ON THE STRUCTURE OF THE NAIMARK DILATION II

by

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I INTRODUCTION

In this paper we are concerned with a discrete time stationary process, namely a sequence of operators $\mathcal{V}=\{V_n\}_{n=-\infty}^{\infty}$, $V_n \in \mathcal{L}(\mathcal{H},\mathcal{K})$ where $\mathcal{H}$ and $\mathcal{K}$ are complex Hilbert spaces. Together with this process, a unique semispectral measure F on the unit circle is associated and let W be the unitary operator (the Naimark dilation) determining the minimal spectral dilation of F.

The operator W can be also viewed as the evolution operator of a discrete time dinamycal system (generally of infinite order).

When the obtained system is of first order, a transfer function is associated to it and this function characterizes the given system. These aspects constitute the content of the Sz.-Nagy-Foias theory (for infinite-state systems).

The aim of this paper is to notice several similar aspects in the general case of infinite order systems; as the main technical point we note the connection between the representation (1.14) of the Naimark dilation and the dual representation (1.23). Then we will minutely exploit the energy conservation law in the context of the Schur analysis of the process.

We will eventualy continue this paper with several aspects regarding the details in the third section and several kinds of "inverse problems".

II PRELIMINARIES

In this section we recall the way W was described in [9] and then we will get thorougly into the considerations made in the introduction.

Let $\mathcal{H}$ and $\mathcal{K}'$ be two Hilbert spaces and $T \in \mathcal{L}(\mathcal{K},\mathcal{K}')$ a contraction ($\|T\| \leq 1$). As usually, $D_T=(I-T^*T)^{\frac{1}{2}}$ and $\mathcal{D}_T=\overline{D_T\mathcal{H}}$ are, respectively, the defect operator and the defect space of T. Moreover, with T we associate the unitary operator:

$$(1.1)\quad \begin{cases} J(T): \mathcal{H} \oplus \mathcal{D}_{T^*} \longrightarrow \mathcal{K}' \oplus \mathcal{D}_T \\ J(T)= \begin{pmatrix} T & , D_{T^*} \\ D_T & , -T^* \end{pmatrix} . \end{cases}$$

As a generalization of a contraction on a Hilbert space $\mathcal{H}$, we will consider in this paper a semispectral measure F on the unit circle $\mathbb{T}$, i.e. a linear positive map $F:C(\mathbb{T}) \longrightarrow \mathcal{L}(\mathcal{H})$, where $C(\mathbb{T})$ means the set of the continuous functions on $\mathbb{T}$; we will suppose $F(1)=I$. The Fourier coefficients of F are $S_n=F(\chi_n)$, where $\chi_n(e^{it})=e^{int}$. In [8] a one-to-one correspondence between the set of the semispectral measures on $\mathbb{T}$ with $F(1)=I$ and the set of the sequences of contractions $\mathcal{G}=\{\Gamma_n\}_{n=1}^{\infty}$, $\Gamma_1 \in \mathcal{L}(\mathcal{H})$, $\Gamma_n \in \mathcal{L}(\mathcal{D}_{\Gamma_{n-1}}, \mathcal{D}_{\Gamma_{n-1}^*})$ is established. Such kind of parameter appears in the study of some classical extrapolation problems and, in general form (operators in Hilbert spaces) it appears under the name of choice sequence in [4] .

In order to point out the above mentioned connection we need more notation. First, for simplifying the writing of some formulas we take $\Gamma_0:\mathcal{H} \rightarrow \mathcal{H}$, $\Gamma_0=0$, the zero operator, so $D_{\Gamma_0}=I_{\mathcal{H}}=D_{\Gamma_0^*}$, where $I_{\mathcal{H}}$ is the identity on the corresponding space. We define the spaces:

$$\mathcal{K}_n^{(p)}=\bigoplus_{m=p-1}^{n-1} \mathcal{D}_{\Gamma_m} \quad , p \geqslant 1,\ n \geqslant k$$

$$\mathcal{K}_+^{(p)}=\bigoplus_{m=p-1}^{\infty} \mathcal{D}_{\Gamma_m} \quad , p \geqslant 1$$

and the contractions (see Lemma 1.2 and Lemma 1.3 [9]):

$$(1.2)\quad \begin{cases} X_n^{(p)}: \mathcal{K}_n^{(p)} \longrightarrow \mathcal{H} \quad , p \geqslant 1,\ n \geqslant k \\ X_n^{(p)}=(\Gamma_p, D_{\Gamma_p^*}\Gamma_{p+1}, \ldots, D_{\Gamma_p^*}\ldots D_{\Gamma_{n-1}^*}\Gamma_n) . \end{cases}$$

$$(1.3)\quad \begin{cases} X_{\infty}^{(p)}:\mathcal{K}_{+}^{(p)}\longrightarrow \mathcal{H} & ,p\geq 1 \\ X_{\infty}^{(p)}=\text{s-}\lim_{n\to\infty} X_n^{(p)}P_n^{(p)} & \end{cases}$$

where $P_n^{(p)}$ is the orthogonal projection of $\mathcal{K}_{+}^{(p)}$ onto $\mathcal{K}_n^{(p)}$

and let us remark that in order not to complicate the formulas, $\mathcal{K}_n^{(p)}$ will be viewed as embedded in $\mathcal{K}_{+}^{(p)}$. In a similar way, we consider the spaces:

$$\widetilde{\mathcal{K}}_n^{(p)}=\bigoplus_{m=p-1}^{n-1}\mathcal{D}_{\Gamma_m^*} \qquad ,p\geq 1,n\geq k$$

$$\widetilde{\mathcal{K}}_{+}^{(p)}=\bigoplus_{m=p-1}^{\infty}\mathcal{D}_{\Gamma_m^*} \qquad ,p\geq 1$$

and the contractions:

$$(1.4)\quad \begin{array}{ll} Y_n^{(p)}:\mathcal{H}\longrightarrow \widetilde{\mathcal{K}}_n^{(p)} & ,p\geq 1,n\geq k \\ Y_n^{(p)}=(\Gamma_p,\Gamma_{p+1}D_{\Gamma_p},\dots,\Gamma_n D_{\Gamma_{n-1}}\dots D_{\Gamma_p})^t & \end{array}$$

("t" standing for the matrix transpose),

$$(1.5)\quad \begin{cases} Y_{\infty}^{(p)}:\mathcal{H}\longrightarrow \widetilde{\mathcal{K}}_{+}^{(p)} & ,p\geq 1 \\ Y_{\infty}^{(p)}=\text{s-}\lim_{n\to\infty} Y_n^{(p)} & \end{cases}$$

Then we consider the unitary operators used in [4] : $n\geq 1$, $1\leq k\leq n$, $k\geq p\geq 1$,

$$(1.6)\quad \begin{cases} J_{nk}^{(p)}:\mathcal{D}_{\Gamma_{p-1}}\oplus\dots\oplus\mathcal{D}_{\Gamma_{k-2}}\oplus(\mathcal{D}_{\Gamma_{k-1}}\oplus\mathcal{D}_{\Gamma_k^*})\oplus\mathcal{D}_{\Gamma_{k+1}}\oplus\dots\oplus\mathcal{D}_{\Gamma_n}\longrightarrow \\ \qquad\longrightarrow\mathcal{D}_{\Gamma_{p-1}}\oplus\dots\oplus\mathcal{D}_{\Gamma_{k-2}}\oplus(\mathcal{D}_{\Gamma_{k-1}^*}\oplus\mathcal{D}_{\Gamma_k})\oplus\mathcal{D}_{\Gamma_{k+1}}\oplus\dots\oplus\mathcal{D}_{\Gamma_n} \\ J_{nk}^{(p)}=I\oplus J(\Gamma_k)\oplus I. \end{cases}$$

Now , we define $V_o=I_{\mathcal{H}}$,

$$(1.7)\quad \begin{cases} V_n:\mathcal{H}\oplus\mathcal{D}_{\Gamma_1}\oplus\dots\oplus\mathcal{D}_{\Gamma_{n-1}}\oplus\mathcal{D}_{\Gamma_n^*}\longrightarrow\mathcal{K}_{n+1}^{(1)} & ,n\geq 1 \\ V_n=J_{n1}^{(1)}\dots J_{nn}^{(1)} & \end{cases}$$

$U_o=I_{\mathcal{H}}$,

$$(1.8)\quad \begin{cases} U_n:\widetilde{\mathcal{K}}_{n+1}^{(1)}\longrightarrow\widetilde{\mathcal{K}}_{n+1}^{(1)} \\ U_n=V_n(U_{n-1}\oplus I) \end{cases}$$

and the algorithm establishing the connection between the Fourier coefficients of F and $\mathcal{G}$ is the following one (Theorem 1.2 in [8]):

$$(1.9)\qquad \begin{cases} S_1=\Gamma_1 \\ S_n=X_{n-1}U_{n-2}Y_{n-1}+D_{\Gamma_1^*}\dots D_{\Gamma_{n-1}^*}\Gamma_n D_{\Gamma_{n-1}}\dots D_{\Gamma_1} & ,n\geq 2 \end{cases} .$$

The next step is to describe the Naimark dilation of F in terms of the parameter $\mathcal{G}$ .For this reason, we first identify the defect spaces of the operators $X_\infty^{(p)}$ (Propositions 1.4 and 1.6 in [9]

We have the unitary operator :

$$(1.10)\qquad \begin{cases} \alpha_+^{(p)}: \mathcal{D}_{X_\infty^{(p)}} \longrightarrow \mathcal{K}_+^{(p+1)} \\ \alpha_+^{(p)} D_{X_\infty^{(p)}} = \begin{pmatrix} D_{\Gamma_p}, & -\Gamma_p^*\Gamma_{p+1}, & -\Gamma_p^* D_{\Gamma_{p+1}^*}\Gamma_{p+2}, & \dots \\ 0, & D_{\Gamma_{p+1}}, & -\Gamma_{p+1}^*\Gamma_{p+2}, & \dots \\ 0, & 0, & D_{\Gamma_{p+2}}, & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix} \end{cases}$$

In order to identify $\mathcal{D}_{X_\infty^{(p)*}}$ we define

$$(1.11)\qquad \begin{cases} G_n^{(p)}: \mathcal{H} \longrightarrow \mathcal{L} \\ G_n^{(p)} = D_{\Gamma_n^*}\dots D_{\Gamma_p^*} \end{cases}$$

and according to Lemma1.5 in [9] , let be

$$(1.12)\qquad H^{(p)}=\text{s-}\lim_{n\to\infty} G_n^{(p)}G_n^{(p)}$$

;then there exists the unitary operator:

$$(1.13)\qquad \begin{cases} \tilde{\alpha}_+^{(p)}: \mathcal{D}_{(X_\infty^{(p)})^*} \longrightarrow \overline{\operatorname{Ran} H^{(p)}} & \text{(Ran means the range)} \\ \tilde{\alpha}_+^{(p)} D_{(X_\infty^{(p)})^*} h = H^{(p)\frac{1}{2}}h & , n\in\mathcal{H} . \end{cases}$$

We define the spaces $\mathcal{D}_*^{(p)}=\overline{\operatorname{Ran}H^{(p)}}$ and

$$\mathcal{K}= \dots \oplus \mathcal{D}_*^{(1)} \oplus \mathcal{D}_*^{(1)} \oplus \mathcal{K}_+^{(1)}$$

and the operator

$$(1.14)\qquad \begin{cases} W: \mathcal{K}\left(=(\dots\oplus\mathcal{D}_*^{(1)})\oplus(\mathcal{D}_*^{(1)}\oplus\mathcal{K}_+^{(1)})\right) \longrightarrow \mathcal{K}\left(=(\dots\oplus\mathcal{D}_*^{(1)})\oplus\mathcal{K}_+^{(1)}\right) \\ W=I\oplus W_{red} \end{cases}$$

where $W_{red}: \mathcal{D}_*^{(1)}\oplus\mathcal{K}_+^{(1)} \longrightarrow \mathcal{K}_+^{(1)}$ is defined by

$$(1.15)\qquad W_{red}= \begin{pmatrix} I, & 0 \\ 0, & \alpha_+^{(1)} \end{pmatrix} J(X_\infty^{(1)}) \begin{pmatrix} 0, & I \\ \tilde{\alpha}_+^{(1)}, & 0 \end{pmatrix} .$$

According to Theorem 2.5 [9], W defined by (1.14) is the Naimark dilation of F. We can write W in a matricial form as follows:

$$(1.16)\qquad W=\begin{pmatrix} & \vdots & \vdots & \vdots & \vdots & \\ \ldots, & I, & 0, & 0, & 0, & \ldots \\ \ldots, & 0, & H^{(1)\frac{1}{2}}, & \boxed{\Gamma_1}, & D_{\Gamma_1^*}\Gamma_2, & \ldots \\ \ldots & 0, & -Z_1, & D_{\Gamma_1}, & -\Gamma_1^*\Gamma_2, & \ldots \\ \ldots & 0, & -Z_2, & 0, & D_{\Gamma_2}, & \ldots \\ & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

where

$$(1.17)\qquad \begin{cases} Z_n: \mathcal{D}_*^{(1)} \longrightarrow \mathcal{D}_{\Gamma_n} & ,n\geqslant 1 \\ Z_n=\Gamma_n^* D_{*n+1} & \end{cases}$$

and $D_{*1}=H^{(1)^{\frac{1}{2}}}$,

$$(1.18)\qquad \begin{cases} D_{*n}: \mathcal{D}_*^{(1)} \longrightarrow \mathcal{D}_{\Gamma_{n-1}^*} & ,n\geqslant 2 \\ D_{*n}=H^{(n)^{\frac{1}{2}}}C_{n-1}\ldots C_1 . & \end{cases}$$

Here, C_n are the unitary operators:

$$(1.19)\qquad \begin{cases} C_n: \mathcal{D}_*^{(n)} \longrightarrow \mathcal{D}_*^{(n+1)} & ,n\geqslant 1 \\ C_n(H^{(n)})^{\frac{1}{2}}=(H^{(n+1)})^{\frac{1}{2}}D_{\Gamma_n^*} & \end{cases}$$

and let us also define the unitary operators:

$$(1.20)\qquad \begin{cases} R_n: \mathcal{D}_*^{(1)} \longrightarrow \mathcal{D}_*^{(n)} \\ R_n=C_{n-1}\ldots C_1 . \end{cases}$$

A similar construction can be made starting with $Y_\infty^{(1)}$ instead of $X_\infty^{(1)}$. In this respect, we have:

$$(1.21)\qquad \begin{cases} \tilde{G}_n^{(p)}: \mathcal{H} \longrightarrow \mathcal{H} \\ \tilde{G}_n^{(p)}= D_{\Gamma_n}\ldots D_{\Gamma_p} \end{cases}$$

and

$$(1.22)\qquad \tilde{H}_p=\text{s-}\lim_{n\to\infty}\tilde{G}_n^{(p)}\tilde{G}_n^{(p)*}.$$

Then, $\mathcal{D}^{(1)}=\overline{\text{Ran}\tilde{H}_1}$, $D_1=\tilde{H}_1^{\frac{1}{2}}$ and

$$\tilde{\mathcal{K}} = \ldots \oplus \mathcal{D}^{(1)}\oplus \mathcal{D}^{(1)}\oplus \boxed{\mathcal{H}} \oplus \mathcal{D}_{\Gamma_1^*} \oplus \mathcal{D}_{\Gamma_2^*} \oplus \ldots$$

If we define the unitary operator:

$$(1.23) \qquad \tilde{W} = \begin{pmatrix} & \vdots & \vdots & \vdots & \vdots & \\ \cdots & I, & O & & & \\ & O & D_1, & -\tilde{Z}_1, & -\tilde{Z}_2, & \cdots \\ \cdots & O, & \boxed{\Gamma_1}, & D_{\Gamma_1^*}, & O, & \cdots \\ \cdots & O, & \Gamma_2 D_{\Gamma_1}, & -\Gamma_2\Gamma_1^*, & D_{\Gamma_2^*}, & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

where $\tilde{Z}_n$ has analogous definition as Z_n, then $\tilde{W}$ also realizes the Naimark dilation of F. We call this reprezentation of the Naimark dilation of F the dual reprezentation and from the general dilation theory we know that there exists a unitary operator $\Omega : \mathcal{K} \to \tilde{\mathcal{K}}$ such that $\Omega h = h$ for $h \in \mathcal{H}$ and $\Omega^* \tilde{W} \Omega = W$.

Now, we pass further on and we consider a discrete time stationary process $\mathcal{V} = \{\mathcal{V}_n\}_{n=-\infty}^{\infty}$ with $\mathcal{V}_n \in \mathcal{L}(\mathcal{H}, \mathcal{K})$. We this process we have the covariance matrix(consequently, a semispectral measure on $\mathbb{T}$, and we will suppose in the following that $F(1)=I$) and W beeing the Naimark dilation of F, then $\mathcal{K}$ may be identified with the space of the dilation and the process itself may be realized as $\mathcal{V}_n h = W^{*n} h$, $h \in \mathcal{H}$ (for all these classical facts see [13]).

Choosing the realization (1.14) of W , this operator can be viewed as the evolution operator of a certain dinamycal system in the following sense: let us consider the system:

$$(1.24) \qquad \begin{cases} y(n) = D_{*1} x_1(n) - \sum_{k=1}^{\infty} Z_k^* x_{k+1}(n) \\ x_k(n+1) = \Gamma_k^* D_{\Gamma_{k-1}^*} \cdots D_{\Gamma_1^*} x_1(n) + \cdots + D_{\Gamma_k} x_{k+1}(n) & , k = \overline{1,\infty} \\ \qquad y(n) \in \mathcal{D}_*, \quad x_1(n) \in \mathcal{H}, \quad x_k(n) \in \mathcal{D}_{\Gamma_{k-1}} & , k \geq 2 \end{cases}$$

then the positive powers of W^* describe the evolution of this system.

If we restrict to the case $\Gamma_k = 0$, $k \geq 2$, then (1.14) becomes the Schäffer reprezentation of the minimal unitary dilation of the contraction Γ_1 and the system becomes

$$(1.25)\qquad \begin{cases} y(n)=D_{\Gamma_1^*}x_1(n)-\Gamma_1 x_2(n) \\ x_1(n+1)=\Gamma_1^* x_1(n)+D_{\Gamma_1}x_2(n) \\ x_k(n+1)=x_{k+1}(n) & ,k\geq 2 \end{cases}$$

or

$$(1.26)\qquad \begin{cases} y(n)=D_{\Gamma_1^*}x(n)-\Gamma_1 u(n) \\ x(n+1)=\Gamma_1^* x(n)+D_{\Gamma_1}u(n) \end{cases}.$$

This system is described by the transfer function (the input/output map) and by the zero state input/output map (for system theory see [10]).The energy conservation law (the unitarity of W) assures that the evolution of (1.26) is described only by the transfer function (which is the characteristic function of Γ_1) and this is one of the main points in Sz.-Nagy -Foias theory of contractions [12] (and in the realization theory of linear systems [10]).

Generally, we cannot expect such kind of results. We will show that on the basis of the energy conservation low the state space $\mathcal{H}$ still plays a privileged role.

III THE DESCRIPTION OF Ω

In this section we describe the operator Ω in terms of the parameter $\mathcal{G}$ .For this aim we need some of the considerations regarding the structure of the block-contractions as they were made in [14] . In this last mentioned paper it was remarked that to every block-contraction a double indexed sequence of parameters can be associated in an algorithmic way (for other remarks on this problem see also [6])

Let $K_n^{(p)}$ be the contraction acting between $\mathcal{R}_n^{(p)}$ and $\tilde{\mathcal{R}}_n^{(p)}$, $n\geq p$ associated by the algorithm in [14] to the following sequence of parameters: $G_{11}^{(p)}=\Gamma_p$, $G_{21}^{(p)}=G_{12}^{(p)}=\Gamma_{p+1}$,....,$G_{n-1,1}^{(p)}=G_{n-2,2}^{(p)}=\ldots=$

$=\ldots=G^{(p)}_{1,n-1}=\Gamma_n$ and zero in rest. It easy to see that

(2.1) $K^{(p)}_n=(I\oplus\Gamma_n)J^{(p)}_{n,n-1}(I\oplus\Gamma_n)J^{(p)}_{n,n-2}J^{(p)}_{n,n-1}(I\oplus\Gamma_n)\ldots$

$$\ldots J^{(p)}_{n,p}\ldots J^{(p)}_{n,n-1}(I\oplus\Gamma_n).$$

In this form, the operators $K^{(p)}_n$ appear in [5] and they have a technical role. In the following we also consider the operators $K^{(p)}_\infty:\mathcal{K}^{(p)}_\infty\longrightarrow\tilde{\mathcal{K}}^{(p)}_\infty$ obtained by means of the algorithm in [14] starting from the sequence of parameters: $G^{(p)}_{11}=\Gamma_p$, $G^{(p)}_{m1}=G^{(p)}_{m-1,2}=\ldots$ $=\ldots=G^{(p)}_{1m}=\Gamma_{p+m-1}$ $,m\geqslant 2$. It is not difficult to see that

(2.2) $$\operatorname*{s-lim}_{n\to\infty}K^{(p)}_n P^{(p)}_n=K^{(p)}_\infty .$$

Then, denoting the matrices in the right side of (1.10) by $D^{(p)}_\infty$ and those corresponding to $Y^{(p)}_\infty$ by $\tilde{D}^{(p)}_\infty$, we have the equalities:

(2.3) $K^{(p)}_\infty=(X^{(p)}_\infty,X^{(p+1)}_\infty D^{(p)}_\infty,\ldots,X^{(m)}_\infty D^{(m-1)}_\infty\ldots D^{(p)}_\infty,\ldots)^t$

and

(2.4) $K^{(p)}_\infty=(Y^{(p)}_\infty,\tilde{D}^{(p)*}_\infty Y^{(p+1)}_\infty,\ldots,\tilde{D}^{(p)*}_\infty\ldots\tilde{D}^{(m-1)*}_\infty Y^{(m)}_\infty,\ldots).$

According to (2.3) and (2.4) we can obtain $K^{(p)}_\infty$ in the form of (1.3) and (1.5) for certain parameters and then we can obtain identifications for the defect spaces of $K^{(p)}_\infty$. Thus, (2.3) shows that the first parameter for writing $K^{(p)}_\infty$ as in (1.5) is $X^{(p)}_\infty$; then,

$$X^{(p+1)}_\infty D^{(p)}_\infty=X^{(p+1)}_\infty\alpha^{(p)}_+ D_{X^{(p)}_\infty}$$

consequently, the second parameter is $X^{(p+1)}_\infty\alpha^{(p)}_+$; we obtain by induction that the sequence of parameters cho sen in order that $K^{(p)}_\infty$ has the reprezentation (1.5) is $\{X^{(p+m-1)}_\infty\alpha^{(p+m-2)}_+\ldots\alpha^{(p)}_+\}_{m=1}^\infty$

We define now $G_1=X^{(p)}_\infty,\ldots,G_n=X^{(p+n-1)}_\infty\alpha^{(p+n-2)}_+\ldots\ldots\alpha^{(p)}_+$ and using the dual form of (1.10) we obtain the unitary operator:

$$\begin{cases}\tilde{\gamma}'^{(p)}:\mathcal{D}_{K^{(p)*}_\infty}\longrightarrow\mathcal{D}_{G_1^*}\oplus\mathcal{D}_{G_2^*}\oplus\ldots\\[2mm] \tilde{\gamma}'^{(p)}D_{K^{(p)*}_\infty}=\begin{pmatrix}D_{G_1^*}, & -G_1G_2^*, & -G_1D_{G_2^*}G_3^*, & \ldots\\ 0, & D_{G_2^*}, & -G_2G_3^*, & \ldots\\ 0, & 0, & D_{G_3^*}, & \ldots\\ \vdots & \vdots & \vdots & \end{pmatrix}\end{cases}$$

; but $\mathcal{D}_{G_k^*} = \mathcal{D}_{X_\infty^{(p+k-1)*}}$ and using (1.13) we have the unitary operator:

$$(2.5)\quad \begin{cases} \tilde{\gamma}^{(p)} : \mathcal{D}_{K_\infty^{(p)*}} \longrightarrow \mathcal{D}_*^{(p)} \oplus \mathcal{D}_*^{(p+1)} \oplus \dots \\ \tilde{\gamma}^{(p)} D_{K_\infty^{(p)*}} = \begin{pmatrix} \tilde{\alpha}_+^{(p)}, & 0 & \dots \\ 0, & \tilde{\alpha}_+^{(p+1)}, & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} DG_1^*, & -G_1 G_2^*, & \dots \\ 0, & DG_2^*, & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \end{cases}$$

In a similar way we obtain the identification of $\mathcal{D}_{K_\infty^{(p)}}$ by the unitary operator:

$$(2.6)\quad \begin{cases} \gamma^{(p)} : \mathcal{D}_{K_\infty^{(p)}} \longrightarrow \mathcal{D}^{(p)} \oplus \mathcal{D}^{(p+1)} \oplus \dots \\ \gamma^{(p)} D_{K_\infty^{(p)}} = \begin{pmatrix} \beta_+^{(p)}, & 0 & \dots \\ 0, & \beta_+^{(p+1)}, & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} D\tilde{G}_1, & -\tilde{G}_1^* \tilde{G}_2, & -\tilde{G}_1^* D\tilde{G}_2^* \tilde{G}_3, & \dots \\ 0, & D\tilde{G}_2, & -\tilde{G}_2^* \tilde{G}_3, & \dots \\ 0, & 0, & D\tilde{G}_3, & \dots \\ \vdots & \vdots & & \ddots \end{pmatrix} \end{cases}$$

where $\beta_+^{(p)}$ are the duals of $\alpha_+^{(p)}$ and $\tilde{G}_1 = Y_\infty^{(p)}$, $\tilde{G}_n = \tilde{\beta}_+^{(p)*} \dots \tilde{\beta}_+^{(p+n-2)*} Y_\infty^{(p+n-1)}$, $\tilde{\beta}_+^{(p)}$ beeing the dual of $\alpha_+^{(p)}$.

Now, we define

$$\Omega' : \dots \oplus \mathcal{D}_*^{(3)} \oplus \mathcal{D}_*^{(2)} \oplus \boxed{\mathcal{K}} \oplus \mathcal{D}_{\Gamma_1} \oplus \mathcal{D}_{\Gamma_2} \oplus \dots \longrightarrow$$
$$\longrightarrow \dots \oplus \mathcal{D}^{(3)} \oplus \mathcal{D}^{(2)} \oplus \boxed{\mathcal{K}} \oplus \mathcal{D}_{\Gamma_1^*} \oplus \mathcal{D}_{\Gamma_2^*} \oplus \dots$$

by the formula

$$\Omega' = \begin{pmatrix} \text{"} -\gamma^{(2)} K_\infty^{(2)*} \tilde{\gamma}^{(2)*} \text{"} & \vdots & \text{"} \gamma^{(2)} D_{K_\infty^{(2)}} \text{"} \\ & 0 & \\ \dots\ 0,\ 0, & \boxed{I}, & 0,\ 0, \dots \\ & 0 & \\ \text{"} D_{K_\infty^{(2)*}} \tilde{\gamma}^{(2)*} \text{"} & 0 & K_\infty^{(2)} \\ & \vdots & \end{pmatrix}$$

where " " means that we applied one more rotation to the operator in " " in order that it acts between the corresponding spaces.

Finally, let us define the unitary operator:

$$(2.7)\quad \begin{cases} \Omega : \mathcal{K} \longrightarrow \tilde{\mathcal{K}} \\ \Omega = \begin{pmatrix} \ddots & & & \\ & \tilde{R}_3^* & & \\ & & \tilde{R}_2^* & \\ & & & \boxed{I} & 0 \\ & & & 0, & I \\ & & & & & \ddots \end{pmatrix} \cdot \Omega' \cdot \begin{pmatrix} R_3 & & & \\ & R_2 & & \\ & & \boxed{I} & 0 \\ & & 0 & I, \\ & & & & \ddots \end{pmatrix} \end{cases}$$

where $\tilde{R}_n$ is the dual of (1.20).

2.1 THEOREM The operator Ω defined by (2.7) is the coupling operator of W and $\tilde{W}$ in the sense that $\Omega h=h$, $h\in \mathcal{H}$ and $\tilde{W}\Omega=\Omega W$.

PROOF We have to prove that

$$(2.8) \qquad \tilde{W}\Omega=\Omega W=\begin{pmatrix} \text{"}-\gamma^{(1)}K_\infty^{(1)*}\tilde{\gamma}^{(1)*}\text{"}, & \text{"}\gamma^{(1)}D_{K_\infty^{(1)}}\text{"} \\ \text{"}D_{K_\infty^{(1)}}\tilde{\gamma}^{(1)*}\text{"}, & K_\infty^{(1)} \end{pmatrix}.$$

For this aim, we eventually use (2.2) and Lemma 4.1 in [5] .

Actually, (2.8) mainly explains the definition and the desired properties of Ω . ∎

2.2 REMARK May be useful to point out the following simple cases:

(a) $\Gamma_k=0$, $k\geqslant 2$, consequently, W is the Schäffer reprezentation of the Sz.-Nagy dilation of Γ_1 ,

$$W=\begin{pmatrix} I & 0 & & \\ 0 & D_{\Gamma_1^*}, & \boxed{\Gamma_1} & \\ & -\Gamma_1^*, & D_{\Gamma_1} & 0 \\ & & 0 & I \\ & & & & \ddots \end{pmatrix}$$

;moreover, we have $\tilde{W}=\begin{pmatrix} I & 0 & & \\ 0 & D_{\Gamma_1}, & -\Gamma_1^* & \\ & \boxed{\Gamma_1}, & D_{\Gamma_1^*} & 0 \\ & & 0 & I \\ & & & & \ddots \end{pmatrix}$ and $\Omega=\begin{pmatrix} & & & \vdots \\ & 0 & I & \cdots \\ 0 & \boxed{I} & 0 & \\ I & 0 & & \\ \cdot^{\cdot^{\cdot}} & & & \end{pmatrix}$

; in this context Ω consists of a single elementary rotation.

(b) $\Gamma_k=0$, $k\geqslant 3$; in a certain sense , this case is generic for our considerations. We have(with an obvious change in the definitions):

$$W=\begin{pmatrix} \ddots & & \vdots & & \vdots & \\ I, & 0, & 0, & 0, & 0 & \cdots \\ 0, & D_{\Gamma_1^*}D_{\Gamma_2^*}, & \boxed{\Gamma_1}, & D_{\Gamma_1^*}\Gamma_2, & 0 & \cdots \\ 0, & -\Gamma_1^*D_{\Gamma_2^*}, & D_{\Gamma_1}, & -\Gamma_1^*\Gamma_2, & 0 & \cdots \\ 0 & -\Gamma_2^*, & 0, & D_{\Gamma_2}, & 0 & \cdots \\ 0, & 0, & 0, & 0, & I & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$\tilde{W} = \begin{pmatrix} I, & 0 & & & \\ 0, & D_{\Gamma_2}D_{\Gamma_1}, & -D_{\Gamma_2}\Gamma_1^*, & \Gamma_2^* & \\ & \boxed{\Gamma_1}, & D_{\Gamma_1^*}, & 0 & \\ & \Gamma_2 D_{\Gamma_1}, & -\Gamma_2\Gamma_1^*, & D_{\Gamma_2^*} & 0 \\ & & & 0 & I \end{pmatrix} \text{ and}$$

$$\Omega = \begin{pmatrix} & & & 0 & I \\ & -\Gamma_2^*, & 0, & D_{\Gamma_2} & 0 \\ \cdots & ,0, & \boxed{I}, & 0 & \\ 0, & D_{\Gamma_2^*}, & 0 & \Gamma_2 & \\ I, & 0 & & & \end{pmatrix}. \text{ Now}$$

$$\tilde{W}\Omega = \Omega W = \begin{pmatrix} & & & 0, & I \\ & -\Gamma_2^* & 0, & D_{\Gamma_2}, & 0 \\ 0, & -D_{\Gamma_2}\Gamma_1^* D_{\Gamma_2^*}, & D_{\Gamma_2}D_{\Gamma_1}, & -D_{\Gamma_2}\Gamma_1^*\Gamma_2, & \cdots \\ -\Gamma_2^*, & D_{\Gamma_1^*}D_{\Gamma_2^*}, & \boxed{\Gamma_1}, & D_{\Gamma_1^*}\Gamma_2 & \\ \cdots\; 0, & & & & \\ 0, D_{\Gamma_2^*}, & -\Gamma_2\Gamma_1^* D_{\Gamma_2^*}, & \Gamma_2 D_{\Gamma_1}, & -\Gamma_2\Gamma_1^*\Gamma_2 & \\ I, 0 & & & & \end{pmatrix}.$$

IV THE CHARACTERISTIC OPERATOR AND ITS ROLE IN THE GEOMETRY OF THE NAIMARK DILATION

In this section we define the characteristic operator of the $\mathcal{K}$ system (1.24) and we describe with its aid the structure of the in the same way the characteristic function is used in Sz.-Nagy-Foias theory. We consider a stationary process $\mathcal{V} = \{\mathcal{V}_n\}_{n=-\infty}^{\infty}$ and F its spectral measure with F(1)=I; $\mathcal{G}$ is the sequence of parameter associated to F by (1.9). The spaces $\mathcal{K}_+ = \bigvee_{n=0}^{\infty} W^n \mathcal{H}$ and $\mathcal{K}_- = \bigvee_{n=0}^{\infty} W^{*n}\mathcal{H}$ are t past and the future of the process. For describing the evolution i the future we use the reprezentation (1.14) and for describing the "evolution in the past" it is more convenient the reprezentation (1.23). In order to connect the two parts of the evolution of the system we have at hand the description of Ω given in Section III.

Let us define $W_{\pm} = W^{\pm 1}/\mathcal{K}_{+}$ and $\widetilde{W}_{\pm} = \widetilde{W}^{\pm 1}/\widetilde{\mathcal{K}}_{-}$ where $\widetilde{\mathcal{K}}_{\pm} = \bigvee_{n=0}^{\infty} \widetilde{W}^{\pm n}\mathcal{K}$; these operators are isometries and let be $\mathcal{K}_{+} = M_{+}(\mathcal{L}_{+}) \oplus \mathcal{R}_{+}$ the Wold decomposition of W_{+}, then , according to Corollary 2.6 in

(3.1) $$\mathcal{L}_{+} = W(\ldots 0 \oplus \mathcal{D}_{*}^{(1)} \oplus \boxed{0} \oplus 0 \oplus \ldots)$$

;analogously, $$\widetilde{\mathcal{K}}_{-} = M_{+}(\widetilde{\mathcal{L}}_{-}) \oplus \widetilde{\mathcal{R}}_{-}$$

(3.2) $$\widetilde{\mathcal{L}}_{-} = \widetilde{W}^{*}(\ldots \oplus 0 \oplus \mathcal{D}^{(1)} \oplus \boxed{0} \oplus 0 \oplus \ldots).$$

Now, we define the following spaces:

$$\mathcal{K}_{i+} = \ldots \oplus 0 \oplus \boxed{0} \oplus \mathcal{D}_{\Gamma_1} \oplus \mathcal{D}_{\Gamma_2} \oplus \ldots$$
$$\mathcal{K}_{i-} = \Omega^{*}(\widetilde{\mathcal{L}}_{-} \oplus \widetilde{W}^{*}\widetilde{\mathcal{L}}_{-} \oplus \ldots)$$
$$\mathcal{K}_{i} = \mathcal{K}_{i-} \vee \mathcal{K}_{i+}$$

$$\mathcal{K}_{o+} = \mathcal{L}_{+} \oplus W\mathcal{L}_{+} \oplus W^{2}\mathcal{L}_{+} \oplus \ldots$$
$$\mathcal{K}_{o-} = \Omega^{*}(\ldots \oplus 0 \oplus \boxed{0} \oplus \mathcal{D}_{\Gamma_1^{*}} \oplus \mathcal{D}_{\Gamma_2^{*}} \oplus \ldots)$$
$$\mathcal{K}_{o} = \mathcal{K}_{o-} \vee \mathcal{K}_{o+}$$

and the following operator will be called the characteristic operato of the process :

(3.3) $$\begin{cases} Q: \mathcal{K}_{i} \longrightarrow \mathcal{K}_{o} \\ Q = P^{\mathcal{K}}_{\mathcal{K}_{o}}/\mathcal{K}_{i} \end{cases}$$

The first problem is whether $\mathcal{K} = \mathcal{K}_{i} \vee \mathcal{K}_{o}$. It is well-known (see for instance [15]) that for a semispectral measure F on $\mathbb{T}$ there exists a maximal subspace $\mathcal{H}_{s}$ of $\mathcal{H}$ reducing F to a spectral measure. For the case $\Gamma_{k}=0$, $k \geqslant 2$, it is also well-known (see [12]) that $\mathcal{K} \ominus (\mathcal{K}_{i} \vee \mathcal{K}_{o}) = \mathcal{R}_{+} \cap \mathcal{R}_{-} = \mathcal{H}_{S}$ ($\mathcal{R}_{-} = \Omega^{*}\widetilde{\mathcal{R}}_{-}$).

But for the general case the last equalities fail. Regarding th space $\mathcal{K} \ominus (\mathcal{K}_{i} \vee \mathcal{K}_{o})$, let us begin by considering $k=(\ldots, d_{*}^{(2)}, d_{*}^{(1)}, \boxed{h}, d_{1}, d_{2}, \ldots)$ in this space and as $k \perp \mathcal{K}_{i+}$, we have $d_{k}=0$, $k \geqslant 1$.

Then, $k \perp \mathcal{K}_{r-}$, so,

$$(k, \Omega^*(\ldots,0,\boxed{0},\tilde{d}_1,\tilde{d}_2,\ldots))=0 \qquad , \tilde{d}_k \in \mathcal{D}_{\Gamma_k^*} \quad , k \geq 1$$

or

$$(3.4) \qquad (\Omega k, (\ldots 0,\boxed{0},\tilde{d}_1,\tilde{d}_2,\ldots))=0 \qquad ,\tilde{d}_k \in \mathcal{D}_{\Gamma_k^*} \quad ,k \geq 1.$$

But, taking into account the form of Ω,

$$\Omega k = \begin{pmatrix} & & \overset{*}{\boxed{h}} & & \\ \text{"} D_{K_\infty^{(2)*}} \,\gamma^{(2)*}\text{"} & \begin{pmatrix} \ddots & & \\ & R_3 & \\ & & R_2 \end{pmatrix} & \begin{pmatrix} \vdots \\ d_*^{(2)} \\ d_*^{(1)} \end{pmatrix} \end{pmatrix}$$

and returning in (3.4), we have

$$(3.5) \qquad \left(\text{"} D_{K_\infty^{(2)*}} \,\gamma^{(2)*}\text{"} \begin{pmatrix} \ddots & & \\ & R_3 & \\ & & R_2 \end{pmatrix} \begin{pmatrix} \vdots \\ d_*^{(2)} \\ d_*^{(1)} \end{pmatrix} , \begin{pmatrix} \tilde{d}_1 \\ \tilde{d}_2 \\ \vdots \end{pmatrix} \right)=0 \; ,\tilde{d}_k \in \mathcal{D}_{\Gamma_k^*} \quad ,k \geq$$

From (3.5) we remark that when $\ker D_{K_\infty^{(2)*}}=0$, we have $d_*^{(k)}=0$, $k \geq$ consequently $\mathcal{K} \ominus (\mathcal{K}_i \vee \mathcal{K}_o) \subset \mathcal{H}$.

So, we have to consider the purity of $K_\infty^{(2)}$. More precisely, take T a contraction in $\mathcal{L}(\mathcal{H},\mathcal{L})$, then

$$T= \begin{pmatrix} T_u , & 0 \\ 0 , & T_p \end{pmatrix} : \begin{matrix} \ker D_T \\ \oplus \\ \mathcal{D}_T \end{matrix} \longrightarrow \begin{matrix} \ker D_{T^*} \\ \oplus \\ \mathcal{D}_{T^*} \end{matrix}$$

where T_u is unitary and T_p is pure ($\|T_p h\| < \|h\|$,$h \neq 0$) (see [12]).

<u>3.1 PROPOSITION</u> Let F be such that Γ_1 is a completely non-unitary contraction and $K_\infty^{(2)}$ is pure. Then $\mathcal{K} = \mathcal{K}_i \vee \mathcal{K}_o$.

<u>PROOF</u> We saw that when $K_\infty^{(2)}$ is pure, then $\mathcal{K} \ominus (\mathcal{K}_i \vee \mathcal{K}_o) \subset \mathcal{H}$. Let $k \in \mathcal{K} \ominus (\mathcal{K}_i \vee \mathcal{K}_o)$, $k=(\ldots 0,\boxed{h},0,\ldots)$; using (3.1) we write that $k \perp \mathcal{L}_+$, so, $(k,W(\ldots 0,d_*,\boxed{0},0,\ldots)^t)=0$,for $d_* \in \mathcal{D}_*^{(1)}$ and using (1.14) we obtain that $D_*^{(1)}h=0$; but the purity of $K_\infty^{(2)}$ implies the purity of $X_\infty^{(2)}$. According to (1.13) this means that $\ker D_*^{(2)}=0$. But ${D_*^{(1)}}^2 = D_{\Gamma_1^*} {D_*^{(2)}}^2 D_{\Gamma_1^*}$, so , it results that $D_{\Gamma_1} h=0$. Computing now the powers of W on the element $(\ldots 0,d_*,\boxed{0},0\ldots)$ we obtain by induction that $D_{\Gamma_1^*}\Gamma_1^{*n} h=0, n \geq 0$; analogously, it results that $D_{\Gamma_1}\Gamma_1^n h=0$, $n \geq 0$. Consequently, we obtained that $k \in \mathcal{K} \ominus (\mathcal{K}_i \vee \mathcal{K}_o)$ if and only if

$h \in \mathcal{H}_u(\Gamma_1)$, where $\mathcal{H}_u(\Gamma_1)$ is the maximal subspace of $\mathcal{H}$ reducing Γ_1 to a unitary operator.

For the space $\mathcal{H}_s$ there are two known descriptions (see for instance [15]):

$$\mathcal{H}_s = \{h \in \mathcal{H} \,/\, \|S_n h\| = \|h\| \,,\, n \in \mathbb{Z}\} = \{h \in \mathcal{H} \,/\, F(\chi_n) h \in \mathcal{H} \,,\, n \in \mathbb{Z}\} .$$

But, (using for instance the Schur analysis of S_n) it is easy to see that $\mathcal{H}_s$ depends only on the first Fourrier coefficient of F. Namely, $\mathcal{H}_s = \mathcal{H}_u(\Gamma_1)$. With this remark, we can end the proof.

A semispectral measure on $\mathbb{T}$ will be called pure if Γ_1 is a copletely non-unitary contraction and $K^{(2)}_\infty$ is pure.

3.2 THEOREM Let F be a pure semispectral measure on $\mathbb{T}$. Then

$$\mathcal{K} = \mathcal{K}_0 \oplus (I-Q)\mathcal{K}_i \tag{3.6}$$

and

$$\mathcal{L} = \mathcal{K} \ominus (\mathcal{K}_{0-} \vee \mathcal{K}_{i+}) \tag{3.7}$$

PROOF ∎

3.3 REMARK When $\Gamma_k = 0$, $k \geq 2$, then Q is a Toeplitz operator having as symbol the characteristic function of Γ_1. $K^{(2)}_\infty = 0$, then F is pure if and only if Γ_1 is a completely non-unitary contraction. Finally, (3.6) and (3.7) are two of the main results in the structure of the unitary dilation of Γ_1. ∎

3.4 REMARK Let us consider the situation when there exists $N \in \mathbb{N}$ such that $\Gamma_k = 0, k > N$ (for simplicity we take $N=3$). We also define the operators: $Q_1 = P^{\mathcal{K}}_{\mathcal{K}_{0+}} / \mathcal{K}_{i+}$, $Q_2 = P^{\mathcal{K}}_{\mathcal{K}_{0+}} / \mathcal{K}_{i-}$, $Q_3 = P^{\mathcal{K}}_{\mathcal{K}_{0-}} / \mathcal{K}_{i+}$, $Q_4 = P^{\mathcal{K}}_{\mathcal{K}_{0-}} / \mathcal{K}_{i-}$, $Q^0 = P^{\mathcal{K}}_{\mathcal{K}_{0+}} / \mathcal{K}_{0-}$, $Q^i = P^{\mathcal{K}}_{\mathcal{K}_{i+}} / \mathcal{K}_{i-}$.

Having the formula for $\mathcal{L}_+$, we consider instead of Q_1 the operator $\tilde{Q}_1$,

$$\tilde{Q}_1 : \mathcal{D}_{\Gamma_1} \oplus \mathcal{D}_{\Gamma_2} \oplus \mathcal{D}_{\Gamma_3} \oplus \cdots \longrightarrow \mathcal{D}^{(1)}_* \oplus \mathcal{D}^{(1)}_* \oplus \mathcal{D}^{(1)}_{R} \oplus \cdots$$

$$\tilde{Q}_1 = \begin{pmatrix} W^*, & 0, & 0 \\ 0, & W^{*2}, & \\ & , & \ddots \end{pmatrix} Q_1$$

A simple computation (using Remark 2.2) shows:

$$\tilde{Q}_1 = \begin{pmatrix} -D_{\Gamma_2^*}\Gamma_1 , & D_{\Gamma_2^*}D_{\Gamma_1^*}D_{\Gamma_1} + D_{\Gamma_2^*}\Gamma_1\Gamma_2^*\Gamma_1 , & \cdots & \\ -\Gamma_2 , & -D_{\Gamma_2^*}\Gamma_1 D_{\Gamma_2} , & \cdots & \\ 0 , & -\Gamma_2 , & -D_{\Gamma_2^*}\Gamma_1 D_{\Gamma_2} , & \cdots \\ 0 , & 0 , & -\Gamma_2 & \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

So , the operator $\tilde{Q}_{1,red}$ obtained by the elimination of the first column in $\tilde{Q}_1$ is a Toeplitz operator having as symbol the function

$$\Theta(z) = -\Gamma_2 + zD_{\Gamma_2^*}\Theta_{\Gamma_1}(z)\left(I - z\Gamma_2^*\Theta_{\Gamma_1}(z)\right)^{-1}D_{\Gamma_2} : \mathcal{D}_{\Gamma_2} \longrightarrow \mathcal{D}_{\Gamma_2^*}$$

where Θ_{Γ_1} is the characteristic function of Γ_1 . We immediately remark that this function is the transfer function of the first order system:

$$\begin{cases} \begin{pmatrix} x_1(n+1) \\ x_2(n+1) \end{pmatrix} = \begin{pmatrix} \Gamma_1^* , & D_{\Gamma_1} \\ \Gamma_2^* D_{\Gamma_1^*} , & -\Gamma_2^*\Gamma_1 \end{pmatrix} \begin{pmatrix} x_1(n) \\ x_2(n) \end{pmatrix} + \begin{pmatrix} 0 \\ D_{\Gamma_2} \end{pmatrix} u(n) \\ y(n) = \left(D_{\Gamma_2^*}D_{\Gamma_1^*} , -D_{\Gamma_2^*}\Gamma_1\right) \begin{pmatrix} x_1(n) \\ x_2(n) \end{pmatrix} - \Gamma_2 u(n) \end{cases}$$

Moreover, we can establish the connection with the Schur algorithm (the operatorial version as described in [7]). Thus, for a contraction $T \in \mathcal{L}(\mathcal{H},\mathcal{H}')$ we define

$$U_T(z) = \begin{pmatrix} T , & zD_{T^*} \\ D_T , & -zT^* \end{pmatrix}$$

and for an analytic contractive function in the unit disc $\mathbb{D}$ and having values in $\mathcal{L}(\mathcal{H},\mathcal{H}')$ we define the linear fractional map

$$C_{U_T}(f)(z) = T/\mathcal{D}_T + zD_{T^*}f(z)(I+zT^*f(z))^{-1}D_T/\mathcal{D}_T : \mathcal{D}_T \longrightarrow \mathcal{D}_{T^*}$$

Then, Θ is obtained by two succesive applications of transformations of type C_U (this is in essence the Schur algorithm- for details see [7] ; for other facts about Schur algorithm and system theory see [11]).

Similar considerations can be made for the others operators Q; one more remark on Q_3 is that considering instead of it the operator

$$\tilde{Q}_3 : \begin{matrix} \mathcal{D}_{\Gamma_1} \\ \oplus \\ \mathcal{D}_{\Gamma_2} \\ \oplus \\ \mathcal{D}_{\Gamma_3} \\ \oplus \\ \vdots \end{matrix} \longrightarrow \begin{matrix} \mathcal{D}_{\Gamma_1^*} \\ \oplus \\ \mathcal{D}_{\Gamma_2^*} \\ \oplus \\ \mathcal{D}_{\Gamma_3^*} \\ \oplus \\ \vdots \end{matrix}$$

$$\tilde{Q}_3 = \Omega_F Q_3,$$

we have $\tilde{Q}_3 = K_{\infty}^{(2)}$. ■

V FINAL COMMENTS

By our considerations in the precedent sections we tryed to extend several facts in Sz.-Nagy-Foias theory from a system theory point of view, having the realizations (1.14) and (1.23) of the Naimark dilation in terms of the Schur analysis of S_n. Thus, we followed the "dynamic" of the system (1.24) but we lost the functional model idea. Actually, this last problem constitute the subject of many attempts on extending Sz.-Nagy-Foias theory (see for instance [15]). For our context we can continue as follows: let $\mathcal{K}_{\pm} = M_{+}(\mathcal{L}_{\pm}) \oplus \mathcal{R}_{\pm}$ be the Wold decomposition of $W_{\pm}$ and

$$\mathcal{K} = (M(\mathcal{L}_{+}) \vee M(\mathcal{L}_{-})) \oplus \mathcal{R}$$

We remarked in Section 3 that $\mathcal{R}$ may be very large and does not depend on the space $\mathcal{H}_s$. Therefore, we are forced to suppose $\mathcal{R} = 0$, an unpleasant condition. Then, we consider the operator:

$$\mathring{Q} = P^{\mathcal{K}}_{M(\mathcal{L}_{+})} / M(\mathcal{L}_{-})$$

and we have $\mathring{Q}(W/M(\mathcal{L}_{-})) = (W/M(\mathcal{L}_{+}))\mathring{Q}$, consequently, by the use of the Fourrier reprezentation lemma ([12]), $\mathring{Q}$ is the multiplication operator with a contractive function $\Theta_F \in L^{\infty}(\mathcal{L}_{+}, \mathcal{L}_{-})$ and

$$\Theta_F(e^{it}) = \sum_{k=-\infty}^{\infty} e^{ikt} P^{\mathcal{K}}_{\mathcal{L}_{+}} W^{*k} P^{\mathcal{K}}_{\mathcal{L}_{-}} .$$

Define $\Delta_F(e^{it}) = (I - \Theta'_F(e^{it})\, \Theta^{*}_F(e^{it}))^{\frac{1}{2}}$, $\Delta_F \in L^{\infty}(\mathcal{L}_{-})$

then, there exists a unitary operator $\psi : \mathcal{K} \longrightarrow L^2(\mathcal{L}_+) \oplus \Delta_F L^2(\mathcal{L}_-$ such that W becomes the multiplication by χ_1.

In this way, we obtained a functional model for the Naimark dilation , but we have not at hand the position of $\mathcal{H}$ in this model.

The above model appeared in [2] in the following form: there are considered two isometries $V_\pm$ on $\mathcal{D}_\pm$ with $\bigcap_{n=0}^{\infty} V_\pm^n \mathcal{D}_\pm = \{0\}$, then, we define $\mathcal{L}_\pm = \mathcal{D}_\pm \ominus V_\pm \mathcal{D}_\pm$, $\mathcal{D}_\pm = \bigoplus_{k=0}^{\infty} V_\pm^k \mathcal{L}_\pm$.

A unitary operator $U \in \mathcal{L}(\mathcal{K})$ is called unitary coupling for the pair $(V_\pm)$ when $\mathcal{D}_\pm \subset \mathcal{K}$ and $U^{\pm 1} d_\pm = V_\pm d_\pm$, $d_\pm \in \mathcal{D}_\pm$. Define $\mathcal{K}_\pm = \bigvee_{n=0}^{\infty} U^n \mathcal{D}_\pm$ then U is called minimal when $\mathcal{K} = \mathcal{K}_+ \vee \mathcal{K}_-$

Finally , one considers

$$S(\zeta, U)h = 2\pi \zeta^{-1} \frac{d}{d\theta} P_{\mathcal{L}_-}^{k} E_\theta h \;, h \in \mathcal{L}_+ \;, |\zeta| = 1$$

the scattering suboperator, where E_θ is the spectral measure of U.

It is easy to see that $S(\cdot, U) = \Theta_F$ and the functional model of $\mathcal{D}_\pm$ and U is now obtained.

This model is succesfully used in the study of the Nehari problem in [3], [1].

One more remark is that when $\Gamma_k = 0$, $k > 2$, then Θ_F is essencially the function Θ in Remark 3.4.

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