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1. Introduction

Starting with the works of J.P.Yvon [8] and F.Mignot [4] much interest developped for the optimal control problems governed by variational inequalities, both from theoretional and numerical point of view. See the recent survey of V.Barbu [1] for the elliptic and parabolic case, as well as the works of D.Tiba [5], [6], [7] for the hyperbolic case.

A classical remark is that variational inequalities are, generally, equivalent with minimization problem with constraints. In this paper we prove that, similarly, there is a close connection between constrained control problems and problems governed by variational inequalities. In a special case we even have equivalence between the two types of problems.

This gives a new interpretation of optimal control problems governed by variational inequalities and provides a new approximation of constrained control problems.

In order to make clear the above ideas we study the following simple model problem (P):

Let V, H, U be Hilbert spaces with dense and compact imbedding $V \subset H \subset V^*$ and $A : V \to V^*$, $B: U \to H$ be linear, continuous operators. We assume that A is positive and symmetric:

(1.1) $(Au, u) \geq \omega \| u \|^2, \ \omega > 0, \quad u \in V,$

(1.2) $(Au, v) = (u, Av), \quad u, v \in V,$

where $(\cdot, \cdot)$ is the pairing between V and V^*(if v_1, v_2 H then (v_1, v_2) is the inner product in H) and $\| \cdot \|$ is the norm in V.

Consider the control problem

(P) Minimize $\{ g(y) + h(u) \}$

subject to

(1.3) $y' + Ay = Bu + f$ in $[0,T]$,

(1.4) $y(0) = y_0$

(1.5) $y(t) \in C$ in $[0,T]$.

Above $C \subset H$ is a closed, convex subset, $y_0 \in C$, $Ay_0 \in H$, $f \in L^2(0,T;H)$, $g : L^2(0,T; H) \to R$ is convex, continuous, majorized from below by a constant c and $h : L^2(0,T;U) \to]-\infty, +\infty]$ is convex, lower semicontinuous, proper, satisfying

(1.6) $\lim_{|u|\to\infty} h(u) = +\infty.$

Under the above hypotheses, equation (1.3), (1.4) has a unique solution $y \in C(0,T;H)$, $y' \in L^2(0,T;H)$ and (1.5) makes sense.

If we also have control constraints $u \in U_0$ (a close, convex subset of $L^2(0,T;U)$) this may be implicitly expressed by adding to h the indicator function of U_0.

We assume the existence of an admissible pair $[\bar{y}, \bar{u}]$ for (P). Then it is easy to show the existence of at least one optimal pair $[y^*, u^*]$.

The plan of the paper is as follows. Section 2 contains the main result on the approximation and the equivalence properties of the associated problems. In the last section we give an algorithm for the solution of (P) in the case of a more specific example.

Throughout this paper we shall denote by the same symbol $|\cdot|$ the norms in H, U, $L^2(0,T;H)$ or $L^2(0,T;U)$ as necessary.

2. The associated problem

Let $\varphi : H \to]-\infty, +\infty]$ be the lower semicontiuous, convex, proper function

(2.1) $$\varphi(y) = \begin{cases} 0 & \text{if } y \in C, \\ +\infty & \text{otherwise.} \end{cases}$$

With (P) we associate the approximate problem (P_ε), $\varepsilon > 0$:

(P_ε) Minimize $\{ g(y) + h(u) + 1/2\, |w|^2 \}$

subject to

(2.2) $y' + Ay + \varepsilon w = Bu + f,\ w \in \partial\varphi(y)$

and (1.4).

Proposition 2.1. There is at least one optimal pair $[y_\varepsilon, u_\varepsilon]$ for (P_ε).

Proof.

Let $\{u_n\}$ be a minimizing sequence for (P_ε) and y_n the corresponding solutions of (2.2), (1.4). Then

$$g(y_n) + h(u_n) + 1/2\, |w_n|^2 \le ct., \qquad w_n \in \partial\varphi(y_n).$$

Since $g(y_n) \ge c$, by (1.6) we see that $\{u_n\}$ is bounded in $L^2(0,T;U)$. Also $\{w_n\}$ is bounded in $L^2(0,T;H)$. Next (2.2) and (1.1) give $\{y_n\}$ bounded in $L^\infty(0,T;V)$, $\{y'_n\}$ bounded in $L^2(0,T;H)$.

On a subsequence $u_n \to \tilde{u}$ weakly in $L^2(0,T;U)$, $y_n \to \tilde{y}$ strongly in $C(0,T;H)$, $w_n \to \tilde{w} \in \partial\varphi(\tilde{y})$ weakly in $L^2(0,T;H)$ because $\partial\varphi$ is demiclosed.

Passing to the limit we get that $\tilde{y}$ is the solution of (2.2), (1.4) corresponding to $\tilde{u}$ and

$$g(\tilde{y}) + h(\tilde{u}) + 1/2\, |\tilde{w}|^2 \le \liminf_{n\to\infty} \{ g(y_n) + h(u_n) + 1/2\, |w_n|^2 \} = \inf(P_\varepsilon).$$

Therefore $[\tilde{y},\tilde{u}]$ is an optimal pair for (P_ε) which we denote $[y_\varepsilon, u_\varepsilon]$.

Remark 2.2. The idea to penalize the nonlinear term in the cost functional comes from the unstable systems control theory J.L. Lions [3], J.F. Bonnans [2].

Denote by J, J_ε the cost functional for (P), (P_ε) and by J^*, J^*_ε their minimum values.

Proposition 2.3. When $\varepsilon \to 0$ we have on a subsequence:

(2.3) $\quad u_\varepsilon \to \hat{u}$ weakly in $L^2(0,T;U)$,

(2.4) $\quad y_\varepsilon \to \hat{y}$ strongly in $C((0,T;H)$,

(2.5) $\quad J^*_\varepsilon \to J^*$, $J^*_\varepsilon \leq J^*$,

where $[\hat{y},\hat{u}]$ is an optimal pair for(P)

Proof

For any admissible pair $[\bar{y},\bar{u}]$ for (P), $0 \in \partial\varphi(\bar{y})$ since $\bar{y}(t) \in C$, $t \in [0,T]$ and $\bar{y}$ may be viewed as the solution corresponding to $\bar{u}$ of (2.2), (1.4) with $\bar{w} = 0$. Therefore $J_\varepsilon(\bar{y},\bar{u}) = J(\bar{y},\bar{u})$ and $J_\varepsilon(y_\varepsilon, u_\varepsilon) = J^*_\varepsilon \leq J^*$ for any $\varepsilon > 0$.

As in the proof of **Proposition 2.1** we get $\{u_\varepsilon\}$ bounded in $L^2(0,T;U)$, $\{w_\varepsilon\}$ bounded in $L^2(0,T;H)$, $\{y_\varepsilon\}$ bounded in $L^\infty(0,T;V)$, $\{y'_\varepsilon\}$ bounded in $L^2(0,T;H)$.

On a subsequence $u_\varepsilon \to \hat{u}$ weakly in $L^2(0,T;U)$, $y_\varepsilon \to \hat{y}$ strongly in $C(0,T;H)$.

We remark that $y_\varepsilon(t) \in \mathrm{dom}(\partial\varphi) = C$ for all $t \in [0,T]$, so $\hat{y}(t) \in C$, $t \in [0,T]$.

One may easily pass to the limit and check that $[\hat{y},\hat{u}]$ is an admissible pair for (P).

Moreover

(2.6) $\quad J^* \geq \liminf_{\varepsilon \to 0} J_\varepsilon(y_\varepsilon, u_\varepsilon) \geq J(\hat{y},\hat{u}) + 1/2\, |\hat{w}|^2 \geq J^*.$

Then $\hat{w} = 0$ and $[\hat{y},\hat{u}]$ is an optimal pair for (P).

By (2.6) we get that $\lim_{\varepsilon \to 0} J_\varepsilon(y_\varepsilon, u_\varepsilon) = J^*$.

Let y^ε denote the solution of (1.3), (1.4) corresponding to u_ε. The pair $[y^\varepsilon, u_\varepsilon]$ is not necessarily admissible for (P), but we can compute $J(y^\varepsilon, u_\varepsilon)$ and prove the following suboptimality corollary:

Corollary 2.4. We have

(2.7) $\quad \lim_{\varepsilon \to 0} J(y^\varepsilon, u_\varepsilon) = J^*$

(2.8) $\quad \mathrm{dist}(y_\varepsilon(t), C) \leq k \cdot \varepsilon$

where k depends on J^*, g, h only.

Proof.

Denote $z_\varepsilon = y^\varepsilon - y_\varepsilon$. Then it satisfies

(2.9) $\quad z'_\varepsilon + Az_\varepsilon = \varepsilon\, w_\varepsilon \quad$ in $[0,T]$,

$\quad z_\varepsilon(0) = 0$

By the assumptions on h we see that it is also majorized from below by a constant, say c_1. Then, by (2.6) it yields

$1/2\,|w_\varepsilon|^2 \leq J^* - c - c_1$

Taking into account (2.9) we obtain that $|z_\varepsilon(t)| \leq k\cdot\varepsilon$ and since $y_\varepsilon(t) \in C$, $t \in [0,T]$, we prove (2.8).

By (2.4) and (2.9) we infer that on a subsequence $y^\varepsilon \to \hat{y}$ strongly in $C(0,T;H)$ and from the continuity of g that $g(y^\varepsilon) \to g(\hat{y})$.

On the other hand $g(y_\varepsilon) \to g(\hat{y})$ too and by (2.6) $h(u_\varepsilon) \to h(\hat{u})$.

Finally, $J(y^\varepsilon, u_\varepsilon) = g(y^\varepsilon) + h(u_\varepsilon) \to J^*$ on the initial sequence.

Remark 2.5. By the above result, it is enough to find the solution u_ε of (P_ε) for ε sufficiently small. However (P_ε) is a nondifferentiable optimization problem and may be difficult to handle. In order to overcome this difficulty we replace $\partial\varphi$ by β^λ, $\lambda > 0$, a smooth approximation of $(\partial\varphi)_\lambda$, the Yosida approximate of $\partial\varphi$. This is a usual procedure in the control of variational inequalities, Barbu [1]. See also the last section for more details in an example.

(P_λ) Minimize $\{ g(y) + h(u) + 1/2\, |\beta^\lambda(y)|^2 \}$

subject to

(2.10) $y' + Ay + \varepsilon\beta^\lambda(y) = Bu + f$

and (1.4).

Obviously, (P_λ) has at least one optimal pair which we denote $[y_\lambda, u_\lambda]$. Let y^λ denote the solution of (1.3), (1.4) corresponding to u_λ.

Corollary 2.6. <u>We have:</u>

(2.11) $\text{dist}\,(y^\lambda(t), C) \leq k\,\varepsilon + \eta(\lambda)$, $t \in [0,T]$

<u>where</u> $\eta(\lambda) \to 0$ <u>as</u> $\lambda \to 0$.

(2.12) $J(y^\lambda, u_\lambda) \leq J^* + \sigma(\varepsilon)$, $\lambda > 0$.

<u>where</u> $\sigma(\varepsilon) \to 0$ <u>as</u> $\varepsilon \to 0$.

Proof.

Since $y^*(t) \in C$, $t \in [0,T]$ we see that $\beta^\lambda(y^*(t)) = 0$ and $[y^*, u^*]$ is an admissible pair for (P_λ) with $J_\lambda(y^*, u^*) = J^*$. Then $J_\lambda(y_\lambda, u_\lambda) \leq J^*$, $\lambda > 0$ and this shows that $\{u_\lambda\}$ is bounded in $L^2(0,T;U)$, $\{\beta^\lambda(y_\lambda)\}$ is bounded in $L^2(0,T;H)$. By (2.10) it yields $\{y_\lambda\}$ bounded in $L^\infty(0,T;V)$, $\{y'_\lambda\}$ bounded in $L^2(0,T;H)$.

On a subsequence, we get

$u_\lambda \to \tilde{u}_\varepsilon$ weakly in $L^2(0,T;U)$,

$y_\lambda \to \tilde{y}_\varepsilon$ strongly in $C(0,T;H)$,

$\beta^\lambda(y_\lambda) \to \partial\varphi(\tilde{y}_\varepsilon)$ weakly in $L^2(0,T;H)$.

Passing to the limit in (2.10) we obtain that $[\tilde{y}_\varepsilon, \tilde{u}_\varepsilon]$ is an admissible pair for (P_ε), therefore $\tilde{y}_\varepsilon(t) \in C$ for $t \in [0,T]$.

We conclude that $\lim_{\lambda \to 0} \sup_{t \in [0,T]} \text{dist}(y_\lambda(t); C) = 0$

and we take $\eta(\lambda) = \sup_{t \in [0,T]} \text{dist}(y_\lambda(t); C)$.

Denote $z_\lambda = y^\lambda - y_\lambda$. By an argument similar to (2.6), (2.9) we see that $|z_\lambda(t)| \leq k\,\varepsilon$, $t \in [0,T]$ and this finishes the proof of (2.11).

As concerns (2.12) we remark that:

$$J(y^\lambda,u_\lambda) = J_\lambda(y_\lambda,u_\lambda) - 1/2|\beta^\lambda(y_\lambda)|^2 + g(y^\lambda) - g(y_\lambda) \le J_\lambda(y_\lambda,u_\lambda) + g(y^\lambda) - g(y_\lambda) \le J^* + g(y^\lambda) - g(y_\lambda).$$

Using once more the estimate for z_λ and the continuity of g we finish the proof.

Remark 2.7. By the above result u_λ is also suboptimal for problem (P). To compute it one may use a gradient method in (P_λ). Since the estimates for (P_λ) are independent of ε, we may chose ε smaller than λ. Then we avoid the usual troubles in computations related to the penalization method, Yvon [8]. Equation (2.10) as well as the adjoint equation will be well conditioned.

Now we turn to the special case when (P) is equivalent with a problem of type (P_ε), slightly modified.

We assume that U = H, B is the identity operator and $h(u) = |u|$. We associate with (P) the problem

(Pa) Minimize $\{ g(y) + |u| + |w| \}$

(2.13) $y' + Ay + w = u + f$, $w \in \partial\varphi(y)$,

$y(0) = y_0$.

The equivalence result is:

Proposition 2.8.

i) Any optimal pair for (P) is optimal for (Pa).

ii) For any optimal pair for (Pa) $[\hat{y},\hat{u}]$ let $\hat{w} \in \partial\varphi(\hat{y})$ be associated through (2.13). Then $[\hat{y},\hat{u} - \hat{w}]$ is an optimal pair for (P) too.

Proof.

Let $[y^*,u^*]$ be optimal for (P). We have $y^*(t) \in C$ for $t \in [0,T]$, that is $0 \in \partial\varphi(y^*)$ and $[y^*,u^*]$ is admissible for (Pa) with $J_a(y^*,u^*) = J(y^*,u^*)$. It yields $J_a^* \le J^*$.

Since $\hat{w} \in \partial\varphi(\hat{y})$ we have that $\hat{y}(t) \in dom(\partial\varphi) = C$ for all $t \in [0,T]$ and by (2.13) $[\hat{y},\hat{u} - \hat{w}]$ is admissible for (P).

We have:

$$J^* \le J(\hat{y},\hat{u} - \hat{w}) = g(\hat{y}) + |\hat{u} - \hat{w}| \le g(\hat{y}) + |\hat{u}| + |\hat{w}| = J_a(\hat{y},\hat{u}) = J_a^* .$$

Therefore $J^* = J_a^*$ and the proof is finished.

3. An Algorithm

In this section we detail the application of the gradient method to (P_λ) in the case of an example.

We take $H = L^2(\Omega)$, $V = H_0^1(\Omega)$, Ω a bounded domain in R^N.

$B : U \to L^2(\Omega)$ is a linear, continuous operator and $A : H_0^1(\Omega) \to H^{-1}(\Omega)$ is the Laplace operator with Dirichlet boundary conditions.

We consider

(3.1) $g(v) = h(v) = 1/2\,|v|^2$

and the constraints set:

(3.2). $C = \{ y \in L^2(\Omega); y \geq 0 \text{ a.e. } \Omega \}$.

We define φ as the indicator function of C on $L^2(\Omega)$ and let $\beta: R \to R$ be the maximal monotone graph

$$\beta(r) = \begin{cases} 0 & r > 0 \\ ]-\infty, 0] & r = 0 \\ \emptyset & r < 0. \end{cases} \tag{3.3}$$

Then, we have:

$$\partial \varphi(y) = \{ w \in L^2(\Omega); w(x) \in \beta(y(x)) \text{a.e.} \Omega \} \quad , y \in L^2(\Omega). \tag{3.4}$$

The smooth approximation β^λ is obtained as the realization in $L^2(\Omega)$ of:

$$\tilde{\beta}^\lambda(r) = \int_{-\infty}^{\infty} \beta_\lambda(r + \lambda - \lambda\tau)\rho(\tau)d\tau, \; r \in R. \tag{3.5}$$

Above β_λ is the Yosida approximation of β and ρ is a Friedrichs mollifier, i.e. $\rho \geq 0$, $\rho(-\tau) = \rho(\tau)$, $\text{supp}\rho \subset [-1,1]$, $\rho \in C^\infty(R)$, $\int_{-\infty}^{\infty} \rho(\tau)d\tau = 1$.

We remark that β^λ has the properties used in Corollary 2.6.

Now, the problem (P_λ) is completely defined and it is quite standard to obtain the optimality conditions:

Proposition 3.1. There is $p_\lambda \in W^{1,2}(0,T;H) \cap L^\infty(0,T;H_0^1(\Omega))$ such that it satisfies together with y_λ, u_λ:

$$y'_\lambda - \Delta y_\lambda + \beta^\lambda(y_\lambda) = Bu_\lambda + f,$$
$$-p'_\lambda - \Delta p_\lambda + \nabla\beta^\lambda(y_\lambda)p_\lambda = y_\lambda + \beta^\lambda(y_\lambda) \cdot \nabla\beta^\lambda(y_\lambda),$$
$$y_\lambda(t,x) = p_\lambda(t,x) = 0 \text{ on } \partial\Omega \times]0,T[,$$
$$y_\lambda(0,x) = y_0(x) \quad \text{in } \Omega,$$
$$p_\lambda(T,x) = 0 \quad \text{in } \Omega,$$
$$B^* p_\lambda + u_\lambda = 0.$$

Proof.

This is based on the fact that the Gateaux differential of J_λ, as a function of the control u only, exists and equals

$B^* p_\lambda + u_\lambda$ (in the point u_λ).

Above $\nabla\beta^\lambda$ is the usual derivative of function β^λ.

We are prepared to give the algorithm:

STEP 1. Let u_0 be given and set $n = 0$.

STEP 2. Compute y_n from the state equation.

STEP 3. Test if the pair $[y^n, u_n]$ is satisfactory.

If YES then STOP, otherwise GOTO step 4.

STEP 4. Compute p_n from the adjoint equation.

STEP 5. Compute u_{n+1} by

$u_{n+1} = u_n - \sigma_n(B^* p_n + u_n)$.

STEP 6. $n := n+1$ and GO TO step 2.

Above y^n is the solution of (1.3), (1.4) corresponding to u_n. The test involved in step 3 concerns the violation of the constraints and the value of the cost functional and may be suitable choosen. In step 5 σ_n is a real parameter which may be obtained via a line search.

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