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EXTENSIONS OF THE (BCP) - TECHNIQUES

by

Andrei HALANAY

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Andrei HALANAY*)

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*) Mathematical Chair 1, Polytechnical Institute, Splaiul Independentei 313
79590 Bucharest, Romania.

EXTENSIONS OF THE (BCP) - TECHNIQUE

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§ 1

In [2] it is proved that, for a completely nonunitary contraction of the class $C_o.$, whose left essential spectrum dominates the unit circle, every element in the predual of $\mathcal{A}_A$, the weak*-closed algebra generated by A in $\mathcal{L}(\mathcal{H})$, is of the form $[x \otimes y]$ with x and y in $\mathcal{H}$. In subsequent works Apostol, Bercovici, Chevreau, Foiaş, Pearcy, Robel, Sz.Nagy generalized this result in several directions (see [1] for a full bibliographical account).

So, it is proved that for an absolutely continuous contraction A ([1] Ch.IV) of the class $C_o.$, whose essential spectrum dominates the unit circle, for every infinite matrix $[L_{ij}]_{i,j\geq 1}$, of elements in the predual of $\mathcal{A}_A$ there exist sequences $\{x_i\}_{i\geq 1}$, $\{y_j\}_{j\geq 1}$ in $\mathcal{H}$ such that

$$[L_{ij}]=[x_i \otimes y_j] \quad \text{for every } i,j \geq 1.$$

In this paper we observe that the same result is true for some weak*-closed subspaces of $\mathcal{A}_A$ using only the presence in the left essential spectrum of A of an interpolating sequence for H^∞ (see [6], [7]). Such contractions can be produced by an obvious modification of the construction in [5] . The proof relies on a theorem of a general nature which is what we need from the generalizations of the (BCP)-technique.

The Hilbert space is assumed to be separable and the notations and definitions will be those from [1].

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§ 2

Definition 2.1. Let $\mathcal{M} \subset \mathcal{L}(\mathcal{H})$ be a weak*-closed subspace and let n be any cardinal number such that $1 \leq n \leq \aleph_0$. $\mathcal{M}$ will be said to have property (A_n) provided every n x n system of simultaneous equations of the form

$$[x_i \otimes y_j] = [L_{ij}], \qquad 0 \leq i,j < n$$

with L_{ij} arbitrary, fixed elements from $Q_{\mathcal{M}}$, the predual of $\mathcal{M}$, has a solution $\{x_i\}_{0 \leq i < n}$, $\{y_j\}_{0 \leq j < n}$ consisting of a pair of sequences of vectors from $\mathcal{H}$.

Theorem 2.2. Let A be an absolutely continuous contractions in $\mathcal{L}(\mathcal{H})$, $\sigma(A) = \sigma_{le}(A)$, $\sigma(A) \cap \mathbb{D} \neq \emptyset$ ($\mathbb{D}$ is the open unit disc in $\mathbb{C}$) and $A^n \to 0$ for $n \to \infty$ in the strong operator topology (that is $A \in C_{0\cdot}$). Let $\mathcal{M}$ be a weak*-closed subspace of H^∞ with the following property:

there exists $M > 0$ such that for $h \in \mathcal{M}$

$$\|h\|_\infty \leq M \sup_{\lambda \in \sigma(A) \cap \mathbb{D}} |\hat{h}(\lambda)| \tag{2.1}$$

Let $\mathcal{A}_A$ be the ultraweakly closed algebra generated by A in $\mathcal{L}(\mathcal{H})$ (see [1]) and $\Phi : H^\infty \longrightarrow \mathcal{A}_A$ the canonical homomorphism $\Phi(h)=h(A)$, and let $\Phi(\mathcal{M})$ be denoted by $\mathcal{M}_A$. Then $\mathcal{M}_A$ is a weak*-closed subspace of $\mathcal{L}(\mathcal{H})$ with property $(\mathbb{A}_{\aleph_0})$.

<u>Proof</u>. $\|\Phi(h)\| \leq \|h\|_\infty \leq M \sup_{\lambda \in \sigma(A) \cap \mathbb{D}} |\hat{h}(\lambda)| \leq$

$\leq M \|h(A)\| = M\|\Phi(h)\|$ for every $h \in \mathcal{M}$ (see [1], [3] Lemma 3.1).

$\mathcal{M}$ is closed in H^∞ and from the previous inequalities $\mathcal{M}_A$ results norm closed in $\mathcal{A}_A$. Φ is weak*-continuous so from [2] Theorem 2.7 and Corollary 2.4, Φ induces a weak* homeomorphism between $\mathcal{M}$ and $\mathcal{M}_A$.

For any fixed $\lambda \in \mathbb{D}$ the map $h \mapsto \hat{h}(\lambda)$ is a weak*-continuous linear functional on $\mathcal{M}$.

Since the map $h(A) \mapsto h$ is a weak* homeomorphism of $\mathcal{M}_A$ onto $\mathcal{M}$, the map $h(A) \mapsto \hat{h}(\lambda)$ is a weak* continuous linear functional on $\mathcal{M}_A$, so there exists an element $[C_\lambda]$ in $Q_{\mathcal{M}_A}$, the predual of $\mathcal{M}_A$, such that

$$\langle [C_\lambda], h(A) \rangle = tr(h(A)C_\lambda) = \hat{h}(\lambda) \quad \text{for all } h \text{ in } \mathcal{M}$$

Condition (2.1) implies that

$\overline{aco}\,\{[C_\lambda] \mid \lambda \in \sigma(A) \cap \mathbb{D}\}$ contains the closed ball of radius $\frac{1}{M}$ about the origin in $Q_{\mathcal{M}_A}$ ([1], Proposition 1.21).

By [2] Lemmas 4.3, 4.4 and 4.5,

$\overline{aco}\,\{[C_\lambda] \mid \lambda \in \sigma(A) \cap \mathbb{D}\} \subset \mathcal{X}_0(\mathcal{M}_A)$ (see [1] Definition 2.7) so $\mathcal{M}_A$ has property $X_{0,\frac{1}{M}}$ ([1] Definition 2.8). The conclusion of the theorem follows using the same technique that leads to Theorem 3.7 from [1] or to Theorem 3.14 from [9].

<u>Remark</u>. The same theorem is true for an absolutely continuous contraction A of class C_{00} with $\sigma(A)=\sigma_e(A)$, $\sigma(A)\cap \mathbb{D}\neq\emptyset$.

§ 3

To apply theorem 2.2 we consider A an absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$, $A\in C_{0\cdot}$, $\sigma(A)=\sigma_{le}(A)$ and $\sigma(A)\cap\mathbb{D}$ contains an interpolating sequence of exponential type, $\{z_j\}_{j=0}^{\infty}$, that is

$$|z_j|\to 1 \quad \text{for } j\to\infty \quad \text{and}$$

$$\frac{1-|z_{j+1}|}{1-|z_j|}<a<1 \quad \text{for every } j\geq 0 \ ([7]).$$

We construct two weak*-closed subspaces of H^{∞}, $\mathcal{M}_1$ and $\mathcal{M}_2$ for which condition (2.1) is fulfilled. The first one $\mathcal{M}_1$, is isomorphic and weak* homeomorphic with l^1, the space of summable complex sequences. It is generated by a sequence $\{h_n\}_{n=0}^{\infty}$ in H^{∞} interpolating a properly chosen sequence in l^{∞}, the space of bounded complex sequences. The construction relies on the properties of some Sidon set.

The second subspace, $\mathcal{M}_2$, is isomorphic and weak*-homeomorphic with l^{∞}, being generated by the sequence $\{f_j\}_{j=0}^{\infty}$ given in [4]. We proceed now to detail the construction of the two subspaces and begin with $\mathcal{M}_1$.

Let $C(\mathbb{T})$ be the space of complex valued continuous functions on the unit circle $\mathbb{T}$, denote by $\hat{f}(n)$ the n-th Fourier coefficient $(n\in\mathbb{Z})$ of f and, for P a subset of $\mathbb{Z}$, let

$$C_P(\mathbb{T})=\{f\in C(\mathbb{T})\,|\,\hat f(n)=0 \text{ for every } n \text{ in } \mathbb{Z}-P\}$$

Let $E=\{2^k\,|\,k\geqslant 1\}\cup\{0\}$.

E is a Sidon set (see [8], [10]) and we denote by C the positive constant for which

(3.1) $$\sum_{n=0}^{\infty}|\hat f(n)|\leqslant C\,\|f\|_\infty \quad \text{for every } f\in C_E(\mathbb{T})$$

We identify $C_E(\mathbb{T})$ with a weak*-closed subspace of H^∞ and subsequently refer to it as a subspace of H^∞.

Define $f_k(z)=\frac{1}{2}(z^{2^{2k-1}}-z^{2^{2k}})$ for $k\geqslant 1$

$f_o(z)=1$ for every $z\in\mathbb{D}$

and denote by F the closed linear subspace generated by $\{f_k\}_{k\geqslant 0}$ in H^∞. It is easy to see that there exists an isomorphism U between l^1 and F such that $U(e_k)=f_k$, where $\{e_k\}_{k\geqslant 0}$ is the canonical basis in l^1. Moreover, F is weak*-closed in H^∞ and U is a weak* homeomorphism of $l^1=(c_o)^*$ onto F (c_o is the space of complex sequences converging to 0).

Define $\mathcal{E}_k=\{\mu\in\mathbb{T}\,|\,f_k(\mu)=0\}$ for $k\geqslant 1$.

Obviously $\mathcal{E}_k\subset\mathcal{E}_{k+1}$. Put $\mathcal{E}=\bigcup_{k\geqslant 1}\mathcal{E}_k$. $\mathcal{E}$ is a countable dense subset of T and let $\mathcal{E}=\{\mu_j\}_{j=0}^{\infty}$ be an ennumeration of $\mathcal{E}$ such that for $\mu_j\in\mathcal{E}_{k_j}$, $\mu_i\in\mathcal{E}_{k_i}$ and $j\leqslant i$ we have $\mathcal{E}_{k_j}\subset\mathcal{E}_{k_i}$.

<u>Proposition 3.1</u>. Let $w_n\in l^\infty$ with components $w_{n,j}=f_n(\mu_j)$, $j\geqslant 0$, $n\geqslant 0$. There exists $\{h_n\}_{n=0}^{\infty}$ in H^∞ such

that $h_n(z_j)=w_{n,j}$ for $j \geq 0$, $n \geq 0$, $\|h_n\|_\infty \leq M_1$ for every $n \geq 0$ and

$$h_n \xrightarrow{w^*} 0 \quad \text{as } n \to \infty$$

<u>Proof</u>. Observe that for $\mu \in \mathcal{E}_k$, $f_n(\mu)=0$ for every $n \geq k$. Then $w_{n,j}=f_n(\mu_j)=0$ for $j \leq 2^{2n-1}$, $n \geq 1$.

Let $u_j=1-|z_j|$, $j \geq 0$. Then $0<|z_j|<1$ implies $0<u_j<1$ for every $j \geq 0$.

$\dfrac{1-|z_{j+1}|}{1-|z_j|}<a<1$ implies $u_j<a^j.u_o$ for every $j \geq 1$. Let $j>n$. $|z_j|^n=(1-u_j)^n \geq (1-a^j.u_o) \geq (1-a^n.u_o)^n \to 1$ as $n \to \infty$. Then $\left\{\dfrac{w_{n,j}}{z_j^n}\right\}_{j=0}^{\infty}$ is in l^∞ for every $n \geq 0$ and there exists $M_1' > 0$ such that

$$\left\|\left\{\frac{w_{n,j}}{z_j^n}\right\}_{j=0}^{\infty}\right\|_{l^\infty} \leq M_1' \quad \text{for every } n \geq 0$$

Let $g_n \in H^\infty$ be such that $g_n(z_j)=\dfrac{w_{n,j}}{z_j^n}$, $n \geq 1$ and $g_o(z)=1$ for every $z \in \mathbb{D}$.

[7] § 10 or [6] Chap.VII implies that there exists $M_1>0$ such that $\|g_n\|_\infty \leq M_1$ for every $n \geq 0$. The sequence

$$h_n(z)=z^n g_n(z) \quad \text{defined for all } n \geq 0$$

satisfies the requirements of the proposition.

Let $\mathcal{M}_1$ be the closed linear subspace generated by $\{h_n\}_{n=0}^{\infty}$ in H^∞.

Proposition 3.2. $\mathcal{M}_1$ is isomorphic and weak* homeomorphic with l^1 (so $\mathcal{M}_1$ is weak*-closed in H^∞).

Proof. Let $\tilde{U} : l^1 \longrightarrow \mathcal{M}_1$

$$\tilde{U}\{\alpha_n\}_{n=0}^{\infty} = \sum_{n=0}^{\infty} \alpha_n h_n$$

$\|\tilde{U}\{\alpha_n\}_{n=0}^{\infty}\|_\infty \leq M_1 \|\{\alpha_n\}_{n=0}^{\infty}\|_{l^1}$ (M_1 is the constant from Proposition 3.1).

Observe that $\{w_n\}_{n=0}^{\infty}$ is basic in l^∞ ([12] I §7) so $\tilde{U}$ is injective.

It is easy to prove $\tilde{U}$ is weak*-continuous (use $h_n \xrightarrow{w^*} 0$ for $n \to \infty$). Since for an element $h = \sum_{n=0}^{\infty} \alpha_n h_n$ in $\mathcal{M}_1$ we have

$$h(z_j) = \sum_{n=0}^{\infty} \alpha_n h_n(z_j) = \sum_{n=0}^{\infty} \alpha_n f_n(\mu_j) \text{ then } \sum_{n=0}^{\infty} |\alpha_n| \leq C\|h\|_\infty$$

(C is the constant in (3.1)) and we deduce $\{h_n\}_{n=0}^{\infty}$ is a basis of $\mathcal{M}_1$ and $\tilde{U}$ is onto. The proposition follows using [2] Th.2.7.

Propositon 3.3. Theorem 2.2 holds for $\mathcal{M}_1$.

Proof. All we have to prove is inequality (2.2). Let h be in $\mathcal{M}_1$, $h = \sum_{n=0}^{\infty} \alpha_n h_n$. Then $\{\alpha_n\}_{n=0}^{\infty}$ is in l^1 and

$$\|h\|_\infty \leq M_1 \|\{\alpha_n\}_0^\infty\|_{l^1} \leq M_1 C \left\|\sum_{n=0}^{\infty} \alpha_n f_n\right\|_{C(\mathbb{T})} =$$

$$= M_1 C \sup_{\mu_j \in \mathcal{E}} \left|\sum_{n=0}^{\infty} \alpha_n f_n(\mu_j)\right| = M_1 C \sup_j \left|\sum_{n=0}^{\infty} \alpha_n h_n(z_j)\right| =$$

$$= M_1 C \sup_j |h(z_j)| \leq M_1 C \sup_{\lambda \in \sigma(A) \cap \mathbb{D}} |h(\lambda)|$$

since $\{z_j\}_{j=0}^{\infty}$ is in $\sigma(A)\cap\mathbb{D}$.

So we have (2.1) with $M=M_1C$.

Corollary 3.4. There exist x and y in $\mathcal{H}$ such that $(x,y)=1$ and $(h_n(A)x,y)=0$ for every $n\geqslant 1$.

Proof. Take $[L]=[C_o]$ the evaluation at 0 and apply theorem 2.2 to $\mathcal{M}_1$.

To construct $\mathcal{M}_2$ we follow [4], [6]. Let

$$(3.2)\qquad M_2=\sup_{\|\{a_j\}_j\|_{l^\infty}\leqslant 1}\inf\{\|f\|_\infty \mid f\in H^\infty, f(z_j)=a_j, j\geqslant 0\}$$

There exists $\{f_j\}_{j=0}^{\infty}$ in H^∞ such that

$$(3.3)\qquad f_j(z_j)=1,\ f_j(z_k)=0 \text{ for } j\neq k \text{ and}$$

$$(3.4)\qquad \sum_{j=0}^{\infty}|f_j(z)|\leqslant M_2 \quad \text{for every } z\in\mathbb{D}$$

Let $\mathcal{M}_2$ be the closed linear subspace generated by $\{f_j\}_{j=0}^{\infty}$ in H^∞. By [6] Ch.VII $\mathcal{M}_2$ is a complemented subspace of H^∞ isomorphic with l^∞ through the canonical operator

$\{a_j\}_{j=0}^{\infty}\mapsto\sum_{j=0}^{\infty}a_jf_j$, the norm of this operator being M_2.

Condition (3.4) insures the weak* continuity of this operator so by [2] Th.2.7 $\mathcal{M}_2$ is weak*-closed in H^∞.

Propositon 3.5. Theorem 2.2 holds for $\mathcal{M}_2$.

Proof. Let h be in $\mathcal{M}_2$, $h=\sum_{j=0}^{\infty}a_jf_j$ with $\{a_j\}_{j=0}^{\infty}$ in l^∞

$$\|h\|_\infty \leq M_2 \|\{a_j\}_{j=0}^{\infty}\|_{l^\infty} = M_2 \sup_j |a_j| =$$

$$= M_2 \sup_j |h(z_j)| \leq M_2 \sup_{\lambda \in \sigma(A) \cap \mathbb{D}} |h(\lambda)|$$

since $\{z_j\}_{j=0}^{\infty}$ is in $\sigma(A) \cap \mathbb{D}$.

(2.1) holds with $M=M_2$ and the conclusion follows.

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Andrei HALANAY
Mathematical Chair 1
Polytechnical Institute
Splaiul Independenţei 313
79590 Bucharest, Romania.