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by
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For each positive integer $n \geq 1$, let $C^*(S(n))$ be the C^* - algebra generated by all weighted shifts (with respect to some fixed orthonormal basis $(e_m)_{m \geq 0}$ of the Hilbert space H) of period n. Given a strictly increasing sequence of positive integers $\underline{p} = (p_k)$, with p_k dividing p_{k+1} for all $k \geq 1$, the Bunce - Deddens algebra, denoted in this paper by $A(\underline{p})$, is $\overline{\bigcup_k \nu(C^*(S(p_k)))}$ (see [2]), where $\nu : B(H) \to B(H)/K(H)$ is the canonical surjection onto the Calkin algebra.

In this paper we give necessary and sufficient conditions for two C^* - algebras, each of which is the (spatial) tensor product of two Bunce - Deddens algebras, to be stably isomorphic (see Theorem 2.5. below) or * - isomorphic (see Theorem 2.6. below).

The interest in this problem is motivated by the study of certain inductive limit C^* - algebras (see [7]).

Let $\underline{p} = (p_k)$ (resp. $\underline{q} = (q_k)$) be two strictly increasing sequences of positive integers, with p_k (resp. q_k) dividing p_{k+1} (resp. q_{k+1}) for all $k \geq 1$. We consider:

$$C(T^2) \otimes M_{p_1 q_1} \xrightarrow{\Lambda_1} C(T^2) \otimes M_{p_2 q_2} \xrightarrow{\Lambda_2} \cdots$$

where each Λ_k is an isometric * - homomorphism such that $\Lambda_k(f \circ \varphi_k \otimes 1_{p_k q_k}) = f \otimes 1_{p_{k+1} q_{k+1}}$, $f \in C(T^2)$ (where $T^2 \ni (u,v) \overset{\varphi_k}{\mapsto} (u^{p_{k+1}/p_k}, v^{q_{k+1}/q_k}) \in T^2$).

Denote by $A(\underline{p},\underline{q},(\Lambda_k))$ the corresponding inductive limit. In [7] it was shown that $A(\underline{p},\underline{q},(\Lambda_k))$ is * - isomorphic to the (spatial) C^* - tensor product $A(\underline{p}) \otimes A(\underline{q})$. So that, the theorems proved in this paper can be used to obtain necessary and sufficient conditions for C^* - algebras of the type $A(\underline{p},\underline{q},(\Lambda_k))$ to be * - isomorphic or stably isomorphic.

§ 1.

Let $\underline{p} = (p_n)$ be a strictly increasing sequence of strictly positive integers, with p_n dividing p_{n+1} for all $n \geq 1$. Throughout this paper we shall denote by underlined characters such kind of sequences (e.g. $\underline{q}, \underline{r}, \underline{p_1}, \underline{p_2}, \ldots$)

Given $\underline{p} = (p_n)$, we shall introduce the notation :

$G(\underline{p}) = \{ m/p_n \mid m \in \mathbb{Z}, n = 1,2, \ldots \}$

a subgroup of the rationals.

For $p = (p_n), q = (q_n)$ we shall write p/q if for every k there is a l such that p_k divides q_l.

Whenever p/q and q/p we shall use the notation:

$p \sim q$.

Let P denote the set of prime integers. We shall note by $f(p)$ the generalized natural number canonicaly associated with p, i.e.:

$$P \ni x \overset{f(p)}{\longmapsto} \sup\{ n / (\exists) i \text{ such that } x^n \text{ divides } p_i \} \in \{0,1,2,\dots\} \cup \{+\infty\}$$

It is easily seen that:

$$f(p \cdot q) = f(p) + f(q);\ f(p) \leq f(q) \Leftrightarrow p/q$$

In particular:

$$p \sim q \Leftrightarrow f(p) = f(q)$$

We shall write:

$$f(p) \prec f(q)$$

if there are strictly positive integers a and b such that $f(a.p) \leq f(b.q)$.

If $f(p) \prec f(q)$ and $f(q) \prec f(p)$ we shall use the notation:

$f(p) \sim f(q)$

Every strictly positive integer m will be also considered as a map $\tilde{m}: P \to \{0,1,2,\dots\} \cup \{+\infty\}$, defined in the obvious way. Note that $f(p) + \tilde{m} = f(m \cdot p)$.

If A and B are C^* - algebras, we shall denote $A \otimes B = A \otimes_{min} B$.

For a unital C^* - algebra A which has an unique trace state τ, we shall use the notation:

$$R(A) = \{ \tau(p) / p = p^* = p^2 \in A \}.$$

We shall denote by K the C^* - algebra of compact operators on a complex separable infinite - dimensional Hilbert space.

The following theorem will be used in the sequel with no reference:

Theorem ([4], [5])

$K_0(A(p)) \simeq G(p)$

$K_1(A(p)) \simeq Z$

§2.

2.1. Proposition

$K_0(A(p) \otimes A(q)) \simeq G(p \cdot q) \oplus Z$

$K_1(A(p) \otimes A(q)) \simeq G(p) \oplus G(q)$.

Proof:

Since $K_0(A(r)) \simeq G(r)$ and $K_1(A(r)) \simeq Z$, $K_*(A(r))$ is torsion free.

On the other hand, it is easily seen that $A(\underline{p})$ ($\underline{p} = (p_n)$) is * - isomorphic with the inductive limit of a system:

$$C(\mathbb{T}) \otimes M_{p_1} \xrightarrow{\phi_1} C(\mathbb{T}) \otimes M_{p_2} \xrightarrow{\phi_2} \cdots$$

where each ϕ_k is a certain unital isometric * - homomorphism (see [2], proof of Theorem 2.)

So that, using these facts, by Proposition 2.4. and Proposition 2.11. from [8], it follows that the Künneth formula holds for $A(\underline{p}) \otimes A(\underline{q})$:

$$K_0(A(\underline{p}) \otimes A(\underline{q})) \simeq$$

$$\simeq (K_0(A(\underline{p})) \otimes K_0(A(\underline{q}))) \oplus (K_1(A(\underline{p})) \otimes K_1(A(\underline{q}))) \simeq G(\underline{p} \cdot \underline{q}) \oplus Z$$

$$K_1(A(\underline{p}) \otimes A(\underline{q})) \simeq$$

$$\simeq (K_0(A(\underline{p})) \otimes K_1(A(\underline{q}))) \oplus (K_1(A(\underline{p})) \otimes K_0(A(\underline{q}))) \simeq G(\underline{p}) \oplus G(\underline{q}),$$ Q.E.D.

2.2. Corollary

$A(\underline{p}) \otimes A(\underline{q})$ is not homotopy equivalent with a W^* - algebra or with an A.F. - algebra.

Proof:

By Proposition 2.1., $K_1(A(\underline{p}) \otimes A(\underline{q}))$ is not trivial.

2.3. Proposition

$A(\underline{p}) \otimes A(\underline{q})$ has an unique trace state and:

$R(A(\underline{p}) \otimes A(\underline{q})) = G(\underline{p} \cdot \underline{q}) \cap [0,1]$

Proof:

Since every Bunce - Deddens algebra has an uniquetrace state (see [1]), by a joint result of J.Cuntz and G.K.Pedersen (see [3], Corollary 6.13) it follows that the same is true for $A(\underline{p}) \otimes A(\underline{q})$.

We denote $p_{k+1} q_{k+1} / p_k q_k = s_{k+1}$ for $k \geq 1$ and $p_1 q_1 = s_1$. Using a result of P.G. Ghatage and W.J.Phillips (see [5], Lemma 2.3.) one obtains that there is an imbedding of $A(\underline{p}) \otimes A(\underline{q})$ into the U.H.F. - algebra $\bigotimes_{k=1}^{\infty} M_{s_k}$. It turns out that:

$$R(A(\underline{p}) \otimes A(\underline{q})) \subset R\left(\bigotimes_{k=1}^{\infty} M_{s_k}\right) = G(\underline{p} \cdot \underline{q}) \cap [0,1].$$

Since the reverse inequality is obvious (for each $\underline{r}$ there is an imbedding of a U.H.F.-algebra of type $\underline{r}$ into $A(\underline{r})$ (see [2])), the proof is complete.

2.4. Remark.

Let us observe that:

$A(\underline{p}) \otimes K \simeq A(\underline{q}) \otimes K$ iff $f(\underline{p}) \sim f(\underline{q})$.

First of all, since $K_0(A(\underline{p})) \simeq G(\underline{p})$, $K_0(A(\underline{q})) \simeq G(\underline{q})$, we remark that $K_0(A(\underline{p})) \simeq K_0(A(\underline{q}))$ iff $f(\underline{p}) \sim f(\underline{q})$."

Suppose $A(p)\otimes K \simeq A(q)\otimes K$. It follows that $K_0(A(p)) \simeq K_0(A(q))$ and, by the above remark, $f(p) \sim f(q)$.

If $f(p) \sim f(q)$, then there are strictly positive integers m,n such that:

$m \cdot p \sim n \cdot q.$

which is equivalent with:

$A(m \cdot p) \simeq A(n \cdot q)$

Using ([7], Remark 3.3.), it easily seen that:

$A(m \cdot p) \simeq A(p)\otimes M_m$, $A(n \cdot q) \simeq A(q)\otimes M_n$

We deduce that:

$A(p)\otimes M_m \simeq A(q)\otimes M_n$

which implies that:

$A(p)\otimes K \simeq A(q)\otimes K.$

2.5. Theorem

We consider $A_i = A(p_i)$, $B_i = A(q_i)$ $(i = 1,2)$. Then, the following are equivalent:

(i) $A_1\otimes A_2\otimes K \simeq B_1\otimes B_2\otimes K$

(ii) $K_1(A_1\otimes A_2) \simeq K_1(A_1\otimes A_2)$

(iii) there are a permutation σ of $\{1,2\}$ and strictly positive integers m_i, $n_i (i = 1,2)$ such that:

$m_i \cdot p_i \sim n_i \cdot q_{\sigma(i)}$ $(i = 1,2)$.

(iv) there is a permutation σ of $\{1,2\}$ such that:

$A_i\otimes K \simeq B_{\sigma(i)}\otimes K$ $\qquad (i = 1,2)$

Proof:

(ii)$\Rightarrow$(iii). Using Proposition 2.1. it follows that there is a group isomorphism $\emptyset = (a_{ij})^2_{i,j=1}: G(p_1)\oplus G(p_2) \to G(q_1)\oplus G(q_2)$, where the a_{ij}'s belong to Q. We consider $\emptyset^{-1} = (b_{ij})^2_{i,i=1}$, where the b_{ij}'s belong to Q. It is easily seen that there exists a permutation σ of $\{1,2\}$ such that:

$a_{\sigma(i)i}, b_{i\sigma(i)} \neq 0$ $\qquad (i = 1,2)$

It follows that:

$f(p_i) \sim f(q_{\sigma(i)})$ $\qquad (i = 1,2)$

or, equivalently, there exist strictly positive integers m_i, $n_i (i = 1,2)$ such that:

$m_i \cdot p_i \sim n_i \cdot q_{\sigma(i)}$ $\qquad (i = 1,2)$

(iii)$\Rightarrow$(iv) is a consequence of Remark 2.4.

Since the implications (i)$\Rightarrow$(ii) and (iv)$\Rightarrow$(i) are obvious, the proof of Theorem 2.5. is complete.

2.6. Theorem

The following are equivalent:

(i) $A(p_1)\otimes A(p_2) \simeq A(q_1)\otimes A(q_2)$

(ii) $\underline{p}_1 \cdot \underline{p}_2 \sim \underline{q}_1 \cdot \underline{q}_2$

and there are a permutation σ of $\{1,2\}$ and strictly positive integers m_i, n_i $(i = 1,2)$ such that:

$$m_i \cdot \underline{p}_i \sim n_i \underline{q}_{\sigma(i)} \qquad (i = 1,2)$$

Proof:

(i)$\Rightarrow$(ii) follows from Proposition 2.3. and Theorem 2.5.

(ii)$\Rightarrow$(i). Suppose (ii) is true. Then, there are F_i $(i = 1,2)$, finite subsets of P, such that:

$$f(\underline{p}_i)(x) = f(\underline{q}_{\sigma(i)})(x),\ x \in P \setminus F_i \ (i = 1,2)$$

$$f(\underline{p}_i)(x),\ f(\underline{q}_{\sigma(i)})(x) < +\infty,\ x \in F_i \ (i = 1,2)$$

Let a_i, b_i $(i = 1,2)$ be strictly positive integers such that:

$$\tilde{a}_1(x) - \tilde{b}_1(x) = \begin{cases} f(\underline{p}_1)(x) - f(\underline{q}_{\sigma(1)})(x), x \in F_1 \\ - f(\underline{p}_2)(x) + f(\underline{q}_{\sigma(2)})(x),\ x \in F_2 \setminus F_1 \\ 0,\ x \in P \setminus (F_1 \cup F_2) \end{cases}$$

$$\tilde{a}_2(x) - \tilde{b}_2(x) = \begin{cases} f(\underline{p}_2)(x) - f(\underline{q}_{\sigma(2)})(x), x \in F_2 \\ - f(\underline{p}_1)(x) + f(\underline{q}_{\sigma(1)})(x),\ x \in F_1 \setminus F_2 \\ 0,\ x \in P \setminus (F_1 \cup F_2) \end{cases}$$

It is easily seen that:

$$f(\underline{p}_i) = f(\underline{q}_{\sigma(i)}) + \tilde{a}_i - \tilde{b}_i \ (i = 1,2)$$

and:

$$a_1 \cdot a_2 = b_1 \cdot b_2.$$

Replacing each $\underline{q}_i$ with a subsequence, we may suppose that:

$$\underline{p}_i \sim r_i \cdot \underline{q}_{\sigma(i)} \qquad (i = 1,2)$$

where $r_i := a_i b_i^{-1}$ $(i = 1,2)$, which implies that:

$$A(\underline{p}_1) \otimes A(\underline{p}_2) \simeq A(r_1 \cdot \underline{q}_{\sigma(1)}) \otimes A(r_1^{-1} \cdot \underline{q}_{\sigma(2)})$$

But, by the Theorem proved in [7], one can easily obtain:

$$A(r_1 \cdot \underline{q}_{\sigma(1)}) \otimes A((r_1^{-1} \cdot \underline{q}_{\sigma(2)}) \simeq A(\underline{q}_1) \otimes A(\underline{q}_2)$$

In conclusion:

$$A(\underline{p}_1) \otimes A(\underline{p}_2) \simeq A(\underline{q}_1) \otimes A(\underline{q}_2) .$$

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