}:\varphi\to\overline{L(\varphi)}$ is a lattice anti-homomorphism and a complete inf-anti-homomorphism of $\overline{R}^{X\times W}$ onto CL, with kernel

$$\operatorname{Ker}\overline{L}=\{\varphi\in\overline{R}^{X\times W}\mid \varphi(x,w)\geq -1 \quad ((x,w)\in X\times W)\}. \tag{5.86}$$

b) The mapping $\overline{L}^r=\overline{L}\mid_{\{0,-\infty\}^{X\times W}}$ is a complete Boolean algebra anti-isomorphism onto CL.

Proof. a) The first statement follows from theorems 5.4 and 4.4 and corollary 4.2 a), since we can write $\overline{L}$ as the composition of the mappings

$$\varphi\xrightarrow{\Delta_\cdot}\overline{\Delta_\varphi}=\Delta_{\overline{\varphi^O}}\xrightarrow{(\Delta^r_\cdot)^{-1}}\overline{\varphi^O}\xrightarrow{L^r}L(\overline{\varphi^O})=\overline{L(\varphi)}\qquad(\varphi\in\overline{R}^{X\times W}),$$

where the equalities hold by (4.94) and (5.65). The proof of (5.86) is similar to that of (5.51), using the first equality of (5.65).

The proof of b) is similar, using theorem 5.4.

Remark 5.6. Corresponding to remark 4.6, let us note, for example, that

$$\overline{L(\max(\varphi_1,\varphi_2))}=\overline{L(\varphi_1)}\wedge\overline{L(\varphi_2)}\qquad(\varphi_1,\varphi_2\in\overline{R}^{X\times W}), \tag{5.87}$$

$$\overline{L(\inf_{j\in J}\varphi_j)}=\bigvee_{j\in J}\overline{L(\varphi_j)}\qquad(\{\varphi_j\}_{j\in J}\subseteq\overline{R}^{X\times W}). \tag{5.88}$$

Theorem 5.5. The restriction of the complete lattice isomorphism $c\in C(\overline{R}^X,\overline{R}^W)\to c^*\in C^*=C(\overline{R}^W,\overline{R}^X)$ (see theorem 5.2) to $CL=CL(\overline{R}^X,\overline{R}^W)$, i.e., the mapping $L(\varphi)\to L(\varphi)^*$, is a complete Boolean algebra isomorphism of $CL=CL(\overline{R}^X,\overline{R}^W)$ onto $(CL)^*=\{L(\varphi)^*\mid\varphi\in\overline{R}^{X\times W}\}=CL(\overline{R}^W,\overline{R}^X)$, the complete lattice of all conjugations of type Lau from $\overline{R}^W$ into $\overline{R}^X$, and we have (where φ^- is given by (5.43))

$$L(\varphi)^{*}=L(\varphi^{-})=L((\varphi^{o})^{-}) \qquad (\varphi\in\overline{R}^{X\times W}). \tag{5.89}$$

<u>Proof.</u> By (4.81) and (5.43), we have

$$(\varphi^{o})^{-}=-\chi_{\{(w,x)\in W\times X \mid \varphi(x,w)\geq -1\}}=(\varphi^{-})^{o} \qquad (\varphi\in\overline{R}^{X\times W}), \tag{5.90}$$

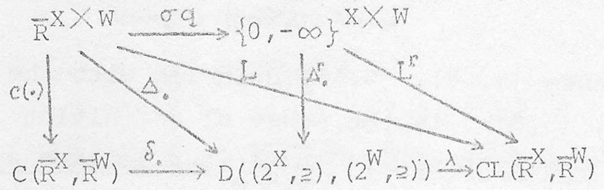
whence, by (5.47), (5.44) (applied to $c=c(\varphi^{o})$), (5.48) and (5.63), we obtain

$$L(\varphi)^{*}=c(\varphi^{o})^{*}=c((\varphi^{o})^{-})=c((\varphi^{-})^{o})=L((\varphi^{-})^{o})=L(\varphi^{-}) \quad (\varphi\in\overline{R}^{X\times W}),$$

which proves (5.89) and that $(CL)^{*}\subseteq CL(\overline{R}^{W},\overline{R}^{X})$. Hence, if $L(\gamma)\in CL(\overline{R}^{W},\overline{R}^{X})$, where $\gamma\in\overline{R}^{W\times X}$, then $L(\gamma)^{*}\in CL(\overline{R}^{X},\overline{R}^{W})=CL$ and $L(\gamma)=L(\gamma)^{**}\in(CL)^{*}$, so $(CL)^{*}=$ $=CL(\overline{R}^{W},\overline{R}^{X})$. Thus, by theorem 5.2, the conclusion follows.

Combining the diagrams (4.90) and (5.73), one obtains further results, for example, the following

<u>Theorem 5.6. The diagram</u>



(5.91)

<u>is commutative, where, for each</u> $c\in C(\overline{R}^{X},\overline{R}^{W})$ <u>and</u> $\Delta\in D((2^{X},\supseteq),(2^{W},\supseteq))$,

$$\delta_{c}(G)=\{w\in W \mid (\chi_{\{x\}})^{c}(w)\geq -1 \quad (x\in G)\} \qquad (G\subseteq X), \tag{5.92}$$

$$f^{\lambda(\Delta)}(w)=-\inf_{\substack{x\in X\\ w\in\Delta(\{x\})}} f(x)=-\inf f(\Delta^{*}(\{w\})) \qquad (f\in\overline{R}^{X},\ w\in W). \tag{5.93}$$

<u>Hence,</u> $\delta_\circ$ <u>is a lattice anti-homomorphism and a complete inf-anti-homomorphism of</u> $C(\overline{R}^{X},\overline{R}^{W})$ <u>onto</u> $D((2^{X},\supseteq),(2^{W},\supseteq))$, <u>with</u>

$$\operatorname{Ker}\delta_\circ=c(\operatorname{Ker}\Delta_\circ) \tag{5.94}$$

(see (4.68)), <u>and</u> λ <u>is a complete Boolean algebra anti-isomorphism of</u> $D(2^{X},\supseteq),(2^{W},\supseteq))$ <u>onto</u> $CL(\overline{R}^{X},\overline{R}^{W})$, <u>satisfying</u>

$$\lambda^{-1}=\delta_\circ^{r}=\delta_\circ|_{CL(\overline{R}^{X},\overline{R}^{W})}. \tag{5.95}$$

<u>Proof.</u> By (5.92), (1.38), (4.63), (5.93) and (5.46), we have

$$\delta_{c(\varphi)}(G)=\{w\in W \mid \varphi(x,w)\geq -1 \ \ (x\in G)\}=\Delta_{\varphi}(G) \qquad (\varphi\in\overline{R}^{X\times W},\ G\subseteq X), \tag{5.96}$$

$$f^{\lambda(\Delta_{\varphi})}(w)=-\inf_{\substack{x\in X\\ \varphi(x,w)\geq -1}} f(x)=f^{L(\varphi)}(w) \qquad (\varphi\in\overline{R}^{X\times W}, f\in\overline{R}^{X},\ w\in W), \tag{5.97}$$

so the diagram (5.91) is commutative. Hence, using also (5.63) and (5.47), we obtain

$$\lambda^{-1}(L(\varphi))=\lambda^{-1}(L(\varphi^{o}))=\Delta_{\varphi^{o}}=\delta_{c(\varphi^{o})}=\delta_{L(\varphi)} \qquad (\varphi\in\overline{R}^{X\times W}),$$

so (5.95) holds. The other statements follow from theorems 4.4 and 5.4 .

Remark 5.7. δ_c and $\lambda(\Delta)$ have been introduced in [22], definition 4.1 and remark 4.2 a) (where they have been denoted by Δ_c and $L(\Delta)$ and called "the duality associated to the conjugation c" and, respectively, "the conjugation of type Lau associated to Δ "). Formulae (5.95)-(5.97) and some related results, have been proved in [22].

Corollary 5.2. The mapping $\bar{\lambda}:\Delta\rightarrow\overline{\lambda(\Delta)}$ is a complete Boolean algebra isomorphism of $D((2^X,\supseteq),(2^W,\supseteq))$ onto $CL(\overline{R}^X,\overline{R}^W)$, and for each $\Delta\in D((2^X,\supseteq),(2^W,\supseteq))$ we have

$$\bar{\lambda}(\Delta)=\overline{\lambda(\Delta)}=L(\overline{\varphi_\Delta})=c(\overline{\varphi_\Delta})=L(\varphi_{\bar{\Delta}})=\lambda(\bar{\Delta}), \tag{5.98}$$

$$f^{\bar{\lambda}(\Delta)}(w)=-\inf_{\substack{x\in X\\ w\in W\setminus\Delta(\{x\})}} f(x)=-\inf f(X\setminus\Delta^*(\{w\})) \quad (f\in\overline{R}^X, w\in W), \tag{5.99}$$

where $\overline{\varphi_\Delta}$ and $\bar{\Delta}$ are, respectively, the complementary coupling functional (4.95) and the complementary duality (4.39).

Proof. By $\Delta=\Delta_{\varphi_\Delta}$, (5.97), (5.65), (5.48) and (4.95), we have (5.98). Hence, $\bar{\lambda}=\lambda\eta$ (where $\eta(\Delta)=\bar{\Delta}$), and thus the first statement follows from theorems 5.6 and 4.1. Finally, by (5.98), (5.93) (applied to $\bar{\Delta}$), (4.38) and (4.2), we obtain (5.99).

From theorems 5.6 and 5.5 there follows

Corollary 5.3. The mapping $\lambda(\Delta)\rightarrow\lambda(\Delta)^*$ is a complete Boolean algebra isomorphism of $CL(\overline{R}^X,\overline{R}^W)$ onto $(CL)^*=CL(\overline{R}^W,\overline{R}^X)$, and for each $\Delta\in D((2^X,\supseteq),(2^W,\supseteq))$ we have

$$\lambda(\Delta)^*=L(\varphi_{\Delta^*})=c(\varphi_{\Delta^*})=L((\varphi_\Delta)^-)=\lambda(\Delta^*), \tag{5.100}$$

$$g^{\lambda(\Delta)^*}(x)=-\inf_{\substack{w\in W\\ x\in\Delta^*(\{w\})}} g(w)=-\inf g(\Delta(\{x\})) \quad (g\in\overline{R}^W, x\in X), \tag{5.101}$$

$$f^{\lambda(\Delta)\lambda(\Delta)^*}(x)=\sup_{w\in\Delta(\{x\})} \inf f(\Delta^*(\{w\})) \quad (f\in\overline{R}^X, x\in X). \tag{5.102}$$

Proof. We prove only (5.102). By (5.101) for $g=f^{\lambda(\Delta)}$ and (5.93), we have

$$f^{\lambda(\Delta)\lambda(\Delta)^*}(x)=-\inf f^{\lambda(\Delta)}(\Delta(\{x\}))=\sup_{w\in\Delta(\{x\})}(-f^{\lambda(\Delta)}(w))=$$

$$=\sup_{w\in\Delta(\{x\})}\inf f(\Delta^*(\{w\})) \quad (f\in\overline{R}^X, x\in X).$$

Using also corollary 5.2, one obtains

Corollary 5.4. The mapping $\Delta\rightarrow\overline{\lambda(\Delta)}^*$ is a complete Boolean algebra isomorphism of $D((2^X,\supseteq),(2^W,\supseteq))$ onto $(CL)^*=CL(\overline{R}^W,\overline{R}^X)$, and for each $\Delta\in D((2^X,\supseteq),(2^W,\supseteq))$ we have

$$\bar{\lambda}(\Delta)^*=\overline{\lambda(\Delta)}^*=L(\overline{\varphi_{\Delta^*}})=c(\overline{\varphi_{\Delta^*}})=L((\overline{\varphi_\Delta})^-)=\lambda(\bar{\Delta}^*), \tag{5.103}$$

$$g^{\overline{\lambda}(\Delta)^*}(x)=-\inf_{x\in X\setminus\Delta^*(\{w\})} g(w)=-\inf g(W\setminus\Delta(\{x\}))\quad (g\in\overline{R}^W, x\in X), \tag{5.104}$$

$$f^{\overline{\lambda}(\Delta)\overline{\lambda}(\Delta)^*}(x)=\sup_{w\in W\setminus\Delta(\{x\})} \inf f(X\setminus\Delta^*(\{w\}))\quad (f\in\overline{R}^X, x\in X). \tag{5.105}$$

Remark 5.8. M. Volle has considered the "conjugation" $\alpha=\alpha(\Delta):\overline{R}^X\to\overline{R}^W$ defined ([24], formula (17)) by

$$f^{\alpha}(w)=-\inf f(X\setminus\Delta^*(\{w\}))\qquad (f\in\overline{R}^X, w\in W), \tag{5.106}$$

and has observed that $\alpha=c(\nu)$ (in the sense of (1.37)), where $\nu=\nu_\Delta\in\{0,-\infty\}^{X\times W}$ is defined by

$$\nu(x, w)=-\chi_{X\setminus\Delta^*(\{w\})}(x)\qquad (x\in X, w\in W); \tag{5.107}$$

also, he has observed [24] that

$$g^{\alpha^*}(x)=-\inf g(W\setminus\Delta(\{x\}))\qquad (g\in\overline{R}^W, x\in X). \tag{5.108}$$

Note that, by (5.106), (5.99), (5.107) and (4.95), we have

$$\alpha=\overline{\lambda}(\Delta)=\overline{\lambda(\overline{\Delta})},\ \nu=\varphi_{\overline{\Delta}}=\overline{\varphi_{\Delta}}, \tag{5.109}$$

so (5.108) is a part of (5.104).

Let us observe now that if $\Delta\in D((2^X,\supseteq),(2^W,\supseteq))$, then for the family $\mathcal{M}\subseteq 2^X$ defined by (4.22) we have, by (2.22) applied to $S_d(h)$ $(h\in\overline{R}^X, d\in R)$ of (4.24) (in $E=(2^X,\supseteq)$, i.e., with $\mathcal{C}(\mathcal{M})(S_d(h))$ of (4.17)),

$$f_{Q(\Delta^*\Delta)}=f_{Q(\mathcal{M})}\qquad (f\in\overline{R}^X), \tag{5.110}$$

where $f_{Q(\Delta^*\Delta)}$ (respectively, $f_{Q(\mathcal{M})}$) denotes the greatest functional $h:X\to\overline{R}$ such that $S_d(h)=\Delta^*\Delta(S_d(h))$ for all $d\in R$ (respectively, such that $S_d(h)=\mathcal{C}(\mathcal{M})(S_d(h))$ for all $d\in R$), majorized by f; in [20], $f_{Q(\mathcal{M})}$ is called "the $\mathcal{M}$-quasi-convex hull" of f.

Now we can express $f_{Q(\Delta^*\Delta)}$ as follows:

Theorem 5.7. *For each $\Delta\in D((2^X,\supseteq),(2^W,\supseteq))$, we have*

$$f_{Q(\Delta^*\Delta)}=f^{\overline{\lambda}(\Delta)\overline{\lambda}(\Delta)^*}\qquad (f\in\overline{R}^X). \tag{5.111}$$

Proof. By [20], theorem 2.2, for any $\mathcal{M}\subseteq 2^X$ we have

$$f_{Q(\mathcal{M})}(x)=\sup_{\substack{M\in\mathcal{M}\\ x\in X\setminus M}} \inf f(X\setminus M)\qquad (f\in\overline{R}^X, x\in X). \tag{5.112}$$

Hence, by (5.110), (5.112) (applied to $\mathcal{M}$ of (4.22)), (4.2) and (5.105), we obtain

$$f_{Q(\Delta^*\Delta)}(x)=f_{Q(\mathcal{M})}(x)=\sup_{\substack{w\in W\\ x\in X\setminus\Delta^*(\{w\})}} \inf f(X\setminus\Delta^*(\{w\}))=$$
$$=f^{\overline{\lambda}(\Delta)\overline{\lambda}(\Delta)^*}(x)\qquad (f\in\overline{R}^X, x\in X). \tag{5.113}$$

Remark 5.9. a) Alternatively, one can also deduce theorem 5.7 from [20], theorem 4.2, applied to $\mathcal{M}\subseteq 2^X$ of (4.22).

b) The equality

$$f_{Q(\Delta^*\Delta)}(x) = \sup_{w\in W} \inf_{x\in X\setminus\Delta^*(\{w\})} f(X\setminus\Delta^*(\{w\})) \qquad (f\in\overline{R}^X, x\in X)$$

(which is part of (5.113)) has been also obtained by M.Volle ([24], theorem I.1.5 and formula (18)), who has been unaware of [20] - [22]; in fact, a number of other results of [24] can be also obtained as the particular case $\mathcal{M}$=(4.22) of some results of [20].

Corollary 5.5. For each $\Delta\in D((2^X,\supseteq),(2^W,\supseteq))$ we have

$$f_{Q(\overline{\Delta}^*\overline{\Delta})} = f^{\lambda(\Delta)\lambda(\Delta)^*} = f_{Q(\widetilde{\mathcal{M}})} \qquad (f\in\overline{R}^X), \qquad (5.114)$$

where

$$\widetilde{\mathcal{M}} = \{X\setminus\Delta^*(\{w\}) \mid w\in W\}. \qquad (5.115)$$

Proof. This follows from theorem 5.7 and formulae (5.98), (3.64) and (5.110), (4.44).

Remark 5.10. By [22], remark 2.3 c), it is possible to modify the above definition (4.63) of Δ_φ (and then to change φ_Δ of (4.67) accordingly), for various purposes. Now, the above results suggest to replace (4.63), for each $\varphi\in\overline{R}^{X\times W}$, by

$$\Gamma_\varphi(G) = \{w'\in W \mid \varphi(x,w')\leq -1 \ (x\in G)\} \qquad (G\subseteq X); \qquad (5.116)$$

correspondingly, for each $\Delta\in D((2^X,\supseteq),(2^W,\supseteq))$, the unique coupling functional $\varphi'=\varphi'_\Delta$ of type $\{0,-\infty\}$, satisfying $\Delta=\Gamma_{\varphi'}$, will be

$$\varphi'_\Delta(x,w) = -\chi_{W\setminus\Delta(\{x\})}(w) = -\chi_{X\setminus\Delta^*(\{w\})}(x) \qquad (x\in X, w\in W), \qquad (5.117)$$

whence, by (4.95) (see also (5.109)),

$$\varphi'_\Delta = \varphi_{\overline{\Delta}} = \overline{\varphi_\Delta}\ . \qquad (5.118)$$

Indeed, let us mention some advantages of Γ_φ.

a) If X is a locally convex space and $W=X^*$, and if we define

$$\varphi_1(x,\Phi) = \Phi(x) - 2 \qquad (x\in X,\ \Phi\in X^*), \qquad (5.119)$$

then, for any set $G\subseteq X$,

$$\Gamma_{\varphi_1}(G) = \{\Phi'\in X^* \mid \Phi'(x)\leq 1 \ (x\in G)\} = G^0, \qquad (5.120)$$

the usual polar of G.

b) If X is a set and $W\subseteq\overline{R}^X$ and if we define $\varphi_2 : X\times(W\times R)\to\overline{R}$ by

$$\varphi_2(x,(w,d)) = -\chi_{\{y\in X \mid w(y)>d\}}(x) \qquad (x\in X,\ w\in W, d\in R), \qquad (5.121)$$

then, by (5.116),

$$\Gamma_{\varphi_2}(G) = \{(w',d')\in W\times R \mid -\chi_{\{y\in X\mid w'(y)>d'\}}(x)\leq -1 \quad (x\in G)\} =$$

$$= \{(w',d')\in W\times R \mid \sup w'(G)\leq d'\} = \Delta(G) \qquad (G\subseteq X),$$

where $\Delta\in D((2^X,\supseteq),(2^W,\supseteq))$ is the duality defined by (4.27).

c) "The hull operator $H_\varphi : 2^X \to 2^X$ associated to a coupling functional" $\varphi: X\times W \to \overline{R}$ can now be defined (modifying [22] , formula (2.17)) by

$$H_\varphi(G)=\{x'\in X \mid \varphi(x',w)\leq d \ ((w,d)\in W\times R, \sup_{x\in G} \varphi(x,w)\leq d)\} \qquad (G\subseteq X), \qquad (5.122)$$

which generalizes (4.29) above.

d) By (5.98) and (5.118), we have

$$\overline{\lambda}(\Delta)=c(\overline{\varphi_\Delta})=c(\varphi'_\Delta) \qquad (\Delta\in D((2^X,\supseteq),(2^W,\supseteq)), \qquad (5.123)$$

so λ will now be replaced by $\overline{\lambda}$ of corollary 5.2, which has the advantage that (5.111) and (5.110) contain explicitly only Δ (while (5.114) contains simultaneously Δ and $\overline{\Delta}$).

e) By (5.99) and (5.116), we have

$$f^{\overline{\lambda}(\Gamma_\varphi)}(w)=-\inf_{\substack{x\in X\\ w\in W\setminus\Gamma_\varphi(\{x\})}} f(x)=-\inf_{\substack{x\in X\\ \varphi(x,w)>-1}} f(x) \qquad (f\in\overline{R}^X, w\in W), \qquad (5.124)$$

which also contains (as does $\lambda(\Delta_\varphi)$, too), as a particular case, the usual Greenberg-Pierskalla quasi-conjugate [10] defined, for a locally convex space X, by

$$f^c((\Phi,d))=-\inf_{\substack{x\in X\\ \Phi(x)\geq d}} f(x) \qquad (f\in\overline{R}^X, (\Phi,d)\in W=X^*\times R); \qquad (5.125)$$

indeed, if we define $\varphi_3\in\{0,-\infty\}^{X\times W}$ by

$$\varphi_3(x,(\Phi,d))=-\chi_{\{y\in X \mid \Phi(y)\geq d\}}(x) \qquad (x\in X, (\Phi,d)\in W), \qquad (5.126)$$

then, by (5.124) for $\varphi=\varphi_3$ and (5.125),

$$f^{\overline{\lambda}(\Gamma_{\varphi_3})}((\Phi,d))=f^c((\Phi,d)) \qquad (f\in\overline{R}^X, \ (\Phi,d)\in W). \qquad (5.127)$$

<u>Theorem 5.8.</u> a) <u>The mapping $\Gamma_\cdot : \varphi\to\Gamma_\varphi$ is a lattice homomorphism and a complete sup-homomorphism of $\overline{R}^{X\times W}$ onto</u> $D((2^X,\supseteq),(2^W,\supseteq))$, <u>with kernel</u>

$$\mathrm{Ker}\,\Gamma_\cdot=\{\varphi\in\overline{R}^{X\times W} \mid \varphi(x,w)\leq -1 \ ((x,w)\in X\times W)\}. \qquad (5.128)$$

b) <u>We have</u>

$$\Gamma_\varphi=\overline{\Delta_\varphi} \qquad (\varphi\in\{0,-\infty\}^{X\times W}), \qquad (5.129)$$

<u>and hence</u> $\Gamma^r_\cdot=\Gamma_\cdot \big|_{\{0,-\infty\}^{X\times W}}$ <u>is a complete Boolean algebra isomorphism onto</u> D.

<u>Proof</u>. The proof of a) is similar to that of theorem 4.3 on $\Delta_\cdot$, mutatis mutandis.

b) If $\varphi\in\{0,-\infty\}^{X\times W}$, then, by (5.116) and (4.94), we have

$$\Gamma_\varphi(G)=\{w'\in W \mid \varphi(x,w')=-\infty \ (x\in G)\}=\overline{\Delta_\varphi}(G) \qquad (G\subseteq X),$$

so (5.129) holds. Hence, by corollary 4.2 b), we obtain the last statement.

<u>Remark 5.12.</u> In the general case (i.e., with φ not necessarily of

type $\{0,-\infty\}$), by (5.116) and (4.94) we have $\Gamma_\varphi \neq \overline{\Delta_\varphi}$; however, $\Gamma'_\varphi = \overline{\Delta_\varphi}$ would not be a good replacement for Δ_φ of (4.63), since it is not of "closed set" type and it does not contain the usual polar set (5.120) of G as a particular case.

§6. Appendix: Relations between dualities $\Delta : E \to F$ and coupling functionals $\varphi : Y \times T \to \overline{R}$

We shall now show, briefly, a possible way of extending some of the preceding results to the general case.

Definition 6.1. Let E, F be two complete lattices, with families of infimal generators $Y \subseteq E$, $T \subseteq F$, let $\varphi : Y \times T \to \overline{R}$ be a coupling functional and assume that the mapping $\Delta^{\circ}_{\varphi} : Y \to F$ defined by

$$\Delta^{\circ}_{\varphi}(y) = \inf \{t \in T \mid \varphi(y,t) \geq -1\} \qquad (y \in Y) \qquad (6.1)$$

is antitone. If there exists a duality $\Delta_\varphi : E \to F$ such that $\Delta_\varphi|_Y = \Delta^{\circ}_{\varphi}$, i.e., that

$$\Delta_\varphi(y) = \inf \{t \in T \mid \varphi(y,t) \geq -1\} \qquad (y \in Y), \qquad (6.2)$$

then, by theorem 1.2, it is (unique and) given by

$$\Delta_\varphi(x) = \sup \{\inf \{t \in T \mid \varphi(y,t) \geq -1\} \mid y \in Y,\ x \leq y\} \quad (x \in E), \qquad (6.3)$$

and we shall call it the duality associated to the coupling functional φ. Also, instead of writing "Δ_φ exists", we shall simply write: $\Delta_\varphi \in D$.

Remark 6.1. a) If $E = (2^X, \supseteq)$, Y = the family of all singletons $\{x\}$ $(x \in X)$ and F is any complete lattice, with a family of infimal generators $T \subseteq F$, then we have

$$\Delta_\varphi \in D \qquad (\varphi \in \overline{R}^{Y \times T}), \qquad (6.4)$$

$$\Delta_\varphi(G) = \sup_{x \in G} \Delta^{\circ}_{\varphi}(\{x\}) \qquad (\varphi \in \overline{R}^{Y \times T},\ G \subseteq X); \qquad (6.5)$$

indeed, this follows from corollary 1.4 applied to $\Delta_0 = \Delta^{\circ}_{\varphi}$ of (6.1). In the particular case when $E = (2^X, \supseteq)$, $F = (2^W, \supseteq)$ (where X and W are two sets) and $Y \subseteq E$, $T \subseteq F$ are the family of all singletons in X and W respectively, definition 6.1 yields the duality Δ_φ of (4.63).

b) If $E = \overline{R}^X$, $F = \overline{R}^W$ (where X and W are two sets) and

$$Y = \{\chi_{\{x\}} \dot{+} d \mid x \in X, d \in R\}, \quad T = \{\chi_{\{w\}} \dot{+} e \mid w \in W,\ e \in R\} \qquad (6.6)$$

(see example 1.6 b)), and if $\varphi : Y \times T \to R$ satisfies

$$\varphi(\chi_{\{x\}} \dot{+} d, \chi_{\{w\}} \dot{+} e) = \varphi(\chi_{\{x\}}, \chi_{\{w\}}) + d + e \qquad (x \in X,\ w \in W; d, e \in R), \qquad (6.7)$$

then (6.1) becomes

$$(\chi_{\{x\}} \dot{+} d)^{\Delta^{\circ}_{\varphi}}(w) = \inf \{\chi_{\{w'\}}(w) \dot{+} e \mid w' \in W, e \in R, \varphi(\chi_{\{x\}} \dot{+} d, \chi_{\{w\}} \dot{+} e) \geq -1\} =$$

$$= \inf \{e \in R \mid \varphi(\chi_{\{x\}}, \chi_{\{w\}}) + d + 1 \geq -e\} =$$

$$= -\varphi(\chi_{\{x\}}, \chi_{\{w\}}) - d - 1 \qquad (x \in X,\ w \in W,\ d \in R), \qquad (6.8)$$

so $\Delta_\varphi^o : Y \to F$ is antitone, since $\chi_{\{x_1\}} \dot{+} d_1 \leqslant \chi_{\{x_2\}} \dot{+} d_2$ implies $x_1 = x_2$ and $d_1 \leqslant d_2$. If we identify Y with $X \times R$ as in remark 1.4 b), and, similarly, T with $W \times R$, then, by (6.7), φ is an extension of $\varphi|_{X\times W} : X \times W \to R$ (here we have also identified $(X\times\{0\})\times(W\times\{0\})$ with $X\times W$). Conversely, given any coupling functional $\gamma : X\times W \to R$, one can define an extension $\varphi : (X\times R)\times(W\times R) \to R$ of γ, by

$$\varphi((x,d),(w,e)) = \gamma(x,w) + d + e \qquad (x\in X, w\in W;\ d,e\in R); \tag{6.9}$$

then, with the above identifications, φ satisfies (6.7). Furthermore, if we identify $\overline{R}^X$ with the family $\mathcal{E}_o$ of all epigraphic subsets of $X\times R$, as in remark 1.4 b), then $\chi_{\{x\}} \dot{+} d = \{(x,d)\}$ is identified with $\mathrm{Epi}\,(\chi_{\{x\}} \dot{+} d) = \{(x,d') \mid d' \in R, d \leqslant d'\}$, and Y of (6.6) is not only a family of infimal generators of $\mathcal{E}_o$, but also of $(2^{X\times R}, \supseteq)$.

c) Similarly to remark 5.10, one can replace the mappings $\Delta_\varphi^o, \Delta_\varphi$ of definition 6.1 by the mappings $\Gamma_\varphi^o : Y \to F$, $\Gamma_\varphi : E \to F$ defined by

$$\Gamma_\varphi^o(y) = \inf\ \{t\in T \mid \varphi(y,t) \leqslant -1\} \qquad (y\in Y), \tag{6.10}$$

$$\Gamma_\varphi(x) = \sup\ \{\inf\ \{t\in T \mid \varphi(y,t)\leqslant -1\} \mid y\in Y, x\leqslant y\} \qquad (x\in E). \tag{6.11}$$

Then, for E, F, Y and T as at the end of a) above, we obtain the duality $\Gamma_\varphi : E \to F$ of (5.116). On the other hand, for E, F, Y and T as in b) above, and for $\varphi : Y\times T \to R$ satisfying

$$\varphi(\chi_{\{x\}} \dot{+} d, \chi_{\{w\}} \dot{+} e) = \varphi(\chi_{\{x\}}, \chi_{\{w\}}) - d - e \qquad (x\in X, w\in W; d,e\in R), \tag{6.12}$$

(6.10) becomes (using (1.38) and (1.8))

$$(\chi_{\{x\}} \dot{+} d)^{\Gamma_\varphi^o}(w) = \inf\ \{e\in R \mid \varphi(\chi_{\{x\}},\chi_{\{w\}}) - d + 1 \leqslant e\} =$$
$$= \varphi(\chi_{\{x\}},\chi_{\{w\}}) - d + 1 = (\chi_{\{x\}} \dot{+} d)^{c(\varphi)}(w) + 1 \qquad (x\in X, w\in W, d\in R), \tag{6.13}$$

so Γ_φ^o can be extended to the duality $\Gamma_\varphi : \overline{R}^X \to \overline{R}^W$ given by

$$f^{\Gamma_\varphi} = f^{c(\varphi)} + 1 \qquad (f\in \overline{R}^X). \tag{6.14}$$

Note also that if we replace $\leqslant -1$ by $\leqslant 0$ in (6.10) and (6.11), then the term +1 in (6.13) and (6.14) disappears, so $\Gamma_\varphi = c(\varphi)$; thus, one obtains again a result of M.Volle ([24], theorem I.2.4).

<u>Definition 6.2.</u> Let E be a complete lattice and Y a family of infimal generators of E. For any $x\in E$, the <u>generalized indicator functional of</u> x is the functional $\Sigma_x : Y \to \{0,+\infty\}$ defined by

$$\Sigma_x(y) = \chi_{\{y'\in Y \mid x\leqslant y'\}}(x) = \begin{cases} 0 & \text{if } x\leqslant y \\ +\infty & \text{if } x\not\leqslant y \end{cases}. \tag{6.15}$$

<u>Remark 6.2.</u> a) If E and Y are as in remark 1.4 a), then $\Sigma_G = \chi_G$ $(G\subseteq X)$.

b) If E and Y are as in remark 1.4 b), then, with the identifications of that remark, $\Sigma_f = \chi_{\mathrm{Epi}\, f}$ $(f\in \overline{R}^X)$.

Now we can generalize φ_Δ of (4.67) as follows:

Theorem 6.2. Let E, F be two complete lattices, with families of infimal generators $Y\subseteq E$, $T\subseteq F$. Then, for any duality $\Delta:E\to F$ there exists a unique coupling functional $\varphi=\varphi_\Delta:Y\times T\to\bar{R}$ of type $\{0,-\infty\}$, such that all sets

$$M_{y,\varphi}=\{t'\in T\mid \varphi(y,t')\geq -1\} \qquad (y\in Y) \tag{6.16}$$

are upper conical subsets of T and that $\Delta=\Delta_\varphi$, namely,

$$\varphi_\Delta(y,t)=-\Sigma_{\Delta(y)}(t)=-\chi_{\{t'\in T\mid \Delta(y)\leq t'\}}(t) \qquad (y\in Y, t\in T). \tag{6.17}$$

Proof. By (6.16) and (6.17), we have

$$M_{y,\varphi_\Delta}=\{t\in T\mid \varphi_\Delta(y,t)\geq -1\}=\{t\in T\mid \Delta(y)\leq t\} \qquad (y\in Y), \tag{6.18}$$

and thus each M_{y,φ_Δ} is an upper conical subset of T (see definition 1.1). Furthermore, by (6.1), (6.18) and (1.19),

$$\Delta^{o}_{\varphi_\Delta}(y)=\inf\ \{t\in T\mid \Delta(y)\leq t\}=\Delta(y) \qquad (y\in Y),$$

whence, by definition 6.1, we obtain $\Delta_{\varphi_\Delta}=\Delta$.

Assume, now that $\varphi\in\{0,-\infty\}^{Y\times T}$ satisfies $\Delta=\Delta_\varphi$ and that all sets (6.16) are upper conical subsets of T. Then, by $\Delta=\Delta_\varphi$, (6.2) and (6.16), we have

$$\Delta(y)=\Delta_\varphi(y)=\inf\ \{t'\in T\mid \varphi(y,t')\geq -1\}=\inf M_{y,\varphi} \qquad (y\in Y),$$

whence, by our assumption on the sets (6.16) and by proposition 1.2, we obtain

$$\{t\in T\mid \varphi(y,t)\geq -1\}=M_{y,\varphi}=\{t\in T\mid \inf\ M_{y,\varphi}\leq t\}=\{t\in T\mid \Delta(y)\leq t\}.$$

Hence, by $\varphi\in\{0,-\infty\}^{Y\times T}$, it follows that $\varphi=\varphi_\Delta$ of (6.17).

Remark 6.3. If $F=(2^W,\supseteq)$ and T is the family of all singletons $\{w\}$, where $w\in W$, then every set $M\subseteq T$ is an upper conical subset of T.

The further extension of the preceding results, along these lines, remains still open.

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