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by

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C_p - ESTIMATES FOR CERTAIN KERNELS: THE CASE $0 < p < 1$

1. The purpose of this work is to show that the results obtained in [15] concerning certain quite general Hankel-like kernels also hold in the more refined setting of non-locally convex Schatten-von Neumann classes C_p , $0 < p < 1$. This extends the one-dimensional results for Hankel operators of [10] and [14]. Our methods are closer to those of Peller [10].

Let A be a bounded measurable function on $R^d \times R^d$; usually A will be supposed C^∞ outside the origin (but see remark 1). Also, we have the homogeneity condition

$$A(gx,gy) = A(x,y) \quad \text{for any} \quad x,y \in R^d \quad \text{and} \quad g \in G \qquad (1)$$

where G is a fixed discrete multiplicative subgroup of R_+. We shall assume in the sequel that G is generated by $g_0 > 1$.

We denote, as in [15], by T(A) the operator (on $L^2(R^d)$) with kernel A(x,y) and with $T(A,\phi)$ the operator with kernel $A(x,y)\,\hat{\phi}(x-y)$, where $\hat{\phi}$ is the Fourier transform of ϕ. Our main purpose is to obtain, under some suitable supplementary conditions on A, a precise criterion for the belonging of the operator $T(A,\phi)$ to the Schatten-von Neumann classes C_p, for $0 < p < 1$ (the best reference for the results we need about these classes is [6]). The typical result is: $T(A,\phi) \in C_p$ if and only if $\phi \in \dot{B}^{d/p}_{pp}$ (homogeneous Besov space; see [7] for reference). This result has been obtained for $1 \le p \le 2$ in [15], where the necesity of condition $\phi \in \dot{B}^{d/p}_{pp}$ is also obtained for $2 \le p < \infty$.

More recently, Janson and Peetre [4] have further extended this by relaxing the homogeneity condition (1) and proving also the sufficiency of $\phi \in \dot{B}^{d/p}_{pp}$ for $2 \le p < \infty$. (The "Fourier transform" of operators of type $T(A,\phi)$ are called in [4] paracommutators). For the history of the subject, which starts with the work of Peller on Hankel operators (see [8], [9]).

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2. The main tool which allows us to adapt to our case the methods of [10] is the following (almost classical) theorem of Plancherel and Polya [12].

THEOREM A. *Let* $p, a > 0$. *For any* $a' > a/\pi$ *there exist two universal constants* $C_1(a',p) > C_2(a',p) > 0$, *such that for any entire function* F *of exponential type* a *we have*

$$C_1(a',p)\int_{\mathbb{R}^d} |F(x)|^p dx \leq \sum_{m\in\mathbb{Z}^d} |F(m/a')|^p \leq C_2(a',p)\int_{\mathbb{R}^d} |F(x)|^p dx \quad (2)$$

Note that in the sequel we always use the formula $\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i\xi\cdot x} dx$ for the Fourier transform.

We have to say a few words about the definition of the operator $T(A,\phi)$. We will always suppose that $\hat{\phi}$ is locally integrable on $\mathbb{R}^d\setminus\{0\}$. If $\hat{\phi}$ happens to be locally integrable on all $\mathbb{R}^d$, it is clear how to define the operator with kernel $A(x,y)\hat{\phi}(x-y)$ on $C_0^\infty(\mathbb{R}^d)$; when it is bounded it may be extended to all $L^2(\mathbb{R}^d)$ and we denote the extension by $T(A,\phi)$. In the general case, take a C^∞ - function θ, supported in the unit ball of $\mathbb{R}^d$ and equal to 1 in a neighbourhood of the origin; then $\hat{\phi}(s)(1-\theta(s/\varepsilon))$ is locally integrable on $\mathbb{R}^d$, and we denote the corresponding operator T_ε. If all T_ε are bounded, and tend to a limit when $\varepsilon \to 0$. we denote this limit by $T(A,\phi)$ (it can then be seen that the definition does not depend on the choice of θ). This discussion will be tacitly assumed in the sequel.

We shall repeatedly use the fact (see [6]) that for $0<p<1$ $(S,T) \to \|S-T\|_p^p$ is a distance invariant under translations on the quasi-Banach space C_p; that is

$$\|T_1+T_2\|_p^p \leq \|T_1\|_p^p + \|T_2\|_p^p \quad (3)$$

Finally, the letter C will denote a constant that may not be the same in different inequalities.

We may state now an analogue of lemma 1 of [15].

LEMMA 1. *Let* $0<p<1$, *and* $E\subset R^d\times R^d$ *be such that, for some* $a>0$, $E\supset\{(x,y)|\, g_0^{-1}a\leq|x-y|\leq g_0a|$, *and denote by* $\chi(x,y)$ *the characteristic function of* E. *Then*

$$\|T(A,\phi)\|_p\leq C\|T(\chi A)\|_p\|\phi\|_{\dot{B}_{pp}^{d/p}}.$$

PROOF. Let $\psi\in S(R^d)$ be such that $\operatorname{supp}\hat{\psi}\subset\{g_0^{-1}a\leq|x|\leq g_0a\}$, and $\sum_{m\in\mathbb{Z}}\hat{\psi}_k=1$ on $R^d\setminus\{0\}$, where $\hat{\psi}_k(x)=\hat{\psi}(g_0^{-k}x)$. Since

$$A(x,y)\hat{\phi}(x-y)=\sum_{m\in\mathbb{Z}}\hat{\psi}_k(x-y)A(x,y)\hat{\phi}(x-y)$$

we have, using (3),

$$\|T(A,\phi)\|_p^p\leq\sum_{k\in\mathbb{Z}}\|T(A,\phi*\psi_k)\|_p^p=\sum_{k\in\mathbb{Z}}\|T(\chi_kA,\phi*\psi_k)\|_p^p \tag{4}$$

where $\chi_k(x,y)=\chi(g_0^{-k}x,g_0^{-k}y)$.

Now, suppose $\hat{\eta}$ is some smooth function with support in $\{x\in R^d\,|\ |x|\leq R'<R\}$. We may write

$$\hat{\eta}(y)=1/R^d\sum_{m\in\mathbb{Z}^d}\eta(m/R)e^{-2\pi iym} \tag{5}$$

and

$$B(x,y)\hat{\eta}(x-y)=(1/R^d)\sum_{m\in\mathbb{Z}^d}\eta(m/R)B_m(x,y) \tag{6}$$

where

$$B_m x,y)=e^{-2\pi ix\cdot m}B(x,y)e^{2\pi iy\cdot m}$$

and it is obvious that $\|T(B_m)\|_p=\|T(B)\|_p$.

Applying these remarks to the terms appearing in (4), we have

$$(\chi_kA)(x,y)(\hat{\phi}\hat{\psi}_k)(x-y)=-[1/g_0^{k+2})^d]\sum_{m\in\mathbb{Z}^d}(\phi*\psi_k)(m/g_0^{k+2}a)(\chi_kA)_m(x,y)$$

and therefore

$$\| T(\chi A, \phi * \psi_k) \|_p^p \leq C(1/g_0^{kdp}) \sum_{m \in \mathbb{Z}^d} |(\phi * \psi_k)(m/g_0^{k+2}a)|^p \| T(\chi_k A) \|_p^p .$$

By (2), and noting that, by homogeneity, $\| T(\chi_k A) \|_p = g_0^{kd} \cdot \| T(\chi A) \|_p$ we obtain

$$\| T(\chi A, \phi * \psi_k) \|_p^p \leq C g_0^{kd} \| \phi * \psi_k \|_p^p \cdot \| T(\chi A) \|_p^p .$$

Therefore, by (4),

$$\| T(A, \phi) \|_p^p \leq C(\sum_{k \in \mathbb{Z}} g_0^{kd} \| \phi * \psi_k \|_p^p) \cdot \| T(\chi A) \|_p^p$$

and the proof is finished.

Using the preceeding lemma and arguments similar to those in [15], we can now prove the following theorem.

THEOREM 1. *Suppose* A *is* C^∞ *on* $R^{2d} \setminus \{0\}$, *satisfies (1), and, together with its derivatives of order* $\leq N - 1$, *vanishes on the diagonal* $\Delta = \{(x,y) \in R^{2d} \mid x = y\}$. *Suppose also* $N > d/p$, $0 < p < 1$. *Then there is a constant* C_p, *depending on* A, *such that for any* ϕ *we have*

$$\| T(A, \phi) \|_p \leq C_p \| \phi \|_{\dot{B}_{pp}^{d/p}}$$

PROOF. Since A vanishes of order N on the diagonal it may be written as a sum of terms of the form

$$[((x_{i_j} - y_{i_j}) \ldots (x_{i_N} - y_{i_N}))/(|x|^2 + |y|^2)^{N/2}] \cdot \tilde{A}(x,y)$$

where $\tilde{A}$ is C^∞ and satisfies (1). Also, by polarization, we may consider only kernels of the form

$$\mu(x,y)^N (|x|^2 + |y|^2)^{-N/2} \tilde{A}(x,y) \hat{\phi}(x - y)$$

where μ is a linear function on R^{2N}, vanishing on Δ.

If we denote by $\tilde{\mu}(x,y) = \mu(x,y)(|x|^2 + |y|^2)^{-1/2}$, then $A = \tilde{\mu}\tilde{A}$, $\tilde{\mu}$ is R^*_+ - -homogeneous (therefore G-homogeneous) and

$$|\tilde{\mu}(x,y)| \leq C \cdot [|x-y|/(|x|^2 + |y|^2)^{1/2}] \qquad (7)$$

We denote, for $m \in \mathbb{Z}^d$, $Q_m = \prod_{i=1}^{d} [m_i, m_i + 1]$ and $\Xi \subset \mathbb{Z}^d \times \mathbb{Z}^d$, $\Xi = \{(m,n) \in \mathbb{Z}^d \times \mathbb{Z}^d \mid 2d^{\frac{1}{2}} \leq |m-n| \leq (3g_0^2 + 1)d^{\frac{1}{2}}\}$ and we choose $E = \bigcup_{(m,n)\in\Xi} Q_m \times Q_n$. Then E fulfills the condition in Lemma 1, with $a = 3d^{\frac{1}{2}}g_0$; we must therefore seek to estimate $\|T(\chi A)\|_p$. Denote by P_Q the orthogonal projection (in $L^2(R^d)$) onto $L^2(Q)$.

We have, by (3)

$$\|T(\chi A)\|_p^p \leq \sum_{(m,n)\in\Xi} \|P_{Q_m} T(\chi A) P_{Q_n}\|_p^p \qquad (8)$$

Now, $P_{Q_m} T(\chi A) P_{Q_n}$, considered as an operator from $L^2(Q_n)$ to $L^2(Q_m)$, has a C^∞ kernel; moreover, for $|\alpha| \leq N-1$, using (7) and the G - homogeneity of $\tilde{A}$ and $\tilde{\mu}$ we obtain that $|D^\alpha A(x,y)|$ is majorized by a sum of terms of the form

$$C|x-y|^{N-k}(|x|^2 + |y|^2)^{-1/2(N+|\alpha|-k)}$$

where $0 \leq k \leq |\alpha|$.

Since, for $(x,y) \in E$, $|x-y|$ is bounded, while $|x|^2 + |y|^2$ is bounded from below and, moreover, of the same order as $|m|^2 + |n|^2$ if $(x,y) \in Q_m \times Q_n$, we obtain:

$$|D^\alpha A(x,y)| \leq C(|m|^2 + |n|^2)^{-N/2}$$

for $0 \leq |\alpha| \leq N-1$ and $(x,y) \in Q_m \times Q_n$.

The estimate is actually valid for all α, since for $|\alpha| \geq N$ it is an immediate consequence of the homogeneity of A. It follows, by [2.XI.9], that $P_{Q_m} T(\chi A) P_{Q_n}$ is in C_p, and

$$\|P_{Q_m} T(\chi A) P_{Q_n}\|_p^p \leq C \cdot (|m|^2 + |n|^2)^{-Np/2}$$

whence, by (8),

$$\| T(\chi A)\|_p^p \le C \sum_{(m,n)\in \Xi} (|m|^2 + |n|^2)^{-Np/2}$$

and, therefore, $\| T(\chi A)\|_p$ is finite ($Np > d$). Then lemma 1 yields the result of the theorem.

3. The reverse problem may be treated in a slightly more general from. Before stating the theorem, let us remark the following multiplier-type property: suppose T is an operator on $L^2(R^d)$ with kernel $k(x,y)$ and $\| T\|_p$ finite; $\alpha(x,y)$ is a function in $C_0^\infty(R^d)$. Then the operator S with kernel $\alpha(x,y)k(x,y)$ is also of class C_p, and $\| S\|_p \le C\| T\|_p$, where C depends only on α. This is easily proved by using (5) for $\alpha = \hat{\eta}$ (in R^{2d}); then $\alpha(x,y)k(x,y)$ may be developed similarly to formula (6) and we have again to apply (3) to obtain the desired result.

THEOREM 2. *Suppose* A *is* C^∞ in $R^{2d} \setminus \{0\}$, *satisfies (1) as well as the following property:*

(9) *for every* $u \in R^d$, *there is* $x \in R^d$, *such that* $A(x + u,x) \neq 0$.

Then there is a constant C_p, *depending only on* A, *such that for any* ϕ, *with* $\hat{\phi}$ *locally integrable on* $R^d \setminus \{0\}$, *we have*

$$\|\phi\|_{\dot{B}_{pp}^{d/p}} \le C_p \| T(A\phi)\|_p$$

PROOF. The proof is rather intricate, but the main idea (similar to that in [10]) is simpler: certain "parts" of $T(A\phi)$ are used to estimate operators unitarily equivalent to periodic convolutions (on T^d); the C_p-norms of such convolution operators are easily calculated, since the exponentials form a complete set of eigenvectors. Then theorem A is called again to estimate the L^p-norm of truncations of ϕ.

Let us pass now to the details. Consider the set $F = \{ u \in R^d \mid 1 \le u \le g_0\}$.

The condition (6) allows us to find a real number $\varepsilon > 0$ as well as $m_1, \dots m_N$, $m'_1, \dots m'_N \in \mathbb{Z}^d$, with properties a), b) listed below. In order to state them (and for further use) we make first the following notations: for any $m \in \mathbb{Z}^d$, Q_m, $\tilde{Q}_m$, R_m, $\tilde{R}_m$ will dénote the cubes with center εm amd having the length of the side respectively equal to ε, $\varepsilon/2$, $(4d^{\frac{1}{2}}+1)\varepsilon$, $(4d^{\frac{1}{2}}+2)\varepsilon$. Then we may assume that the following assertions are valid:

a) $\tilde{Q}_{m_1-m'_1}, \dots, \tilde{Q}_{m_N-m'_N}$ cover F.

b) The origin does not belong to any $\tilde{R}_{m_j}$, $\tilde{R}_{m'_j}$; moreover, $A(x,y) \neq 0$ on $\tilde{R}_{m_j} \times \tilde{R}_{m'_j}$.

We choose then functions $\beta_j \in \mathcal{S}(R^d)$, such that $\hat{\beta}_j$ is positive, with support equal to $Q_{m_j-m'_j}$. Also, for any cube Q, P_Q will be the orthogonal projection on $L^2(Q)$ (as in the proof of Theorem 1).

Now, for any $j = 1, \dots, N$, let α_j be a function in $C_0^\infty(R^{2d} \setminus \{0\})$, supported on $\tilde{R}_{m_j} \times \tilde{R}_{m'_j}$, and identically 1 on $R_{m_j} \times R_{m'_j}$. By the remark preceeding the theorem, since $\alpha'_j(x,y) = (1/A(x,y))\alpha_j(x,y t)\beta_j(x-y)$ is a C_0^∞ function on R^d, we have

$$\| P_{R_{m_j}} T(1,\beta_j * \phi) P_{R_{m'_j}} \|_p \leq C \cdot \| P_{\tilde{R}_{m_j}} T(A,\phi) P_{R_{m'_j}} \|_p . \qquad (10)$$

Consider now the functions $\phi^{(j)} = \phi * \beta_j$; define $\phi_m^{(j)}$ by $\hat{\phi}_m^{(j)}(x) = \hat{\phi}^{(j)}(x + \varepsilon m)$, and $\psi^{(j)}$ by $\hat{\psi}^{(j)}(x) = \sum_{m \in \mathbb{Z}^d} \hat{\phi}_m^{(j)}(x)$ (the sum is locally finite). Obviously, $\hat{\psi}^{(j)}$ is $\varepsilon\mathbb{Z}^d$-periodic. For $(x,y) \in Q_{m_j} \times Q_{m'_j}$, we have

$$\hat{\psi}^{(j)}(x-y) = \sum_{|m| \leq 2d^{\frac{1}{2}}} \hat{\phi}_m^{(j)}(x-y)$$

and, therefore, using (3),

$$\| P_{Q_{m_j}} T(1,\psi^{(j)}) P_{Q_{m'_j}} \|_p^p \leq \sum_{|m| \leq 2d^{\frac{1}{2}}} \| P_{Q_{m_j}} T(1,\phi_m^{(j)}) P_{Q_{m'_j}} \|_p^p = \qquad (11)$$

$$= \sum_{|m| \leq 2d^{\frac{1}{2}}} \| P_{Q_{m_j+m}} T(1,\phi^{(j)}) P_{Q_{m'_j}} \|_p^p \leq C \cdot \| P_{R_{m_j}} T(1,\phi^{(j)}) P_{R_{m'_j}} \|_p^p .$$

Now, by the periodicity of $\hat{\psi}^{(j)}$,

$$\|P_{Q_{m_j}} T(1,\psi^{(j)}) P_{Q_{m'_j}}\|_p = \|P_{Q_0} T(1,\psi^{(j)}) P_{Q_0}\|_p .$$

But the operator $P_{Q_0} T(1,\psi^{(j)}) P_{Q_0}$ is easily seen to have the complete set of eigenvectors $\{e^{i(2\pi/\varepsilon)k\cdot}\}\ k \in \mathbb{Z}^d$; the corresponding eigenvalues are

$$\lambda_k = \int_{Q_o} \hat{\psi}^{(j)}(s) e^{-i(2\pi/\varepsilon)k\cdot s} ds = \int_{Q_o} \Big(\sum_{m\in\mathbb{Z}^d} \hat{\phi}^{(j)}(s+\varepsilon m)\Big) e^{-i(2\pi/\varepsilon)k\cdot s} ds =$$

$$= \int_{\mathbb{R}^d} \hat{\phi}^{(j)}(s+\varepsilon(m-m'_j)) e^{-i(2\pi/\varepsilon)k\cdot s} ds = e^{2\pi i k\cdot(m_j-m'_j)} (\phi * \beta_j)(-k/\varepsilon)$$

Therefore

$$\|P_{Q_0} T(1,\psi^{(j)}) P_{Q_0}\|_p^p = \sum_{k\in\mathbb{Z}^d} |(\phi * \beta_j)(k/\varepsilon)|^p \tag{12}$$

But, since the support of the Fourier transform of the function $e^{2\pi i\varepsilon(m_j-m'_j)}(\phi * \beta_j)$ is obtained in $\tilde{Q}_o$, this function is of exponential type $\varepsilon/2$; we may apply theorem A to deduce that

$$\sum_{k\in\mathbb{Z}^d} |(\phi * \beta_j)(k/\varepsilon)|^p \geq C \cdot \|\phi * \beta_j\|_p^p \tag{13}$$

Combining (10), (11), (12) and (13) we obtain the basic estimate

$$\|\phi * \beta_j\|_p^p \leq \|P_{\tilde{R}_{m_j}} T(A,\phi) P_{\tilde{R}_{m'_j}}\|_p^p$$

If $\Phi = \sum \beta_j$, we get

$$\|\phi * \Phi\|_p^p \leq C \cdot \sum_{j=1}^{N} \|P_{\tilde{R}_{m_j}} T(A,\phi) P_{\tilde{R}_{m'_j}}\|_p^p . \tag{14}$$

Note that condition a) guarantees that $\hat{\Phi}$ is a positive function supported in $C_0^\infty(\mathbb{R}^d \setminus \{0\})$, which is strictly positive on F. Its dilations are therefore convenient for the calculation of the Besov norms; we define $\Phi_k \in S(\mathbb{R}^d)$ by $\hat{\Phi}_k(x) = \hat{\Phi}(g_0^{-k}x)$, and apply the preceeding result to the function $\phi_k(x) = g_0^{-kd}\phi(g_0^{-k}x)$.

(Note that the whole construction above depends only on A, and not on ϕ.) The homogenity of A yields

$$g_0^{-kd(p-1)} \| \phi * \Phi_k \|_p^p \leq C \cdot g_0^{-kdp} \cdot \sum_{j=1}^{N} \| P_{\tilde{R}^k_{m_j}} T(A,\phi) P_{\tilde{R}^k_{m'_j}} \|_p^p$$

where $\tilde{R}^k_{m_j} = g_0^k \tilde{R}_{m_j}$, $\tilde{R}^k_{m'_j} = g_0^k \tilde{R}_{m'_j}$.

Summing for $k \in \mathbb{Z}$, we obtain

$$\sum_{k\in\mathbb{Z}} g_0^{kd} \| \phi * \Phi_k \|_p^p \leq C \cdot \sum_{k\in\mathbb{Z}} \sum_{j=1}^{N} \| P_{\tilde{R}^k_{m_j}} T(A,\phi) P_{\tilde{R}^k_{m'_j}} \|_p^p \leq C \cdot \| T(A,\phi) \|_p^p$$

the last inequality following from fact that, for $|k - \ell|$ large, $\tilde{R}^k_{m_j}$ and $\tilde{R}^\ell_{m_j}$, $\tilde{R}^k_{m'_j}$ and $\tilde{R}^\ell_{m'_j}$ are disjoint. The theorem is thus proved.

4. Final remarks. 1. In both theorems 1 and 2 the hypothesis $A \in C^\infty(R^{2d} \setminus \{0\})$ is a convenient one, but not necessary. First, it is obvious that only a fixed finite number of derivatives are necessary. What is more important from the point of view of applications, we may allow some lower-dimensional sets of non diferentiability.

Thus, we may prove, by similar methods, the following two statements, which extend some previous work ([3], [5], [13], [14]).

PROPOSITION 1. *If* $N > d/p$, *and* $\phi \in \dot{B}^{d/p}_{pp}$, *then the N-times iterated commutator* $[\ldots[M_\phi, K_1], K_2]\ldots K_N]$ *is in* C_p (M_ϕ *is multiplication by* ϕ; $K_1, \ldots, K_N$ *are singular Calderon-Zygmund transforms - see* [3].) *The converse also holds, under the nondegeneracy condition (9) applied to* $A(x,y) = \prod_{i=1}^{N} (\hat{K}_j(x) - \hat{K}_j(y))$.

PROPOSITION 2. *If* $s,t > -d/2$, *and* $A(x,y)$ *satisfies the hypothesis of thoerem 1, with* $N > s + t + d/p$, *then, if* $\phi \in \dot{B}^{d/p+s+t}_{pp}$, *then the operator with kernel* $A(x,y)|x|^s|y|^t\hat{\phi}(x-y)$ *is in* C_p. *The converse also holds, if* $A(x,y)$ *satisfies (9).*

For the case $p \geq 1$, more general conditions on A are given in [4].

2. The fact that condition $N > d/p$ in the statement of theorem 1 (or $N > s + t + d/p$ in proposition 2) is sharp can be proved similarly to theorem 3 of [3].

3. Certain types of para-products (see, for instance, [11]) are also covered by theorems 1 and 2.

4. Since it is known (interpolation theory: see [1]) that $(C_{p_0}, C_{p_1})_{\theta,q} = C_{pq}$ if $[(1-\theta)/p_0)] + [\theta/p_1)] = 1/p$, it follows that, in the hypothesis of theorem 1, if $\phi \in (\dot{B}^{d/p_0}_{p_0p_0}, \dot{B}^{d/p_1}_{p_1p_1})_{\theta,q}$ where $0 < p_0 < p < p_1$, and $\theta = (1/p_0 - 1/p)/(1/p_0 - 1/p_1)$, $0 < q \leq \infty$, then $T(A,\phi) \in C_{pq}$.

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