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by

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Abstract

For systems of conservation laws [(1.1), (2.1), (3.1), (4.1)] one discusses the manner in which the number of space dimensions and/or the number of equations influences the structure of the set of concepts/restrictions connected with the linearized well-posedness (see also [1], [3]- [8]); the points of this discussion are gathered up in the table on page 27. Moreover (see [4], [8]), the remarks of §3 show that in adiabatic gasdynamics 2D in space, it is possible to formulate, for certain equations of state, an (exponential) criterion of linearized stability/well-posedness. This criterion doesn't work (for instance) for equation of state (4.3). In such a case the possibility of linearized stability /well-posedness should be studied by starting directly from the solution.

CONTENTS

1. LINEARIZED PROBLEM FOR A SINGLE CONSERVATION LAW, 1D IN SPACE

1.1. Wording up of linearized porblem

Let us consider the Riemann problem

(1.1) $$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0 \quad , -\infty < x < \infty \, , \quad t > 0$$

(1.2) $$u(x,0) = \begin{cases} u_\ell & \text{for } x < 0 \\ u_r & \text{for } x > 0 \end{cases}$$

where u_ℓ and u_r are constants, $u_r \neq u_\ell$, and $f'' \neq 0$ on a certain domain in which u takes values.

A discontinuous solution of (1.1) satisfies, at the points of a discontinuity line, the relation

(1.3) $$[\![f(u)]\!] = D \, [\![u]\!]$$

where $[\![f(u)]\!] = f(u_r) - f(u_\ell)$, $[\![u]\!] = u_r - u_\ell$ and D denotes the speed with which the discontinuity propagates.

A discontinuous solution of Riemann problem is called <u>admissible</u> if it can be regarded as a non-dissipative limit ($\nu \to 0$) of a solitary wave solution of Burgers equation $\partial_t u + \partial_x f(u) = = \nu \partial^2_{xx} u$. In this case we say that the discontinuity involved in the solution has a <u>structure</u>.

We can show (see, for example, [3]) that a discontinuity has a structure iff the conditions

(*) $$f'(u_r) < D < f'(u_\ell)$$

are fulfilled.

If in (1.1) $f'' > 0 / f'' < 0$ then the conditions (*) demand, in particular, that $u_r < u_\ell / u_\ell < u_r$ in (1.2).

We shall abbreviate the admissibility conditions (*) by CEL (entropy conditions of Lax) and assume that, on the afore - mentioned discontinuity, these conditions are fulfilled. In fact CEL are conditions of determinacy (through initial data and jump relation) imposed to the piecewise constant solution of the Riemann problem[1].

In the perturbation theory we shall present hereinafter in this paragraph, this solution plays the part of the "zeroth order".

Let ε be a parameter of the problem, small in comparison with the constant states adjacent to the discontinuity and also small in comparison with the magnitude of the jump through discontinuity

(1.4) $$0 < \varepsilon \ll |u_d - u_s|$$

In a perturbation theory, to the initial data

(1.5) $$\overset{\varepsilon}{u}_0(x) = \begin{cases} \overset{\varepsilon}{u}_{\ell 0}(x) & \text{for } x < 0 \\ \overset{\varepsilon}{u}_{r0}(x) & \text{for } x > 0 \end{cases}$$

the hereinbelow solution corresponds

(1.6) $$\overset{\varepsilon}{u}(x,t) = \begin{cases} \overset{\varepsilon}{u}_\ell(x,t) & \text{for } x - Dt - \overset{\varepsilon}{\Psi}(t) < 0 \\ \overset{\varepsilon}{u}_r(x,t) & \text{for } x - Dt - \overset{\varepsilon}{\Psi}(t) > 0 \end{cases}$$

The data (1.5) evolve according to the equations

1) According to the method of characteristics.

$$(1.7)\quad \begin{cases} \dfrac{\partial \overset{\varepsilon}{u}_\ell}{\partial t} + a(\overset{\varepsilon}{u}_\ell)\,\dfrac{\partial \overset{\varepsilon}{u}_\ell}{\partial x} = 0 & \text{for } \bar{x} = x - Dt - \overset{\varepsilon}{\Psi}(t) < 0 \\ \dfrac{\partial \overset{\varepsilon}{u}_r}{\partial t} + a(\overset{\varepsilon}{u}_r)\,\dfrac{\partial \overset{\varepsilon}{u}_r}{\partial x} = 0 & \text{for } \bar{x} > 0 \end{cases}$$

where we denote $a(u)=f'(u)$, and on a line of discontinuity the jump relation

$$(1.8)\quad f(\overset{\varepsilon}{u})\Big|_{\bar{x}=0+} - f(\overset{\varepsilon}{u})\Big|_{\bar{x}=0-} = \left[D + \overset{\varepsilon}{\Psi}'(t)\right]\left(\overset{\varepsilon}{u}\Big|_{\bar{x}=0+} - \overset{\varepsilon}{u}\Big|_{\bar{x}=0-}\right)$$

obtained from (1.3) is satisfied. By mapping

$$(1.9)\quad \bar{x} = x - Dt - \overset{\varepsilon}{\Psi}(t)\ , \quad \bar{t} = t$$

(1.7) passes into

$$(1.10)\quad \begin{aligned} &\frac{\partial}{\partial \bar{t}}\overset{\varepsilon}{U}_\ell + \left[a(\overset{\varepsilon}{U}_\ell) - D\right]\frac{\partial}{\partial \bar{x}}\overset{\varepsilon}{U}_\ell = \overset{\varepsilon}{\Psi}'(t)\,\frac{\partial}{\partial \bar{x}}\overset{\varepsilon}{U}_\ell\ , \quad \text{for } \bar{x} < 0 \\ &\frac{\partial}{\partial \bar{t}}\overset{\varepsilon}{U}_r + \left[a(\overset{\varepsilon}{U}_r) - D\right]\frac{\partial}{\partial \bar{x}}\overset{\varepsilon}{U}_r = \overset{\varepsilon}{\Psi}'(t)\,\frac{\partial}{\partial \bar{x}}\overset{\varepsilon}{U}_r\ , \quad \text{for } \bar{x} > 0 \end{aligned}$$

where

$$(1.11)\quad \overset{\varepsilon}{U}(\bar{x},\bar{t}) = \overset{\varepsilon}{u}(x,t)$$

For separating the first order in ε we shall assume that $\overset{\varepsilon}{U}_{\ell 0}$, $\overset{\varepsilon}{U}_{r0}$, $\overset{\varepsilon}{U}_\ell$, $\overset{\varepsilon}{U}_r$, $\overset{\varepsilon}{\Psi}$ depend smoothly on ε, then differentiate (1.10) and (1.8) with respect to ε and take into account

$$\left[\overset{\varepsilon}{U}_{\ell,r}(\bar{x},\bar{t})\right]_{\varepsilon=0} = u_{\ell,r} \quad , \quad \left[\overset{\varepsilon}{\Psi}(\bar{t})\right]_{\varepsilon=0} = 0$$

$$\frac{d}{d\varepsilon}\left[\overset{\varepsilon}{U}(\bar{x},\bar{t})\right]\Big|_{\varepsilon=0} = \tilde{U}(\bar{x},\bar{t}) \quad , \quad \frac{d}{d\varepsilon}\left[\overset{\varepsilon}{\Psi}(\bar{t})\right]\Big|_{\varepsilon=0} = \Psi(\bar{t}) \ ,$$

$$\frac{d}{d\varepsilon}\left[\overset{\varepsilon}{U}_0(\bar{x})\right]\Big|_{\varepsilon=0} = \tilde{u}_0(\bar{x}) .$$

ignoring, in (1.9), the dependence of $\bar{x}$ on ε. It thus results

$$\text{(1.12)} \quad \begin{aligned} &\frac{\partial}{\partial \bar{t}} \tilde{U}_\ell + A_\ell \frac{\partial}{\partial \bar{x}} \tilde{U}_\ell = 0 \quad , \quad \bar{x} < 0 \\ &\frac{\partial}{\partial \bar{t}} \tilde{U}_r + A_r \frac{\partial}{\partial \bar{x}} \tilde{U}_r = 0 \quad , \quad \bar{x} > 0 \end{aligned}$$

$$\text{(1.13)} \qquad A_r\tilde{U}_r = A_\ell\tilde{U}_\ell + [\![u]\!]\,\Psi' \quad \text{for} \quad \bar{x} = 0$$

where $A_{\ell,r} = a(u_{\ell,r}) - D$, and

$$\text{(1.14)} \qquad \tilde{U}(\bar{x},0) = \tilde{u}_0(\bar{x}) \ , \ \bar{x}\in R \ ; \ \Psi(0) = 0$$

The equations of the following (allowed) orders are obtained similarly.

DEFINITION 1.1. The problem (1.12)-(1.14) is called the linearized problem associated with the Riemann problem.

1.2. Determinacy.

Since we have ignored - to separate the first order in ε - the dependence of $\bar{x}$ on ε in (1.9), in the solution of the linearized problem-depending on the nature of initial data-secular terms will appear. Therefore, as we shall show through the theorem 1.1, the method described in 1.1. "linearizes" - at the first order in ε - the problem (1.7), (1.5) only for certain classes of initial data.

Let us thus consider the class of initial data

(1.15) $C_0 = \{\tilde{u}_0 | \tilde{u}_{\ell 0}$ and $\tilde{u}_{r0}$ are smooth functions with compact support $\}$

and, correspondingly, the class of functions $\tilde{U}(x,t)$ with the properties

(a) for each $\bar{x} \in R$, $\tilde{U}$ is a Laplace original (abbreviated fo) with respect to $\bar{t}$,

(b) for each $\bar{t} < \infty$, $\tilde{U}_\ell$ and $\tilde{U}_r$ are smooth functions with compact support with respect to $\bar{x}$.

Let us denote

(1.16) $C = \{\tilde{U}, \Psi | \tilde{U}$ with the properties (a) and (b), Ψ is fo$\}$.

For data in C_0 we shall seek in C for the solution of the linearized problem. We shall thus suppose a certain type for time growing of the solution, and the study of the problem will show that this assumption is justified.

Applying the Laplace transform to (1.12), (1.13) and putting $\overset{*}{f}(\bar{x},\omega) = L[\tilde{f}] = \int_0^\infty \tilde{f} e^{-\omega \bar{t}} d\bar{t}$, we find

$$(1.17) \quad \begin{cases} A_1 \dfrac{d\overset{*}{U}_\ell}{d\bar{x}} + \omega \overset{*}{U}_\ell - \tilde{u}_0 = 0 & \text{for } \bar{x} < 0 \\ A_r \dfrac{d\overset{*}{U}_r}{d\bar{x}} + \omega \overset{*}{U}_r - \tilde{u}_0 = 0 & \text{for } \bar{x} > 0 \end{cases} \qquad \mathcal{R}e\, \omega > 0$$

$$(1.18) \quad A_r \overset{*}{U}_r = A_\ell \overset{*}{U}_\ell + \omega [\![u]\!] \overset{*}{\Psi} \qquad \text{for } \bar{x} = 0$$

The solution of the system (1.17) can be represented as

$$(1.19)\quad \overset{*}{U}(\bar{x},\omega) = \begin{cases} \left[\overset{*}{U}_-(\omega) + A_\ell^{-1}\int_0^{\bar{x}} \tilde{u}_0(\xi)e^{(\omega/A_\ell)\xi}d\xi\right]e^{-(\omega/A_\ell)\bar{x}} & \text{, for } \bar{x} < 0 \\ \left[\overset{*}{U}_+(\omega) + A_r^{-1}\int_0^{\bar{x}} \tilde{u}_0(\xi)e^{(\omega/A_r)\xi}d\xi\right]e^{-(\omega/A_r)\bar{x}} & \text{, for } \bar{x} > 0 \end{cases}$$

where we have put

$$\overset{*}{U}_-(\omega) = \overset{*}{U}_\ell(0,\omega) \quad , \quad \overset{*}{U}_+(\omega) = \overset{*}{U}_r(0,\omega)$$

In order to divide, formally, the considerations concerning the well-posedness of linearized problem into parts reflecting the extension corresponding to the passage from §1, through §2, to §§ 3 and 4 , we shall introduce hereinbelow the concepts of determinacy (through the initial data and jump relations), evolutionary conditions and stability.

In the context of §1, the exposition of these concepts is trivial and will be used only to support the analogy considered at page 27 .

Let us take

$$(1.20)\qquad A_\ell > 0 \ , \ A_r < 0$$

in (1.19) and put, correspondingly, the coefficients of $\exp\left[-(\omega/A_\ell)\bar{x}\right]$ and $\exp\left[-(\omega/A_r)\bar{x}\right]$ equal to zero

$$(1.21)\quad \begin{cases} \overset{*}{U}_-(\omega) = -A_\ell^{-1}\int_0^{-\infty} \tilde{u}_0(\xi)e^{(\omega/A_\ell)\xi}d\xi \\ \overset{*}{U}_+(\omega) = -A_r^{-1}\int_0^{\infty} \tilde{u}_0(\xi)e^{(\omega/A_r)\xi}d\xi \end{cases}$$

We shall use the two relations thus obtained, together with the relation (1.18) to determine the three unknowns $\overset{*}{U}_-$, $\overset{*}{U}_+$ and

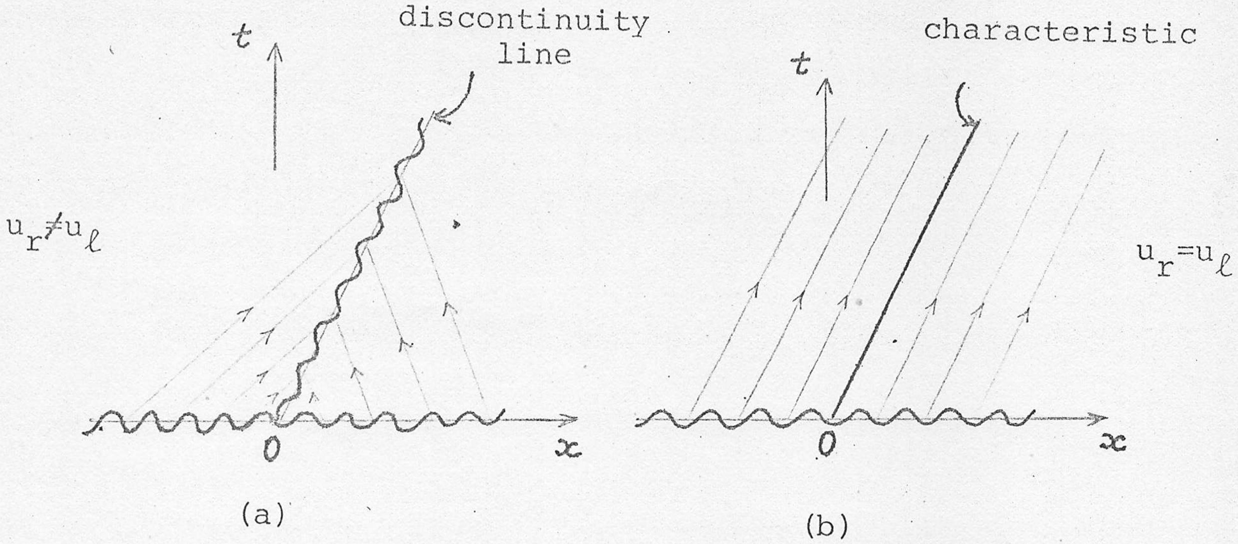


Figure 1

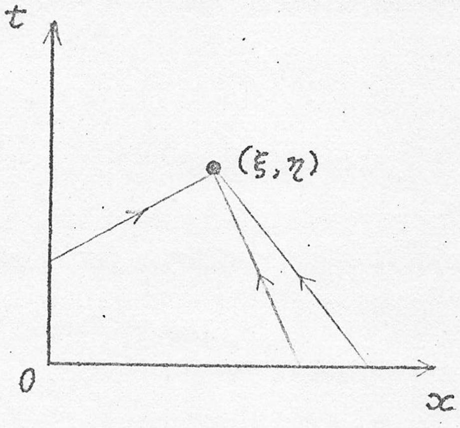


Figure 2

$\overset{*}{\psi}$ 1).

From (1.18) we find, by (1.21),

$$(1.22)\qquad -\omega[\![u]\!]\overset{*}{\psi}(\omega) = \int_0^\infty \left[A_\ell \tilde{u}_0(-A_\ell\tau) - A_r\tilde{u}_0(-A_r\tau)\right]e^{-\omega\tau}d\tau$$

The formal procedure described hereinabove becomes (as we shall see immediately) <u>effective</u> if the data are, for instance, in C_0.

The relations (1.21) are called <u>relations of determinacy.</u> The conditions (1.20) are called <u>conditions of determinacy.</u> If these conditions are fulfilled we say that the liniarized problem is <u>determinate.</u>

Carrying (1.21) into (1.19) we obtain

$$(1.23)\qquad \tilde{U}(\bar{x},\bar{t}) = \begin{cases} \tilde{u}_0(\bar{x} - A_\ell\bar{t}) & , \quad \bar{x} < 0 \\ \tilde{u}_0(\bar{x} - A_r\bar{t}) & , \quad \bar{x} > 0 \end{cases}$$

and from (1.22) it results

$$(1.24)\qquad -[\![u]\!]\psi(\bar{t}) = \int_0^{\bar{t}} \left[A_\ell\tilde{u}_0(-A_\ell\tau) - A_r\tilde{u}_0(-A_r\tau)\right]d\tau.$$

Also, (1.24) can be put into the form

$$(1.25)\qquad \psi(\bar{t}) = -\frac{1}{[\![u]\!]}\int_{-A_\ell\bar{t}}^{-A_r\bar{t}} \tilde{u}_0(\tau)d\tau$$

From (1.23) and (1.25) we see that for data in C_0 the solution of the linearized problem does not contain secularities. On the other hand, if $\tilde{u}_0(\bar{x}) \equiv \cos k\bar{x}$ in (1.25) then for each $\bar{t} < \infty$ we have $\lim_{k\to 0} \psi(\bar{t}) = \bar{t}\{(A_\ell - A_r)/[\![u]\!]\}$ and so, for $|k| \approx 0$, the nonli-

1) In fact, the method of characteristics points that the determinacy considerations of the zeroth order are completely analogue to those corresponding to the first order.

nearity slinks even from the first order and the procedure of isolating the linearized problem is not justified any more (in the absence of its uniform validity) for $\tilde{t} \sim 0(\varepsilon^{-1})$. Something similar happens wherein the data do not tend quickly enough to zero when $|\vec{x}| \to \infty$, because in that case the Laplace images $L[\tilde{u}_0(-A_\ell \tau)]$ and/or $L[\tilde{u}_0(-A_r \tau)]$ in (1.22) have a singularity in $\omega = 0$.

REMARK 1.1

(i) From (1.4) we can see that the picture of fig.1b cannot be obtained as a limit - when $|u_r - u_\ell| \to 0$ - from the picture of fig.1a. A relation can be established only between the zeroth orders of the two pictures because the small parameter of the perturbation expansion which leads to the linearization in fig.1b is free of restriction (1.4).

(ii) The determinacy conditions (1.20) (associated to the first order of the perturbation theory) can be transcribed

$$(1.26) \qquad a(u_r) < D < a(u_\ell)$$

and so they coincide with CEL (see the foot note on page 9). The restrictions (1.26) correspond to the requirement that every characteristic line, extended in the direction of incresing time, should meet the discontinuity thereby being continued through the latter one. From (1.25) we see that if these conditions are fulfilled the evolution of distortion Ψ depends on the data on the whole $\tilde{x}$ axis.

1.3. Linearized stability. Linearized well-posedness

DEFINITION 1.2. A solution - consisting of $\tilde{U}$ and Ψ - of the linearized problem is called stable/unstable if it is kept bounded/grows boundlessly when $\tilde{t} \to \infty$. We say, correspondingly, that the discontinuous solution considered for the Riemann problem is (linearized) stable/unstable. The linearized problem with data in the class K_0 is said to be well-posed in the class K if (it attaches to each element in K_0 a unique and stable solution in K, that is,) it is determined and stable in K.

THEOREM 1.1. If the conditions (1.20) are fulfilled then the linearized problem with data in C_0 is well-posed in the class C.

◀ According to (1.23) and (1.25). ▶

REMARK 1.2. The hypotheses of the theorem 1.1 do not impose on the values u_ℓ and u_r but the ordering restriction $u_r < u_\ell$.

2. Linearized problem for a system of conservation laws, 1D in space

2.1. Wording up of linearized problem

Let us extend now, in case of systems of conservation laws, the results of §1. The Riemann problem takes then the form

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0 \ , \quad -\infty < x < \infty \ , \quad t > 0 \tag{2.1}$$

$$u(x,0) = \begin{cases} u_\ell & \text{for } x < 0 \\ u_r & \text{for } x > 0 \end{cases} \tag{2.2}$$

where u and f are vector functions with n components and u_ℓ, u_r are constant arbitrary vectors, $u_r \neq u_\ell$.

In 2.1-2.3 we shall suppose that u_ℓ, u_r are sufficiently close in R^n and so related that the solution of Riemann problem should contain only one discontinuity together with the constant regions adjacent to it. We also suppose that the system (2.1) is genuinely nonlinear with respect to (the family of characteristics associated with) this discontinuity.

DEFINITION 2.1 ([7]). We say that the discontinuity-involved in the solution of Riemann problem - is a j - shock and call the afore - mentioned discontinuous solution admissible if the conditions

$$\lambda_j(u_r) < D < \lambda_j(u_\ell)$$

(**)

$$\lambda_{j-1}(u_\ell) < D < \lambda_{j+1}(u_r)$$

hold (λ are the eigenvalues of matrix $a(u)=(\partial f_i/\partial u_j)$).

These conditions should be abbreviated CEL, too. In fact, CEL are conditions of determinacy (through initial data and jump relations) imposed to the piecewise constant solution of the Riemann problem[1)]. In the perturbation theory we shall present hereinafter in this paragraph, this solution plays the part of the "zeroth order".

According to [2] and [7] we can formulate

LEMMA 2.1. Given u_ℓ as a state on the left (of a discontinuity), the set of vectors u_r which can be joined (as states on the right) with u_ℓ by a j-shock can be represented as $u_r=u(\bar{\varepsilon})$, $\bar{\varepsilon} < 0$ and $\bar{\varepsilon}$ sufficiently small, where particularly $u(0)=u_\ell$, $\dot{u}(0)=R_\ell^j$ (R is an eigenvector of matrix a).

1) According to the method of characteristics.

In the proof of this lemma, the fact that $\bar{\varepsilon} < 0$ results when the restrictions (**) work.

We shall further assume that, on the considered discontinuity, CEL are fulfilled.

Let ε be a small parameter of the problem, characterized the same as in 1.1.

The relation (1.3), the expressions (1.5) and (1.6) and the notations of §1 have a vectorial analogue here. In particula motivating as in 1.1 we find for the linearized problem the follo wing form

$$(2.3)\qquad \begin{cases} \frac{\partial}{\partial \bar{t}} \tilde{U}_\ell + A_\ell \frac{\partial}{\partial \bar{x}} \tilde{U}_\ell = 0 , & \bar{x} < 0 \\ \frac{\partial}{\partial \bar{t}} \tilde{U}_r + A_r \frac{\partial}{\partial \bar{x}} \tilde{U}_r = 0 , & \bar{x} > 0 \end{cases}$$

$$(2.4)\qquad A_r \tilde{U}_r = A_\ell \tilde{U}_\ell + [\![u]\!] \Psi' \quad \text{for } \bar{x} = 0$$

$$(2.5)\qquad \tilde{U}(\bar{x},0) = \tilde{u}_0(\bar{x}), \ \bar{x} \in \mathbb{R}; \quad \Psi(0) = 0$$

with $A(u)=a(u)-DI$, I the unit matrix.

2.2. Determinacy. Evolutionary conditions

We suppose that the matrices A_ℓ and A_r are nonsingular and have distinct eigenvalues, $\lambda_1 < \lambda_2 < \ldots < \lambda_n$.

We shall use here the classes C_0 and C introduced the same as in 1.2.

Using the Laplace transformin (2.3) and (2.4) we find

$$(2.6)\qquad \begin{cases} A_\ell \frac{d\overset{*}{U}_\ell}{d\bar{x}} + \omega \overset{*}{U}_\ell - \tilde{u}_0 = 0 & \text{for } \bar{x} < 0 \\ A_r \frac{d\overset{*}{U}_r}{d\bar{x}} + \omega \overset{*}{U}_r - \tilde{u}_0 = 0 & \text{for } \bar{x} > 0 \end{cases}$$

(2.7) $$A_r \overset{*}{U}_r = A_\ell \overset{*}{U}_\ell + \omega [\![U]\!] \overset{*}{\Psi} \quad \text{for} \quad \bar{x} = 0.$$

The systems (2.6) can be put in the form

(2.8) $$\frac{d\overset{*}{U}}{d\bar{x}} = P\overset{*}{U} + f \quad , \quad P = -\omega A^{-1} \quad , \quad f = A^{-1}\tilde{u}_0$$

Since the matrices A and P have the same eigenvectors and the eigenvalues $\tilde{\lambda}$ of A, the eigenvalues $\hat{\lambda}$ of P and the eigenvalues λ of a are related by

(2.9) $$\hat{\lambda}_i = -\frac{\omega}{\tilde{\lambda}_i} \quad , \quad \tilde{\lambda}_i = \lambda_i - D$$

it follows that the solution of (2.8) can be (formally) represented by

(2.10) $$\overset{*}{U}(\bar{x};\omega) = \sum_{i=1}^{n} \overset{i}{R}\left\{\overset{i}{L}.\overset{*}{U}(0) + \int_0^{\bar{x}} \overset{i}{L}.f(\xi)e^{-\hat{\lambda}_i \xi} d\xi\right\} e^{\hat{\lambda}_i \bar{x}}$$

where R,L are right/left eigenvectors of A. Then, by (2.9) we get

(2.11) $$\overset{*}{U}(\bar{x};\omega) = \begin{cases} \sum_{i=1}^{n} \overset{i}{R}_\ell \left\{\overset{i}{L}_\ell.\overset{*}{U}_- + \overset{i}{L}_\ell.A_\ell^{-1}\int_0^{\bar{x}} \tilde{u}_0(\xi) e^{\frac{\omega\xi}{\lambda_i(u_\ell)-D}} d\xi\right\} e^{-\frac{\omega\bar{x}}{\lambda_i(u_\ell)-D}} & ,\bar{x}<0 \\ \sum_{i=1}^{n} \overset{i}{R}_r \left\{\overset{i}{L}_r\overset{*}{U}_+ + \overset{i}{L}_r.A_r^{-1}\int_0^{\bar{x}} \tilde{u}_0(\xi) e^{\frac{\omega\xi}{\lambda_i(u_r)-D}} d\xi\right\} e^{-\frac{\omega\bar{x}}{\lambda_i(u_r)-D}} & ,\bar{x}>0 \end{cases}$$

(2.12) $$A_r \overset{*}{U}_+ = A_\ell \overset{*}{U}_- + \omega [\![u]\!] \overset{*}{\Psi} \quad \text{for} \quad \bar{x} = 0$$

Let us extend now the formal procedure introduced in 1.2 (see (1.20)-(1.25)). When $\lambda_i(u_\ell) > D/\lambda_i(u_r) < D$ we shall annul the coefficient of $\exp\{-[\omega\bar{x}/(\lambda_i(u_\ell)-D)]\}$ / $\exp\{-[\omega\bar{x}/(\lambda_i(u_r)-D)]\}$ in (2.11) obtaining for a given i , $1 \le i \le n$, <u>a relation of determinacy.</u>

DEFINITION 2.2. We say that the linearized problem is determined if the number of linear algebraic equations - having $\overset{*}{\Psi}$ and the components of $\overset{*}{U}_-$, $\overset{*}{U}_+$ as unknowns - of the system which consists of determinacy relations and jump relations (2.12) is equal to 2n+1.

If the system (2.1) is genuinely nonlinear with respect to a given label j, $1 \leq j \leq n$, then we have

THEOREM 2.1. The linearized problem is determined iff the conditions

$$\text{(2.13)} \qquad \begin{cases} \lambda_j(u_r) < D < \lambda_j(u_\ell) \\ \lambda_{j-1}(u_\ell) < D < \lambda_{j+1}(u_r) \end{cases}$$

are fulfilled.

◀ When

$$\text{(2.14)} \qquad \begin{cases} \lambda_{j_\ell}(u_\ell) < D < \lambda_{j_\ell+1}(u_\ell) \\ \lambda_{j_r}(u_r) < D < \lambda_{j_r+1}(u_r) \end{cases}$$

then we obtain $n-j_\ell+j_r$ determinacy relations. Since (2.12) offers n relations for 2n+1 unknowns $\overset{*}{U}_-$, $\overset{*}{U}_+$ and $\overset{*}{\Psi}$ we have to impose that $n-j_\ell-j_r=n+1$. Denoting $j_r=j$, we find $j_\ell=j-1$ and (2.14) can be re-arranged as (2.13). ▶

We call (2.13) determinacy conditions.

REMARK 2.1. The determinacy conditions (associated to the first order) coincide with CEL (which correspond to the zeroth order of the perturbation theory); see, again, the foot note on page 9. Also, see [1].

Using the notations (2.9) we can re-write (2.13) as

$$(2.15)\qquad \begin{cases} \hat{\lambda}_j(u_r) > 0 > \hat{\lambda}_j(u_\ell) \\ \hat{\lambda}_{j-1}(u_\ell) > 0 > \hat{\lambda}_{j+1}(u_r) \end{cases} .$$

The conditions (2.13) extend (1.26).

If (2.13) are satisfied then we have to pose see (2.11)

$$(2.16)\qquad \begin{cases} \overset{j}{L}_\ell . \overset{*}{U}_- = -\overset{j}{L}_\ell A_\ell^{-1} \int_0^{-\infty} \tilde{u}_0(\xi) e^{\frac{\omega\xi}{\lambda_j(u_\ell)-D}} d\xi \\ \cdots\cdots\cdots\cdots \\ \overset{n}{L}_\ell . \overset{*}{U}_- = -\overset{n}{L}_\ell . A_\ell^{-1} \int_0^{-\infty} \tilde{u}_0(\xi) e^{\frac{\omega\xi}{\lambda_n(u_\ell)-D}} d\xi \\ \overset{1}{L}_r . U_+ = -\overset{1}{L}_r . A_r^{-1} \int_0^{\infty} \tilde{u}_0(\xi) e^{\frac{\omega\xi}{\lambda_1(u_r)-D}} d\xi \\ \cdots\cdots\cdots\cdots \\ \overset{j}{L}_r . U_+ = -\overset{j}{L}_r . A_r^{-1} \int_0^{\infty} \tilde{u}_0(\xi) e^{\frac{\omega\xi}{\lambda_j(u_r)-D}} d\xi . \end{cases}$$

The relations (2.12) and (2.16) make up a linear algebraic system of 2n+1 equations for 2n+1 unknowns, $\overset{*}{U}_-$, $\overset{*}{U}_+$ and $\overset{*}{\Psi}$. After an easy re-arrangement, taking into account that $(\lambda-D)L=LA$, we can give to (2.16), (2.12) the form

$$(2.17)\qquad \begin{cases} \overset{j}{L}_\ell \overset{*}{U}_- = \overset{j}{L}_\ell \cdot \int_0^{\infty} \tilde{u}_0 \{-[\lambda_j(u_\ell)-D]\tau\} e^{-\omega\tau} d\tau = \overset{j}{g}_\ell \\ \cdots\cdots\cdots \qquad \cdots\cdots \\ \overset{n}{L}_\ell \overset{*}{U}_- = L_\ell \cdot \int_0^{\infty} \tilde{u}_0 \{-[\lambda_n(u_\ell)-D]\tau\} e^{-\omega\tau} d\tau = \overset{n}{g}_\ell \\ \overset{1}{L}_r \overset{*}{U}_+ = \overset{1}{L}_r \cdot \int_0^{\infty} \tilde{u}_0 \{-[\lambda_1(u_r)-D]\tau\} e^{-\omega\tau} d\tau = \overset{1}{g}_r \\ \cdots\cdots\cdots \qquad \cdots\cdots \\ \overset{j}{L}_r \overset{*}{U}_+ = \overset{j}{L}_r \cdot \int_0^{\infty} \tilde{u}_0 \{-[\lambda_j(u_1)-D]\tau\} e^{-\omega\tau} d\tau = \overset{j}{g}_r \\ \overset{*}{U}_+ = A_r^{-1} A_\ell \overset{*}{U}_- + \omega A_r^{-1} [\![u]\!] \overset{*}{\Psi} \end{cases}$$

We shall use the lemma 2.1 in order to prove

LEMMA 2.2. If in the problem (2.1), (2.2) u_ℓ and u_r are linked by a j-shock and are conveniently close[1], then for the system (2.17) there exist a unique solution.

◀ By expressing, from the last n relations (2.17), $\overset{*}{U}_+$ with respect to $\overset{*}{U}_-$ and $\overset{*}{\Psi}$ we can find $\overset{*}{U}_-$ and $\overset{*}{\Psi}$ from the system of n+1 equations

$$(2.18)\qquad \begin{cases} \overset{k}{L}_\ell . \overset{*}{U}_- = \overset{k}{g}_\ell \quad, k=j,\dots,n \\ \overset{\ell}{L}_r A_r^{-1} A_\ell \overset{*}{U}_- + \omega\overset{*}{\Psi} \overset{\ell}{L}_r A_r^{-1} [\![U]\!] = \overset{\ell}{g}_r \quad, \ell=1,\dots,j \end{cases}$$

The proof comes to an end if, denoting by Δ the discriminant of the system (2.19), we show that $\Delta \neq 0$ when $[\![u]\!] \neq 0$ and u_ℓ, u_r are close enough.

According to lemma 2.1 we write

$$[\![u]\!] = \bar{\varepsilon}\,\frac{u(\bar{\varepsilon})-u(0)}{\bar{\varepsilon}}$$

and

$$(2.19)\qquad \Delta = \bar{\varepsilon}\Delta_1$$

where

$$(2.20)\quad \Delta_1 = \begin{vmatrix} \overset{j}{L}_{\ell 1} & \overset{j}{L}_{\ell 2} & \dots & \overset{j}{L}_{\ell n} & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \overset{n}{L}_{\ell 1} & \overset{n}{L}_{\ell 2} & \dots & \overset{n}{L}_{\ell n} & 0 \\ (\overset{1}{L}_r A_r^{-1} A_\ell)_1 & (\overset{1}{L}_r A_r^{-1} A_\ell)_2 & \dots & (\overset{1}{L}_r A_r^{-1} A_\ell)_n & \overset{1}{L}_r A_r^{-1} \frac{u(\bar{\varepsilon})-u(0)}{\bar{\varepsilon}} \\ \dots & \dots & \dots & \dots & \dots \\ (\overset{j}{L}_r A_r^{-1} A_\ell)_1 & (\overset{j}{L}_r A_r^{-1} A_\ell)_2 & \dots & (\overset{j}{L}_r A_r^{-1} A_\ell)_n & \overset{j}{L}_r A_r^{-1} \frac{u(\bar{\varepsilon})-u(0)}{\bar{\varepsilon}} \end{vmatrix}$$

---

1) For example, $\bar{\varepsilon}$ is small enough in the parametrization of lemma 2.1.

Let us denote by Δ_2 the determinant obtained from Δ_1 by deleting the last row and the last column. From $\overset{i}{L}_r = \overset{i}{L}[u(\bar{\varepsilon})] = \overset{i}{L}_\ell + 0(\bar{\varepsilon})$ it appears, since $\overset{1}{L}_\ell, \ldots, \overset{n}{L}_\ell$ are independent, that $\lim_{\bar{\varepsilon}\to 0} \Delta_2 \neq 0$ and therefore

(2.21) $\Delta_2 \neq 0$ for u_ℓ and u_r close enough

According to lemma 2.1,

$$\lim_{\bar{\varepsilon}\to 0} A_r^{-1} \tilde{\lambda}_j(u_r) \frac{u(\bar{\varepsilon})-u(0)}{\bar{\varepsilon}} = \overset{j}{R}_\ell$$

and, when $\overset{i}{L}\overset{k}{R} = \delta_{ik}$,

$$\lim_{\bar{\varepsilon}\to 0} [\lambda_j(u_r)-D] \overset{k}{L}_r A_r^{-1} \frac{u(\bar{\varepsilon})-u(0)}{\bar{\varepsilon}} = \delta_{kj}. \tag{2.22}$$

From (2.21) and (2.22) we find that for a j-shock we have

$$\lim_{\bar{\varepsilon}\to 0} [\lambda_j(u_r)-D]\Delta_1 = \lim_{\bar{\varepsilon}\to 0} \Delta_2 \neq 0 \tag{2.23}$$

The fact that - for u_ℓ and u_r conveniently close - we have $\Delta \neq 0$, results from (2.19) and (2.23). ▶

REMARK 2.2

(i) It is easy to formulate an analogue of remark 1.1 (i). When $\|u_r - u_\ell\| \to 0$, the matrices A_ℓ and A_r become singular and in (2.4) we have $[\![u]\!] \to 0$-though, usually, $|\Psi'|$ does not tend to zero - and $\|\tilde{U}_\ell\| \to 0$, $\|\tilde{U}_r\| \to 0$. The linearized problem (2.3)-(2.5) tends, in this case, to the zeroth order of the problem which corresponds to the circumstance $u_r = u_\ell$. This fact shows that a relation can only be established between the zeroth orders of the problems which correspond to $u_r \neq u_\ell$ and $u_r = u_\ell$ respectively (see remark 1.1 (i)).

(ii) Generally, the requirement that u_ℓ , u_r be closed (as imposed by lemma 2.2) must be added, when the first order is considered, to the similar restriction imposed by lemma 2.1. This fact does not appear in the context of §1 (see remark 1.2)

DEFINITION 2.3. The requirements which guarantee the existence of a unique solution for the system (2.17) are called evolutionary conditions.

In the context of §2 the set of evolutionary conditions contains the determinacy conditions together with the (possible) demand that u_ℓ and u_r should be close (see lemma 2.2). From (1.2 and (1.25) it appears that in case of a single conservation law 1D in space, the evolutionary conditions come down to determinacy conditions.

2.3. Linearized stability. Linearized well-posedness

According to lemma 2.2 we can express

$$(2.24)\quad \begin{aligned}\omega\overset{*}{\Psi} = {}& \overset{j}{b}_\ell \int_0^\infty \tilde{u}_0\{-[\lambda_j(u_\ell)-D)]\tau\}e^{-\omega\tau}d\tau + \ldots + \overset{n}{b}_\ell \int_0^\infty \tilde{u}_0\{-[\lambda_n(u_\ell)-D]\tau\}e^{-\omega\tau}d\tau \\ &+ \overset{1}{b}_r \int_0^\infty \tilde{u}_0\{-[\lambda_1(u_r)-D]\tau\}e^{-\omega\tau}d\tau + \ldots + \overset{j}{b}_r \int_0^\infty \tilde{u}_0\{-[\lambda_j(u_r)-D]\tau\}e^{-\omega\tau}d\tau\end{aligned}$$

and thus find

$$(2.25)\quad \Psi(\bar{t}) = \sum_{k=j}^{n} \overset{k}{b}_\ell \int_0^{\bar{t}} \tilde{u}_0\{-[\lambda_k(u_\ell)-D]\tau\}\,d\tau + \sum_{k=1}^{j} \overset{k}{b}_r \int_0^{\bar{t}} \tilde{u}_0\{-[\lambda_k(u_r)-D]\tau\}\,d\tau$$

It is easy to formulate an analogue of definition 1.2. We shall now extend the theorem 1.1 by

THEOREM 2.2. If the evolutionary conditions are fulfilled then the linearized problem with data in C_0 is well-posed in the class C.

◀ According to (2.25) and Haar estimates (see appendix). ▶

REMARK 2.3. With the notations (and according to assumptions) of lemma 2.1, from lemmas 2.1 and 2.2 it appears (see remark 2.2 (ii)) that we can find a convenient neighbourhood in R_- of $\bar{\varepsilon}$ so that the solution - which contains only a j-shock - of Riemann problem should be (linearized) stable.

In the proof of theorem 2.1 we have denoted j_ℓ / j_r the number of eigenvalues of matrix a for which we have $\lambda > D / \lambda < D$ in the left handed/right handed region of discontinuity.

In the context of § 3, any of these (integer) labels can be taken as an index (according to the theorem 3.2). For example,

DEFINITION 2.4 The (integer) label j_r will be called the index of discontinuity (see remark 2.1).

In case of a j-shock we have $j_r = j$.

3. LINEARIZED PROBLEM FOR A SYSTEM OF CONSERVATION LAWS, 2D IN SPACE

3.1. Wording up of linearized problem

Let us now extend, in case of two space dimensions, the considerations of § 2.

Instead of the problem (2.1), (2.2) we have here

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} + \frac{\partial g(u)}{\partial y} = 0 \tag{3.1}$$

$$u(x,y,0) = \begin{cases} u_\ell & \text{for } x < 0 \\ u_r & \text{for } x > 0 \end{cases} \tag{3.2}$$

where u , f and g are vector functions with n components and u_ℓ, u_r are constant arbitrary vectors, $u_\ell \neq u_r$.

In 3.1 - 3.3 we shall assume that u_ℓ, u_r are conveniently close in $\mathbb{R}^n$ and so related that the solution of the problem should contain only one shock together with the constant regions adjacent to it. On the shock line the jump conditions

$$(3.3) \qquad [\![u]\!]\frac{\partial\phi}{\partial t} + [\![f(u)]\!]\frac{\partial\phi}{\partial x} + [\![g(u)]\!]\frac{\partial\phi}{\partial y} = 0$$

are fulfilled.

REMARK 3.1. In case of steady and normal shock discontinuity, the possibility of such a solution/the nature of demands formulated above is similar to that presented in §2. Let us consider, indeed, the system

$$(3.1)' \qquad \frac{\partial}{\partial \tilde{t}}u + \frac{\partial}{\partial \tilde{x}}\tilde{f}(u) + \frac{\partial}{\partial \tilde{y}}\tilde{g}(u) = 0$$

and denote by D the speed, in $\tilde{x}$ direction, of a plane discontinuity normal to that direction. In the frame of mentioned discontinuity, $x = \tilde{x} - D\tilde{t}$, $y = \tilde{y}$, $t = \tilde{t}$, (3.1)' becomes (3.1) with $f(u) = \tilde{f}(u) - Du$, $g = \tilde{g}$ and on a steady discontinuity the jump conditions have the form $0 = [\![f(u)]\!] = [\![\tilde{f}(u)]\!] - D[\![u]\!]$.

The small parameter ε of the problem has to be characterized the same as in § 2.

Proceeding as in 1.1 and 2.1 but using instead of (1.9) the mapping

$$(3.4) \qquad \bar{x} = x - Dt - \overset{\varepsilon}{\psi}(y,t) \quad , \quad \bar{t} = t \quad , \quad \bar{y} = y$$

we find for the linearized problem the following form

$$(3.5)\quad \begin{cases} \dfrac{\partial}{\partial \bar{t}}\tilde{U}_\ell + A_\ell \dfrac{\partial}{\partial \bar{x}}\tilde{U}_\ell + b(u_\ell)\dfrac{\partial}{\partial \bar{y}}\tilde{U}_\ell = 0 \ , & \bar{x} < 0 \\ \dfrac{\partial}{\partial \bar{t}}\tilde{U}_r + A_r \dfrac{\partial}{\partial \bar{x}}\tilde{U}_r + b(u_r)\dfrac{\partial}{\partial \bar{y}}\tilde{U}_r = 0 \ , & \bar{x} > 0 \end{cases}$$

$$(3.6)\qquad A_r\tilde{U}_r = A_\ell\tilde{U}_\ell + [\![u]\!]\frac{\partial}{\partial \bar{t}}\Psi + [\![g(u)]\!]\frac{\partial}{\partial \bar{y}}\Psi \qquad \text{for } \bar{x} = 0$$

$$(3.7)\qquad \tilde{U}(\bar{x},\bar{y},0) = \tilde{u}_0(\bar{x},\bar{y}), \quad \Psi(\bar{y},0)=0$$

with notations similar to those of §§1,2.

3.2. Determinacy. Evolutionary conditions

We assume that the system (3.1) is strictly hyperbolic with $\bar{t}$ time-like which means, in 2D, imposing - in either adjacent region of discontinuity - the condition that for every $\lambda,\gamma \in R$ we are able to find n real distinct roots $\omega(\lambda,\gamma)$ of the equation

$$(3.8)\qquad \det\left[\omega I + \lambda A + b\gamma\right] = 0$$

Since the discontinuity is normal, we shall consider solutions of the form

$$(3.9)\qquad \left[\tilde{U}(\bar{x},\bar{y},\bar{t}),\ \Psi(\bar{y},\bar{t})\right] = e^{-i\alpha\bar{y}}\left[\tilde{U}(\bar{x},\bar{t})\ ,\ \Psi(\bar{t})\right]$$

Carrying (3.9) into (3.5) and (3.6) we obtain

$$(3.10)\quad \begin{cases} \dfrac{\partial}{\partial \bar{t}}\tilde{U}_\ell + A_\ell \dfrac{\partial}{\partial \bar{x}}\tilde{U}_\ell - i\alpha b_\ell\tilde{U}_\ell = 0 \ , & \bar{x} < 0 \\ \dfrac{\partial}{\partial \bar{t}}\tilde{U}_r + A_r \dfrac{\partial}{\partial \bar{x}}\tilde{U}_r - i\alpha b_r\tilde{U}_r = 0 \ , & \bar{x} > 0 \end{cases}$$

$$(3.11)\qquad A_r\tilde{U}_r = A_\ell \tilde{U}_\ell + [\![u]\!]\psi'(t) - i\alpha[\![g(u)]\!]\psi \qquad \text{for } \bar{x} = 0$$

Using the Laplace transform we find, as in (2.8), for either system (3.10) the form

$$(3.12)\qquad \frac{d}{d\bar{x}}\overset{*}{U} = P\overset{*}{U} + f$$

where

$$(3.13)\qquad P = -A^{-1}\left[\omega I - i\alpha b\right], \quad f = A^{-1}\tilde{u}_0$$

to which we add, according to (3.11), the jump relations

$$(3.14)\qquad A_r\overset{*}{U}_r = A_\ell\overset{*}{U}_\ell + \{\omega[\![u]\!] - i\alpha[\![g(u)]\!]\}\overset{*}{\psi} \qquad \text{for } \bar{x} = 0$$

As in 2.2, we denote by $\hat{\lambda}$ the eigenvalues of matrix P. Using the remark 3.1, we can prove an analogue of the theorem 2.1.

THEOREM 3.1. The linearized problem is determined iff the conditions

$$(3.15)\qquad \begin{aligned} Re\,\hat{\lambda}_j(u_r) > 0 > Re\,\hat{\lambda}_j(u_\ell) \\ Re\,\hat{\lambda}_{j-1}(u_\ell) > 0 > Re\,\hat{\lambda}_{j+1}(u_r) \end{aligned}$$

are fulfilled [see (2.15)].

Let us consider the number j_r of the eigenvalues $\hat{\lambda}$ for which we have $Re\,\hat{\lambda} > 0$ in the right-handed region of discontinuity.

Since the eigenvalues $\hat{\lambda}$ depend on ω and α , j_r might depend on ω and α .

THEOREM 3.2 ([6]). The number j_r is independent on ω and α.

◀ If the number j_r depends on ω and α, we can find (ω_0, α_0) so that $Re\,\hat{\lambda}(\omega_0, \alpha_0)=0$. The eingevalues $\hat{\lambda}$ can be determined, according to (3.13), by

$$\det\left[\omega I + \hat{\lambda} A - i\alpha b\right] = 0 \tag{3.16}$$

with the restriction $Re\,\omega > 0$ imposed by the Laplace transform. In (ω_0, α_0), (3.16) gives

$$\det\left[-i\omega I + (Im\,\hat{\lambda}) A - \alpha b\right] = 0$$

and, since the system is hyperbolic, we have (according to (3.8)) $Im(i\omega)=Re\,\omega=0$ for every $Im\,\hat{\lambda} \in R$ and $\alpha \in R$. ▶

We can now give

DEFINITION 3.1. We call the (integer) label j_r the index of discontinuity.

REMARK 3.2

(i) An analogue, easy to formulate, of remark 2.2 works.

(ii) In case of a steady and normal shock lemma 2.1 keeps valid (according to the remark 3.1; the formulation of the analogue of lemma 2.1 depends only on the nature of f in (3.1)) and lemma 2.2. can be easily extended (however its formulation depends on the nature of g in (3.1)). The remark 2.3 keeps valid too.

(iii) When $\alpha = 0$, the definition 3.1 comes down - in view of theorem 3.2 - to the definition 2.4. This circumstance gathers up definition 2.1, remark 2.1, definition 2.4 and definition 3.1.

(iv) The index of a discontinuity is related to the facts of determinacy and (according to (iii)) it does not depend on the order (of the perturbation theory) considered. This assertion is trivial in 1D but requires a proof (given by theorem 3.2) in 2D.

(v) The conditions (3.15) are determinacy conditions; these conditions, together with the (possible) restriction (mentioned in (ii)) that u_ℓ, u_r are close, constitute the set of evolutionary conditions.

3.3. Linearized stability. Linearized well-posedness

DEFINITION 3.2. We say that the discontinuous solution considered is unstable if we can find, at least for a value of $\alpha \in \mathbb{R}$, a solution (3.9) - consisting of $\tilde{U}$ and Ψ - of linearized problem which grows boundlessly when $\bar{t} \to \infty$. The discontinuous solution is called stable if the solution (3.9) of the linearized problem is kept bounded, for every $\alpha \in \mathbb{R}$, when $\bar{t} \to \infty$.

To prove the well-posedness of linearized problem we have to show again that this problem is evolutionary and stable.

From theorems 3.1 and 3.2 it appears that the passing from 1D to 2D keeps unchanged the form/the nature of evolutionary conditions.

On the other hand, the stability result of theorem 2.2 cannot be obtained any more whithout a new restriction. Indeed, the Haar estimates (see appendix) show that the stability of solution of linearized problem depends on the stability of Ψ. In 2D the distribution of singularities of Ψ depends on α. Let $\Psi = [d(\omega,\alpha)/ /L(\omega,\alpha)]$ be the expression obtained according to the analogue of lemma 2.2. The function g (see (3.1)) contributes by $\hat{\lambda}(\omega,\alpha)$ (see (3.13)) to the evolutionary conditions and by $L(\omega,\alpha)$ to the stability conditions. When $\alpha = 0$ this contribution vanishes [together with the dependence on $\bar{y}$; according to (3.9)]. $L(\omega,0)$ has only one zero in $\omega = 0$. However, when $\alpha \neq 0$, it is possible - depending on the form of f and g in (3.1) - that some of the zeros of $L(\omega,\alpha)$ be placed in the region $Re\ \omega > 0$ thus implying instability even for data in C_0.

For each $\alpha \in R \setminus \{0\}$ the solution of the linearized problem is (exponentially) stable when the zeros of $L(\omega,\alpha)$ are all placed in $Re\ \omega < 0$.

REMARK 3.3

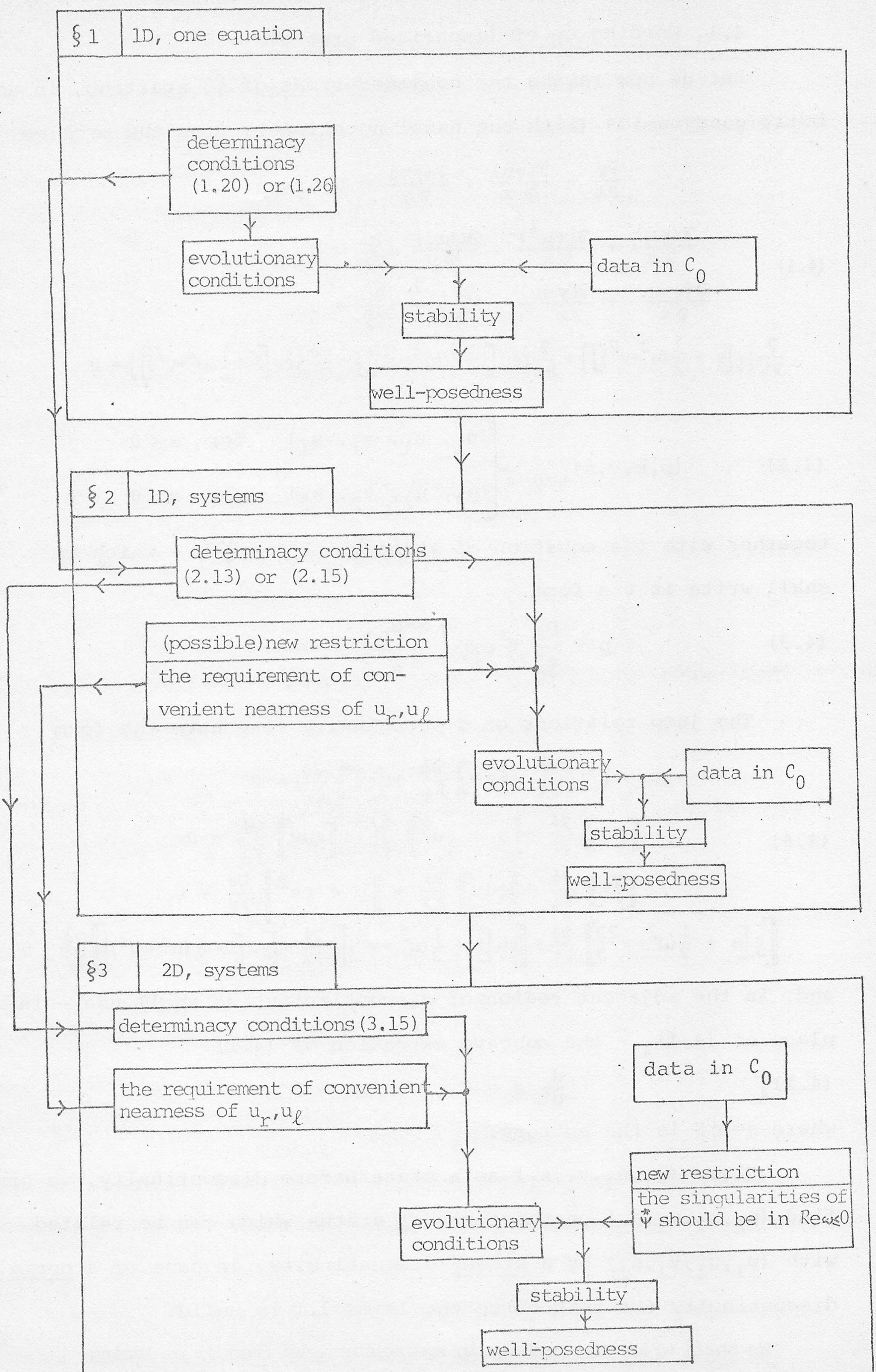
(i) In 2D we require stability for all $\alpha \in R$, particularly for $\alpha = 0$. Then we shall take data in C_0.

(ii) When $\alpha \neq 0$ we have to find the conditions for which the zeros of $L(\omega,\alpha)$ are all placed in $Re\ \omega < 0$. This is the new restriction we mentioned hereinabove.

In the context of gasdynamics it can be shown that these conditions do not depend on α and are related, as we have already mentioned, only on the form of f and g in (3.1). This form depends in its turn on the equation of state considered. Such (exponential) conditions/criteria of stability or instability are given in [4] and [8] (see [5] for magnetodynamics). A stability criterion removes the exponentially unstable evolutions. We should also remark that $L(\omega,\alpha)$ is an even function of α.

(iii) For certain equations of state, the condition $Re\ \omega < 0$ cannot be fulfilled strictly, under stability requirements, by the set of zeros of $L(\omega,\alpha)$. In such a case, when (a part of) zeros of $L(\omega,\alpha)$ are placed on the line $Re\ \omega = 0$, we have explicitly to study the possibility of (nonexponential) stability. Such a study is presented in §4.

The table on page 27 compares the facts of §§ 1,2,3.


§ 1
1D, one equation
determinacy conditions (1.20) or (1.26)
evolutionary conditions
data in C_0
stability
well-posedness
§ 2
1D, systems
determinacy conditions (2.13) or (2.15)
(possible) new restriction
the requirement of convenient nearness of u_r, u_ℓ
evolutionary conditions
data in C_0
stability
well-posedness
§3
2D, systems
determinacy conditions (3.15)
the requirement of convenient nearness of u_r, u_ℓ
data in C_0
new restriction
the singularities of $\hat{\psi}$ should be in Reω≤0
evolutionary conditions
stability
well-posedness

4. LINEARIZED PROBLEM FOR THE SYSTEM OF CONSERVATION LAWS, 2D IN SPACE , OF ADIABATIC GASDYNAMICS.

4.1. Wording up of linearized problem

Let us now remake the considerations of §3 starting, in adiabatic gasdynamics (with the usual notations), from the problem

$$\frac{\partial \varrho}{\partial t} + \frac{\partial(\varrho u)}{\partial x} + \frac{\partial(\varrho v)}{\partial y} = 0$$

$$(4.1) \qquad \frac{\partial(\varrho u)}{\partial t} + \frac{\partial(\varrho u^2)}{\partial x} + \frac{\partial(\varrho uv)}{\partial y} + \frac{\partial p}{\partial x} = 0$$

$$\frac{\partial(\varrho v)}{\partial t} + \frac{\partial(\varrho uv)}{\partial x} + \frac{\partial(\varrho v^2)}{\partial y} + \frac{\partial p}{\partial y} = 0$$

$$\frac{\partial}{\partial t}\left\{\varrho\left[e + \frac{1}{2}(u^2+v^2)\right]\right\} + \frac{\partial}{\partial x}\left\{\varrho u\left[i + \frac{1}{2}(u^2+v^2)\right]\right\} + \frac{\partial}{\partial y}\left\{\varrho v\left[i + \frac{1}{2}(u^2+v^2)\right]\right\} = 0$$

$$(4.2) \qquad (p,u,v,s)_{t=0} = \begin{cases} (p_1, u_1, v_1, s_1) & \text{for } x < 0 \\ (p_2, u_2, v_2, s_2) & \text{for } x > 0 \end{cases}$$

together with the equation of state $e = c_v T = \frac{p\tau}{\gamma-1}$ - which we shall write in the form

$$(4.3) \qquad p = \frac{p_0}{\varrho_0^\gamma}\, \varrho^\gamma \exp\left\{\frac{S-S_0}{c_v}\right\}$$

The jump relations on discontinuity line have the form

$$[\![\varrho]\!]\frac{\partial\phi}{\partial t} + [\![\varrho u]\!]\frac{\partial\phi}{\partial x} + [\![\varrho v]\!]\frac{\partial\phi}{\partial y} = 0$$

$$(4.4) \qquad [\![\varrho u]\!]\frac{\partial\phi}{\partial t} + [\![p + \varrho u^2]\!]\frac{\partial\phi}{\partial x} + [\![\varrho uv]\!]\frac{\partial\phi}{\partial y} = 0$$

$$[\![\varrho v]\!]\frac{\partial\phi}{\partial t} + [\![\varrho uv]\!]\frac{\partial\phi}{\partial x} + [\![p + \varrho v^2]\!]\frac{\partial\phi}{\partial y} = 0$$

$$\left[\!\!\left[\varrho\left[e + \frac{1}{2}(u^2+v^2)\right]\right]\!\!\right]\frac{\partial\phi}{\partial t} + \left[\!\!\left[\varrho u\left[i + \frac{1}{2}(u^2+v^2)\right]\right]\!\!\right]\frac{\partial\phi}{\partial x} + \left[\!\!\left[\varrho v\left[i + \frac{1}{2}(u^2+v^2)\right]\right]\!\!\right]\frac{\partial\phi}{\partial y} = 0$$

and, in the adjacent regions of discontinuity, we shall use - in place of $(4.1)_4$ - the concave extension of (4.1)

$$(4.1)'_4 \qquad \frac{d}{dt}\, s = 0$$

where $s = \varrho S$ is the entropy.

Given (p_1, u_1, v_1, s_1) as a state before discontinuity, we can find (p_2, u_2, v_2, s_2) on the curve of states which can be related with (p_1, u_1, v_1, s_1) by a steady discontinuity. In case of a normal discontinuity for this curve the lemma 2.1 is valid.

We shall write the problem in a dimensionless from by replacing

$$t \to \frac{t}{[t]} \ , \quad x \to \frac{x}{[x]} \ , \quad y \to \frac{y}{[x]} \ , \quad \rho \to \frac{\rho}{[\rho]} \ ,$$

$$u \to \frac{u}{[u]} \ , \quad v \to \frac{v}{[u]} \ , \quad p \to \frac{p}{[p]} \ , \quad s \to \frac{s}{[s]} \ , \quad \psi \to \frac{\psi}{[x]}$$

where

$$[t] = \frac{L}{c_2} \ , \ [x] = L \ , \ [\rho] = \rho_2 \ , \ [u] = c_2 \ ,$$

$$[p] = \rho_2 c_2^2 \ , \ [s] = c_p \ , \quad c^2 = \gamma \frac{p}{\rho}$$

and taking

$$M = \frac{u_2}{[u]} \ , \ \bar{M} = \frac{u_1}{[u]} \ , \ M_y = \frac{v}{[u]} \ , \ \bar{\rho} = \frac{\rho_1}{[\rho]}$$

$$(4.5) \quad P = \frac{p_2}{[p]} \ , \ \bar{P} = \frac{p_1}{[p]} \ , \ \bar{c} = \frac{c_1}{[u]}$$

$$\tilde{\rho}_i = \frac{\rho_i'}{[\rho]} \ , \ \tilde{u}_i = \frac{u_i'}{[u]} \ , \ v_i = \frac{v_i'}{[u]} \ , \ p_i = \frac{p_i'}{[p]} \ , \ s_i = \frac{s_i'}{[s]}$$

(furthermore we shall ignore the labels of perturbations which correspond to the region after discontinuity).

The zeroth order of the jump relations gives

$$(4.6) \quad M = \bar{\rho}\bar{M}, \ P - \bar{P} = M(\bar{M} - M), \ M_y = \bar{M}_y, \ \frac{\gamma}{\gamma - 1}(MP - \bar{M}\bar{P}) = \frac{1}{2}M(\bar{M}^2 - M^2).$$

From (4.6) we can obtain, in particular,

$$(4.7) \qquad (\gamma - 1)M^2 - (\gamma + 1)M\bar{M} + 2 = 0$$

For a normal discontinuity we have $M_y = 0$ and, for a 3-shock, $0 < M < 1$.

In this context, the system (3.10) may be written

$$(4.8)\quad \begin{cases} \dfrac{1}{\bar{c}^2}\left(\dfrac{\partial}{\partial t} + \bar{M}\dfrac{\partial}{\partial x}\right)\tilde{p}_1 + \dfrac{\partial \tilde{u}_1}{\partial x} - i\alpha\tilde{v}_1 = 0 \\ \bar{\rho}\left(\dfrac{\partial}{\partial t} + \bar{M}\dfrac{\partial}{\partial x}\right)\tilde{u}_1 + \dfrac{\partial \tilde{p}_1}{\partial x} = 0 \\ \bar{\rho}\left(\dfrac{\partial}{\partial t} + \bar{M}\dfrac{\partial}{\partial x}\right)\tilde{v}_1 - i\alpha\tilde{p}_1 = 0 \\ \left(\dfrac{\partial}{\partial t} + \bar{M}\dfrac{\partial}{\partial x}\right)\tilde{s}_1 = 0 \end{cases} \quad \text{for } x < 0$$

where

$$(4.9)\quad \tilde{p}_1 = \bar{c}^2\tilde{\rho}_1 + \bar{\rho}\bar{c}^2\tilde{s}_1$$

and

$$(4.10)\quad \begin{cases} \left(\dfrac{\partial}{\partial t} + M\dfrac{\partial}{\partial x}\right)\tilde{p} + \dfrac{\partial \tilde{u}}{\partial x} - i\alpha\tilde{v} = 0 \\ \left(\dfrac{\partial}{\partial t} + M\dfrac{\partial}{\partial x}\right)\tilde{u} + \dfrac{\partial \tilde{p}}{\partial x} = 0 \\ \left(\dfrac{\partial}{\partial t} + M\dfrac{\partial}{\partial x}\right)\tilde{v} - i\alpha\tilde{p} = 0 \\ \left(\dfrac{\partial}{\partial t} + M\dfrac{\partial}{\partial x}\right)\tilde{s} = 0 \end{cases} \quad \text{for } x > 0$$

where

$$(4.11)\quad \tilde{p} = \tilde{\rho} + \tilde{s}$$

Also, the relations (3.11) become

$$(4.12)\quad \begin{cases} \tilde{s}_+ = a_{11}\tilde{s}_- + a_{12}\tilde{p}_- + a_{13}\tilde{u}_- + a_{14}\tilde{v}_- + b_1\psi' - i\alpha c_1\psi \\ \tilde{p}_+ = a_{21}\tilde{s}_- + a_{22}\tilde{p}_- + a_{23}\tilde{u}_- + a_{24}\tilde{v}_- + b_2\psi' - i\alpha c_2\psi \\ \tilde{u}_+ = a_{31}\tilde{s}_- + a_{32}\tilde{p}_- + a_{33}\tilde{u}_- + a_{34}\tilde{v}_- + b_3\psi' - i\alpha c_3\psi \\ \tilde{v}_+ = a_{41}\tilde{s}_- + a_{42}\tilde{p}_- + a_{43}\tilde{u}_- + a_{44}\tilde{v}_- + b_4\psi' - i\alpha c_4\psi \end{cases} \quad \text{for } x = 0$$

where +/- labels the after/front side of discontinuity and we have

$$a_{11} = 1 - \frac{\gamma-1}{2}(M-\bar{M})^2$$

$$a_{12} = -\frac{1}{2}(\gamma^2-1)^2 M\bar{M}\,\frac{(\bar{M}-M)^2}{\left[(\gamma-1)M^2+2\right]\left[2\gamma M^2-(\gamma-1)\right]}$$

$$a_{13} = -b_1,\ a_{14}=0,\ a_{21} = -\frac{2}{\gamma+1}M\bar{M}$$

(4.13)

$$a_{22} = \frac{(\gamma+1)-2(\gamma-1)M\bar{M}}{2\gamma M^2-(\gamma-1)},\quad a_{23} = -b_2,\ a_{24}=0=a_{42}$$

$$a_{31} = M-\frac{\gamma-1}{\gamma+1}\bar{M},\ a_{32}=2\,\frac{\gamma-1}{\gamma+1}\,\frac{1}{M},\ a_{33}=2\,\frac{\gamma-1}{\gamma+1}-\frac{M}{\bar{M}}$$

$$a_{34}=a_{41}=a_{43}=0,\ a_{44}=1,$$

$$c_1=M_y b_1,\ c_2=M_y b_2,\ c_3=M_y b_3,\ c_4=\bar{M}-M$$

$$b_1 = -\frac{2}{M}\left(1-\frac{M}{\bar{M}}\right)(1-M\bar{M}),\ b_2 = -\frac{4}{\gamma+1}M,$$

$$b_3 = \frac{3-\gamma}{\gamma+1}+\frac{M}{\bar{M}},\ b_4 = 0$$

The initial conditions are

$$(4.14)\quad \begin{cases} (\tilde{s}_1,\tilde{p}_1,\tilde{u}_1,\tilde{v}_1)_{t=0} = (\tilde{s}_{10}(x),\ \tilde{p}_{10}(x),\ \tilde{u}_{10}(x),\ \tilde{v}_{10}(x)),\ x<0 \\ (\tilde{s},\tilde{p},\tilde{u},\tilde{v})_{t=0} = (\tilde{s}_0(x),\tilde{p}_0(x),\ \tilde{u}_0(x),\ \tilde{v}_0(x)),\ x>0 \\ \Psi(0)=0 \end{cases}$$

4.2. The expression of distortion of the discontinuity line

Taking

$$(4.15)\qquad \bar{\omega} = \frac{\omega}{M},\quad \sigma(\bar{\omega}) = \left[M^2\bar{\omega}^2+\alpha^2(1-M^2)\right]^{1/2}$$

we find for the four eivenvalues of matrix P (see (3.13)), which correspond to the region after discontinuity, the following expressions:

(4.16) $\lambda_1 = (1-M^2)^{-1}\left[M^2\bar{\omega} + \sigma(\bar{\omega})\right]$, $\lambda_2 = (1-M^2)^{-1}\left[M^2\bar{\omega} - \sigma(\bar{\omega})\right]$, $\lambda_3 = \lambda_4 = -\bar{\omega}$

($\lambda_1, \lambda_2, \lambda_3$ are distinct eigenvalues of reduced matrix obtained from P by deleting the row and the column corresponding to s). Furthermore we shall not write - in order to simplify the notation - the bar on $\bar{\omega}$.

In is easy to verify that the discontinuity is a 3-shock because, for $M < 1$ it appears from (4.16) that

$$\lambda_1 > 0, \ \lambda_2 < 0, \ \lambda_3 < 0 \tag{4.17}$$

(see (3.15)).

According to the remark 3.1 we can obtain, in the context of this paragraph, without the restriction that u_r, u_ℓ should be close:

$$\overset{*}{\psi}(\omega) = \frac{\gamma+1}{2} M \frac{d_1(\omega) + d_2(\omega)}{L(\omega)} \tag{4.18}$$

where

$$d_1(\omega) = \frac{\sigma(\omega)}{M}\left[a_{21}\overset{*}{s}_- + a_{22}\overset{*}{p}_- + a_{23}\overset{*}{u}_-\right] - \omega\left[a_{31}\overset{*}{s}_- + a_{32}\overset{*}{p}_- + a_{33}\overset{*}{u}_-\right] - i\alpha a_{44}\overset{*}{v}_- \tag{4.19}$$

$$-d_2(\omega) = \int_0^\infty \left\{ \frac{\sigma(\omega)}{M}\left[\tilde{p}_0 \frac{M}{M^2-1} - \tilde{u}_0 \frac{1}{M^2-1}\right] + \omega\left[\tilde{p}_0 \frac{1}{M^2-1} - \tilde{u}_0 \frac{M}{M^2-1}\right] - i\alpha\tilde{v}_0 \frac{1}{M}\right\} e^{-\lambda_1(\omega)\xi} \, d\xi \tag{4.20}$$

$$L(\omega) = 2M^2\omega\left[\sigma(\omega) + \omega\right] + (1-M^2)(\alpha^2 - \xi\omega^2) \tag{4.21}$$

For the considerations which follow it is convenient to put

$$\frac{1}{L(\omega)} = \frac{N_1(\omega)}{N_2(\omega)} \tag{4.22}$$

where

$$N_1(\omega) = \omega^2 \left[2M^2 - \bar{\rho}(1-M^2)-2M^3\right] + \alpha^2(1-M^2)-2M^2\omega \left[\sigma(\omega) - M\omega\right]$$

$$N_2(\omega) = \omega^4 \{\left[2M^2- \bar{\rho}(1-M^2)\right]^2 -4M^6\} + 2\omega^2\alpha^2(1-M^2)\cdot$$
$$\cdot\left[2M^2- \bar{\rho}(1-M^2)-2M^4\right] + \left[\alpha^2(1-M^2)\right]^2.$$

Since $1 < \gamma < \frac{5}{3}$, for $M < 1$ we obtain easily from (4.7)

$$2M^2 - \bar{\rho}(1-M^2) > 2M^3 > 2M^4 > 0 \tag{4.23}$$

By seeking for the roots of $N_2(\omega)$, we shall remark that the discriminant Δ is strictly positive :

$$\Delta = \left[\alpha^2(1-M^2)\right]^2 \{\left[2M^2- \bar{\rho}(1-M^2)-2M^4\right]^2- \left[(2M^2- \bar{\rho}(1-M^2))^2 -4M^6\right]\} =$$
$$= (2M^2\alpha^2)^2 (1-M^2)^3 (\bar{\rho}-M^2) = \bar{\rho}(2M^2\alpha^2)^2 (1-M^2)^3 (1-M\bar{M}) =$$
$$= \left[\text{by } (4.7)\right] = 4\,\frac{\gamma-1}{\gamma+1}\,\bar{\rho}\left[\alpha M(1-M^2)\right]^4 > 0$$

Then we have

$$\frac{1}{L(\omega)} = \varkappa\,\frac{(\omega^2+\omega_3^2) - k^2\omega\left[\sigma(\omega) - M\omega\right]}{(\omega^2+\omega_1^2)(\omega^2 + \omega_2^2)} \tag{4.24}$$

where

$$\begin{cases} \varkappa=\left[2M^2-\bar{\rho}(1-M^2)+2M^3\right]^{-1}, \quad k= \{2M^2/\left[2M^2-\bar{\rho}(1-M^2)-2M^3\right]\}^{1/2} \\ \omega_{1,2}=\left\{\dfrac{\alpha^2(1-M^2)}{\bar{\rho}} \cdot \dfrac{(2M\bar{M}-1)\pm 2(\frac{\gamma-1}{\gamma+1})^{1/2}\bar{\rho}^{1/2}M\bar{M}}{(2M\bar{M}-1)^2 - M^2}\right\}^{1/2} \\ \omega_3 = \{\left[\alpha^2(1-M^2)\right]/\left[2M^2- \bar{\rho}(1-M^2)-2M^3\right]\}^{1/2} \end{cases} \tag{4.25}$$

and, according to (4.23), in (4.25) the expressions under the radical are positive.

REMARK 4.1 The gasdynamic context related to (4.3) has the following peculiarities

- we do not restrict u_r and u_ℓ to be close
- for every $\alpha \in R$ the singularities of $\overset{*}{\psi}$ are placed on the line $Re\ \omega = 0$.

LEMMA 4.1 If f is fo and $F(\omega) = L[f]$ then

(4.26) $$F\left[M\omega+(\omega^2+1)^{1/2}\right] = \int_0^\infty e^{-\omega t}\left\{\frac{1}{1+M}\, f\left(\frac{t}{1+M}\right) - (1-M^2)^{-3/2}\int_{Mt}^{t}\frac{J_1\left[\left(\frac{t^2-s^2}{1-M^2}\right)^{1/2}\right]}{(t^2-s^2)^{1/2}}(s-Mt)\, f\left(\frac{s-Mt}{1-M^2}\right)ds\right\}dt$$

◀ Since

$$J_{1/2}(z) = \frac{e^z - e^{-z}}{(2\pi z)^{1/2}}, \quad K_{1/2}(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}$$

we can calculate [1)]

(4.27) $$\frac{1}{v}\left\{e^{-\omega v} - e^{-v(\omega^2+1)^{1/2}}\right\} = I_{1/2}\left\{\tfrac{1}{2}v\left[(\omega^2+1)^{1/2} - \omega\right]\right\}\cdot$$

$$\cdot K_{1/2}\left\{\tfrac{1}{2}v\left[(\omega^2+1)^{1/2}+\omega\right]\right\} = \int_0^\infty J_1(u)\,\frac{e^{-\omega(u^2+v^2)^{1/2}}}{(u^2+v^2)^{1/2}}\,du =$$

$$= \int_v^\infty \frac{J_1\left[(t^2-v^2)^{1/2}\right]}{(t^2-v^2)^{1/2}}\, e^{-\omega t}\, dt$$

1) Y.S.Gradstein, Y.M.Rijik - Tables of integrals, summs, series and products. 5th edition. Moscow, Nauka, 1971 (in Russian) ; pag. 733, 6.637.

and further

$$\int_0^\infty vf(v)\ \{ \int_v^\infty \frac{J_1\left[(t^2-v^2)^{1/2}\right]}{(t^2-v^2)^{1/2}} e^{-\omega(t+Mv)} dt\}dv = \{t= \frac{\tau-Ms}{1-M^2}, v= \frac{s-M\tau}{1-M^2}\} =$$

$$=(1-M^2)^{-3/2} \int_0^\infty e^{-\omega\tau}\ \{ \int_{M\tau}^{\tau} \frac{J_1\left[(\frac{\tau^2-s^2}{1-M^2})^{1/2}\right]}{(t^2-s^2)^{1/2}} (s-M\tau) \cdot f(\frac{s-M\tau}{1-M^2}) ds\ \}\ d\tau$$

However, let us remark, according to (4.27), that

$$F\left[M\omega+(\omega^2+1)^{1/2}\right] = \int_0^\infty f(v)\ e^{-v\left[M\omega+(\omega^2+1)^{1/2}\right]} dv =$$

$$= \int_0^\infty f(v) e^{-vM\omega} \{ e^{-v\omega} - v \int_v^\infty \frac{J_1\left[(t^2-v^2)^{1/2}\right]}{(t^2-v^2)^{1/2}} \cdot e^{-\omega t} dt\} dv.$$

By lemma 4.1 it follows easily that

$$(4.28)\quad F_f(t) = L^{-1}\left[\int_0^\infty f(t)\ e^{-\lambda_1(\omega)t} dt\right] = \frac{1-M}{M}\ f(\frac{1-M}{M}\ t) -$$

$$- \frac{1}{M} \int_{|\alpha|t}^{\frac{|\alpha|t}{M}} \frac{J_1\left[(\frac{\alpha^2t^2}{M^2} - u^2)^{1/2}\right]}{\left(\frac{\alpha^2t^2}{M^2} - u^2\right)^{1/2}} (u-|\alpha|t)\, f\left(\frac{u-|\alpha|t}{|\alpha|}\right) du$$

The expression of ψ then comes, using the tables, from (4.18), (4.24), (4.19), (4.20) and (4.28).

4.3. Linearized stability. Linearized well-posedness

THEOREM 4.1 The linearized problem (4.8) - (4.13) with data from C_0 is well-posed in the class C.

◀ From (4.8) it appears, using (4.28), that for data in C_0 ψ and ψ' are bounded. The theorem then follows by the Haar estimates. ▶

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APPENDIX. The Haar estimates (see, for example, [9])

Let us consider the system

$$\frac{\partial}{\partial t} q + A \frac{\partial}{\partial x} q + B_1 q = 0 \tag{1}$$

with A and B_1 constant matrices [see (2.3)/(3.10)/(4,8), (4.10)]. The eigenvalues $\lambda_1,\ldots,\lambda_n$ of matrix A are real and distinct. Let P be a matrix which diagonalizes A :

$$P^{-1}AP = D = \operatorname{diag}\{\lambda_1,\ldots,\lambda_n\} \tag{2}$$

and put $v=P^{-1}q$. Multiplying (1) by P^{-1} to the left we find

$$\frac{\partial}{\partial t} v + D\frac{\partial}{\partial x} v + Bv = 0, \quad B=P^{-1}B_1P \tag{3}$$

Let us take the point (ξ,η), $\eta>0$. By integrating (3) along the characteristics we obtain

$$v_i(\xi,\eta)=g_i\left[X_i(0\ ;\xi,\eta)\right]-\int_0^{\eta}\sum_{j=1}^{n}B_{ij}v_j(X_i,t)dt;\ 1\leq i\leq n;\ g=P^{-1}q_0 \tag{4}$$

where the points $\Xi(\xi,\eta)$ and $\Xi_i\left[X_i(0;\xi,\eta),0\right]$ are in correspondence as belonging to the characteristic $x=X_i(t;\xi,\eta)$.

Let now $[x_1,x_2]$ be a compact interval of the real axis. We denote by $\mathcal{D}_\eta$ the closure of the intersection of the determinacy domain of this interval with the strip $0\leq t\leq\eta$ and put

$$H = \max_{Q\in\mathcal{D}_\eta}|v(Q)|, \quad |v|\leq\max_{1\leq i\leq n}|v_i|$$

Let $R(x_R, t_R)\in\mathcal{D}_\eta$ be a point at which $|v(Q)|$ reaches the value H . Denoting

$$\|g\|_{[x_1,x_2]} = \max_{1\leq i\leq n} \sup_{x\in[x_1,x_2]} |g_i(x)| \ , \quad K = \max_{i,j} |B_{ij}|$$

we obtain from (4)

$$|v(P)| \leq |v(R)| = H \leq \max_{1\leq i\leq n} |v_i(R)| =$$

$$= \max\left|g_i(R_i) - \int_0^{t_R} \left(\sum_{j=1}^{n} B_{ij}v_j\right)dt\right| \leq \|g\|_{[x_1,x_2]} + n\eta KH$$

and further

$$H \leq \tilde{C}(\eta)\|g\|_{[x_1,x_2]} \ , \quad \tilde{C}(\eta) = \frac{1}{1-n\eta K} \quad \text{for} \quad \eta < \frac{1}{nK} \ .$$

When $\eta > (1/nK)$ the procedure has to be repeated. Let us advance, in this case, by strips of breadth $1/2nK$ and parallel to axis $t=0$. In such a strip $\tilde{C}(\eta) \leq 2$ so that

(5) $$H \leq 2\|g\|_{[x_1,x_2]} \leq 2\|g\|$$

where the constant $\|g\|$ majorizes the initial data (on a given interval).

The mentioned procedure can be applied directly to the problem (4.8), $(4.14)_1$ because the determinacy domain of the interval $x < 0$, $t = 0$ is the whole region $x < 0$, $t > 0$. If the initial data are bounded, then from (5) the solution (corresponding to them, in the domain of determinacy) is bounded.

To study the <u>mixed problem</u> (4.10), (4.12), $(4.14)_{2,3}$ we need, moreover, the expression of Ψ. In fig.2 we depict, in such a case, the curve which carries the initial ($\tilde{u}_0$) or boundary (Ψ)

data and its domain of determinacy. The procedure expounded above keeps valid if one makes certain minor and obvious modifications related to the estimates corresponding to the points of discontinuity. The boundedness of solution depends now, moreover, on the boundedness of ψ and ψ'.

The estimate (5), and the analogue estimates which correspond to the mixed problem, have to be regarded as Haar estimates because they allow to evaluate the solution by means of initial and boundary data.