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by T.Constantinescu

I INTRODUCTION

The realization theory of time-variant linear systems is by now wounded up and almost complete (the Sz.-Nagy-Foias model(see [8]) and the Ho-Kalman algorithm(see [6])-moreover, for other remarks concerning the operator theoretic approach to system and network theory see for instance [1], [6], [7]). The Sz.-Nagy-Foias model explicitely uses the energy conservation law through the notion of unitary dilation of a contraction in Hilbert space and gives a geometrical interpretation for the dynamic of the system (essentiall for the transfer function of the system).

In recent years, several attempts at formalisms working in both time-variant and time-invariant cases appeared in system theory and one of the most illustrative paper in this area is [5]. Along this line, a formalism (in [2] we call it Schur analysis) can be developed, where the shift operator (which implies function theory) is replaced by "marking operators" and the differences between the two cases become a matter of notation.

The aim of this paper is to develop the time-variant analog of the Sz.-Nagy-Foias theory, as a model for discrete time, time-variant systems with a double infinite time lenght. Consequently, we obtain the geometric interpretation for the transfer operator(recently, a series of papers on time-variant systems with boundary conditions was begun with [3] , and this paper is our starting point, also for definitions and references) and a " canonical model" for such kind of systems. Moreover, for an upper triangular contractive operator we construct the time-variant system having as transfer operator the given one. Several connections with [4] are to be noticed and we also discuss some aspects regarding " the embedding of a nonstationary system into a stationary one".

II PRELIMINARIES

In this paper we are concerned with time-variant linear systems in the following state space reprezentation:

$$(1.1)\qquad \begin{cases} x_{n+1} = T_n^* x_n + B_n u_n \\ y_n \ = C_n x_n + D_n u_n \end{cases} \qquad n \in \mathbb{Z}$$

where $x_n \in \mathcal{H}_n$, $\mathcal{H}_n$ beeing Hilbert spaces (the state spaces), $u_n \in \mathcal{U}_n$, $y_n \in \mathcal{Y}_n$, $\mathcal{U}_n$ and $\mathcal{Y}_n$ beeing the input spaces and respectively the output spaces, also Hilbert spaces. The coefficients are bounded operators such that the matrices

$$\begin{pmatrix} T_n & , B_n \\ C_n & , D_n \end{pmatrix} : \begin{matrix} \mathcal{H}_{n+1} \\ \oplus \\ \mathcal{U}_n \end{matrix} \longrightarrow \begin{matrix} \mathcal{H}_n \\ \oplus \\ \mathcal{Y}_n \end{matrix}$$

are unitary operators, $n \in \mathbb{Z}$. Consequently, T_n are contractions and we can suppose that the system (1.1) is of the form:

$$(1.2)\qquad \begin{cases} x_{n+1} = T_n^* x_n + D_{T_n} u_n \\ y_n \ = D_{T_n^*} x_n - T_n u_n \end{cases} \qquad n \in \mathbb{Z}$$

; here, as in the rest of this paper we use the notation from [8].

Thus , for a contraction $T \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$ we note $D_T = (I - T^* T)^{\frac{1}{2}}$ and $\mathcal{D}_T = \overline{\operatorname{Ran} D_T}$ and if we consider the unitary operator

$$(1.3)\qquad \begin{cases} J(T) : \mathcal{H} \oplus \mathcal{D}_T \longrightarrow \mathcal{H}' \oplus \mathcal{D}_{T^*} \\ J(T) = \begin{pmatrix} T & , D_{T^*} \\ D_T & , -T^* \end{pmatrix} \end{cases}$$

then we can write the system (1.2) in the form

$$(1.4)\qquad \begin{pmatrix} x_{n+1} \\ y_n \end{pmatrix} = J(T_n)^* \begin{pmatrix} x_n \\ u_n \end{pmatrix}$$

Now we consider the positive form $\mathcal{F}$ on $\mathbb{Z}$ associated by the algorithm in [2] (Theorem 2.4) to the parameters: $G_{i,i+1} = T_i$, $i \in \mathbb{Z}$.

and zero in rest. Using Theorem 3.2 in [2], we describe the Kolmogorov decomposition of $\mathcal{F}$ in the following way:

(1.5)
$$\begin{cases} W_n : \mathcal{K}_{n+1} \longrightarrow \mathcal{K}_n \\ W_n = \begin{pmatrix} \ddots & \vdots & \vdots & & \\ I & 0 & & \cdots & \\ 0, & D_{T_n^*}, & \boxed{T_n}, & 0 & \cdots \\ & -T_n^*, & D_{T_n} & & \cdots \\ & & 0, & I & \cdots \\ & & \vdots & \vdots & \end{pmatrix} \end{cases}$$

where $\mathcal{K}_n = \dots \oplus \mathcal{D}_{T_{n-2}^*} \oplus \mathcal{D}_{T_{n-1}^*} \oplus \mathcal{H}_n \oplus \mathcal{D}_{T_n} \oplus \mathcal{D}_{T_{n+1}} \oplus \dots$

and the entries marked by I in the definition of W_n are the identit operators on the corresponding spaces.

By a direct computation (according to [3]) we have the transfer operator of the system (1.2):

(1.6)
$$\begin{cases} \Theta : \bigoplus_{n=-\infty}^{\infty} \mathcal{D}_{T_n} \longrightarrow \bigoplus_{n=-\infty}^{\infty} \mathcal{D}_{T_n^*} \\ \Theta = \begin{pmatrix} \ddots & \vdots & & & \\ -T_{-1}, & D_{T_{-1}^*}D_{T_0}, & \cdots & & \\ & -T_0, & D_{T_0^*}D_{T_1}, & D_{T_0^*}T_1^*D_{T_2}, & \cdots \\ \cdots & 0, & -T_1, & D_{T_1^*}D_{T_2}, & \cdots \\ & \cdots, & 0, & & \vdots \end{pmatrix} \end{cases}$$

Finally, let us remark that when $T_n = T$, $n \in \mathbb{Z}$, the system (1.2) is time-invariant, $W_n = W$ is the unitary dilation of T and Θ is a Toeplitz operator having as symbol the characteristic function of T

III THE GEOMETRICAL MODEL OF THE SYSTEM (1.2).

We begin this section by describing the nonstationary variant of the Wold decomposition. Thus, let $\{V_n\}_{n=0}^{\infty}$ be a family of isometries, $V_n \in \mathcal{L}(\mathcal{E}_{n+1}, \mathcal{E}_n)$. Define the spaces:

$$\mathcal{L}_n = \mathcal{E}_n \ominus V_n \mathcal{E}_{n+1} \qquad , n \geq 0$$

then, a simple computation shows that:

(3.1) $\mathcal{L}_p \oplus V_p\mathcal{L}_{p+1}\oplus\ldots\oplus V_pV_{p+1}\ldots V_{p+k-1}\mathcal{L}_{p+k}=\mathcal{E}_p\ominus V_pV_{p+1}\ldots V_{p+k}\mathcal{E}_{p+k+1}$

for $p\geqslant 0$, $k\geqslant 0$.

If we define the spaces $\mathcal{R}_p=\bigcap_{k=0}^{\infty}V_pV_{p+1}\ldots V_{p+k-1}\mathcal{E}_{p+k}$ then we obtain the decompositions:

(3.2) $$\mathcal{E}_p=\bigoplus_{k=0}^{\infty}V_p\ldots V_{p+k-1}\mathcal{L}_{p+k}\oplus\mathcal{R}_p \quad .$$

Consequently, the operators V_p have the decompositions:

(3.3) $$V_p=V_p^m\oplus V_p^u$$

with respect to (3.2) , where $V_p^u:\mathcal{R}_{p+1}\longrightarrow\mathcal{R}_p$ are unitary operato[rs] and for V_p^m there are unitary operators

$$\Phi_p:\bigoplus_{k=0}^{\infty}V_pV_{p+1}\ldots V_{p+k-1}\mathcal{L}_{p+k}\longrightarrow\bigoplus_{k=0}^{\infty}\mathcal{L}_{p+k}$$

such that

(3.4) $$\Phi_pV_p^m=S_{+,p}\Phi_{p+1}$$

where

(3.5) $$\begin{cases} S_{+,p}:\bigoplus_{k=0}^{\infty}\mathcal{L}_{p+k+1}\longrightarrow\bigoplus_{k=0}^{\infty}\mathcal{L}_{p+k} \\ S_{+,p}(l_{p+1},l_{p+2},\ldots)=(0,l_{p+1},l_{p+2},\ldots) \end{cases}$$

Using (3.2)-(3.5) for the family of isometries $\{W_{+,k}\}_{k\geqslant n}$, $n\in\mathbb{Z}$, the corresponding $\mathcal{L}$ spaces are $\mathcal{L}_{+,n}=W_n\mathcal{D}_{T_n^*}^{(-1)}$ where $\mathcal{D}_{T_n^*}^{(-1)}$ is a notation for the space $\ldots\oplus 0\oplus\mathcal{D}_{T_n^*}\oplus\boxed{0}\oplus 0\oplus\ldots$ in $\mathcal{L}_{n+1}$ and then

(3.6) $$\mathcal{K}_{+,n}=\bigoplus_{k=0}^{\infty}W_n\ldots W_{n+k}\mathcal{D}_{T_{n+k}^*}^{(-1)}\oplus\mathcal{R}_{+,n}$$

where $\mathcal{R}_{+,n}=\bigcap_{k=0}^{\infty}W_n\ldots W_{n+k-1}\mathcal{K}_{+,k}$.

One more remark is that if we define the space:

$$\mathcal{K}_n^{out}=\ldots\oplus W_{n-1}^*W_{n-2}^*\mathcal{D}_{T_{n-3}^*}^{(-1)}\oplus W_{n-1}^*\mathcal{D}_{T_{n-2}^*}^{(-1)}\oplus\mathcal{D}_{T_{n-1}^*}^{(-1)}\oplus\bigoplus_{k=0}^{\infty}W_n\ldots W_{n+k}\mathcal{D}_{T_{n+k}^*}^{(-1)}$$

then we have the equality:

(3.7) $$\mathcal{K}_n=\mathcal{K}_n^{out}\oplus\mathcal{R}_{+,n}.$$

Similar considerations for the family $\{W_{-,k}\}_{k\leq n-1}$ give us that the corresponding $\mathcal{L}$ spaces are $\mathcal{L}_{-,n} = W_n^* \mathcal{D}_{T_n}^{(1)}$ where $\mathcal{D}_{T_n}^{(1)}$ is a notation for the space $\ldots \oplus \boxed{0} \oplus \mathcal{D}_{T_n} \oplus 0 \oplus \ldots$ and we have

(3.8) $$\mathcal{K}_{-,n} = \bigoplus_{k=1}^{\infty} W_{n-1}^* \ldots W_{n-k}^* \mathcal{D}_{T_{n-k}}^{(1)} \oplus \mathcal{R}_{-,n}$$

where $\mathcal{R}_{-,n} = \bigcap_{k=1}^{\infty} W_{n-1}^* \ldots W_{n-k}^* \mathcal{K}_{-,n-k}$. We also consider the space

$$\mathcal{K}_n^{inp} = \bigoplus_{k=1}^{\infty} W_{n-1}^* \ldots W_{n-k}^* \mathcal{D}_{T_{n-k}}^{(1)} \oplus \mathcal{D}_{T_n}^{(1)} \oplus W_n \mathcal{D}_{T_{n+1}}^{(1)} \oplus W_n W_{n+1} \mathcal{D}_{T_{n+2}}^{(1)} \oplus \ldots$$

and we have the equality:

(3.9) $$\mathcal{K}_n = \mathcal{L}_n^{inp} \oplus \mathcal{R}_{-,n} .$$

Now , we define the family of characteristic operators of the system (1.2) by the formulas:

(3.10) $$\begin{cases} Q_n : \mathcal{K}_n^{inp} \longrightarrow \mathcal{L}_n^{out} \\ Q_n = P_{\mathcal{K}_n^{out}}^{\mathcal{K}_n} / \mathcal{K}_n^{inp}. \end{cases}$$

The space $\mathcal{K}_n^{out}$ is the output space over all time (with respect to the channel of index n) and $\mathcal{K}_n^{inp}$ is the input space over all time (with respect to the same channel of index n). A usual condition in system theory is that this two spaces generate $\mathcal{K}_n$ (see for instance [7]). In the time-invariant case, this condition is connected with the controllability and observability of the system and this last condition means that T is a completely nonunitary contraction. Similar computations hold in time-variant case. Thus, using the form (1.5) of the Kolmogorov decomposition of $\mathcal{F}$ we immediately get:

$$\mathcal{K}_n \ominus (\mathcal{K}_n^{inp} \vee \mathcal{K}_n^{out}) =$$

$$= \{ h \in \mathcal{K}_n \,/ \ldots = \|T_{n-2} T_{n-1} h\| = \|T_{n-1} h\| = \|h\| = \|T_n^* h\| = \|T_{n+1}^* T_n^* h\| = \ldots \}$$

and the condition $\mathcal{K}_n = \mathcal{K}_n^{inp} \vee \mathcal{K}_n^{out}$ can be connected with an adequate notion of controllability and observability with respect to the channel of index n(moreover, in certain circumstances we can connect these with the unitary part of the family $\{T_n\}_{n=-\infty}^{\infty}$).

From now on we will suppose that $\mathcal{K}_n=\mathcal{K}_n^{inp}\vee\mathcal{K}_n^{out}$ and in thi situation the system (1.2) willbe called canonical. We can write no the main result of this section.

3.1 THEOREM For a canonical system (1.2) we have the equalities:

$$\mathcal{K}_n=\mathcal{K}_n^{out}\oplus\overline{(I-Q_n)\mathcal{K}_n^{inp}}$$

$$\mathcal{H}_n=\mathcal{K}_{+,n}\ominus\{Q_nu\oplus(I-Q_n)u/u\in\ldots O\oplus\mathcal{D}_{T_n}^{(1)}\oplus W_n\mathcal{D}_{T_{n+1}}^{(1)}\oplus\cdots\}.$$

IV THE TRANSFER OPERATOR AND ITS GEOMETRICAL ROLE

In this section we describe the nonstationary variant of the Sz.-Nagy-Foias model of a contraction. This model is based on Theorem 3.1 and on the following natural identifications.

Let us define the spaces

$$\mathcal{M}_{+,n}=\bigoplus_{k=-\infty}^{\infty}\mathcal{D}_{T^*_{n+k-1}}=\ldots\oplus\mathcal{D}_{T^*_{n-2}}\oplus\boxed{\mathcal{D}_{T^*_{n-1}}}\oplus\mathcal{D}_{T^*_n}\oplus\cdots$$

(the spaces $\mathcal{M}_{+,n}$ coincide , but we differently marked the initial position). We consider the marking operators:

$$(4.1)\qquad\begin{cases} M_{+,n}:\mathcal{M}_{+,n+1}\longrightarrow\mathcal{M}_{+,n}\\ M_{+,n}(\ldots,d_{n-1},\boxed{d_n},d_{n+1},\ldots)=(\ldots,d_{n-2},\boxed{d_{n-1}},d_n,\ldots).\end{cases}$$

Similarly, define the spaces

$$\mathcal{M}_{-,n}=\bigoplus_{k=-\infty}^{\infty}\mathcal{D}_{T_{n+k}}=\ldots\oplus\mathcal{D}_{T_{n-1}}\oplus\boxed{\mathcal{D}_{T_n}}\oplus\mathcal{D}_{T_{n+1}}\oplus\cdots$$

and the marking operators:

$$(4.2)\qquad M_{-,n}:\mathcal{M}_{-,n+1}\longrightarrow\mathcal{M}_{-,n}$$

It is clear that the marking operators are unitary. Then , we define the identification operators:

$$(4.3)\qquad\begin{cases}\Phi_{+,n}:\mathcal{K}_n^{out}\longrightarrow\mathcal{M}_{+,n}\\ \Phi_{+,n}=\begin{pmatrix} \vdots & & & \\ \cdots W_{n-1}, & 0 & & \\ \cdots & 0 & \boxed{I}, & 0, \\ & & 0, & W_n^*, \cdots \\ & & & \vdots\end{pmatrix}\end{cases}$$

and

$$(4.4) \quad \begin{cases} \Phi_{-,n}: \mathcal{K}_n^{inp} \longrightarrow \mathcal{M}_{-,n} \\ \Phi_{-,n} = \begin{pmatrix} \vdots & & \\ \cdots W_{n+1}, & O & \\ O & \boxed{I}, & O \\ & O, & W_n^*, \cdots \\ & & \vdots \end{pmatrix} \end{cases}$$

where we do not write the natural identification of $\mathfrak{D}^{(-1)}_{T_n^*}$ with [$\mathfrak{D}_{T_n^*}$] and of $\mathfrak{D}^{(1)}_{T_n}$ with [$\mathfrak{D}_{T_n}$]. The identification operators are also unitary. A direct computation based on (1.5) shows that

$$(4.5) \quad \Phi_{+,n} Q_n \Phi^*_{-,n} = \Theta$$

(acting naturally between $\mathcal{M}_{-,n}$ and $\mathcal{M}_{+,n}$). Moreover, considering the family of unitary operator $\{\tilde{W}_{+,n}: \mathcal{K}^{out}_{n+1} \longrightarrow \mathcal{K}^{out}_n\}_{n=-\infty}^{\infty}$ obtained by the natural action of each W_n on $\mathcal{K}^{out}_{n+1}$, we get

$$(4.6) \quad \Phi_{+,n} \tilde{W}_{+,n} \Phi^*_{+,n+1} = M_{+,n}$$

and similarly, for the family $\{\tilde{W}_{-,n}: \mathcal{K}^{inp}_{n+1} \longrightarrow \mathcal{K}^{inp}_n\}_{n=-\infty}^{\infty}$ obtained by the action of each W_n on $\mathcal{K}^{inp}_{n+1}$, we get:

$$(4.7) \quad \Phi_{-,n} \tilde{W}_{-,n} \Phi^*_{-,n+1} = M_{-,n} \quad .$$

Now , we can define the unitary operators:

$$(4.8) \quad \begin{cases} \Phi_{\mathcal{Q}_{+,n}}: \mathcal{Q}_{+,n} \longrightarrow \overline{D_{\Theta}\mathcal{M}_{-,n}} \\ (I-Q_n)k = D_{\Theta} \Phi_{-,n} k \qquad , \; k \in \mathcal{K}^{inp}_n \end{cases}$$

and

$$(4.9) \quad \begin{cases} \Psi_n: \mathcal{K}_n \longrightarrow \mathcal{M}_{+,n} \oplus \overline{D_{\Theta}\mathcal{M}_{-,n}} \\ \Psi_n = \Phi_{+,n} \oplus \Phi_{\mathcal{Q}_{+,n}} \end{cases}$$

Using Theorem 3.1 and these unitary operators we obtain the main result of this section.

<u>4.1 THEOREM</u> For a canonnical system (1.2) we have the following model in terms of the transfer operator of the system:

$$\mathcal{H}_n = \left(\bigoplus_{k=0}^{\infty} \mathfrak{D}_{T^*_{n+k}} \oplus \overline{D_{\Theta}\mathcal{M}_{-,n}}\right) \ominus \left\{\Theta u \oplus D_{\Theta} u \,/\, u \in \bigoplus_{k=0}^{\infty} \mathfrak{D}_{T_{n+k}}\right\}$$

$$T_n(u_+ \oplus v_-) = M_{+,n} u_+ \oplus M_{-,n} v_- , \quad u_+ \oplus v_- \in \mathcal{H}_n. \qquad \blacksquare$$

V THE INVERSE PROBLEM

Having Theorem 4.1 there is only a matter of notation to obtain the realization of an upper triangular contraction acting between $\bigoplus_{n=-\infty}^{\infty}\mathcal{U}_n$ and $\bigoplus_{n=-\infty}^{\infty}\mathcal{Y}_n$, where $\{\mathcal{U}_n\}_{n=-\infty}^{\infty}$ and $\{\mathcal{Y}_n\}_{n=-\infty}^{\infty}$ are given families of Hilbert spaces, as the transfer operator of a system (1.2). Thus, let be

$$\Theta : \bigoplus_{n=-\infty}^{\infty}\mathcal{U}_n \longrightarrow \bigoplus_{n=-\infty}^{\infty}\mathcal{Y}_n$$

an upper triangular contractive operator and we consider the marked spaces

$$\mathcal{M}_{+,n} = \bigoplus_{k=-\infty}^{\infty}\mathcal{Y}_{n+k-1} = \dots \oplus \mathcal{Y}_{n-2} \oplus \boxed{\mathcal{Y}_{n-1}} \oplus \mathcal{Y}_n \oplus \cdots$$

$$\mathcal{M}_{-,n} = \bigoplus_{k=-\infty}^{\infty}\mathcal{U}_{n+k} = \dots \oplus \mathcal{U}_{n-1} \oplus \boxed{\mathcal{U}_n} \oplus \mathcal{U}_{n+1} \oplus \cdots$$

and the marking operators:

$$M_{+,n} : \mathcal{M}_{+,n+1} \longrightarrow \mathcal{M}_{+,n}$$

$$M_{-,n} : \mathcal{M}_{-,n+1} \longrightarrow \mathcal{M}_{-,n} \quad .$$

Now , we define the spaces

$$\mathcal{K}_n = \mathcal{M}_{+,n} \oplus \overline{D_{\Theta}\,\mathcal{M}_{-,n}}$$

and the unitary operators:

$$W_n : \mathcal{K}_{n+1} \longrightarrow \mathcal{K}_n$$

$$W_n = M_{+,n} \oplus M_{-,n} \; .$$

Finally, we define

$$\mathcal{K}_{+,n} = \bigoplus_{k=0}^{\infty}\mathcal{Y}_{n+k} \oplus \overline{D_{\Theta}\,\mathcal{M}_{-,n}}$$

$$\mathcal{G}_n = \{\Theta u \oplus D_{\Theta}u \,/\, u \in \bigoplus_{k=0}^{\infty}\mathcal{U}_{n+k}$$

and $\mathcal{H}_n = \mathcal{K}_{+,n} \ominus \mathcal{G}_n$, $T_n = P^{\mathcal{K}_n}_{\mathcal{H}_n} W_n / \mathcal{H}_{n+1} : \mathcal{H}_{n+1} \longrightarrow \mathcal{H}_n$.

Adapting the computations in Sec3, Chap. VI [8] , we obtain:

5.1 THEOREM For an upper triangular pure contractive operator Θ (pure means that $\|\Theta_{nn}u_n\| < \|u_n\|,\, u_n \in \mathcal{H}_n, u_n \neq 0$), the above construction gives a canonical system (1.2) having Θ as transfer operator. ∎

VI FINAL COMMENTS

The first question we deal with regards the connection with the paper [4] on the models for noncommuting operators. The differ ce is clear enough. Thus, we only exploit the Kolmogorov decomposition of a positive kernel, so our model is of "marking operators " type and different channels are modeled through different unitary operators. Instead, in the above mentioned paper, the following situation is of interest: $T_1, T_2 \in \mathcal{L}(\mathcal{H})$, $T_1T_1^* + T_2T_2^* \leq I$ and under certain conditions, one obtains a simultaneous model for T_1 and T_2 (i.e. the model is obtained through the same unitary operator) .

But, we mention that the Schur analysis may be also used in this situation (and even for a family $\{T_n\}_{n=1}^{\infty}$ with $\sum_{n=1}^{\infty} T_nT_n^* \leq I$).

Thus , in the case n=2, we have $T_2 = D_{T_1^*}\Gamma$ where $\Gamma : \mathcal{H} \to \mathcal{D}_{T_1^*}$ is a contraction and define (having in mind the structure of the Naimark dilation as given in [2]):

$$\mathcal{K}_+ = \mathcal{H} \oplus \mathcal{D}_{T_1} \oplus \mathcal{D}_{\Gamma} \oplus \mathcal{D}_{T_1} \oplus \mathcal{D}_{\Gamma} \oplus \cdots$$

$$V_1 = \begin{pmatrix} T_1, & 0, & & & \\ D_{T_1}, & 0, & & & \\ 0, & 0, & & & \\ 0, & I, & 0 & & \\ 0, & 0, & 0, & 0 & \cdots \\ \cdot & 0, & I, & 0 & \end{pmatrix}, \quad V_2 = \begin{pmatrix} D_{T_1^*}\Gamma, & 0, & & \\ -T_1^*\Gamma, & 0, & & \\ D_\Gamma, & 0, & & \\ 0, & 0, & 0, & \cdots \\ 0, & I, & 0 & \\ 0, & 0, & 0 & \cdots \end{pmatrix}$$

then V_1, V_2 are isometries on $\mathcal{K}_+$ such that $\mathcal{H}$ is invariant for V_1^*, V_2^* and $T_1^* = V_1^*/\mathcal{H}$, $T_2^* = V_2^*/\mathcal{H}$ and $V_1V_1^* + V_2V_2^* \leq I$.

Moreover, considering on $\mathcal{K} = \ldots \oplus \mathcal{D}_{\Gamma^*} \oplus \mathcal{D}_{\Gamma^*} \oplus \mathcal{K}_+$

the opertors:

$$U_1 = \left(\begin{array}{cc|c} \vdots & & \\ \cdots 0 & 0 & 0 \\ \cdots I, & 0 & \\ \hline & D_{T^*}D_{\Gamma^*} & \\ & -T^*D_{\Gamma^*} & V_1 \\ & -\Gamma^* & \end{array}\right), \quad U_2 = \left(\begin{array}{cc|c} I, & 0 & 0 \\ 0 & 0 & \\ \hline & 0 & V_2 \end{array}\right)$$

then U_1, U_2 are isometries on $\mathcal{K}$ satisfying the properties of V_1 ,V and $U_1U_1^*+U_2U_2^*=I$. As a consequence, $\mathcal{G} = U_1(\ldots 0 0 \oplus I \oplus 0 \oplus \boxed{0} \ldots)$ ($\mathcal{G}$ beeing the wandering space in the Wold type decomposition in Theorem 1 [4]

The second question regards the embedding of the nonstationary process given by (1.2) into a stationary one. We saw in [2] that the general dilation theory gives easy such kind of constructions, and , moreover, we clarified this operation at the Schur analysis level. Thus, let be the space $\tilde{\mathcal{K}} = \bigoplus_{n=-\infty}^{\infty} \mathcal{H}_n$ and

$$\tilde{G}_1 = \begin{pmatrix} 0, & T_{-1}, & 0, & \\ & \boxed{0}, & T_0 & 0 \\ & & 0, & T_1 \\ & & & \ddots \end{pmatrix} \quad \text{on } \tilde{\mathcal{K}} \quad , \tilde{G}_k=0,\ k>1$$

then $\tilde{G}_1$ is a contraction and we can use the Sz.-Nagy-Foias model.

But, as usual when appear connections between the parameters of the nonstationary process, the associated stationary construction is too large; for this reason may be preferable a direct derivation of the model instead of sift it from the one obtained by the above general embedding.

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