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ISSN 0250 3638

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by

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PREPRINT SERIES IN MATHEMATICS

No.50/1985

BUCURESTI

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September 1985

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ON AUTOMORPHISMS OF SURFACES AND ABELIAN VARIETIES

A.BUIUM

In the present paper we investigate the field of invariants of certain groups acting on function fields of surfaces and abelian varieties .This allows us to prove the equivalence between Kolchin's weak and strong normality [11] in the case of curves, non-ruled surfaces and abelian varieties.

1. Weak and strong normality

Throughout the paper we adopt the conventions in [11], Chapter II, namely $\mathcal{F}\subset\mathcal{G}$ will be a finitely generated differential field extension of finite transcendence degree such that $\mathcal{F}$ and $\mathcal{G}$ have the same field of constants $\mathcal{C}$, $\mathcal{C}$ is algebraically closed and $\mathcal{F}$ is relatively algebraically closed in $\mathcal{G}$;we denote by $n,\varkappa,q$ the transcendence degree, the Kodaira dimension and the irregularity of $\mathcal{G}/\mathcal{F}$ ($\varkappa$ and q are defined via a non-singular projective model of $\overline{\mathcal{F}}\mathcal{G}/\overline{\mathcal{F}}$ where $\overline{\mathcal{F}}$ is an algebraic closure of $\mathcal{F}$,see [21] pp. 68 and 114).

There are two fundamental concepts of normality in Kolchin's Galois theory of differential fields called weak and strong normality. Recall that $\mathcal{F}\subset\mathcal{G}$ is called weakly normal (cf. [11] p.755) if $\mathcal{G}^G=\mathcal{F}$ where $G=G(\mathcal{G}/\mathcal{F})$ is the group of all differential $\mathcal{F}$-automorphisms of $\mathcal{G}$. $\mathcal{F}\subset\mathcal{G}$ is called strongly normal (cf. [11] p.756 or [12] p.393) if for any $\mathcal{F}$-isomorphism σ of $\mathcal{G}$ into an extension of $\mathcal{G}$ the extensions $\mathcal{G}\subset\mathcal{G}\sigma\mathcal{G}$ and $\sigma\mathcal{G}\subset\mathcal{G}\sigma\mathcal{G}$ are generated by constants. By Kolchin's theory [11], [12] if $\mathcal{F}\subset\mathcal{G}$ is strongly normal then it is also weakly normal (and in fact a series of remarkable properties hold, strarting with the fact that $G(\mathcal{G}/\mathcal{F})$ is, in a natural way, an algebraic group of dimension n over $\mathcal{C}$). An important question is: when are weak and strong normality equivalent? We shall prove:

THEOREM.- Weak and strong normality are equivalent in each of the following cases:

A) $n=1$.

B) $n=2$, $\varkappa\neq-\infty$.

C) $n=q$, $\varkappa\neq-\infty$.

Case A) was proved by Kolchin [11] p.809 under the additional assumption that either $\mathcal{F}$ is algebraically closed or $\varkappa=-\infty$; we shall prove it here in general. Since A) for $\varkappa\neq-\infty$ clearly reduces to C), the only things to prove are B) and C). Note also that one cannot drop the assumption $\varkappa\neq-\infty$ in B); indeed one can check, using computations in [11] pp. 792-795, that the purely transcendental extension

$$\mathbb{C}(e^{x},e^{ix})\subset\mathbb{C}(x,e^{x},e^{ix},e^{x^2})$$

(with derivation d/dx) is weakly normal but not strongly normal. Finally let's make the following remark: if $\mathcal{F}\subset\mathcal{G}$ is weakly normal then $\varkappa\leq 0$. Indeed $\overline{\mathcal{F}}\subset\overline{\mathcal{F}}\mathcal{G}$ will be also weakly normal and now apply:

LEMMA.- Let K be an algebraically closed field of characteristic zero and L a finitely generated extension of L. Suppose that $L^G=K$ where G is the group of all K-automorphisms of L. Then L/K has Kodaira dimension ≤ 0.

Proof. Let X be a non-singular projective model of L/K and suppose X has Kodaira dimension $\geqslant 1$. Then for some $m \geqslant 1$ the pluricanonical map $X \dashrightarrow \mathbb{P}(H^0(X,\mathcal{O}_X(mK_X))=\mathbb{P}$ will have an image Y of dimension $\geqslant 1$. By a result of Deligne-Namikawa-Ueno [21] p.182 G acts on $\mathbb{P}$ via some finite group so we get

$$K \subset K(Y)^G \subset K(X)^G=K \quad \text{and} \quad [K(Y):K(Y)^G] < \infty$$

hence $K(Y)=K$, contradiction. The lemma is proved.

Now the above discussion shows that in cases B) and C) in our Theorem we may suppose $\varkappa=0$; in particular by [10] case C) refers essentially to abelian varieties.

The ingredients in the proof of our Theorem will be:

1) the investigation of the group of automorphisms of a surface (or abelian variety) globally invariating a divisor (see Propositions 1 and 3).

2) a "reduction" criterion for Galois cocycles with values in the automorphism group of an abelian variety (see Proposition 2)

3) our previous work on movable singularities [3] [4] [5] (see Proposition 4)

4) an argument of Burns and Wahl [7] (Proposition 5).

2. Automorphisms of surfaces.

This section is devoted to the proof of the following:

PROPOSITION 1.- Let X be a smooth projective non-ruled minimal surface over an algebraically closed field K of characteristic zero. Let D be a divisor on X and put

$$G=\{g\in \mathrm{Aut}(X);\ g^{*}D=D\}$$

Assume that $K(X)^G=K$. Then either $D=0$ or the support of D is a disjoint union of A-D-E curves.

Recall from [1] p.74 that an A-D-E curve on a surface is a connected effective divisor all of whose irreducible components $D_1,\dots,D_p$ are non-singular rational curves with self-intersection -2 and for which the intersection matrix $\|(D_i.D_j)\|$ is negative definite.

Remark. The case $D\neq 0$ may really occur in Proposition 1: take X to be the Kummer surface associated to $E\times E$ where E is an elliptic curve and take D to be the sum of the 16 distinguished (-2)-curves on X (see [1] p.251). One can easily check that $K(X)^G=K$, where G is defined as in the Proposition above.

We start by recalling some well known facts about automorphisms of projective varieties cf.[9].

Given a smooth projective K-variety X (K algebraically closed of characteristic zero) its automorphism group Aut(X/K) (or simply Aut(X)) is a group scheme locally of finite type over K having at most countably many components; the connected component $Aut^{o}(X)$ of the identity is an algebraic group over K. Furthermore $Aut^{o}(X)$ is an abelian variety provided X is non-ruled [17]. If D is a divisor on X the group $Aut(X,D)=\{g \in Aut(X);\ g^{*}D=D\}$ is a closed subgroup of Aut(X) and we denote by $Aut^{o}(X,D)$ its component of the identity. Now if $\lambda \in NS(X)$=Neron-Severi group of X, put $Aut(X,\lambda)= \{g \in Aut(X);\ g^{*}\lambda=\lambda\}$; then $Aut(X,\lambda)$ clearly contains $Aut^{o}(X)$. Moreover the quotient $Aut(X,\lambda)/Aut^{o}(X)$ is finite provided λ is ample.

Now we shall prove a series of lemmas. In what follows X will be a surface (smooth, projective over an algebraically closed field K of characteristic zero; to simplify proofs we shall sometimes tacitly assume that K is the complex field but results hold without this assumption).

LEMMA 1.- Let $\lambda \in NS(X)$ with $(\lambda.\lambda) > 0$. Then the quotient $Aut(X,\lambda)/Aut^{o}(X)$ is finite.

Proof. Let $u: Aut(X,\lambda)/Aut^{o}(X) \longrightarrow O(NS(X),\lambda)$ be the natural homomorphism, the second group being the group of all orthogonal automorphisms of the lattice $(NS(X),(\ .\ ))$ keeping λ fixed. Now $Ker(u) \subset Aut(X,h)/Aut^{o}(X)$ for some (in fact for any) ample $h \in NS(X)$, hence $Ker(u)$ is finite. Finally, by the Hodge Index Theorem, $(\ .,)$ is negative definite on the orthogonal complement of λ in $NS(X)\otimes\mathbb{Q}$ hence $O(NS(X),\lambda)$ must also be finite and we are done.

LEMMA 2.- Let $D \neq 0$ be a divisor on X, $Supp(D)=D_1\cup..\cup D_p$, D_i integral curves. Suppose $Aut(X,D)/Aut^{o}(X,D)$ is infinite. Then the intersection matrix of the divisors $D_1,\ldots,D_p$ is negative semi-definite.

Proof. Suppose there exist $m_1,\ldots,m_p \in \mathbb{Z}$ such that $(\lambda.\lambda) > 0$ where λ is the image of $m_1D_1+\ldots+m_pD_p$ in $NS(X)$ and look for a contradiction. Each $g \in Aut(X,D)$ induces a permutation of $D_1,\ldots,D_p$ so we have a natural group homomorphism $u:Aut(X,D) \longrightarrow S_p$ where S_p is the corresponding symmetric group. Now $Ker(u)=\bigcap_{i=1}^{p} Aut(X,D_i) \subset Aut(X,\lambda)$; since $Aut(X,D)$ meets infinitely many components of $Aut(X)$ the same will hold for $Ker(u)$ and hence for $Aut(X,\lambda)$, contradicting Lemma 1.

LEMMA 3.- Suppose X is minimal, $\varkappa = 0$, $D \neq 0$ is a connected effective divisor on X, $Supp(D)=D_1\cup\ldots\cup D_p$, D_i integral curves. Suppose the intersection matrix of the divisors $D_1,\ldots,D_p$ is negative semi-definite. Then either D is an A-D-E curve or there exists an elliptic fibration([1]p.149)

$f:X \longrightarrow B$ such that Supp(D) is (set-theoretically) a fibre of f.

Proof . It is an easy consequence of [1] p.16, that D is either an A-D-E curve or Supp(D) is the support of an elliptic configuration (see [1]p.273 for the definition of elliptic configurations). By classification of surfaces with $\varkappa=0$([1]p.188) we get four cases:

Case 1: X is hyperelliptic. Let $u_i:X \longrightarrow B_i$, $i=1,2$ be two distinct ellipticvc fibrations and F_1,F_2 fibres of u_1,u_2 respectively $((F_1.F_2) \geqslant 1)$. Since $B_2(X)=2$ (cf. [1] p.148), we get $D \approx a_1F_1+a_2F_2$, $a_i \in \mathbb{Q}$. Since $(D.D) \leqslant 0$ we get $a_1a_2=0$ hence say $a_1=0$. We get $(D.F_2)=0$ hence u_2 contracts D and we are done.

Case 2: X is Enriques. By [1]p.273, if Supp(D)=Supp(E) with E an elliptic configuration then either $|E|$ or $|2E|$ is an ellipti v pencil and we are done.

Case 3: X is K3. Again if Supp(D)=Supp(E) with E an elliptic configuration then by Riemann-Roch $h^0(\mathcal{O}(E)) \geqslant 2$ and $|E|$ will give (via a possible Stein factorisation) an elliptic fibration with the desired property.

Case 4: X is abelian. Then D must be a smooth elliptic curve (which may be assumed to pass through the origin of X) and the quotient map $X \longrightarrow X/D$ is the desired elliptic fibration.

LEMMA 4.- Suppose X is minimal, $\varkappa = 0$ and let $D \neq 0$ be a connected effective divisor such that $\mathrm{Aut}(X,D)$ is infinite. Then either D is an A-D-E curve or there is an elliptic fibration $f: X \longrightarrow B$ such that $\mathrm{Supp}(D)$ is (set-theoretically) a fibre of f.

Proof. If $\mathrm{Aut}^0(X,D)=1$ we are done by Lemmas 2 and 3. If $A=\mathrm{Aut}^0(X,D) \neq 1$ then by [8] or [14] A is an abelian variety of dimension 1 and X has a structure of fibre space $f: X \longrightarrow B$ with A acting transitively on the fibres and we are done again.

LEMMA 5.- Let $f: X \longrightarrow \mathbb{P}^1$ be an elliptic fibration. Suppose f has at most two degenerate fibres and there exists an infinite set $S \subset \mathbb{P}^1$ and an elliptic curve E with $f^{-1}(x) \simeq E$ for all $x \in S$. Then X is ruled.

Proof. Let $\mathcal{L} \in \mathrm{Pic}(X)$ be ample relative to f and put $F_b = f^{-1}(b)$ (scheme-theoretically) for all $b \in B = \mathbb{P}^1$. Let $B_0 = \{b \in B;\ F_b \text{ is smooth}\}$ and $X_0 = f^{-1}(B_0)$. We claim there is an étale surjective map $B^* \longrightarrow B_0$ such that $X^* = X_0 \times_{B_0} B^* \longrightarrow B^*$ is a projective abelian scheme. Indeed it is sufficient to find a divisor B^* on X_0 which is étale over B_0. We shall give here a quick but somewhat rough argument for the existence of B^*; one can of course make the argument precise by making a systematic use of [9]. Choose any divisor C on X_0 which is integral and dominates B_0. Put $m = [K(C):K(B)]$ and let $C \cap F_b$ be the corresponding 0-cycle of degree m on $F_b (b \in B_0)$. Let B^*_b be the set of all $x \in F_b$ for which $mx - C \cap F_b$ is linearly equivalent to zero on F_b. Clearly $\# B_b^* = m^2$ does not depend on b. Now the "locus" of all cycles B_b^* as b runs through B_o is a divisor B^* on X_o which will be étale over B_o.

Now by [18] lecture 10, there is an étale finite map $B^{**} \longrightarrow B^*$ such that $X^{**} = X^* \times_{B^*} B^{**} \longrightarrow B^{**}$ has a level n structure ($n \geqslant 3$) and hence the latter morphism is obtained by base change from the universal family $U \longrightarrow M$ of projective abelian varieties of dimension 1 with polarisation of some fixed degree and with level n structure, via a unique classifying morphism $e: B^{**} \longrightarrow M$. Since each elliptic curve carries finitely many level n structures, e(S) is a finite set so e must be constant hence $X^{**} \simeq E \times B^{**}$, E being an elliptic curve. We shall be done if we prove that B^{**} is a rational curve. But B_o is either $\mathbb{C}$ or $\mathbb{C} \setminus \{0\}$ hence $\pi_1(B_o)$ is either 0 or $\mathbb{Z}$ hence any étale finite covering of B_o is either the identity of B_o or the map $\mathbb{C} \setminus \{0\} \longrightarrow \mathbb{C} \setminus \{0\}$ $z \longrightarrow z^k$, $k \geqslant 1$. Consequently $B^{**} \simeq B_o$ and we are done.

Proof of Proposition 1. By the Lemma in section 1 we have $\varkappa = 0$. Suppose $D \neq 0$ and let $D_1, \ldots, D_p$ be the integral components of D. As in Lemma 2 we have a group homomorphism $u: \mathrm{Aut}(X,D) \longrightarrow S_p$. Clearly $[K(X)^{\mathrm{Ker}(u)} : K(X)^G] < \infty$ hence $K(X)^{\mathrm{Ker}(u)} = K$, in particular if C is a connected component of Supp(D) (with the reduced structure) then $K(X)^{\mathrm{Aut}(X,C)} = K$. Suppose C is not an A-D-E curve and look for a contradiction. By Lemma 4 there exists an elliptic fibration $f: X \longrightarrow B$ such that $C = (F_b)_{red}$ for some $b \in B$ (where as usual F_x denotes $f^{-1}(x)$, $x \in B$). Let $b_1, \ldots, b_q \in B$ be the set of all points in B at which f has a degenerate fibre. We claim that for any $g \in Ker(u)$ we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad g \quad} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{\quad \tilde{g} \quad} & B \end{array}$$

with $\tilde{g} \in Aut(B)$. Since $f_* \mathcal{O}_X = \mathcal{O}_B$ it is sufficient to check the existence of the above diagram set-theoretically. Take $x \in B$ and let's show that $g(F_x)$ is contracted by f. We have $F_x \approx F_b$ ($\approx$ denoting the algebraic equivalence) hence $g(F_x) \approx g(F_b) = F_b$ hence $(g(F_x).F_b)=0$ and we are done. Now we got a homomorphism $Ker(u) \longrightarrow Aut(B)$ whose image H is contained in $Aut(B,b) \cap Aut(B,b_1+\ldots+b_q)$. As in Lemma 2 we get a homomorphism $v: H \longrightarrow S_q$ so $Ker(v) \subset Aut(B,b) \cap Aut(B,b_1) \cap \ldots$ $\ldots \cap Aut(B,b_q)$. Now

$$K \subset K(B)^H \subset K(X)^{Ker(u)} = K \quad \text{and} \quad \left[K(B)^{Ker(v)} : K(B)^H\right] < \infty$$

so $Ker(v)$ must be infinite. Consequently $B = \mathbb{P}^1$ and if $B_s = \{b\} \cup \{b_1, \ldots, b_q\}$ then $1 \leq \# B_s \leq 2$, in particular f has at most two degenerate fibres. To find a contradiction it is sufficient, by Lemma 5, to find an infinite set $S \subset \mathbb{P}^1$ and an elliptic curve E such that $F_x \simeq E$ for all $x \in S$.

If $\# B_s = 1$, take a sequence $\{g_n\}_n$ of distinct elements in $Ker(v)$ and choose an affine coordinate z such that $B_s = \{z = \infty\}$ and $g_n(z) = a_n z + b_n$, $a_n \in K^*$, $b_n \in K$. Supposing K is the complex field there exists $z_o \in K$ not belonging to the field generated by all a_n's and b_n's. Then $g_n(z_o)$ is a sequence of distinct elements in B and $F_{g_n(z_o)} \simeq F_{z_o}$ for all n (we suppose that g_1=identity).

If $\# B_s=2$ choose g_n and z such that $B_s=\{z=0,\infty\}$ and $g_n(z)=a_n z$, $a_n \in K^*$ and conclude in the same way. This closes the proof of Proposition 1.

3. Automorphisms of abelian varieties

First we shall discuss a "reduction" problem for Galois cocycles with values in the automorphism group of an abelian variety. For background and notations we send to [20].

Let C be an algebraically closed field of characteristic zero, K an extension of C and K' a finite Galois extension of K. Let A be an abelian C-variety and identify A with the component $\mathrm{Aut}^o(A/C)$ of the identity in $\mathrm{Aut}(A/C)$. Let $c \in Z^1(K'/K, \mathrm{Aut}(A/C))$ be a cocycle (so c is a map $G(K'/K) \longrightarrow \mathrm{Aut}(A\otimes_C K'/K')$ satisfying the cocycle rule). Define as usual a group homomorphism $\tilde{c}:G(K'/K) \longrightarrow \mathrm{Aut}(A\otimes_C K'/K')$ by $\tilde{c}(g)=c(g)\circ(1\otimes g)$ and put ${}_cA=A\otimes_C K'/(G(K'/K),\tilde{c})$. Then ${}_cA$ is a K-variety such that ${}_cA\otimes_C K'$ is K'-isomorphic to $A\otimes_C K'$. We shall view $K({}_cA)$ as a subfield of the function field $K'(A)$ of $A\otimes_C K'$ via the quotient map $p:A\otimes_C K' \longrightarrow {}_cA$. Note that p is faithfully flat. Finally note that for each C-point $\alpha \in A(C)$, the right translation $\rho_\alpha :A \longrightarrow A$ induces a K'-automorphism $\rho_\alpha^*:K'(A) \longrightarrow K'(A)$. Our "reduction" result i

PROPOSITION 2.- Let S be the set of all $\alpha \in A(C)$ such that $\rho_\alpha^*(K({}_cA)) \subset K({}_cA)$. If S is Zariski dense in $A(C)$ then $c \in \mathrm{Im}(Z^1(K'/K,A) \longrightarrow Z^1(K'/K,\mathrm{Aut}(A))$.

Proof. Suppose there exists $g \in G(K'/K)$ such that $c(g) \notin \mathrm{Aut}^o(A \otimes_C K'/K')$; we shall prove that S is not Zariski dense in $A(C)$.

First we claim that for any $\alpha \in S$ there is a factorisation

$$\begin{array}{ccc} A \otimes_C K' & \xrightarrow{\rho_\alpha \otimes 1} & A \otimes_C K' \\ \downarrow p & & \downarrow p \\ {}_cA & \longrightarrow & {}_cA \end{array}$$

Indeed by faithful flatness of p, $\mathcal{O}_{{}_cA} = \mathcal{O}_{A \otimes_C K'} \cap K({}_cA)$ hence $\rho_\alpha^*(\mathcal{O}_{{}_cA}) \subset \mathcal{O}_{{}_cA}$. Now take $\mathrm{Spec}(K)$ as a base scheme, put $T=\mathrm{Spec}(\bar{K})$ ($\bar{K}$ being an algebraic closure of K) and $B=\mathrm{Spec}(K')$.
Taking the functor of T-points in the above diagram (and writing $A(T)$ instead of $(A \otimes_C K)(T)$) we get:

$$\begin{array}{ccc} A(T) \times B(T) & \xrightarrow{(\rho_\alpha \otimes 1)(T)} & A(T) \times B(T) \\ \downarrow p(T) & & \downarrow p(T) \\ {}_cA(T) & \longrightarrow & {}_cA(T) \end{array}$$

where $p(T)$ is the quotient of the set $A(T) \times B(T)$ by the group $G(K'/K)$ acting via $\tilde{c}(T)$. Note that $B(T)$ is a principal homogenous space for $G(K'/K)$. Now for any $\alpha \in S$ denote the corresponding point in $A(T)$ also by α; then $(\rho_\alpha \otimes 1)(T)$ takes (a,b) into $(a+\alpha, b)$ for all $(a,b) \in A(T) \times B(T)$. On the other hand $c(g)$ being an element in $\mathrm{Aut}(A \otimes_C K'/K')$ may be written as $c(g) = \lambda_{t_g} \sigma_g$ where λ_{t_g} is the left translation with some element $t_g \in A(K')$ and $\sigma_g \neq 1$ is a group automorphism of A/C. Clearly σ_g induces a group automorphism $A(T) \longrightarrow A(T)$ denoted also by σ_g. Moreover if we view t_g as a morphism $B \longrightarrow A$, it will induce a map $B(T) \longrightarrow A(T)$ also denoted by t_g. The map $\lambda_{t_g}(T)$ from $A(T) \times B(T)$ into itself carries (a,b) into $(a+t_g(b),b)$. Consequently the map $c(g)(T)$ from $A(T) \times B(T)$ into itself carries (a,b) into $(\sigma_g(a)+t_g(b),b)$. Finally the map $\tilde{c}(g)(T)$ from $A(T) \times B(T)$ into itself carries (a,b) into $(\sigma_g(a)+t_g(gb),gb)$. Now by commutativity of the above diagram of T-points we get that for any $(a,b) \in A(T) \times B(T)$ the pairs $(a+\alpha, b)$ and $(\sigma_g(a)+t_g(gb)+\alpha, gb)$ must be in the same $G(K'/K)$-orbit hence there exists $h \in G(K'/K)$ such that

$$(\sigma_h(a)+\sigma_h(\alpha)+t_h(hb),hb) = (\sigma_g(a)+t_g(gb)+\alpha, gb)$$

By $hb=gb$ we get $h=g$ hence $\sigma_g(\alpha)=\alpha$. Since $\sigma_g \neq 1$ the closed subgroup $\{\alpha \in A(C);\ \sigma_g(\alpha)=\alpha\}$ is different from $A(C)$ and we are done.

We close this section by proving an analog of Proposition 1 for abelian varieties;

PROPOSITION 3.- Let X be an abelian K-variety (K algebraically closed of characteristic zero) and D a divisor on it such that $K(X)^{Aut(X,D)}=K$. Then $D=0$.

Proof. Suppose $D \neq 0$. Considering permutations induced by $Aut(X,D)$ on the irreducible componenets of $Supp(D)$ we may suppose that D is an integral effective divisor. Identify X with $Aut^{o}(X)$. Then $Y=X \cap Aut(X,D)$ is a closed subgroup of X distinct from X. Put $A=X/Y$ and let $p:X \longrightarrow A$ be the natural projection and $E=p(D)$. Since Y is a normal subgroup in $Aut(X,D)$, $Aut(X,D)$ will still act on A via some subgroup of $Aut(A,E)$. Note there is no $a \in A$, $a \neq 0$ such that $E+a=E$ hence by [13] pp.87 and 94, E is ample. Since

$$K \subset K(A)^{Aut(A,E)} \subset K(X)^{Aut(X,D)}=K$$

we shall be done if we prove that $Aut(A,E)$ is finite. Now ampleness of E implies that $Aut(A,E)$ has finitely many components. On the other hand its connected component is contained in $A \cap Aut(A,E)$ which is finite by [13] p.96 and we are done.

4. Movable singularities. Conclusion.

This section is devoted to the proof of the Theorem stated in the first section. Conventions and notations from the first section remain in force in the present section.

LEMMA 6.- Let V be a normal model of $\mathcal{G}/\mathcal{F}$. Then the set S of all (not necessary closed) points $p \in V$ such that $\delta(\mathcal{O}_{V,p}) \not\subset \mathcal{O}_{V,p}$ for some $\delta \in \Delta$, is the support of some Weil divisor on V (the reduced divisor D whose support equals S will be called the divisor of movable singularities - see [15] p.95 for a motivation of this terminology).

Proof. It is easy to see that $V \setminus S$ is open (see for instance [6]). The fact that S has pure codimension 1 is a trivial consequence of normality of V.

Recall from [4],[5] that $\mathcal{F} \subset \mathcal{G}$ is called a Fuchs extension if $\mathcal{G}/\mathcal{F}$ admits a non-singular projective model for which the divisor of movable singularities is zero (i.e. S is empty in Lemma 6). We will need the following corollary of [5]:

PROPOSITION 4.- Suppose $\mathcal{F}$ is algebraically closed. If $\mathcal{F} \subset \mathcal{G}$ is a Fuchs weakly normal extension then $\mathcal{F} \subset \mathcal{G}$ is strongly normal.

Proof. This is an immediate consequence of Theorem 4 in [5] and of [12] pp.400 and 419.

Remark. One can prove that the converse of Proposition 4 also holds i.e. that any strongly normal extension is a Fuchs extension. We shall not need this fact here.

Here is one more ingredient for the proof of our Theorem; it is essentially due to Burns and Wahl [7]:

PROPOSITION 5.- Suppose $\mathcal{F}$ is algebraically closed, n=2 and V is a non-singular projective (minimal) model of $\mathcal{G}/\mathcal{F}$. Then no connected component of the divisor of movable singularities can be contracted in the category of projective surfaces. In particular no such component is an A-D-E curve.

Proof. Same arguments as in [7]; however, since derivations under consideration need not vanish on $\mathcal{F}$, one will have to use Seidenberg's theorems in Matsumura's form [16].

Now we start proving our Theorem. As already noticed we may suppose $\varkappa=0$; we only have to prove statements B) and C). First we prove them under the assumption that $\mathcal{F}$ is algebraically closed. If n=2 $\varkappa \neq -\infty$, take the minimal non-singular projective model V of $\mathcal{G}/\mathcal{F}$ and let D be the divisor of movable singularities. Then $G(\mathcal{G}/\mathcal{F}) \subset Aut(V,D)$. If $\mathcal{F} \subset \mathcal{G}$ is weakly normal then by Proposition 1 either D=0 or all connected components of D are A-D-E curves. The latter possibility contradicts Proposition 5 so D=0 and we conclude by Proposition 4. If n=q, by [10] $\mathcal{G}$ is the function field of some abelian $\mathcal{F}$-variety V. If D is the divisor of movable singularities we get exactly as above $G(\mathcal{G}/\mathcal{F}) \subset Aut(V,D)$ and we conclude by Proposition 3.

Now suppose $\mathcal{F}$ is not necessarily algebraically closed and let $\overline{\mathcal{F}}$ be an algebraic closure of $\mathcal{F}$; note that the compositum $\overline{\mathcal{F}}\mathcal{G}$ has a natural structure of differential field and its field of constants equals $\mathcal{C}$. Let's prove first the statement C). Start with W a non-singular projective model of $\mathcal{G}/\mathcal{F}$ and consider the natural $\mathcal{F}$-morphism $W \longrightarrow Alb^1(W/\mathcal{F})$ (notations being as in [9]). By compatibility of Alb^1 with base change and by [10] the morphism $W \otimes \overline{\mathcal{F}} \longrightarrow \mathrm{Alb}^1(W/\mathcal{F}) \otimes \overline{\mathcal{F}}$ is birational, so the original morphism is birational; hence $W_1 = \mathrm{Alb}^1(W/\mathcal{F})$ is a model of $\mathcal{G}/\mathcal{F}$. Since $\mathcal{F} \subset \mathcal{G}$ is weakly normal so is $\overline{\mathcal{F}} \subset \overline{\mathcal{F}}\mathcal{G}$ = function field of $W_1 \otimes \overline{\mathcal{F}}$ so by the first part of our proof, $\overline{\mathcal{F}} \subset \overline{\mathcal{F}}\mathcal{G}$ is strongly normal. In particular if $A = G(\overline{\mathcal{F}}\mathcal{G}/\overline{\mathcal{F}})$ then by [12] p.426, the function field of $A \otimes \overline{\mathcal{F}}$ identifies with $\overline{\mathcal{F}}\mathcal{G}$; so $W_1 \otimes \overline{\mathcal{F}}$ is $\overline{\mathcal{F}}$-isomorphic to $A \otimes \overline{\mathcal{F}}$. Choose a finite Galois extension $\mathcal{F}'$ of $\mathcal{F}$ such that $W_1 \otimes \mathcal{F}'$ is $\mathcal{F}'$-isomorphic to $A \otimes \mathcal{F}'$ and such that $\mathcal{F}' \subset \mathcal{F}'\mathcal{G}$ is strongly normal with differential Galois group A. The above isomorphism induces, by [20], a cocycle $c \in Z^1(\mathcal{F}'/\mathcal{F}, \mathrm{Aut}(A/\mathcal{C}))$. We shall be done if we prove that c "comes from" $Z^1(\mathcal{F}'/\mathcal{F}, A)$. Indeed in this case $W_1/\mathcal{F}$ will be a principal homogenous space for A; since $A = G(\mathcal{F}'\mathcal{G}/\mathcal{F}')$ we get $A \subset G(\mathcal{G}/\mathcal{F})$ and we are done by a criterion of Bialynicki-Birula [2].

Now to see that c comes from $Z^1(\mathcal{F}'/\mathcal{F}, A)$ we apply Proposition 2, so we have to check that the set of all $\alpha \in A(\mathcal{C})$ such that $\rho_\alpha^*(\mathcal{G}) \subset \mathcal{G}$ is Zariski dense in $A(\mathcal{C})$. But the Zariski closure of S in $A(\mathcal{C})$ is a closed subgroup A_1 of A. Let $i: G(\mathcal{G}/\mathcal{F}) \longrightarrow G(\mathcal{F}'\mathcal{G}/\mathcal{F}')$ be the injection induced by base change and $H = \mathrm{Im}(i)$. Weak normality of $\mathcal{G}/\mathcal{F}$ and [19] pp.405-406 give:

$$\mathcal{F}' \subset (\mathcal{F}'(A))^{A_1} \subset (\mathcal{F}'(A))^H = (\mathcal{F}'(W_1))^H = (\mathcal{G} \otimes_{\mathcal{F}} \mathcal{F}')^H =$$
$$= \mathcal{G}^{G(\mathcal{G}/\mathcal{F})} \otimes_{\mathcal{F}} \mathcal{F}' = \mathcal{F}'$$

hence $\mathcal{F}' = (\mathcal{F}'\mathcal{G})^{A_1}$ so by the Galois correspondence for $\mathcal{F}' \subset \mathcal{F}'\mathcal{G}$ we get $A_1 = A$ and we are done.

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We close by proving B). If $\mathcal{F}\subset\mathcal{G}$ is weakly normal, so will be $\bar{\mathcal{F}}\subset\bar{\mathcal{F}}\mathcal{G}$ hence by the first part of our proof $\bar{\mathcal{F}}\subset\bar{\mathcal{F}}\mathcal{G}$ must be strongly normal. Now by [12] p.426, $\bar{\mathcal{F}}\mathcal{G}$ is the function field of $G(\bar{\mathcal{F}}\mathcal{G}/\bar{\mathcal{F}})\otimes_{\mathcal{C}}\bar{\mathcal{F}}$ and since $\varkappa=0$, $G(\bar{\mathcal{F}}\mathcal{G}/\bar{\mathcal{F}})$ is an abelian surface. In particular $\varkappa=0$, n=q we so may conclude by statement C). Our Theorem is proved.

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