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The aim of this paper is to give sufficient conditions for two natural numbers d,g such that to exist a smooth, irreducible, nondegenerate curve $C \subset \mathbb{P}^n$, of degree d and genus g

We shall work over the complex field $\mathbb{C}$. We shall use the standard notations (see, for instance [Ha 1], [Ba]). Curve (resp. surface) means a $\mathbb{C}$-algebraic integral scheme of dimension 1 (resp. of dimension 2).

We recall that a curve $C \subset \mathbb{P}^n$ is called nondegenerate if it is not contained in any hyperplane.

We recall the Castelnuovo's bound also (see, for instance [G-H] pag.252):

Let $C \subset \mathbb{P}^n$ be a smooth nondegenerate curve of degree d (necessarily $d \geq n$) and genus g. Then:

$$g \leq \frac{m(m-1)}{2}(n-1)+m\varepsilon$$

with $m=\left[\frac{d-1}{n-1}\right]$ ([ ] denotes the integer part) and $\varepsilon=d-1-m(n-1)$.

§ 1. Nondegenerate curves in $\mathbb{P}^n$, $n \geq 3$

In this § we shall prove the following

Theorem 1:

Let be $n \in \mathbb{N}$, $n \geq 3$ and $d,g \in \mathbb{N}$, $d \geq 2n-1$ such that

$$g \leq \frac{1}{4(n-1)}(d-1)^2.$$

Then there exists a smooth nondegenerate curve $C \subset \mathbb{P}^n$ of degree d and genus g.

The case n=3 of this theorem was proved by Gruson and Peskine (see [G-P] or [Ha2]). The proof of our theorem is a generalization of the proof given by Gruson and Peskine for n=3.

Let $P_1, \ldots, P_9 \in \mathbb{P}^2$ be points satisfying the following conditions:

(1) there exist a unique smooth cubic $\Gamma_o \subset \mathbb{P}^2$ containing $P_1, \ldots, P_9$;

(2) the classes of the points $P_1, \ldots, P_9$ and $\mathcal{O}_{\mathbb{P}^2}(1) \otimes \mathcal{O}_{\Gamma_o}$ are Z-independent in Pic Γ_o.

Let be S the surface obtained by blowing up the points $P_1, \ldots, P_9$ from $\mathbb{P}^2$. Then

$$\mathrm{Pic}\, S \cong \mathbb{Z} \oplus \mathbb{Z}^9$$

with $(\ell, -e_1, \ldots, -e_9)$ a $\mathbb{Z}$-basis (here ℓ is the class of the inverse image in S of a line $L \subset \mathbb{P}^2$ and e_i are the classes of the exceptional divisors $E_i \subset S$ corresponding to the points $P_1, \ldots, P_9$, in Pic S).

We recall that:

$$\ell^2=1, \quad e_i^2=-1, \quad \ell . e_i=0, \quad e_i . e_j=0$$

$(\forall)$ $1 \leq i \neq j \leq 9$ ("." means the intersection form on S).

If $\mathcal{L} = a\ell - \sum_{i=1}^{9} b_i e_i \in \mathrm{Pic}\, S$ we shall write it as $\mathcal{L} = (a; b_1, \ldots, b_9)$.

Let be $C_o \in (1;1,0,\ldots,0)$ smooth

$\Gamma \in (3;1,\ldots,1)$ smooth

$B_o = E_8 \cup E_9$ $(0;0,\ldots,0,-1,-1)$.

<u>Proposition 1:</u>

i) The complete linear system $|C_o + m\Gamma|$, $m \geq 1$ (resp. $|B_o + m\Gamma|$, $m \geq 2$) on S has no base points and defines a morphism

$$\varphi_m : S \to \mathbb{P}^{2m+1} \text{ (resp. } \psi_m : S \to \mathbb{P}^{2m}).$$

ii) $X_m := \varphi_m(S) \subset \mathbb{P}^{2m+1}$ (resp. $X'_m := \psi_m(S) \subset \mathbb{P}^{2m}$) is a surface of degree 4m (resp. 4m-2).

iii) $\varphi_m(\Gamma):=L_m\subset X_m$ (resp. $\psi_m(\Gamma):=L'_m\subset X'_m$) is a line in $\mathbb{P}^{2m+1}$ (resp. $\mathbb{P}^{2m}$) and $\varphi_{m/\Gamma}:\Gamma\to L_m$ (resp. $\psi_{m/\Gamma}:\Gamma\to L'_m$) is a finite morphism of degree 2 with 4 ramification points $Q_1,\dots,Q_4$.

iv) $\varphi_{m/S\setminus\Gamma}:S\setminus\Gamma\to X_m\subset L_m$ (resp. $\psi_{m/S\setminus\Gamma}:S\setminus\Gamma\to X'_m\subset L'_m$) is an isomorphism.

v) $d_x\varphi_m:T_xS\to T_{\varphi_m(x)}\mathbb{P}^{2m+1}$ (resp. $d_x\psi_m:T_xS\to T_{\psi_m(x)}\mathbb{P}^{2m}$) is injective $(\forall)x\in S\setminus\{Q_1,\dots,Q_4\}$.

vi) Let be $Y\subset S$ a smooth curve, $Y\neq\Gamma$ such that

- $Q_i\notin Y$, $i=\overline{1,4}$
- $(\forall)\ P,Q\ ,P\neq Q,\ \varphi_m(P)=\varphi_m(Q)$ (resp. $\psi_m(P)=\psi_m(Q)$) $\Rightarrow\{P,Q\}\not\subset Y$.

Then $\varphi_m(Y)\subset\mathbb{P}^{2m+1}$ (resp. $\psi_m(Y)\subset\mathbb{P}^{2m}$) is a smooth curve.

vii) Let be $Y\subset S$ a curve as in vi) and such that

$$Y.(C_o+m\Gamma)\geq 4m+1\quad(\text{resp. }Y.(B_o+m\Gamma)\geq 4m-1).$$

Then $\varphi_m(Y)\subset\mathbb{P}^{2m+1}$ (resp. $\psi_m(Y)\subset\mathbb{P}^{2m}$) is nondegenerate.

<u>Proof</u>

i) The fact that $|B_o+m\Gamma|$, $m\geq 1$ and $|C_o+m\Gamma|$, $m\geq 0$ has no base points is proved in [Ha 2].

In order to compute $h^o(\mathcal{O}_S(B_o+m\Gamma))$ and $h^o(\mathcal{O}_S(C_o+m\Gamma))$ we use the standard exact sequences of sheaves

$$0\to\mathcal{O}_S(B_o+(m-1)\Gamma)\to\mathcal{O}_S(B_o+m\Gamma)\to\mathcal{O}_\Gamma(B_o+m\Gamma)\to 0$$

$$0\to\mathcal{O}_S(C_o+(m-1)\Gamma)\to\mathcal{O}_S(C_o+m\Gamma)\to\mathcal{O}_\Gamma(C_o+m\Gamma)\to 0$$

and the vanishing of $h^1(\mathcal{O}_S(B_o+m\Gamma))$, $h^1(\mathcal{O}_S(C_o+m\Gamma))$ for $m\geq 0$.

iii) Because of the surjections

$$H^o(\mathcal{O}_S(C_o+m\Gamma))\twoheadrightarrow H^o(\mathcal{O}_\Gamma(C_o+m\Gamma))$$

$$H^o(\mathcal{O}_S(B_o+m\Gamma))\twoheadrightarrow H^o(\mathcal{O}_\Gamma(C_o+m\Gamma))$$

$\psi_{m/\Gamma}$ (resp. $\varphi_{m/\Gamma}$) is given by $\mathcal{O}_\Gamma(C_o+m\Gamma)$ (resp. $\mathcal{O}_\Gamma(B_o+m\Gamma)$)

$h^0\mathcal{O}_\Gamma(C_o+m\Gamma))=h^0(\mathcal{O}_\Gamma(B_o+m\Gamma))=2$

hence $\varphi_{m/\Gamma}:\Gamma\to L_m$ ($\psi_{m/\Gamma}:\Gamma\to L'_m$)

with $g(L_m)=g(L'_m)=0$.

$g(\Gamma)=1\Rightarrow\varphi_m,\psi_m$ are not isomorphisms.

But $2=\deg\Gamma=\deg\varphi_{m/\Gamma}\cdot\deg L_m$ and $\deg\varphi_{m/\Gamma}\geq 2$. It follows $\deg\varphi_{m/\Gamma}=$ $=2$, $\deg L_m=1$ (analogously for ψ_m).

For the ramification we use the Hurwitz' formula.

iii) For φ_m:

The case m=1 is known (see [G-P] or [Ha2]).

Let be $P\in S\setminus\Gamma$. Let be $\bar{S}$ the surface obtained by blowing up S in P and let's denote by E the exceptional divisor on $\bar{S}$ corresponding to P. We must prove that the complete linear system on $\bar{S}$, $|\bar{C}_o+m\bar{\Gamma}-E|$ has no base points $(\forall)m\geq 1$ (here $\bar{C}_o$ and $\bar{\Gamma}$ are, respectively, the proper transformers of C_o and Γ in $\bar{S}$).

For this we use induction on m and the exact sequences

$$0\to\mathcal{O}_{\bar{S}}(\bar{C}_o+(m-1)\bar{\Gamma}-E)\to\mathcal{O}_{\bar{S}}(\bar{C}_o+m\bar{\Gamma}-E)\to\mathcal{O}_{\bar{\Gamma}}(\bar{C}_o+m\bar{\Gamma}-E)\to 0.$$

Because $\mathcal{O}_{\bar{\Gamma}}(\bar{C}_o+m\bar{\Gamma}-E)$ is generated by its global sections and $h^1(\mathcal{O}_{\bar{S}}(\bar{C}_o+(m-1)\bar{\Gamma}-E))=0$ $(\forall)m\geq 0$ it follows that the known case m=1 is sufficient.

• For ψ_m :

The situation can be reduced as before to the case m=2. For m=2, let be $P\in S\setminus\Gamma$. Let be $D:=B_o+\Gamma\in(3,1,\dots,1,0,0)\in\text{Pic } S$. We must show that

$$\mathcal{O}_{\bar{S}}(\bar{B}_o+2\bar{\Gamma}-E)=\mathcal{O}_{\bar{S}}(\bar{D}+\bar{\Gamma}-E)$$ is generated by its global sections.

From the exact sequence

$$0\to H^0(\mathcal{O}_{\bar{S}}(\bar{D}-E))\to H^0(\mathcal{O}_{\bar{S}}(\bar{D}))\to H^0(\mathcal{O}_E)\to H^1(\mathcal{O}_{\bar{S}}(\bar{D}-E))\to 0,$$

because $h^0(\mathcal{O}_{\bar{S}}(\bar{D}-E))=2$, $h^0(\mathcal{O}_E)=1$, $h^0(\mathcal{O}_{\bar{S}}(\bar{D}))=3$ we deduce that $2-3+1-h^1(\mathcal{O}_{\bar{S}}(\bar{D}-E))=0$, hence $h^1(\mathcal{O}_{\bar{S}}(\bar{D}-E))=0$.

Using now the exact sequence

$$0 \to \mathcal{O}_S(\bar{D}-E) \to \mathcal{O}_S(\bar{D}+\bar{\Gamma}-E) \to \mathcal{O}_\Gamma(\bar{D}+\bar{\Gamma}-E) \to 0$$

and the before vanishing we deduce the surjection

$$H^0(\mathcal{O}_S(\bar{D}+\bar{\Gamma}-E)) \twoheadrightarrow H^0(\mathcal{O}_\Gamma(\bar{D}+\bar{\Gamma}-E)).$$

From this and because $\mathcal{O}_\Gamma(\bar{D}+\bar{\Gamma}-E)$ is generated by its global sections it follows that the points P and Q are separated by $|D+\Gamma|=|B_o+2\Gamma|$, $(\forall) Q\in\Gamma$.

Let be now $Q\in S\setminus\Gamma$ $(Q\neq P)$ or $Q\in T_pS$.

a) If Q <u>is not</u> the base point of the cubics containing $P_1,\dots,P_7$, P, then $(\exists) D_o\in(3;1,\dots,1,0,0)$ such that $D_o\ni P$, $D_o\not\ni Q$. Because $Q\notin\Gamma$ it follows that $D_o+\Gamma\in|B_o+2\Gamma|$ contains P and do not contains Q.

b) If Q <u>is</u> the base point of the cubics containing $P_1,\dots,P_7$, P then there exists a smooth cubic $T\in(3;1,\dots,1,0,0)$ such that $T\ni P_1,\dots,P_7$, P, Q (see [De] , corollaire pag.4.1). In order to prove that $|B_o+2\Gamma|$ separates P and Q it is sufficient to prove that $\varphi_2/_T$ is an embedding. Or, using the exact sequence

$$0 \to \mathcal{O}_S(\Gamma) \to \mathcal{O}_S(B_o+2\Gamma) \to \mathcal{O}_T(B_o+2\Gamma) \to 0$$

and taking its cohomology we obtain the following exact sequence

$$0\to H^0(\mathcal{O}_S(\Gamma)) \to H^0(\mathcal{O}_S(B_o+2\Gamma)) \to H^0(\mathcal{O}_T(B_o+2\Gamma)) \to H^1(\mathcal{O}_S(\Gamma)) \to 0$$

Because

$$h^0(\mathcal{O}_S(\Gamma))=1, h^0(\mathcal{O}_S(B_o+2\Gamma))=5,$$

$h^0(\mathcal{O}_T(B_o+2\Gamma))=4$ (use the Riemann-Roch theorem on T)

it follows $1-5+4-h^1(\mathcal{O}_S(\Gamma))=0$.

Hence $h^1(\mathcal{O}_S(\Gamma))=0$, and hence the surjection

$$H^0(\mathcal{O}_S(B_o+2\Gamma)) \twoheadrightarrow H^0(\mathcal{O}_T(B_o+2\Gamma)).$$

It follows that φ_2/T is given by $\mathcal{O}_T(B_o+2\Gamma)$ which is very ample on T.

v) We use iv) if $x\in S\setminus\Gamma$ and the canonical surjections

$$H^0(\mathcal{O}_S(C_o+m\ )) \twoheadrightarrow H^0(\mathcal{O}_\Gamma(C_o+m\Gamma)),\ m\geq 1$$

$$H^0(\mathcal{O}_S(B_o+m\Gamma)) \twoheadrightarrow H^0(\mathcal{O}_\Gamma(B_o+m\Gamma)),\ m\geq 2$$

because Γ and L ($L \in \{L_m, L'_m \,/\, m \in \mathbb{N}^*\}$) are locally diffeomorphic by φ ($\varphi \in \{\psi_m, \psi_{m+1} \,/\, m \in \mathbb{N}^*\}$) in x, if x is not a ramification point (see [Sa], ch.VII, §3, th.1).

vi) follows from iv) and v)

vii) follows from i), ii) and vi).

Observation 1 (see [Ha 2], prop.3,2):

i) For any $m \in \mathbb{N}$ the complete without base points linear system $|C_0 + m\Gamma| \subset S$ contains a nonsingular (irreducible) curve Y of genus $g(Y)=2m$.

ii) For any $m \in \mathbb{N}^*$ the complete without base points linear system $|B_0 + m\Gamma| \subset S$ contains a nonsingular (irreducible) curve Y of genus $g(Y)=2m-1$.

Let's consider now

$$G := \{\sigma : \text{Pic } S \to \text{Pic } S \,\mid\, \sigma \text{ isomorphism of groups, } \sigma(\xi) \cdot \sigma(\eta) = \xi \cdot \eta, \ (\forall)\, \xi, \eta \in \text{Pic } S \text{ and } \sigma(\omega) = \omega\}.$$

("." denotes the intersection form on S and $\omega = \omega_S$ is the canonical sheaf of S).

The proof of the theorem 1:

Lemma 1 ([G-P] or [Ha 2] pag.308):

Let be $\sigma \in G$. Then there exist a morphism $\pi' : S \to \mathbb{P}^2$ which is the blowing up of 9 points $P'_1, \ldots, P'_9 \in \mathbb{P}^2$ satisfying the conditions (1), (2) in the beginning of this §, and such that

$$\pi'^{-1}(P'_i) = \sigma(E_i), \quad i = \overline{1,9}$$

$$\pi'^{*}(\mathcal{O}_{\mathbb{P}^2}(1)) = \sigma(\ell).$$

We divide the proof of the theorem in two parts, in accordance with the parity of $n \in \mathbb{N}$, $n \geq 3$.

If n is odd: $n=2m+1$, $m \geq 1$.

We divide this case in two steps:

Step 1:

For any $d\geq 2n-1=4m+1$, $(\forall)g\in\mathbb{N}$, $g\leq d-n$ there exists $Y\subset S$ a nonsingular (irreducible) curve of genus g satisfying the conditions vi) of the Proposition 1, such that $\varphi_m(Y)\subset\mathbb{P}^{2m+1}=\mathbb{P}^n$ is a nonsingular, nondegenerate curve of degree d.

Step 2: The general case.

The proof of step 1:

Let be $k\in\mathbb{N}$. Using the Observation i) it follows that $|C_o+k\Gamma|$ contains a nonsingular (irreducible) curve of genus $g=2k$.

Let be now $\sigma\in G$. Using Observation i) and Lemma 1 it follows that $\sigma(C_o+k\Gamma)$ contains a nonsingular curve Y of genus $g=2k$ and degree

(3) $d'=\deg\varphi_m(Y)=(C_o+m\Gamma).\sigma(C_o+k\Gamma)=C_o.\sigma(C_o)+2k+2m.$

We use now the point i) in the

Proposition 2:

i) The intersections $C_o.\sigma(C_o)$ take any integer value ≥ 1 when σ cover G.

ii) The intersections $C_o.\sigma(B_o)$ take any integer value $\geq$ (when σ cover G.

iii) The intersections $B_o.\sigma(C_o)$ take any integer value $\geq$ when σ cover G.

iv) The intersections $B_o.\sigma(B_o)$ take any integral value $\geq$ ≥ -1 when σ cover G.

Observation 2. In [Ha 2] are proved i) and iii). iv) and ii) will be proved in the Appendix.

Using proposition 2i) and (3) it follows that we proved the st for g=2k (even) as soon as we proved that the curve Y which g

(3) satisfies the conditions vi) and vii) in Proposition 1.

The first condition in vi) is clear. (in Bertini's theorem used in order to obtain Y it is possible to avoid a finite number of points).

For the second conditions in vi) we observe first that we need d' such that $d' \geq 2n-1=4m+1$. Using (3) this means that

(4) $\quad C_o \cdot \sigma(C_o) \geq 2(m-k)+1.$

Secondly, if Y contains a fibre of $\varphi_{m/\Gamma}$, using the injective map

$$\mathrm{Pic}\ S \hookrightarrow \mathrm{Pic}\ \Gamma$$

(which follows from the condition (2)) it follows that $\sigma(C_o+k\Gamma)=C_o+m\Gamma$, hence (intersecting with C_o)

(5) $\quad C_o \cdot \sigma(C_o)=2(m-k).$

The conditions (4) and (5) are incompatible and because we need σ as in (4) the relation (5) is automatically unsatisfied (hence the second condition in vi) is automatically satisfied).

The condition in Proposition 1 vii) is also satisfied because $d \geq 2n-1=4m+1$ $(d=Y\cdot(C_o+m\Gamma))$.

In order to obtain the odd genus $g=2k-1$ $(k\geq 1)$ too, we proceed in the same manner, using curves $Y\in\sigma(B_o+k\Gamma)$, $k\geq 1$ and the Proposition 2 ii).

<u>The proof of step 2:</u>

Let be d_1, $g_1 \in \mathbb{N}$, $d_1 \geq 4m+1$, $g_1 \leq d_1-(2m+1)$.

Using the step 1 it follows that there exists $Y_1 \subset S$ such that $C_1 := \varphi_m(Y) \subset \mathbb{P}^n = \mathbb{P}^{2m+1}$ has

$$(6) \quad \begin{cases} \bullet\ g(C_1) = g(Y_1) = g_1 \\ \bullet\ \deg C_1 = d_1 \\ \bullet\ C_1 \text{ is nonsingular (irreducible) and nondegenerate} \\ \quad \text{in } \mathbb{P}^n \end{cases}$$

Because $|Y_1|$ and $|r(C_0+m\Gamma)|$, $r\geq 1$ has no base points it follows that $|Y_1+r(C_0+m\Gamma)|$ has no base points. More, using the standard exact sequences we compute

$$h^0(\mathcal{O}_S(Y_1+r(C_0+m\Gamma)))=2mr^2+r+2+2g_1+rd_1$$

hence $\dim|Y_1+r(C_0+m\Gamma)|\geq 2$.

Using Bertini's theorem we find $Y\in|Y_1+r(C_0+m\Gamma)|$ o smooth (irreducible) curve. It can be proved as before that Y satisfies i) and ii) of the Proposition 1. Hence $C:=\varphi_m(Y)$ is nonsingular and nondegenerate in $\mathbb{P}^{2m+1}$.

If $d=\deg C$, $g=g(C)$ we have

$$\begin{cases} d_1 = d-4mr \\ g_1 = g-r(d-2mr-1) \end{cases}$$

The inequality $0\leq g_1\leq d_1-(2m+1)$ can be written as

$$F_d(r-1)\leq g\leq F_d(r)$$

where $F_d(r)=(r+1)(d-2mr-2m-1)$.

Because $\max_r F_d(r)=\frac{1}{8m}(d-1)^2$ it follows that we obtain all genus g with $g\leq\frac{1}{8m}(d-1)^2=\frac{1}{4(n-1)}(d-1)^2$.

<u>If n is even</u>: $n=2m$, $m\geq 2$ the proof is absolutely similar using the morphism ψ_m (instead of the morfisms φ_m).

§2. More on the nondegenerate curves in $\mathbb{P}^4$ and $\mathbb{P}^5$

In the case of $\mathbb{P}^4$ and $\mathbb{P}^5$ we can obtain a better bound as in the Theorem 1, using this theorem and curves on Del Pezzo surfaces. For the general theory of the Del Pezzo surfaces we refer to [De]. The results which we need are listed in [Pa] too.

Let be X a Del Pezzo surface of degree $d\in\{4,5\}$ ($d=\omega_X\cdot\omega_X$). X is obtained using the embedding in $\mathbb{P}^d$ given by the (very ample) anticanonical sheaf of the surface X', obtained by blowing up (9-d) points in <u>general position</u> (i.e. no three of them are colliniar) from $\mathbb{P}^2$.

Hence Pic $X\cong\mathbb{Z}\oplus\mathbb{Z}^{9-d}$ with $\ell, -e_1,\dots,-e_{9-d}$ as a $\mathbb{Z}$-basis (as in §1).

If $D=a\ell-\sum_{i=1}^{9-d} b_ie_i$ we shall write $D=(a; b_1,\dots,b_{9-d})$.

In [G-P] (see [Ha2] too) is proved the following

<u>Proposition A:</u>

For any $d\geq 5$ and $g\in\mathbb{N}$ such that

$$\frac{1}{\sqrt{3}}d^{3/2}-d+1<g\leq\frac{1}{6}d(d-3)+1$$

there exists a smooth (irreducible) curve C of degree d and genus g on a nonsingular cubic surface in $\mathbb{P}^3$ (which is a Del Pezzo surface of degree 3).

Combining the Proposition A with the case n=3 of the theorem 1 (and completing the gaps with curves which lie on the cubic surface in $\mathbb{P}^3$), Gruson and Peskine ([G-P]) obtaind the following

<u>Proposition B</u>

For any $d,g\in\mathbb{N}$, $d\geq 3$ such that

$$0\leq g\leq \frac{1}{6}d(d-3)+1$$

there exists a smooth (irreducible) curbe $C\subset\mathbb{P}^3$ of degree d and genus g.

Here we proceed similarly, using curves which lie on the Del Pezzo surfaces of degrees 4 and 5 and the cases n=4 and n=5 of the Theorem 1.

Proposition 3:

Let be $X\subset\mathbb{P}^4$ a Del Pezzo surface of degree 4. Then, for any $d,g\in\mathbb{N}$, $d\geq 10$ and

$$d\sqrt{d+9}-5d+10\sqrt{d+9}-2g\leq g\leq \frac{1}{8}d(d-4)+1$$

there exists a smooth (irreducible) curve $C\subset X$ of degree d and genus g, which is non degenerate in $\mathbb{P}^4$.

The proof of this proposition is similar with the proof of the proposition A (see [G-P] or [Ha2]) using the followings lemmas:

Lemma 1:

Let be $k\in\mathbb{N}^*$. Then any number $n\in\mathbb{N}^*$ with $n\leq(k+1)^2$ can be written as $n=\sum_{i=1}^{4}x_i^2$, $x_i\in\mathbb{Z}$, $|x_i|\leq k$, $i=\overline{1,4}$.

Lemma 2:

Let be $k\in\mathbb{N}^*$ odd. Then any $n\in\mathbb{N}^*$, $n\equiv 4 \pmod 8$ and $n<2(k^2+2k+3)$ can be written as $n=\sum_{i=1}^{4}x_i^2$, $x_i\in\mathbb{Z}$, x_i odd, $|x_i|\leq k$, $i=\overline{1,4}$.

Lemma 3:

Let be X as in the Proposition 3 and $D=(a;b_1,\ldots,b_5)\in\mathrm{Pic}\,X$, $D\neq 0$ and $D\neq(n;n,0,0,0,0)$, $n>1$.

Then, if

$$(6') \quad \begin{cases} a \geq b_1 + b_2 + b_3 \\ b_1 \geq \dots \geq b_5 \geq 0 \end{cases}$$

the complete linear system $|D| \subset X$ contains a smooth (irreducible) curve C. More,

$$d = \deg C = 3a - \sum_{i=1}^{5} b_i$$

$$g = g(C) = \frac{1}{2}(a^2 - \sum_{i=1}^{5} b_i^2 - d) + 1$$

Making the changement of the variables

$$\begin{cases} r = a - b_1 \\ \alpha_i = \frac{1}{2}r - b_i \end{cases} \qquad i = \overline{2,5}$$

the conditions $(6')$ can be written, equivalently as

$$(E) \quad \begin{cases} \alpha_i \equiv \frac{1}{2}r \pmod 1, \qquad i = \overline{2,5} \\ d + r - \sum_{i=2}^{5} \alpha_i \equiv 0 \pmod 2 \\ |\alpha_2| \leq \alpha_3 \leq \alpha_4 \leq \alpha_5 \leq \frac{1}{2} r \\ -\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 \leq d - 2r \end{cases}$$

where $d = 2a - r + \sum_{i=2}^{5} \alpha_i$. More,

$$g(C) := g = \overline{F}_d(r) - \frac{1}{2} \sum_{i=2}^{5} \alpha_i^2, \quad \overline{F}_d(r) = \frac{1}{2}((r-1)d - r^2) + 1.$$

<u>Lemma 4</u>

Let be $d \geq 10$, $d \in \mathbb{N}$, $r \in \mathbb{Z}$ such that

$$2(\sqrt{d+9}-3)\leq r\leq \frac{d}{2}$$

and $g\in\mathbb{Z}$, $\overline{F}_d(r-1)<g\leq\overline{F}_d(r)$.

Then there exists $\alpha_i\in\frac{1}{2}\mathbb{Z}$, $i=\overline{2,5}$ satisfying (E) and such that $g=\overline{F}_d(r)-\frac{1}{2}\sum_{i=2}^{5}\alpha_i^2$. The proofs of the lemmas 1-4 are inspirated from [Ha2]. As in [Ha2], using these lemmas we prove the Proposition 3.

Corollary:

Let be $g,d\in\mathbb{N}$, $d\geq 5$ and $g\leq\frac{1}{8}d(d-4)+1$. Then there exists a smooth, nondegenerate (irreducible) curve $C\subset\mathbb{P}^4$ of degree d and genus g.

Proposition 4:

Let be $X\subset\mathbb{P}^5$ a Del Pezzo surface of degree 5. Then for any $d,g\in\mathbb{N}$, $d\geq 32$ and

$$(d+25)\sqrt{2d+39}-\frac{21}{2}d-159\leq g\leq\frac{1}{10}d(d-5)-\frac{3}{2}$$

there exists a smooth (irreducible) curve $C\subset X$ of degree d and genus g, which is nondegenerate in $\mathbb{P}^5$.

The proof is similar with the proof of the Proposition 3 using the following lemmas:

Lemma 5:

Let be $k\in\mathbb{N}$ odd, $n\in\mathbb{N}$, $n\equiv 3 \pmod 8$, $n<k^2+4k+6$. Then n can be written as $n=\sum_{i=1}^{3}x_i^2$, x_i odd, $|x_i|\leq k$, $x_i\in\mathbb{Z}$, $i=\overline{1,3}$.

Lemma 6:

Let be X as in the Proposition 4 and

$D=(a;b_1,\dots,b_4)\in\mathrm{Pic}\,X$, $D\neq 0$, $D\neq(n;\,n,0,0,0)$, $n>1$.

Then, if

$$(7)\qquad \begin{cases} a \geq b_1 + b_2 + b_3 \\ b_1 \geq b_2 \geq b_3 \geq b_4 \geq 0, \end{cases}$$

the complete linear system $|D| \subset X$ contains a smooth (irreducible) curve $C \subset X$. More, $d = \deg C = 3a - \sum_{i=1}^{4} b_i$

$$g = g(C) = \frac{1}{2}(a^2 - \sum_{i=1}^{4} b_i^2 - d) + 1.$$

Making the substitutions

$$\begin{cases} r = a - b_1 \\ \alpha_i = \frac{1}{2} r - b_i \ , \quad i = \overline{2,4}, \end{cases}$$

the conditions (7) can be written, equivalently as

$$(\widetilde{E})\qquad \begin{cases} \alpha_i \equiv \frac{1}{2} r \pmod 1, \ i = 2,4 \\ d + \frac{1}{2} r - \sum_{i=2}^{4} \alpha_i \equiv 0 \pmod 2 \\ |\alpha_2| \leq \alpha_3 \leq \alpha_4 \leq \frac{1}{2} r \\ -\alpha_2 + \alpha_3 + \alpha_4 \leq d - \frac{5}{2} r \ (\alpha) \end{cases}$$

where $d = 2a - \frac{1}{2} r + \sum_{i=2}^{4} \alpha_i$. More,

$$g(C) = g = \widetilde{F}_d(r) - \frac{1}{2} \sum_{i=2}^{4} \alpha_i^2 \ , \ \widetilde{F}_d(r) = \frac{1}{2}((r-1)d - \frac{5}{4} r^2) + 1.$$

Lemma 2:

Let be $d \geq 25$, $d \in \mathbb{N}$, $r \in \mathbb{Z}$ odd such that

$$2(\sqrt{39+2d} - 6) \leq r \leq \frac{2d}{5} - 1$$

and $g \in \mathbb{Z}$; $\widetilde{F}_d(r-2) < g \leq \widetilde{F}_d(r)$ $\qquad (\beta)$.

Then there exist $\alpha_i \in \frac{1}{2}\mathbb{Z}$, $i=\overline{2,4}$ satisfying $(\tilde{E})$ such that

$$g=\tilde{F}_d(r)-\frac{1}{2}\sum_{i=2}^{4}\alpha_i^2.$$

Obs: Because (α) and (β) we obtain the boound on the right $\frac{1}{10}d(d-5)-\frac{3}{2}$ and not $\frac{1}{10}d(d-5)+1$.

Corollary:

Let be $d,g\in\mathbb{N}$, $d\geq 7$ and $g\leq\frac{1}{10}d(d-5)-\frac{3}{2}$. Then there exists a smooth, nondegenerate (irreducible) curve $C\subset\mathbb{P}^5$ of degree d and genus g.

APPENDIX: The study of a quadratic form on $\mathbb{Z}^{10}$

Let be $V=\mathbb{Z}^{10}$ with $\{e_0,\dots,e_9\}$ a $\mathbb{Z}$-basis and Q the quadratic form $x_0^2-\sum_{i=1}^{9}x_0^2$. Let's denote by (.) the scalar product defined by Q. Let be $\omega=(-3;-1,-1,-1)$. ($\omega^2=0$). Let be $H\subseteq V$, $H=\{\lambda\in V/(\lambda.\omega)=0\}=\omega^{\perp}$. Let be $W=H/(\omega)$. The quadratic form induced on W is even (see [Ha 2]).

Let be G the group of the isometries σ of V such that ω is invariable by σ.

Let be G_H the group of the isometries τ of H such that ω is invariable by τ. Let be G_W the group of the isometries of W. Let be $G_{\overline{W}}$ the group of the isometries of $\overline{W}=W/2W$ with the quadratic form $\overline{Q}=\frac{1}{2}Q(x)$ (mod 2).

Lemma A1

Let be $k\in\mathbb{Z}$. The elements $x\in V$ such that

(i) $x^2=k$

(ii) $(x.\omega)=-2$

(iii) $\overline{x}\neq 0$ (modω) in $\overline{V}=V/2V$

form one complete orbit under the natural action of G on V.

Proof: The case k=0 is proved in [Ha 2] . The general case is completely similar.

Lemma A2 ([Ha 2], pag.311, [G-P]):

For any $n \in \mathbb{N}^*$ there exists $x \in H$ such that $n = -\frac{1}{2}(x.x)$ and $\overline{x} \neq 0$ in $\mathcal{W}$.

Lemma A3

Let be $n \in \mathbb{N}^*$, $n \equiv 0,4,7 \pmod 8$. Then there exist $\theta_1, \ldots, \theta_6 \in \mathbb{Z}$, not all even , such that

$$\sum_{i=1}^{6} \theta_i = 0, \ \sum_{i=1}^{6} \theta_i^2 = 2n.$$

Proof:

Use the Gauss' theorem on the sum of three squares for n (see [Se] , pag.45). Let be $c = (1; 1,-,0,0) \in V$

$b = (0;0,\ldots,0,-1,-1) \in V.$

Corolary A4

Let be $n \in \mathbb{N}^*$ and $x \in H$, $x^2 = -2n$ and $\overline{x} \neq 0$ in $\mathcal{W}$ (cf. lemma A1).

Then:

i) There exists $\sigma_1 \in G$ such that:

$\overline{\sigma_1(x)} \neq 0$ in $\mathcal{W}$; $\overline{\sigma_1(x)} + \overline{c} \neq 0 \pmod{\omega}$ in $\overline{V} = V/2V$;

$$(c.\sigma_1(x)) \equiv 0 \pmod 2)$$

ii) There exists $\sigma_2 \in G$ such that

$\overline{\sigma_2(x)} \neq 0$ in $\overline{W}$; $\overline{\sigma_2(x)} + \overline{c} \neq 0 \pmod{\omega}$ in $\overline{V}$;

$$(c.\sigma_2(x)) \equiv 1 \pmod 2).$$

iii) There exists $\sigma_3 \in G$ such that:

$\overline{\sigma_3(x)} \neq 0$ in $\mathcal{W}$, $\overline{\sigma_3(x)} + \overline{b} \neq 0 \pmod{\omega}$ in $\overline{V}$;

$$(b.\sigma_3(x)) \equiv 0 \pmod 2).$$

iv) There exists $\sigma_4 \in G$ such that:

$$\overline{\sigma_4(x)} \neq 0 \text{ in } \overline{W}, \quad \overline{\sigma_4(x)} + \overline{b} \neq 0 \pmod{\omega} \text{ in } \overline{V},$$

$$(b.\sigma_4(x)) \equiv 1 \pmod 2.$$

Proof: Use Lemma A3 and the Witt's theorem on quadratic forms (see [Se] pag.31 or [La] pag.360).

Proposition A5 (iii) and (iv) from the Proposition 2, §1)

Let be $C=(1,1,0,\ldots,0)$, $b=(0;0,\ldots,0,-1,-1) \in V$. Then:

i) The intersections $(C.\sigma(b))$ take any integer value ≥ 0 when σ cover G.

ii) The intersections $(b.\sigma(b))$ take any integer value ≥ -1 when cover G.

Proof

i) Let be $n \in \mathbb{N}^*$ and $x \in H$, $\overline{x} \neq 0$ in $\overline{W}$ such that $n = -\frac{1}{2}(x.x)$ (lemma A2).

Using the corollary A4, substituting if necessary x with $\sigma_0(x)$ for some $\sigma_0 \in G$ we can suppose that

$$(C+x)^2 \equiv 2 \pmod 4, \quad \overline{C} + \overline{x} \neq 0 \pmod{\omega} \text{ in } \overline{V} = V/2V.$$ Let be now

$$C' = C + x + \frac{1}{4}((C+x)^2 + 2)\omega.$$

Using lemma A1 (k=-2) it follows that $C' = \sigma(b)$ for some $\sigma \in G$.

Then $(C.\sigma(b)) = (C.C') = n-1 \geq 0$.

ii) We use the same idea. Let be

$$b' = b + x + \frac{1}{4}((b+x)^2 + 2)\omega$$

(use iii) and iv) in the Corollary A4).

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Using lemma A1 ($k=-2$), it follows that

$b'=\sigma(b)$ for some $\sigma\in G$. Then

$$(b.\sigma(b))=(b.b')=n-2\gamma-1.$$

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