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0. INTRODUCTION

In this paper we continue the analysis concerning third order and third grade fluids which has been started in [1] by some considerations on thermodynamics restrictions and on the existence of the free energy. This problem has been already considered by Fosdick and Rajagopal [2] but in their paper they have exmployed a particular hypothesis which finally leads to some different conclusions. This hypothesis relative to the existence of an absolute minimum for the free energy on equilibrium states plays, tacitely, the role of a stability criterion in [2] . In our considerations we shall employ a different and more general stability criterion.

1. STABILITY CRITERION AND CORRESPONDING CONSTITUTIVE RESTRICTIONS

The previous mentioned criterion (see for example I.Müller [3]) is:

$$\frac{d}{dt}\int_{\mathcal{P}} \rho(\overline{\Psi} + \frac{1}{2}v^2)\,dx \leq 0 \tag{1.1}$$

for all $\mathcal{P} \subset \Omega$, where Ω is the fluid domain and $\overline{\Psi}$ is the response function of the free energy.

Generally the notations are the same as in [1]. Employing the balance equation of linear momentum and restricting the analysis to the case of isothermic processes we see that (1.1) is equivalent to the Clausius-Duhem inequality written on homogeneous and isothermic processes. That is, we can write this condition in the local form:

$$\rho\overline{\Psi} - \frac{1}{2}T \cdot A_1 \leq 0 \tag{1.2}$$

Now we shall consider the particular case of the relation (4.18) from [1] which gives the free energy for a third grade fluid, namely when the function g from (4.20) is identically nul:

$$\rho\overline{\Psi}(A_1, A_2, \theta) = \frac{1}{2}\beta_1(\theta)A_1\cdot A_2 - \frac{1}{3}\beta_3(\theta)A_1\cdot A_1^2 + \tag{1.3}$$

$$+ \frac{1}{4}\alpha_1(\theta)A_1\cdot A_1 + \rho\overline{\overline{\Psi}}_o(0;\theta)$$

where as we have denoted in [1], $\overline{\overline{\Psi}}_o : R_+ \times R_+ \longrightarrow R$ is the free energy on viscometric motions.

An elementary calculus shows us that if (1.2) is true then:

$$\frac{1}{2}\beta_1 A_1\cdot A_2 - (\beta_1 + \beta_2 + \beta_3)A_1^2\cdot A_2 - \frac{1}{2}\mu A_1\cdot A_1 \leq 0 \tag{1.4}$$

on isothermic processes and where we have emploied the relation (3.1) and the constitutive restriction $(3.19)_4$ from [1].

The inequality (1.4) must be valid for all $A_1, A_2 \in SLin(\mathcal{V}, \mathcal{V})$. So by a classical way, denoting $A_1 \equiv sY$, $A_2 \equiv lX$, with $s, l \in R$ and $Y, X \in SLin(\mathcal{V}, \mathcal{V})$ and employing the constitutive restrictions (3.19) from [1] we arrive to the following:

THEOREM 1.1. In order that a third grade fluid obays the restriction (1.2) on isothermic processes it is necessary that:

$$\beta_1 + \beta_2 + \beta_3 = 0 \qquad (1.5)$$

REMARK 1.1. The restrictions (1.5) and $(3.19)_{1,2}$ from [1] with the relation (1.3) are also sufficient conditions for the validity of Clausius-Duhem inequality (3.5) from [1] written on isothermic and homogeneous processes

REMARK 1.2. From the above results we see that the stress response function for a third grade fluid is given with necessity by the following relation:

$$T(x,t) = -pI + \mu A_1 + \alpha_1 (A_2 - A_1^2) - \beta_2 [A_3 - (A_2 A_1 + A_1 A_2)] - \beta_3 [A_3 - (\mathrm{tr} A_1^2) A_1] \qquad (1.6)$$

with $\mu \geq 0$ and $\beta_2 + \beta_3 > 0$

2. ASSYMPTOTIC STABILITY FOR A THIRD GRADE FLUID

The fact that the nul flow must be asymptotically stable seems to be a necessity for the mechanical behaviour of a fluid body. In the following of this section we try to give the conditions in which a third grade fluid, as it has been described only from the constitutive point of view, realizes this behaviour. As we have mentioned in the introduction the subject first has been discussed for third grade fluids in the paper [2] and for second grade fluids in the paper of Dunn and Fosdick [4]. The mathematical technics which we employ is (as in the above cited papers) a classical one.

We consider a domain Ω with fixed boundary occupied by the fluid, the body being mechanically isolated andinsulated which implies that the balance equation of linear momentum multiplied by the velocity v and integrated over Ω can be written in the following manner:

$$\frac{1}{2}\int_\Omega \rho\, \dot{v}\cdot v dV + \int_\Omega T\cdot L dV = 0 \tag{2.1}$$

Employing the constitutive relation (1.6), some lengthy but straightforward computation gives us, from (2.1):

$$\frac{d}{dt}\left\{\rho\int_\Omega |v|^2 + \frac{\alpha_1}{2}\int_\Omega |A_1|^2 - \frac{\beta_2+\beta_3}{2}\frac{d}{dt}\int_\Omega |A_1|^2 - \frac{\beta_2+3\beta_3}{3}\int_\Omega tr A_1^3\right\} + \tag{2.2}$$
$$+\mu\int_\Omega |A_1|^2 + \frac{\beta_2+\beta_3}{2}\int_\Omega |A_1|^4 + (\beta_2+\beta_3)\int_\Omega |\dot{A}_1 - (WA_1 - A_1W)|^2 = 0$$

where we have used the incompressibility condition and the adherence of the fluid to the boundary.

We shall denote

$$(2.3) \qquad E(t) \equiv \int_\Omega |v|^2 + \frac{\alpha_1}{2\rho}\int_\Omega |A_1|^2 - \frac{\beta_2+\beta_3}{2\rho}\frac{d}{dt}\int_\Omega |A_1|^2 - \frac{\beta_2+3\beta_3}{3\rho}\int_\Omega \operatorname{tr} A_1^3$$

and we want to think the function E as some energetic measure of the fluid. We shall see at the end of this section the conditions in which this function can be thought as above. For the moment we observe that due to the constitutive restrictions $\mu \geq 0$ and $\beta_2+\beta_3 > 0$ we have:

$$(2.4) \qquad \dot{E}(t) \leq 0$$

for all $t \in (0,\infty)$. It follows that E(t) is a decreasing function and that $E(t) \leq E(0)$ for all $t \in (0,\infty)$.

If we suppose that the initial perturbation has been created with a viscometric motion and that the motion in continuation remains viscometric, than $\operatorname{tr}A_1^3=0$ and $A_2 \cdot A_1=0$ and so

$$(2.5) \qquad E_v(t) = \int_\Omega |v|^2 + \frac{\alpha_1}{2\rho}\int_\Omega |A_1|^2$$

which is an energetic measure (and is similar with those obtained in [2]). Therefore:

$$(2.6) \qquad \dot{E}_v(t) \leq 0$$

for all $t \in [0,\infty)$.

We consider a viscometric motion in the presence of a conservative body forces field in a fixed rigid domain Ω $(v|_{\partial\Omega}=0)$, and we want to investigate if there exists $\xi \in R_+$ so that

$$\dot{E}_V(t) + \xi E_V(t) \leq 0$$

For this we remark first that for all $\xi \in R_+$

(2.7) $$\dot{E}_V(t)+\xi E_V(t)=\frac{1}{\varrho}\left\{-(\beta_2+\beta_3)\int_\Omega |\dot{A}_1-(wA_1-A_2w)|^2 - \right.$$

$$\left. -\frac{\beta_2+\beta_3}{2}\int_\Omega |A_1|^4-\mu\int_\Omega |A_1|^2+\xi\int_\Omega |v|^2+\frac{\xi\alpha_1}{2\varrho}\int_\Omega |A_1|^2\right\}$$

Since $v\Big|_{\partial\Omega}=0$ there result (Friedrics inequality) that there is a constant $c \in R_+$ such that

$$\int_\Omega |v|^2 \leq c\int_\Omega |A_1|^2$$

and from (2.7) we have:

(2.8) $$\dot{E}_V(t)+\xi E_V(t)\leq -\frac{1}{\varrho}\left\{\left[\mu-\frac{\xi}{2}(\alpha_1+\varrho c)\right]\int_\Omega |A_1|^2+\right.$$

$$\left.+(\beta_2+\beta_3)\int_\Omega |\dot{A}_1-(WA_1-A_1W)|^2+\frac{\beta_2+\beta_3}{2}\int_\Omega |A_1|^4\right\}$$

Taking into account the constitutive restrictions it results that the necessary and sufficient condition for the right hand side of the inequality (2.8) be nonpositive is:

(2.9) $$\xi(\alpha_1+\varrho c)\leq 2\mu .$$

Therefore we have proved the following theorem:

<u>THEOREM 2.1</u>. Let the cannister viscometric flow of a third grade fluid be mechanically isolated for all $t \geq 0$. Then there exists $\xi \in R_+$ such that:

(2.10) $$\dot{E}_V(t) + \xi E_V(t) \leq 0 ,$$

where ξ is determined by (2.9) ■

We shall prove that this theorem tells us that the cannister viscometric flows of a third grade fluid, with $\alpha_1 > 0$ (which is an usual hypothesis based on the experimental data given for example in [5 , 6] are asymptotically at rest (that is the null flow is asymptotically stable). For this we observe that (2.10) implies:

(2.11) $$E_V(t) \leq E_V(0)e^{-\xi t}$$

and if $\alpha_1 > 0$, the deffinition (2.5) gives both

(2.12) $$0 \leq \int_\Omega |v|^2 dV \leq E_V(0)e^{-\xi t}$$

and

(2.13) $$0 \leq \int_\Omega |A_1|^2 dV \leq \frac{2\rho}{\alpha_1} E_V(0)e^{-\xi t}$$

As in [2] we can obtain, by means of the same known technique, the lower bound of the positive definite function E(t). For this we consider a positive function $\lambda(t)$ and we evaluate:

(2.14) $$\dot{E}_V(t) + \lambda(t)E_V(t) = -\frac{1}{\rho}\left[\mu - \frac{\lambda(t)\alpha_1}{2}\right]\int_\Omega |A_1|^2 + \lambda(t)\int_\Omega |v|^2 *$$

$$-\frac{\beta_2+\beta_3}{\rho}\int_\Omega |\dot{A}_1 - (WA_1 - A_1W)|^2 - \frac{\beta_2+\beta_3}{2\rho}\int_\Omega |A_1|^4$$

On the other hand we observe that on viscometric flows

$$|\dot{A}_1-(WA_1-A_1W)|^2 \equiv \frac{1}{2}|A_1|^4$$

and so we have

$$\text{(2.15)} \qquad \dot{E}_V(t)+\lambda(t)E_V(t) \geq -\frac{2\mu-\lambda(t)\alpha_1}{2\rho}\int_\Omega |A_1|^2$$

Now, we choose

$$\text{(2.16)} \qquad \lambda(t) \equiv \frac{2}{\alpha_1}(2\mu+\nu(t)) \ ,$$

where $\nu : R_+ \longrightarrow R_+$ is defined by

$$\text{(2.17)} \qquad \int_\Omega |A_1|^4 = |A_1(x^*,t)|^2 \int_\Omega |A_1|^2 \ ,$$

$$\nu(t) \equiv \nu_o |A_1(x^*,t)|$$

x^* which appears in (2.17) is a point in Ω whose existence is guaranted by the virtue of the mean value theorem for integrals, and $\nu_o \in R_+$ is such that $\nu_o-(\beta_2+\beta_3) \geq 0$.

Now we are ready to state the following:

<u>THEOREM 2.2</u>. Let the cannister viscometric flow of a third grade fluid with $\alpha_1 > 0$ be mechanically isolated for all $t \geq 0$. Then

$$\text{(2.18)} \qquad E_V(t) \neq 0$$

for all t and there exists $\lambda(t) \in R_+$ such that:

$$(2.19)\qquad \overset{\circ}{E}_v(t) + \lambda(t) E_v(t) \geq 0 .$$

Proof. With the preceding remarks the proof is immediate because (2.19) results from (2.15), (2.16), (2.17) and the choise for ν_o. From (2.19) by integration we have

$$(2.20)\qquad E_v(t) \geq E_v(0) e^{-\int_0^t \lambda(s)ds}$$

If we observe that from $(2.17)_2$ and (2.13) we have

$$\nu(t) \leq \frac{2\rho\nu_o}{\alpha_1} E_v(0) e^{-\xi_1 t}$$

a simple calculus led us to the following conclusion

$$(2.21)\qquad E_v(t) \geq E_v(0) e^{-\frac{4\rho\nu_o}{\alpha_1^2\xi_1}E_v(0)} e^{-\frac{4\mu}{\alpha_1}t}$$

for all $t \in (0,\infty)$, which in particular, gives (2.18) ∎

REMARK 2.1. Using the Friedrics inequality we have

$$(2.22)\qquad E_v(t) \leq \left(c + \frac{\alpha_1}{2\rho}\right) \int_\Omega |A_1|^2$$

and so from (2.22) and the Theorem 2.2 we can obtain the following appriori bounds for v and A_1:

$$(2.23)\qquad \begin{cases} \dfrac{2\rho}{\alpha_1 + 2\rho c} E_v(0)\, e^{-\frac{4\rho\nu_o}{\alpha_1^2\xi_1}E_v(0)} e^{-\frac{4\mu}{\alpha_1}t} \leq \displaystyle\int_\Omega |A_1|^2 \leq \frac{2\rho}{\alpha_1} E_v(0)\, e^{-\xi_1 t} \\[2ex] 0 \leq \displaystyle\int_\Omega |v|^2 \leq E_v(0)\, e^{-\xi_1 t} \end{cases}$$

These results allow us to conclude that initial disturbances cannot disapear at any finite instant of time in such flows. ■

Now we start the analysis of the general case of interest, which means that we shall consider the general expression (2.3) for E(t) and for $\xi > 0$ we shall evaluate:

$$\dot{E}(t) + \xi E(t)$$

Without the use of the restriction (1.5) but taking into account the constitutive restrictions (3.19) from [1] after some straightforward computation we obtain:

$$\dot{E}(t) + \xi E(t) = \frac{1}{\rho}\left\{\beta_1 \int_\Omega \mathrm{tr}\left[\dot{A}_1 + A_1 W - W A_1\right]^2 - \frac{\beta_1 + 2\beta_2 + 2\beta_3}{2}\int_\Omega |A_1|^4 - \mu \int_\Omega |A_1|^2\right\}$$
$$+ \xi \int_\Omega |v|^2 + \frac{\xi \alpha_1}{2\rho}\int_\Omega |A_1|^2 + \frac{\xi\beta_1}{2\rho}\frac{d}{dt}\int_\Omega |A_1|^2 + \xi\,\frac{3\beta_1 + 2\beta_2}{3\rho}\int_\Omega \mathrm{tr} A_1^3 \leq \tag{2.24}$$
$$\leq -\frac{1}{\rho}\left[\mu - \frac{\xi}{2}(\alpha_1 + \rho C_0)\right]\int_\Omega |A_1|^2 + \frac{\xi\beta_1}{2\rho}\frac{d}{dt}\int_\Omega |A_1|^2 + \xi\,\frac{3\beta_1 + 2\beta_2}{3\rho}\int_\Omega \mathrm{tr} A_1^3$$

and where we have also used the Friedrics inequality. We shall rearrange the terms in (2.24) and we shall introduce the new function

$$F(t) \equiv E(t) - \frac{\xi\beta_1}{2\rho}\int_\Omega |A_1|^2 \tag{2.25}$$

in terms of which (2.24) becomes:

$$\dot{F}(t)+\xi F(t) \leq -\frac{1}{2\rho}\left[2\mu-\xi(\alpha_1+\rho C)+\xi^2\beta_1\right]\int_\Omega |A_1|^2 + \tag{2.26}$$

$$+\xi\,\frac{3\beta_1+2\beta_2}{3\rho}\int_\Omega \mathrm{tr}A_1^3$$

If we restrict the analysis to the class of third grade fluids for which $3\beta_1+2\beta_2=0$ (which is a sufficiently general one) we see that a necessary and sufficient condition for the existance of an upper nul bound for the first term in (2.26) is:

$$\xi \in \left(0,\ \frac{\alpha_1+\rho C-\sqrt{(\alpha_1+\rho C)^2-8\mu\beta_1}}{4\beta_1}\right) \tag{2.27}$$

With this choice for ξ the following inequality holds:

$$\dot{F}(t)+\xi F(t) \leq 0 \tag{2.28}$$

and so, it is an evidence that:

$$F(t) \leq F(0)e^{-\xi t} \tag{2.29}$$

and we have established the following:

THEOREM 2.3. Let the cannister flow of a third grade fluid be mechanically isolated for all $t \geq 0$. If the constitutive moduli β_1, β_2 satisfy the relation $3\beta_1+2\beta_2=0$, then there exists $\xi \in R_+$, satisfying (2.27) such that:

$$\dot{F}(t)+\xi F(t) \leq 0 \quad \blacksquare \tag{2.30}$$

On the other hand we observe that if $g(t) \equiv \int_\Omega |A_1(x,t)|^2 e^{-\xi t}dx$ is a decreasing function of t for sufficiently large t then the following Lemma is true:

LEMMA 2.1. If the third grade fluid under consideration is such that $3\beta_1+2\beta_2=0$, $\alpha_1>0$ and g is a decreasing function of t for large t then F is non-negative for sufficiently large t, and:

$$F(t) \geq \int_\Omega |v|^2 + \frac{\alpha_1}{2\rho}\int_\Omega |A_1|^2 \qquad \blacksquare \tag{2.31}$$

REMARK 2.2. The hypothesis concerning the monotonicity of the function g is quite general. For example if $\int_\Omega |A_1|^2 dx$ is a periodic function of t, this condition cannot be true. We shall consider the inequality (2.29) which can be explicitely written

$$\int_\Omega |v|^2 + \frac{\alpha_1}{2\rho}\int_\Omega |A_1|^2 + \frac{\beta_1}{2\rho} e^{\varepsilon t}\frac{d}{dt}\left(\int_\Omega |A_1|^2 e^{-\varepsilon t} dx\right) \leq F(0)\, e^{-\varepsilon t} \tag{2.32}$$

for all $t>0$. However $\int_\Omega |A_1|^2$ being a periodic function of t, it results that $\int_\Omega |A_1|^2 e^{-\varepsilon t}$ has a periodically decreasing maxima and periodically increasing minima. So if we denote by T the period and if we integrate (2.32) on a period we shall have:

$$\int_{t_0+kT}^{t_0+(k+1)T}\int_\Omega |v|^2 + \frac{\alpha_1}{2\rho}\int_{t_0+kT}^{t_0+(k+1)T}\int_\Omega |A_1|^2 + \frac{\beta_1}{2\rho} e^{\varepsilon t_k^*}\int_{t_0+kT}^{t_0+(k+1)T}\frac{d}{dt}\left(\int_\Omega |A_1|^2 e^{-\varepsilon t} dx\right)dt \leq \tag{2.33}$$

$$\leq -\frac{1}{\varepsilon}F(0)\left\{e^{-\varepsilon[t_0+(k+1)T]} - e^{-\varepsilon[t_0+kT]}\right\}$$

where for the third term on the left we have used a mean value theorem for integrals, $t_k^* \in (t_o+kT, t_o+(k+1)T)$. Now if we observe the third term of the left part of (2.33) we shall see that

$$\int_{t_o+kT}^{t_o+(k+1)T} \frac{d}{dt}\left(\int_\Omega |A_1(x,t)|^2 e^{-\xi t} dx\right) dt = \int_\Omega |A_1(x,t_o+(k+1)T) e^{-\xi_1(t_o+(k+1)T)} dx -$$

$$-\int_\Omega |A_1(x,t_o+kT) e^{-\xi_1(t_o+kT)} dx < 0$$

due to the fact that the maxima of the function $\int_\Omega |A_1(x,t)|^2 e^{-\xi t} dx$ are decreasing values of t. So (2.33) implies with necessity that the following inequality must hold:

$$(2.34) \qquad \frac{\alpha_1}{2\rho} \int_{t_o+kT}^{t_o+(k+1)T} \int_\Omega |A_1(x,t)|^2 dx\, dt \leq \frac{1}{\xi_1} F(0) \left\{ e^{-\xi_1 t_o} - e^{-\xi_1(t_o+T)} \right\} e^{-\xi_1 kT}$$

for all $k>0$. Then taking into account that the intervals of integration are of equal length and that for $k \to \infty$ the left hand side converges to zero the following relation must holds:

$$(2.35) \qquad \lim_{t\to\infty} \int_\Omega |A_1(x,t)|^2 dx = 0$$

which is not true due to the fact that the function $t \longmapsto \int_\Omega |A_1(x,t)|^2 dx$ have been supposed to be periodic.

Then it results that no globally periodically motion can appear in a third grade fluid under above mentioned conditions (which means essentially that we have adherence condition on the boundary for the velocity).

Finally we see that in the conditions of the Lemma 2.1 we have the following:

COROLLARY 2.1. The cannister flows of a third grade fluid with $\alpha_1 > 0$ and $3\beta_1+2\beta_2=0$ are asymptotically at rest.

Proof. The proof is immediate because of the relation (2.29) which implies, if $\alpha_1>0$, $3\beta_1+2\beta_2=0$, the deffinitions (2.3), (2.25) and the relation (2.31), that

$$0 \leq \int_\Omega |v(x,t)|^2 dx \leq F(0)e^{-\xi t} \tag{2.36}$$

$$0 \leq \int_\Omega |A_1(x,t)|^2 dx \leq \frac{2\varrho}{\alpha_1} F(0)e^{-\xi t} \tag{2.37}$$

$$\frac{2\varrho}{\beta_1} F(0)e^{-2\xi t} \leq \frac{d}{dt}\int_\Omega |A_1(x,t)|^2 e^{-\xi t} dx \leq 0 \tag{2.38}$$

for t sufficiently large.∎

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