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GENERALIZED ADJUNCTION AND APPLICATIONS

by

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Introduction. The linear system $|K+C|$ "adjoint" to a curve C on a projective surface was studied by the classical italian geometers. The adjoint system to a hyperplane section H of a smooth projective surface was investigated systematically, in modern terms, by Sommese, see [22] and Van de Ven, see [26]. The map associated to the linear system $|K+(r-1)H|$, where H is a hyperplane section of a smooth variety of arbitrary dimension r was used to classify submanifolds of $\mathbb{P}^n$ with "small invariants" (e.g. degree, sectional genus, etc), see [10]. On the other hand, Sommese, see [23], [24], [25], studied adjoint systems to a smooth ample divisor H on a smooth threefold X and obtained, as applications, many interesting results about the pair (X,H). As noticed independently by several authors (see e.g. [17], [4], [11]) the appearance of Mori's deep contribution [20] (see also [21]) put the subject of adjunction in a new perspective. Accordingly, the present paper - which relies heavily on Mori's results and on the contraction theorem due Kawamata-Shokurov (see [14]) - contains a systematical study of various adjoint systems to an ample (possibly non-effective divisor on a manifold of arbitrary dimension. More precisely, the main result (which is contained in section 1) gives the precise description of polarized pairs (X,H), where X is a complex projective manifold of dimension r and H an ample divisor on it (not necessarily effective), such that K_X+iH is not semiample (respectively ample) for $1\leq i=r+1, r, r-1, r-2$ (respectively $i=r+1, r, r-1$). This theorem was first proved, for surfaces, by Lanteri-Palleschi, see [17], using Mori's results, and by

Beltrametti-Palleschi, see [4] and Lanteri-Palleschi, see [16] for threefolds, using both Mori's results and the Kawamata-Shokurov contraction theorem. Our approach does not make use of the precise description of varieties whose canonical bundles are not numerically effective, which is available only for dimensions ≤ 3. Inspired by [19], Theorem 4, we prove in section 0 a useful estimation for the size of the locus of curves belonging to an extremal ray (Theorem 0.4). This is used at a critical step in the proof of the main result. The following applications of the main theorem are given in Section 2. First we recover and sometimes slightly improve the main results in Sommese's paper [24] (or [25]); we also give an alternative proof of a theorem due to L.Bădescu, see [1], [2], [3], classifying smooth projective threefolds which support a geometrically ruled surface as an ample divisor; more generally, we describe all (smooth, projective) threefolds containing a smooth surface which is not of general type as an ample divisor. Finally, using the idea of [4], we classify the polarized pairs (X,H) of arbitrary dimension r whose sectional genus is "small" with respect to the "degree" (H^r) (see also [10], [11], where we considered the case when H is a very ample divisor).

We would like to thank M.Beltrametti for pointing out a gap in an erlier version of this paper. Working independently, M. Beltrametti and M.Palleschi also obtained partial results of the same kind.

When writing down the present version of this paper, we received a note by T.Fujita entitled "Generalized adjunction mappings" announcing, without proofs, results similar to ours. However, some of the exceptional varieties are missing in his list. We also received a manuscript by A.J.Sommese entitled "On the adjunction theoretic structure of projective varieties", dealing with similar questions about varieties admiting certain kind of singularities, but working with spanned ample line bundles. His techniques are quite different

(0.3) Corollary. Let X be a manifold of dimension r and let R be an extremal ray. Then there exists a normal projective variety Y and a surjective morphism $f=\mathrm{cont}_R: X \longrightarrow Y$ with connected fibres such that for any integral curve C on X, $\dim f(C)=0$ is equivalent to $[C] \in R$.

The next theorem, which seems to be interesting in itself, will play a key role in the proof of the main result. Its proof was largely inspired by Mori's proof of Theorem 4 in [19].

(0.4) Theorem. Let X be a manifold of dimension r defined over any algebraically closed field and let R be an extremal ray of X. Denote by k the codimension in X of the locus of points of curves belonging to R. Put: $b=\min\{-(K_X.C) \mid [C] \in R,\ C \text{ a rational curve}\}$.
Then we have: $2k \leq r+1-b$.

Proof. Let C_o be a rational curve in R such that $b=-(K_X.C_o)$ and let $f:\mathbb{P}^1 \longrightarrow C_o$ be the normalization morphism. Let $\mathrm{Hom}(\mathbb{P}^1,X)$ denote the scheme representing the functor of morphisms from $\mathbb{P}^1$ to X and let U be an irreducible component of maximal dimension containing $[f]$, given the reduced structure. The universal family over $\mathrm{Hom}(\mathbb{P}^1,X)$ induces the commutative diagram:

$$\begin{array}{ccccc} Z & \overset{i}{\hookrightarrow} & \mathbb{P}^1 \times X \times U & \overset{m}{\longrightarrow} & X \times U \\ & \searrow^{v} & \downarrow{\scriptstyle n} & & \\ & & U & & \end{array}$$

where m and n are the projections and i is a closed immersion. Consider the closed (reduced) subscheme Z' of $X \times U$ given by $Z'=m(i(Z))$. By generic flatness we get an open nonempty subset U_o of U such that the restriction of Z' to $X \times U_o$ is flat over U_o. Thus, by the universal property of the Hilbert scheme of X, we get a morphism $\alpha: U_o \longrightarrow \mathrm{Hilb}_X$.
curve is got by

§0. In this section we fix our terminology and notation, recal some results needed in the sequel and prove a useful estimation for the dimension of the locus of curves belonging to an extremal ray.

We shall work over the field of complex numbers unless otherwise specified. A smooth, connected algebraic variety is called simply a manifold. All manifolds are assumed to be projective, unless otherwise stated. A polarized pair, denoted (X,H), means a (projective) manifold X together with an ample divisor H not necessarily effe tive. A divisor D on X is said to be nef if $(D.C)\geq 0$ for any effectiv curve C. A nef divisor D is "big" if $(D^r)>0$, $r=\dim X$. A divisor D is said semi-ample if the linear system $|mD|$ is base-points free, for $m\gg 0$. K_X will denote a canonical divisor of the manifold X. We write "$\sim$" (respectively "$\approx$") for the linear (respectively numerical) equivalence of divisors. If Z is a closed subscheme of X and D is a divisor (class) on X we denote by $D|_Z$ its restriction to Z. A ration curve is an irreducible reduced curve whose normalization is $\mathbb{P}^1$.

We refer the reader to [20] for definitions and properties of extremal rays, extremal rational curves, etc. This paper relies on th following two fundamental results:

(0.1) Mori's Cone Theorem (see [20] Theorem 1.5). Let X be a manifol Then the closed cone of effective curves, denoted $\overline{NE}(X)$, is the smal lest closed convex cone containing the set $\overline{NE}_-(X)=\{C\in\overline{NE}(X) \mid (C.K_X)\geq 0\}$ and all the extremal rays. For any open convex cone U containing $\overline{NE}_-(X)-\{0\}$ there are only finitely many extremal rays that do not lie in $U\cup\{0\}$. Every extremal ray is spanned by an extremal rational cur

(0.2) Kawamata-Shokurov Contraction Theorem (see [14] Theorem 2.6). Let X be a manifold and D a nef divisor on it. Assume that $aD-K_X$ is nef and big for some $a\geq 1$. Then D is semi-ample.

Using [21] Proposition 3.1, the next corollary follows from (0.2).

taking the common image of the maps associated to points in $\alpha^{-1}(t)$.
Denote by T the closure of $\alpha(U_o)$ and let C be any curve correspondin to some closed point in T. We claim:

(1) $C \approx C_o$ and, moreover, C is irreducible, generically reduced.

Indeed, to prove the first part, we may assume that C correspo to a point in $\alpha(U_o)$ associated to some map, say $g:\mathbb{P}^1 \longrightarrow X$. Then, for any $L \in Pic(X)$ we get:
$(C_o.L) = \deg f^*L = \deg g^*L = d(C.L)$, where d is the degree of g. This show that $C_o \approx dC$. But then $[C] \in R$ since R is an extremal ray. As C is a rati nal curve, the minimality of C_o gives d=1. On the other hand, it is not difficult to see that any irreducible component of a curve C cor ponding to a closed point in T is rational. Since $C_o \approx C$, C must be ir reducible and generically reduced again by minimality of C_o.

Returning now to the map α, we remark that the above argument shows that for any closed point $t \in \alpha(U_o)$, $\alpha^{-1}(t)$ identifies to a part of $Aut(\mathbb{P}^1)$. Therefore we get:

(2) $\dim T \geq \dim U_o - 3$

By [19], Proposition 3 and Riemann-Roch theorem, we get:

(3) $\dim U_o = \dim_{[f]} Hom(\mathbb{P}^1, X) \geq \chi(f^*T_X) = b+r$ (where T_X is the tangent bundl of X).

Consider now the commutative diagram:

$$\begin{array}{ccccc} Y & \overset{j}{\hookrightarrow} & X \times T & \overset{p}{\longrightarrow} & X \\ & \searrow^{\nu} & \downarrow q & & \\ & & T & & \end{array}$$

induced by base-change from the universal family over $Hilb_X$, p and q being the projections and j a closed immersion. By (2) and (3) we ha- ve:

(4) $\dim Y \geqslant b+r-2$.

By (1) it follows that $p(Y)$ is contained in the locus of points of curves belonging to R. Therefore we get:

(5) $\dim p(Y) \leqslant r-k$.

If we put $Y_x = p^{-1}(x) \cap Y$, $T_x = w(Y_x) \simeq Y_x$, we get by (4) and (5):

(6) $\dim T_x = \dim Y_x \geqslant b+k-2$, for any $x \in p(Y)$. We get from (6):

$$\dim w^{-1}(T_x) \geqslant b+k-1.$$

Since we have a fortiori $\dim p\ (w^{-1}(T_x)) \leqslant r-k$, we may apply once again the same reasoning with $w^{-1}(T_x)$ replacing Y. Thus, if we let $T_{x,x'} = w(Y_x) \cap w(Y_{x'})$ for $x \in p(Y)$, $x' \in p(w^{-1}(T_x))$, we get:

(7) $\dim T_{x,x'} \geqslant b-r+2k-1$.

Next we claim:

(8) $\dim T_{x,x'} \leqslant 0$, for $x \in p(Y)$, $x' \in p(w^{-1}(T_x))$, $x' \neq x$.

The conclusion of the theorem follows from (7) and (8).
Assume that (8) would be false. Then we may find a complete curve D contained in T, such that the corresponding curves in X all pass through the points x, x'. Moreover, we may choose x, x' to be smooth distinct points of some curve of the family parametrized by D. Let $\tilde{D}$ be the normalization of D. Let S be the reduced scheme structure of the surface got by base-change over $\tilde{D}$ from the map $w: Y \longrightarrow T$ and let $\tilde{S}$ be the normalization of S. By (1) all the fibres of the map $\tilde{w}: S \longrightarrow \tilde{D}$ deduced from w are irreducible rational curves. It follows easily that the sa-

with the normalization morphism. Therefore π is a $\mathbb{P}^1$-bundle and, in particular, $\tilde{S}$ is smooth. But we have at least two disjoint sections E, E' for π, which are mapped to the points x and x' respectively. We get:

$$(E^2)<0, \ (E'^2)<0, \ (E.E')=0 \text{ and } (E-E')^2=0,$$

which is absurd. This contradiction gives (8) and thereby completes the proof of the theorem.

We also need the following statement, which follows from the explicit description of extremal rays in case of threefolds, see [20] or [21] Theorem 2.3 and Theorem 2.5.

(0.5) Corollary. If C is an extremal rational curve on a (smooth, projective) threefold X such that $(K_X.C)=-4$, it follows that X is a Fano threefold with $\mathrm{Pic}(X)\simeq\mathbb{Z}$.

We shall also use the following simple lemma.

(0.6). Lemma. Let (X,H) be a polarized pair. Assume that $K_X+iH\approx 0$ for some integer $i>0$. Then $K_X+iH\sim 0$.

Proof. We have $\chi(\mathcal{O}_X(K_X+iH))=\chi(\mathcal{O}_X)$, see [15], Ch.II, §2, Theorem. The hypothesis implies that $-K_X$ is ample. Therefore $\chi(\mathcal{O}_X(K_X+iH))=h^0(\mathcal{O}_X(K_X+iH))$ and $\chi(\mathcal{O}_X)=h^0(\mathcal{O}_X)=1$ by Kodaira vanishing. Thus we get $h^0(\mathcal{O}_X(K_X+iH))=1$ which gives $K_X+iH\sim 0$.

The following useful characterizations of projective spaces and hyperquadrics will be needed several times.

Theorem (see [16] and [6]). Let X be an integral projective scheme of dimension r and H an ample divisor on it.

(0.7) If $(H^r)=1$ and $h^0(\mathcal{O}_X(H))\geq r+1$, it follows $X\simeq\mathbb{P}^r$, $H\in|\mathcal{O}(1)|$;

(0.8) If $(H^r)=2$ and $h^0(\mathcal{O}_X(H))\geq r+2$, it follows X is isomorphic to a section (we shall write $H\in|\mathcal{O}(1)|$).

Assume moreover that X is a manifold.

(0.9) If $K_X+(r+1)H\sim 0$, it follows $X\simeq\mathbb{P}^r$, $H\in|\mathcal{O}(1)|$;

(0.10) If $K_X+rH\sim 0$, it follows $X\simeq Q^r$, $H\in|\mathcal{O}(1)|$.

(0.11) Let (X,H) be a polarized pair with $\dim X=r$. An effective divisor $E\subset X$ is called exceptional if $E\simeq\mathbb{P}^{r-1}$, $\mathcal{O}_X(H)\otimes\mathcal{O}_E\simeq\mathcal{O}(1)$ and $\mathcal{O}_X(E)\otimes\mathcal{O}_E\simeq\mathcal{O}(-1)$. Note that the set of exceptional divisors contained in X is finite, and, if $r\geqslant 3$, any two such exceptional divisors are disjoint. If (X,H) is a polarized pair and E an exceptional divisor on X, consider the morphism $\sigma:X\longrightarrow X'$ which contracts E (to a smooth point) and the unique divisor H' on the manifold X' such that $\mathcal{O}_X(H)\simeq\sigma^*(\mathcal{O}_{X'}(H'))\otimes\otimes\mathcal{O}_X(-E))$. By [7] Lemma 5.7, it follows that H' is ample on X'. Continuing in this way, we find a new polarized pair (X',H') such that: $u:X\longrightarrow X'$ is the blowing-up of n distinct points $P_1,\dots,P_n\in X'$; $u^{-1}(P_i)=E_i$ is an exceptional divisor for $i=1,\dots,n$; $\mathcal{O}_X(H)\simeq u^*(\mathcal{O}_{X'}(H'))\otimes\mathcal{O}_X(-E_1-\dots-E_n)$; (X',H') does not contain exceptional divisors. Such a pair (X',H') will be called a reduction of (X,H), see [24]. In case $r\geqslant 3$, the contraction u is uniquely determined by (X,H). This is no longer true if $r=2$, but all we shall need is the existence of a reduction.

Let (X,H) be a polarized pair with $\dim X=r\geqslant 2$.

(X,H) is called a scroll if there is a morphism $f:X\longrightarrow Y$ onto some manifold Y with $\dim Y=s>0$, which is a $\mathbb{P}^{r-s}$ - bundle, such that H induces $\mathcal{O}(1)$ on each fibre.

(X,H) is called a hyperquadric fibration if there is a morphism $f:X \longrightarrow C$ onto some smooth curve C such that each (closed) fibre of f is isomorphic to a hyperquadric and H induces $\mathcal{O}(1)$ on it.

As we shall see in section 1, any fibre of f is reduced and, if $r \geq 3$, it is also irreducible.

We shall introduce now several notations for the isomorphism classes of polarized pairs which will appear frequently in what follows.

(0.12) $(X,H) \in \mathcal{A}$ will mean $X \simeq \mathbb{P}^r$, $H \in |\mathcal{O}(1)|$;

(0.13) $(X,H) \in \mathcal{B}$ will mean $X \simeq Q^r$, $H \in |\mathcal{O}(1)|$ or (X,H) is a scroll over a curve;

(0.14) $(X,H) \in \mathcal{B}'$ will mean $(X,H) \in \mathcal{B}$ or $X \simeq \mathbb{P}^2$, $H \in |\mathcal{O}(2)|$;

(0.15) $(X,H) \in \mathcal{C}$ will mean that either:

1) $K_X + (r-1)H \sim 0$ (dim $X=r$); these are called Del Pezzo manifolds, see [8], and their classification is completely known, see [12] and [8];

2) (X,H) is a hyperquadric fibration;

3) (X,H) is a scroll over a surface.

(0.16) $(X,H) \in \mathcal{D}$ will mean that either:

$X \simeq \mathbb{P}^4$, $H \in |\mathcal{O}(2)|$ or $X \simeq \mathbb{P}^3$, $H \in |\mathcal{O}(3)|$, or $X \simeq Q^3$, $H \in |\mathcal{O}(2)|$, or

X is a $\mathbb{P}^2$-bundle over a smooth curve and H induces $\mathcal{O}(2)$ on each fibre.

Finally, we shall need the following simple fact.

(0.17) <u>Lemma.</u> <u>Let</u> (X,H) <u>be a polarized pair with reduction</u> (X',H'). <u>If</u> (X',H') $\in \mathcal{A}\cup\mathcal{B}'\cup\mathcal{C}$, <u>then</u> (X,H) $\in \mathcal{A}\cup\mathcal{B}'\cup\mathcal{C}$.

<u>Proof.</u> If $X' \simeq \mathbb{P}^r$, $H' \in |\mathcal{O}(1)|$ or $X' \simeq Q^r$, $H' \in |\mathcal{O}(1)|$, or (X',H') is a scroll, or $r \geq 3$ and (X',H') is a hyperquadric fibration, it follows easily that we must have X=X'. If (X',H') is a Del Pezzo manifold, the same holds for (X,H). If (X',H') is a two-dimensional hyperquadric fibration, the same holds for (X,H). Finally, if $X' \simeq \mathbb{P}^2$, $H' \in |\mathcal{O}(2)|$, (X,H) has to be a scroll over $\mathbb{P}^1$ (of degree 3), unless X=X'.

§1. This section is devoted to the statement and proof of the main result. Using the definitions and notations given in (0.1[1])-(0.16), our main theorem is the following:

Theorem. Let (X,H) be a polarized pair, with $\dim X=r\geq 1$. Then we have:

(1.1) $K_X + (r+1)H$ is semi-ample;

(1.2) If $(X,H)\notin \mathcal{A}$, $K_X+(r+1)H$ is ample;

(1.3) If $(X,H)\notin \mathcal{A}$, K_X+rH is semi-ample;

(1.4) If $(X,H)\notin \mathcal{A}\cup\mathcal{B}$, K_X+rH is ample;

Assume $r\geq 2$;

(1.5) If $(X,H)\notin \mathcal{A}\cup\mathcal{B}'$, $K_X+(r-1)H$ is semi-ample;

(1.6) If $(X,H)\notin \mathcal{A}\cup\mathcal{B}'\cup\mathcal{C}$, there is a reduction (X',H') such that $K'+(r-1)H'$ is ample (where $K'=:K_{X'}$);

Assume furthermore $r\geq 3$;

(1.7) If $(X,H)\notin \mathcal{A}\cup\mathcal{B}\cup\mathcal{C}$, there is a reduction (X',H') such that either $(X',H')\in\mathcal{D}$ or $K'+(r-2)H'$ is semi-ample.

Remarks 1. Clearly (1.1) and (1.2) follow from (1.3); we stated them separately since we shall prove them in this order, and moreover, this way the result looks more symmetrical.

2. If $|H|$ is moreover assumed to be base-points free, (1.2) was proved(even if not stated) by L.Ein in [5].

3. Using (0.2), we may restate (1.3) as follows: if

4. The above results were first proved by Lanteri-Palleschi, see [17], in case of surfaces, and by Beltrametti-Palleschi, see [4], and Lanteri-Palleschi, see [18], in case of threefolds.

5. In case H is very ample (1.5) can be made more precise: if $(X,H)\notin \mathcal{A}\cup\mathcal{B}'$, $|K_X+(r-1)H|$ is base-points free (see [22], Proposition 1.5, [26] Theorem II for r=2, and [10] Theorem 1.4 for $r\geqslant 3$). Is it true that, with the hypotheses of (1.7) and assuming moreover H to be very ample, $|K'+(r-2)H'|$ is base-points free if $(X',H')\notin\mathcal{D}$?

The proof of the Theorem is based on a lemma which will be stated below.

Let (X,H) be a polarized pair with dim $X=r\geqslant 1$, let $i\geqslant 1$ be an integer and assume that K_X+iH is semi-ample. Therefore, if $m\gg 0$, the morphism $\varphi_i=:\varphi_{|m(K_X+iH)|}:X\longrightarrow\mathbb{P}^N$ has connected fibres and maps X onto some normal variety Y. Keeping these assumptions and notations we have the following:

<u>Lemma</u> a) <u>One has either</u> dim Y=r, <u>or</u> dim $Y\leqslant r+1-i$;

b) <u>Assume that</u> dim Y=r and $i\geqslant r-1\geqslant 1$. <u>If</u> φ_i <u>is not a finite morphism, then</u> i=r-1 <u>and</u> X <u>contains an exceptional divisor</u> (see (0.11));

c) <u>Assume that</u> dim Y<r;

- <u>if</u> i=r+1, $(X,H)\in\mathcal{A}$;
- <u>if</u> i=r, $(X,H)\in\mathcal{B}$;
- <u>if</u> i=r-1, $(X,H)\in\mathcal{C}$.

Assuming for the moment this lemma we shall prove the Theorem.

<u>Proof of the Theorem</u>

(1.1) If V is an effective curve on X such that $(V.K_X)\geqslant 0$

it follows $(K_X+(r+1)H.V)>0$ by ampleness of H. Assume that $K_X+(r+1)H$ is not nef. By (0.1) we may find an extremal ration curve C such that $(K_X+(r+1)H.C)<0$. But $(H.C)\geqslant 1$ by ampleness of H and $(K_X.C)\geqslant -r-1$ since C is an extremal rational curve. We reached a contradiction, which shows that $K_X+(r+1)H$ is nef By (0.2) $K_X+(r+1)H$ is semi-ample.

(1.2) Using (1.1) we may apply the Lemma with $i=r+1$. Since $(X,H)\notin\mathcal{A}$, φ_{r+1} is generically finite by c), hence it is finite, by b). Therefore $K_X+(r+1)H$ is ample.

(1.3) By (0.2) it is enough to prove that K_X+rH is nef Assuming the contrary and using (0.1) as above, we may find a extremal rational curve C such that:

(1) $(K_X+rH.C)<0$.

If we let $a=:(H.C)$, $b=:-(K_X.C)$, we have $a\geqslant 1$, $b\leqslant r+1$; by (1.2) we get:

(2) $(K_X+(r+1)H.C)>0$.

Comparing (1) and (2) it follows:

(3) $ra<b<(r+1)a$.

Thus we must have $a\geqslant 2$ and we get:

$2r\leqslant ar<b\leqslant r+1$,

which is impossible for $r\geqslant 1$.

(1.4) This is, as in the proof of (1.2), a consequence o (1.3) and assertions b) and c) of the Lemma.

(1.5) Assume, as in the proof of (1.3), that $(X,H)\notin\mathcal{A}\cup$ and $K_X+(r-1)H$ is not nef. We shall prove that $X\simeq\mathbb{P}^2$, $H\in|\mathcal{O}(2)|$. Using (0.1) we may find an extremal rational curve C such tha if we let $a=(H.C)$, $b=-(K_X.C)$, it follows as above

Therefore $a \geqslant 2$ and we get:

$2(r-1) \leqslant (r-1)a < b \leqslant r+1$.

This implies $r=2$, $a=2$, $b=3$. Now, using the explicit description of extremal rational curves in case of surfaces, see [20], Theorem 2.1, it follows $X \simeq \mathbb{P}^2$. Alternatively, consider the divisor $H_1 =: K_X + 2H$. H_1 is ample, by (1.4). Since $(K_X + 3H_1 . C) = 0$, it follows by (1.2) $X \simeq \mathbb{P}^2$, $H_1 \in |\mathcal{O}(1)|$, so $H \in |\mathcal{O}(2)|$.

(1.6) The proof is similar to that of (1.4), using (1.5), parts b) and c) of the Lemma and (0.17).

(1.7) Assume that $(X,H) \notin \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ and consider the reduction (X',H'). Then $(X',H') \notin \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$, see (0.17). Suppose that $K' + (r-2)H'$ is not nef. Using (0.1) we find an extremal rational curve C such that:

(4) $(K' + (r-2)H' . C) < 0$.

If we let $a = (H'.C)$, $b = -(K'.C)$, so that $a \geqslant 1$, $b \leqslant r+1$, it follows by (1.6):

(5) $(K' + (r-1)H' . C) > 0$.

Therefore, using (4) and (5), we get:

(6) $(r-2)a < b < (r-1)a$.

This gives $a \geqslant 2$ and:

(7) $2(r-2) \leqslant (r-2)a < b \leqslant r+1$.

Thus, one of the following holds, by (6) and (7):

i) $r=3$, $a=3$, $b=4$;

ii) $r=3$, $a=2$, $b=3$;

iii) $r=4$, $a=2$, $b=5$.

In case i), it follows by (0.5) that $\mathrm{Pic}(X') \simeq \mathbb{Z}$. If we consider the divisor $H_1 =: -K' - H'$, we have $(H_1 . C) = 1$, so H_1 is ample since $\mathrm{Pic}(X') \simeq \mathbb{Z}$. We have $(K' + 4H_1 . C) = 0$, therefore by (1.2), $X' \simeq \mathbb{P}^3$, $H_1 \in |\mathcal{O}(1)|$, hence $H' \in |\mathcal{O}(3)|$.

If we are in case ii), we let $H_1 =: K' + 2H'$. By (1.6) H_1

is ample and we have $(K'+3H_1.C)=0$, so $(X',H_1)\in\mathcal{A}\mathcal{B}$ by (1.4). Clearly we can't have $X'\simeq\mathbb{P}^3$, $H_1\in|\mathcal{O}(1)|$. We are left with two possibilities: either $X'\simeq Q^3$, $H_1\in|\mathcal{O}(1)|$, so $H'\in|\mathcal{O}(2)|$, or (X',H') is a scroll over a curve, so X' is a $\mathbb{P}^2$-bundle and H' induces $\mathcal{O}(2)$ on each fibre. Finally, assume that we are in case iii). Then $H_1=:K'+3H'$ is ample by (1.6). Since $(K'+5H_1.C)=0$, it follows by (1.2) that $X'\simeq\mathbb{P}^4$, $H_1\in|\mathcal{O}(1)|$, hence $H'\in|\mathcal{O}(2)|$.

Thus we proved that either $(X',H')\in\mathcal{D}$ or $K'+(r-2)H'$ is nef, hence semi-ample by (0.2). The proof of the Theorem is complete modulo the key lemma above.

Proof of the Lemma

a) Assume that $\dim Y<r$ and denote by F a general (hence smooth) fibre of φ_i. We have:

$m(K_X+iH)|_F\sim 0$, hence:

(8) $K_F+iH|_F\approx 0$.

Kodaira vanishing theorem gives:

(9) $H^j(\mathcal{O}_F(K_F+nH|_F))=0$ for $j>0$ and $n>0$.

Using (8) we get:

(10) $H^0(\mathcal{O}_F(K_F+nH|_F))=0$ if $n\leq i-1$.

Consider the polynomial $P(n)=\chi(\mathcal{O}_F(K_F+nH|_F))$.

By (9) and (10) it follows that $P(n)=0$ for $1\leq n\leq i-1$. We get:

$\dim F=\deg P\geq i-1$.

Since $\dim F=r-\dim Y$, it follows $\dim Y\leq r-i+1$.

b) If $V\in\overline{NE}(X)$, $(K_X+iH.V)\geq 0$ since we assumed K_X+iH nef. Moreover, $(K_X+iH.V)>0$ if $(K_X.V)\geq 0$, by ampleness of H. As we supposed that φ_i is not finite, there is an effective curve V such that $(K_X+iH.V)=0$. Therefore, by (0.1), we may find an extremal rational curve C such that $(K_X+iH.C)=0$. If we let $a=(H.C)$, $b=-(K_X.C)$, we get:

$ai=b\leq r+1$.

Since, by hypothesis $i\geq r-1$, one of the following holds:

iii) $r\geqslant 3$, $a=1$.

In case i) the conclusion follows easily from the explicit description of extremal rational curves in case of surfaces, see [20] Theorem 2.1. Case ii) does not occur. Indeed, we would have $b=4$, so $\mathrm{Pic}(X)\simeq\mathbb{Z}$ by (0.5) and in this case φ_2 would be finite. In what follows we shall be concerned with the remaining case $r\geqslant 3$, $a=1$, hence $b=i$. If C' is any effective curve belonging to R, we may write $C'\approx \alpha C$ for some positive rational number α. Since we have $a=1$, it follows that α is a natural number. Therefore we have $b=\min\{-(K_X.C')\,|\,C'$ an effective curve of $R\}$. Denote by $E\subset X$ the locus of points of effective curves belonging to R. Since $b=i\geqslant r-1$, it follows from (0.4) that we have $\dim E\geqslant r-1$. As φ_i is generically finite, we must have $E\neq X$, hence $\dim E=r-1$. It follows that E is an irreducible divisor, see [14] Proposition 5.5. Moreover, the argument in [21], Lemma 3.3, shows that we have $c=:-(E.C)>0$. The morphism $f=\mathrm{cont}_R$ (see 0.3) is an isomorphism outside E and $s=:\dim.f(E)<r-1$. Our aim is to prove that E is an exceptional divisor, see (0.11). We first show that $s=0$, so E is contracted by f to a point. Suppose the morphism f is given by the complete linear system $|D|$. Denote by B the intersection of s generic members of $|D|$ (so $B=X$ if $s=0$). By Bertini's theorem, B is smooth and connected. Let f' denote the restriction of f to E and let F be a general fibre of f'. Note that F is a (reduced) connected component of $G=:B\cap E$. We want first to prove that:

(11) $H^j(\mathcal{O}_G(-nH|_G))=0$ for $j<\dim G=r-s-1$ and $n\geqslant c$.

To prove (11), consider the exact sequence:

$$0\to\mathcal{O}_B(-G-nH|_B)\to\mathcal{O}_B(-nH|_B)\to\mathcal{O}_G(-nH|_G)\to 0.$$

By Kodaira vanishing we have:

(12) $H^j(\mathcal{O}_B(-nH|_B))=0$ for $j<\dim B=r-s$ and $n>0$.

On the other hand, we claim that the divisor $G+nH|_B$ on B is nef and big if $n\geqslant c$. Indeed, by ampleness of $H|_B$, we have $(G+nH|_B.V)>0$ for any effective curve $V\subset B$ which is not contained in G. If $V\subset G$, it follows $[V]\in R$ since V is contracted by f. Recalling that $(E.C)=-c$, we get $(G+nH|_B.C)\geqslant 0$ if $n\geqslant c$, so $(G+nH|_B.V)\geqslant 0$ if $[V]\in R$. Therefore $(G+nH|_B.V)\geqslant 0$ for any effective curve $V\subset B$. Moreover, $G+nH|_B$ is also big, since $H|_B$ is ample and G is effective. Hence we deduce, using Kawamata-Viehweg vanishing, see [13] or [27]:

(13) $H^j(\mathcal{O}_B(-G-nH|_B))=0$ for $j<r-s$ and $n\geqslant c$.

By (12), (13) and the exact sequence we get (11). Then, we also have (since F is a reduced, connected component of G):

(14) $H^j(\mathcal{O}_F(-nH|_F))=0$ for $j<r-s-1$ and $n\geqslant c$.

On the other hand:

(15) $K_F\sim(K_X+E)|_F\approx(-b-c)H|_F$.

Indeed, by the properties of f, any effective curve on F belongs to R. Therefore, if D_1 and D_2 are two divisors on X such that $(D_1.C)=(D_2.C)$ we get $D_1|_F\approx D_2|_F$ and the relation (15) follows.

Using Serre duality and (15) we obtain:

(16) $H^{r-s-1}(\mathcal{O}_F(-nH|_F))=H^0(\mathcal{O}_F(K_F+nH|_F))=0$ for $n\leqslant b+c-1$.

Consider now the polynomial $P(n)=:\chi(\mathcal{O}_F(nH|_F))$. By (14) and (16) it follows that:

(17) $P(-n)=0$ for $c\leqslant n\leqslant b+c-1$.

This gives:

$r-s-1=\dim F=\deg P\geqslant b=i\geqslant r-1$, hence $s=0$ and $b=i=r-1$.

Therefore we have $F=E$ and moreover:

(18) $\chi(\mathcal{O}_E(-nH|_E))=0$ for $c\leqslant n\leqslant c+b-1=c+\dim E-1$.

If we let $d=:(H|_E)^{r-1}>0$, the relation (18) gives:

(19) $\chi(\mathcal{O}_E(nH|_E))=\frac{d}{(r-1)!}(n+c)(n+c+1)\ldots(n+c+r-2)$.

Next we want to prove that $\chi(\mathcal{O}_E)=1$.

Indeed, since $K_E\approx(-c-r+1)H|_E$ by (15), we get using [15] Ch.II, §2, Theorem 1:

(20) $\chi(\mathcal{O}_E)=\chi(\mathcal{O}_E(K_E+(c+r-1)H|_E))$.

But, by duality and (14), it follows:

Using (20) and (21) we get:

(22) $\chi(\mathcal{O}_E)=h^0(\mathcal{O}_E(K_E+(c+r-1)H|_E))$, so $\chi(\mathcal{O}_E)=0$ or 1 since $K_E+(c+r-1)H|_E\approx 0$.

By (19) $\chi(\mathcal{O}_E)\neq 0$. It follows that:

(23) $\chi(\mathcal{O}_E)=1$.

Using this relation, (19) gives $d=c=1$ and:

(24) $\chi(\mathcal{O}_E(nH|_E))=\frac{1}{(r-1)!}(n+1)(n+2)\dots(n+r-1)$.

Now, by Serre duality, the relations (14) and (24) give:

(25) $h^0(\mathcal{O}_E(K_E+(r+1)H|_E))=h^{r-1}(\mathcal{O}_E(-(r+1)H|_E))=$

$$=(-1)^{r-1}\chi(\mathcal{O}_E(-(r+1)H|_E))=(-1)^{r-1}(-1)^{r-1}r=r.$$

Since $c=1$, (15) gives $K_E\approx -rH|_E$, so $K_E+(r+1)H|_E\approx H|_E$. Therefore, $K_E+(r+1)H|_E$ is ample, $(K_E+(r+1)H|_E)^{r-1}=(H|_E)^{r-1}=$ $=d=1$ and $h^0(\mathcal{O}_E(K_E+(r+1)H|_E))=r$ by (25). It follows by (0.7) that $E\simeq\mathbb{P}^{r-1}$, $H|_E\in|\mathcal{O}(1)|$. Since C is a line and $(E.C)=-c=-1$, we get $\mathcal{O}_X(E)\otimes\mathcal{O}_E\simeq\mathcal{O}(-1)$, so E is an exceptional divisor, as we wanted.

c) Assume that $\dim Y<r$.

If $i=r+1$, Y is a point by the first assertion of the Lemma and we get $K_X+(r+1)H\approx 0$. By (0.6) and (0.9) we get $(X,H)\in\mathcal{A}$.

If $i=r$, $\dim Y\leq 1$ by statement a) of the Lemma.

If Y is a point, using (0.6) and (0.10) as above we get $X\simeq Q^r$, $H\in|\mathcal{O}(1)|$. If Y is a (smooth) curve, φ_r is flat. If F denotes a smooth fibre of φ_r, we get $F\simeq\mathbb{P}^{r-1}$, $H|_F\in|\mathcal{O}(1)|$ as above. Now, if E is an arbitrary (closed) fibre, it follows $(H^{r-1}.E)=1$ by flatness of φ_r. In particular E is irreducible and generically reduced, hence reduced because it is Cohen-Macaulay. By semicontinuity we get:

$h^0(\mathcal{O}_E(H|_E))\geq h^0(\mathcal{O}_F(H|_F))=r$.

It follows $E\simeq\mathbb{P}^{r-1}$, $H|_E\in|\mathcal{O}(1)|$, by (0.7), so φ_r is a scroll over a curve. Thus $(X,H)\in\mathcal{B}$ if $i=r$.

Assume now that $i=r-1$. Again by the first part of the Lemma we get $\dim Y\leq 2$. If Y is a point, using (0.6) we get tha (X,H) is a Del Pezzo manifold, see (0.1[5]). We are left with the cases when Y is a curve or a surface. First of all, if r and Y is a curve, (X,H) is easily seen to be a hyperquadric fibration, with reduced, possibly reducible fibres. Assume th $r\geq 3$. In order to have better control on the special fibres w replace φ_{r-1} by a contraction of an extremal ray. Indeed, usi (0.1), we may find an extremal rational curve C with $(K_X+$ $+(r-1)H.C)=0$. We let again $a=(H.C)$, $b=-(K_X.C)$ and we get:

$$a(r-1)=b\leq r+1.$$

Therefore, either $a=1$, or $r=3$, $a=2$, $b=4$. This last possibilit is absurd since by (0.5) we would have $\mathrm{Pic}(X)\cong\mathbb{Z}$, so φ_2 would be constant. So we have $a=1$, $b=r-1$. Consider the extremal ray R generated by C. If R is not nef, reasoning exactly as in th proof of part b) of the Lemma, we may find an exceptional div sor on X. Therefore, after replacing (X,H) by its reduction (X',H'), we may assume that R is nef. Consider the morphism $f=:\mathrm{cont}_R$. For a general fibre F of f, we get as in the proof of (15):

$$K_F\sim K_{X'}|_F\approx(1-r)H'|_F.$$

Using this and Kodaira vanishing it follows as before:

$\chi(\mathcal{O}_F(K_F+nH'|_F))=0$ for $1\leq n\leq r-2$; hence $\dim F\geq r-2$.

Since $K'+(r-1)H'\not\approx 0$, $\dim F<r$, or equivalently, $\dim f(X')>0$. Assume that $\dim F=r-1$, hence $f(X')$ is a (smooth)curve. Then f is flat and a simple argument shows that it has reduced, irre ducible fibres, see [21], p.185. It follows by (0.10) that we have $F\cong Q^{r-1}$, $H'|_F\in|\mathcal{O}(1)|$ for a smooth (closed) fibre F of f. For an arbitrary fibre E we get $(H'^{r-1}.E)=2$ by flatness and is also a hyperquadric by (0.8) and (X',H') is a hyperquadric fibration. Actually, since $r\geq 3$, we must have $X=X'$, see the proof of (0.17).

Finally, assume that $\dim F=r-2$ and denote by S the image of f, which is a normal surface. We claim that f is equidimensional and S is smooth. Indeed, let $s\in S$ be a closed point and denote by E the fibre over s. Consider the embedding of X' given by $|mH'|$ for $m\gg 0$. Let $\tilde{S}$ be the smooth surface got by intersecting $r-2$ general members of $|mH'|$. We first prove that $\dim E=r-2$. Assume that $\dim E=r-1$; since $(E.R)=0$, it follows, as in the proof of (15):

(26) $E_{|E}\approx 0$.

If we denote by $\tilde{f}$ the restriction of f to $\tilde{S}$, we see that $\tilde{f}$ contracts the curve $V=:\tilde{S}\cap E$ to a point. Therefore we get using (26):

$$0>(V^2)_{\tilde{S}}=m^{r-2}(H'^{r-2}.E^2)_X=m^{r-2}(H'^{r-2}_{|E}.E_{|E})_E=0.$$

This contradiction shows that $\dim E=r-2$, hence f is equidimensional. Therefore, by construction of $\tilde{S}$, we may assume that $\tilde{S}\cap E'$ is zero-dimensional and reduced, where E' denotes E with reduced structure.On the other hand, for a general fibre F, we get using (0.9) $F\approx\mathbb{P}^{r-2}$, $H'|_F\in|\mathcal{O}(1)|$. Therefore, $\tilde{f}$ has degree m^{r-2}; since the number of points in $\tilde{S}\cap E'$ is $m^{r-2}(E'.H'^{r-2})\geq$ $\geq m^{r-2}$ and S is normal, it follows by a well-known criterion that $\tilde{f}$ is étale over s. Therefore S is smooth at s since $\tilde{S}$ is smooth. Now, since both S and X' are smooth and f is equidimensional, it follows that f is flat. Therefore $(H'^{r-2}.F)=1$ for any fibre F. It follows that F is irreducible and generically reduced, hence reduced since it is Cohen-Macaulay. We may now deduce exactly as before, using semicontinuity and (0.7) that

$F \cong \mathbb{P}^{r-2}$, $H'|_F \in |\mathcal{O}(1)|$. Thus the reduction (X',H') is a scroll ove
a surface. But in this case, see (0.17), we must have $X=X'$.

This ends the proof of our lemma.

§2. This section is devoted to a couple of applications of the main result. In the following four corollaries we consider polarized pairs (X,H) where X is a threefold and H is an effective, smooth, ample divisor on X.

The first application is an improvement of a result due to Sommese, see [24], Theorem 2.4. In case H is very ample it was proved in [1¹], Theorem II.

Corollary 1. Consider a polarized pair (X,H) such that H is a smooth, birationally ruled surface. Then $(X,H)\in\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}$ or there is a reduction $(X',H')\in\mathcal{D}$.

Proof. By adjunction formula $(K_X+H)|_H\sim K_H$. Since H is ruled, K_H, hence also K_X+H, is not nef. The result follows from (1.7).

The second result is due to L.Bădescu, who proved it by direct arguments, see [1] Theorem 5, [2] Theorem 1 and Theorem 3, and [3].

Corollary 2 (Bădescu). Let (X,H) be a polarized pair such that H is a smooth geometrically ruled surface. Then (X,H) is either a scroll over a curve, or, in case $H\simeq\mathbb{P}^1\times\mathbb{P}^1$, there are two further possibilities: $X\simeq\mathbb{P}^3$, $H\in|\mathcal{O}(2)|$ and $X\simeq Q^3$, $H\in|\mathcal{O}(1)|$.

Proof. By Corollary 1, $(X,H)\in\mathcal{B}\cup\mathcal{C}$ or there is a reduction $(X',H')\in\mathcal{D}$. We shall prove that either $(X,H)\in\mathcal{B}$ or $X\simeq\mathbb{P}^3$, $H\in|\mathcal{O}(2)|$. Assume that $(X,H)\in\mathcal{C}$. If (X,H) is a Del Pezzo threefold, it follows that H is a Del Pezzo surface. But, as it is well-known, the only geometrically ruled Del Pezzo surfaces are $\mathbb{P}^1\times\mathbb{P}^1$ and the projective plane blown-up at a point, denoted F_1. Using the classification of Del Pezzo threefolds, see [12] or

[8], it follows that the case $H\simeq F_1$ is impossible, while for $H\simeq\mathbb{P}^1\times\mathbb{P}^1$ we get $X\simeq\mathbb{P}^3$, $H\in|\mathcal{O}(2)|$. Next we prove that (X,H) can n be a hyperquadric fibration. Indeed, in this case, as in Bădescu's original approach, by Lafschetz's theorem on hyperp ne sections we would have $Pic(X)\simeq Pic(H)$ via restriction. Ther fore, if Q denotes a general fibre of the hyperquadric fibrat and $F=:Q\cap H$, we may find an invertible sheaf on X, say $\mathcal{O}_X(D)$, such that $(D|_H.F)=1$. This leads to a contradiction, since we have:

$$1=(D|_H.F)_H=(D.H.Q)_X=(D|_Q.H|_Q)_Q$$

and the last integer has to be even.

Assume now that (X,H) is a scroll over a surface. We shall first prove that this is possible only if $H\simeq F_1$. Indeed recall that a geometrically ruled surface is a minimal model unless it is isomorphic to F_1. Now, if $f:X\longrightarrow S$ is a morphism m king X a scroll over the surface S, the restriction of f to H birational,hence an isomorphism, unless $H\simeq F_1$ and $S\simeq\mathbb{P}^2$, when it is the blowing-up of a point. As before, we must have $Pic(X)\simeq Pic(H)\simeq Pic(S)$, if $H\not\simeq F_1$. But this is clearly absurd sinc X is a $\mathbb{P}^1$-bundle over S. Assume now that $f:X\to\mathbb{P}^2$ gives the scroll structure and f restricted to H is the blowing-up of a point. Denote by $C\subset H$ the exceptional divisor of the blowing-up In this case we shall see that $X\simeq\mathbb{P}^1\times\mathbb{P}^2$, $H\in|\mathcal{O}(1,1)|$, so X is a scroll over $\mathbb{P}^1$, too. If F denotes a fibre of the rulling of H, $\mathcal{O}_H(C)$ and $\mathcal{O}_H(F)$ form a basis for Pic(H), see [9] Ch.V, §2. Write $H|_H\sim aC+bF$, with $a>0$, $b>a$, see [9] loc.cit. If $P\simeq\mathbb{P}^1$ is a fibre of f, we may write:

$$-2=(K_X.P)_X=(K_X|_H.C)_H=(K_H-H|_H.C)_H=a-b-1,$$

$K_H \sim -2C-3F$, see [9] loc.cit. It follows $b=a+1$. If E denotes the inverse image by f of a line in $\mathbb{P}^2$, we get:

$$(H^3)=a(a+2),\ (H^2.E)=a+1,\ (H.E^2)=1,\ (E^3)=0.$$

Since $\mathrm{Pic}(X)\simeq\mathrm{Pic}(H)$, we may find an invertible sheaf on X, say $\mathcal{O}_X(D)$, such that $D|_H=F$. We get easily $D\sim H-aE$.

Consider now the exact sequence:

$$0\longrightarrow\mathcal{O}_X(-aE)\longrightarrow\mathcal{O}_X(D)\longrightarrow\mathcal{O}_H(F)\longrightarrow 0.$$

Since we have $H^1(X,\mathcal{O}_X(-aE))=H^1(\mathbb{P}^2,\mathcal{O}(-a))=0$, we get that $|D|$ is a pencil. Since two distinct members of $|D|$ can meet only outside H, their intersection is finite, hence empty. It follows $(D^3)=(H-aE)^3=0$ and this gives $a=1$. Now it is very easy to see that $X\simeq\mathbb{P}^1\times\mathbb{P}^2$, $H\in|\mathcal{O}(1,1)|$.

Assume now that we have a reduction $(X',H')\in\mathcal{D}$. We shall prove that this is impossible. Since H is minimal unless $H\simeq F_1$, it follows that either $X=X'$ or $H'\simeq\mathbb{P}^2$. This last case is absurd since (as it is well-known, see for instance [7], Corollary 3.10), we would have $X'\simeq\mathbb{P}^3$, $H'\in|\mathcal{O}(1)|$ and this forces $X=X'$. Therefore, we have $(X,H)\in\mathcal{D}$. The cases $X\simeq\mathbb{P}^3$, $H\in|\mathcal{O}(3)|$ and $X\simeq Q^3$, $H\in|\mathcal{O}(2)|$ are not possible since H is geometrically ruled. It remains to exclude the case when X is a $\mathbb{P}^2$-bundle and H induces $\mathcal{O}(2)$ on each fibre. This is done by the same kind of argument as in the case of hyperquadric fibrations. The proof of Corollary 2 is complete.

The next application is the main result from [24].

<u>Corollary 3</u> (Sommese). <u>Let</u> (X,H) <u>be a polarized pair such that</u> H <u>is a smooth non-ruled surface. Then, either</u> (X,H) is a scroll over a surface, or there is a reduction (X',H')

such that H' is a minimal model.

Proof. Assume H to be non-minimal. Since $(K_X+H)|_H \sim K_H$, it follows that K_X+H is not nef. By (1.7), either $(X,H)\in\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}$ or there is a reduction (X',H') such that $(X',H')\in\mathcal{D}$ or H' is minimal. Since we assumed H to be non-ruled, $(X,H)\in\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}$ is possible only if (X,H) is a scroll over a surface. Finally $(X',H')\in\mathcal{D}$ is impossible since H' is non-ruled.

Our main application in dimension three is the following result, which cupled with Corollary 1 describes completely the threefolds supporting a smooth surface which is not of general type as an ample divisor.

Corollary 4. Let (X,H) *be a polarized pair such that* H *is a smooth surface and denote by* $\kappa(H)$ *its Kodaira dimension. If* $\kappa(H)=0$, (X,H) *is either a scroll over a surface, or there is a reduction* (X',H') *such that* H' *is a* K3 *surface and, consequently,* X' *is a Fano manifold with* H'=-K'. *If* $\kappa(H)=1$, *either* (X,H) *is a scroll over a surface, or there is a reduction* (X',H') *and a morphism* $g:X'\longrightarrow C$ *onto some smooth curve* C, *which is a Del Pezzo fibration, i.e.* $H'|_F \sim -K_F$ *for any smooth fibre* F *of* g.

Proof. Assume that (X,H) is not a scroll over a surface. If $\kappa(H)=0$, by Corollary 3, we may find a reduction (X',H') such that H' is minimal. By classification of minimal surfaces of Kodaira dimension zero, H' may be abelian, K3, Enriques or hyper-elliptic. But one may prove (see for instance [2]) using Lefschetz's theorem that an abelian, Enriques or hyperelliptic surface cannot be an ample divisor on a smooth threefold.

Thus we are left with the case when H' is K3. It follows $H'+K'\sim 0$ by Lefschetz's theorem. Assume that $\kappa(H)=1$. Using (1.7)

Let g be the map associated to $|m(K'+H')|$ for $m\gg 0$. Since H' is an elliptic surface, the restriction of g to H' gives a (pluricanonical) map onto some curve and the rest is clear.

To put the preceding corollary into perspective, we mention the following rather general (and not difficult) fact.

Proposition 5. Let (X,H) be a polarized pair with dim $X=r\geq 2$. Assume that H is smooth and $\kappa(H)<r-1$. Then $\kappa(X)=-\infty$.

Proof. Assume that $|mK_X|\neq\emptyset$ for some $m>0$ and let $E\in|mK_X|$. Write $E=aH+E'$, with $a\geq 0$ and $H\not\subset \operatorname{supp}(E')$. Since nH is very ample for $n\gg 0$, we may find $n\gg 0$ and $D\in|nE|$ such that $H\not\subset\operatorname{supp}(D)$. Since we have $(D+nmH)|_H\sim nmK_H$, it follows that $|nmK_H|$ is very ample outside $H\cap\operatorname{supp}(D)$, so $\kappa(H)=r-1$. This contradiction proves the proposition.

In the sequel, we consider polarized pairs (X,H) with dim $X=r\geq 1$. Write the Hilbert polynomial of the pair (X,H) as:

$$\chi(\mathcal{O}_X(nH))=\sum_{i=0}^{r}a_i\binom{n+i-1}{i};$$

we define the sectional genus g of the pair (X,H) by:

$$g=1-a_{r-1}.$$

It is not difficult to prove that the following relation is true:

$$(27)\quad 2g-2=(K_X+(r-1)H.H^{r-1}).$$

Lemma 6. Assume that (X,H) is a scroll over a curve C. Then the sectional genus of (X,H) equals the genus of C.

Proof. Let $f:X\to C$ be the morphism making (X,H) a scro

As it is well-known, we may find a very ample divisor $\bar{H}$ on X such that $\bar{H}|_F \in |\mathcal{O}(1)|$ for any fibre F of f. It follows easily by induction on dim X=r that the sectional genus of the pair $(X,\bar{H})$ is equal to the genus of C. Therefore, it will be enoug using (27), to prove the following:

$$(28)\quad (K_X+(r-1)H.H^{r-1})=(K_X+(r-1)\bar{H}.\bar{H}^{r-1}).$$

Since $\bar{H}|_F \in |\mathcal{O}(1)|$ and $H|_F \in |\mathcal{O}(1)|$ for any fibre F of f, there is some divisor D on C such that:

$$(29)\quad \bar{H} \approx H + f^*(D).$$

Moreover, by a well-known formula, we have:

$$(30)\quad K_X \approx -rH + f^*(E),\ \text{for some divisor } E \text{ on } C.$$

Using (29) and (30) we get after some computations:

$$(K_X+(r-1)\bar{H}\ .\bar{H}^{\ r-1}) = -(H^r)+(H^{r-1}.f^*(E)) = (K_X+(r-1)H.H^{r-1}).$$

Thus (28) is proved and we are done.

The next results were proved by Beltrametti-Palleschi i case of threefolds, see [4]. Using their idea, we extend them to arbitrary dimension (see also [10], [11] for the case where H is very ample).

Lemma 7. For any polarized pair (X,H), the sectional genus g is non-negative.

Proof. By (1.5) we may assume that either (K_X+

when the previous lemma applies.

Corollary 8 (compare with [4], Proposition 3.1 and [10] Proposition 2.3). Let (X,H) be a polarized pair with $g=0$. Then one of the following holds:

$X \simeq \mathbb{P}^r$, $H \in |\mathcal{O}(1)|$, or $X \simeq Q^r$, $H \in |\mathcal{O}(1)|$, or $X \simeq \mathbb{P}^2$, $H \in |\mathcal{O}(2)|$, or (X,H) is a scroll over $\mathbb{P}^1$.

Proof. We argue as in the proof of Lemma 7, using also Lemma 6.

Corollary 9 (compare with [4], Proposition 3.2 and [10] Proposition 2.6).

Let (X,H) be a polarized pair with $g=1$. Then (X,H) is either a Del Pezzo manifold or a scroll over an elliptic curve.

Proof. If $K_X+(r-1)H$ is not nef, as in the proof of the preceding Corollary, it follows that (X,H) is a scroll over an elliptic curve. If $K_X+(r-1)H$ is nef, it has to be trivial since $(K_X+(r-1)H.H^{r-1})=0$ (see [15] Ch.I, §4, Proposition 3 and (0.6)).

Corollary 10 (compare with [4], Theorem 3.3 and [11], Theorem I).

Let (X,H) be a polarized pair with $g \geqslant 2$ and $2g-2 \leqslant (H^r)$. Then (X,H) is of one of the following types:

(2.1) $r=2$ and either X is birationally ruled or $K_X \approx 0$;

$r \geqslant 3$ and either:

(2.2) (X,H) is a scroll over a curve or a surface;

(2.3) (X,H) is a hyperquadric fibration;

(2.4) There is a reduction (X',H') such that $(X',H') \in \mathcal{D}$;

(2.5) $K_X+(r-2)H \sim 0$.

Proof. Assume that $r=2$. Since $2g-2=(H^2)+(H.K_X)$, the hypothesis gives: $(H.K_X)\leq 0$. Thus, either $|mK_X|=\emptyset$ for any $m\geq 1$ and X is ruled by Enriques' criterion, or, if $|mK_X|\neq\emptyset$, for some m it follows $K_X\equiv 0$. Assume $r\geq 3$.

Using (1.7) we deduce that either (X,H) is as stated in (2.2)-(2.4), or $K'+(r-2)H'$ is nef. On the other hand, by direct computation, we find that (X,H) and (X',H') have the same sectional genus; moreover, $(H^r)\leq(H'^r)$, with equality only if X=X'. Using this and the hypothesis $2g-2\leq(H^r)$ we get:

$$(31)\quad 2g-2=(K'+(r-1)H'.H'^{r-1})\leq(H^r)\leq(H'^r).$$

This implies:

$$(K'+(r-2)H'.H'^{r-1})\leq 0.$$

Assuming now $K'+(r-2)H'$ to be nef, by [15] Ch.I, §4, Proposition 3 and (0.6), we get:

$$K'+(r-2)H'\sim 0.$$

Using (31) if follows $(H^r)=(H'^r)$, so X=X' and we are in case (2.5).

The final application is due to Lanteri-Palleschi, see [18].

Corollary 11 (compare with [5]). *Let* $f:X\to Q^r$ *be a finite morphism from a manifold* X *to the smooth* r*-dimensional hyperquadric. Suppose that* f *is not an isomorphism. Then, the ramification divisor* R *is ample, unless* $r=2$, X *is a scroll and* R *is a sum of fibres.*

Proof. Let H be a divisor in $|f^*(\mathcal{O}_Q(1))|$ and assume that R is not ample. Since $R\in|K_X+rH|$, it follows by (1.4) that $(X,H)\in\mathcal{A}\cup\mathcal{B}$. Since $(H^r)=2.\deg f$, we may assume that (X,H) is a scroll over a curve. The restriction of f to a fibre of the scroll gives a $\mathbb{P}^{r-1}$ embedded in Q^r as a linear space. This

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