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by

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§ 1. Introduction

Let $R \hookrightarrow R'$ be an unramified extension of (noetherian) discrete valuation rings inducing separable extensions on fraction and residue fields. Then

(1.1) Theorem (Néron [N]) R' is a filtered inductive union of its smooth sub-R-algebras of finite type.

This theorem gives a method to substitute the solvability in R' of certain polynomial equations over R with the solvability of some equations for which it is possible to apply the Implicit Function Theorem Trying to extend Theorem (1.1) we stated in [P_3] that a morphism of noetherian rings $A \longrightarrow A'$ is regular iff A' is a filtered inductive limit of smooth A-algebras of finite presentation. This result has some nice applications (see [P_3]) and it would be good to have an analog of it for nonnoetherian rings, for instance for (nonnoetherian) valuation rings.

(1.2) Theorem (Zariski [Z]). Let R be a valuation ring containing a field k of characteristic zero. Then R is a filtered inductive union of its smooth sub-k-algebras of finite type.

In [P_1] we state the following result (by mistake the condition (ii) appeared actually in a weaker form).

(1.3) Theorem. Let $R \subseteq R'$ be an extension of valuation rings (i.e. R' dominates R) and k the residue field of R. Suppose that

(i) char $k=0$ and $\dim R<\infty$,

(ii) every prime ideal from R generates in R' a prime ideal.

Then R' is a filtered inductive limit of smooth R-algebras of finite presentation.

Note that in Theorem (1.3) the "smooth R-algebras" can be not "sub-R-algebras" like in Theorem (1.1) and (1.2).

It is the purpose of our paper to try to improve the result in this sense and to investigate the obstructions which appear in positive characteristic (see Theorems (5.3), (6.6) and (6.9)). Ostrowski's "Defektsatz" seems to be behind of some results from §3, §5 (see (3.10.1) and (5.4.1)), though we do not use it. The "completion" considered here (see (2.2.3)) is in fact a weak form of completion which may be not henselian when the rank of the valuation is bigger than one. All the rings are supposed to be commutative with identity.

§ 2. Preliminaries

(2.1) A valued field is a triplet $\mathcal{F}=(F,v,\Gamma)$, where F is a field, Γ a totally ordered group and $v:F^*\to\Gamma$, $F^*=F\setminus\{0\}$ a valuation (here we assume valuations to be surjective). By convention we shall put sometimes $v(0)=\infty$. The ring $R=\{x\in F \mid v(x)\geq 0\}\cup\{0\}$ is the valuation ring of $\mathcal{F}$. A valued field $\mathcal{F}'=(F',v',\Gamma')$ is an extension of $\mathcal{F}$ (shortly we write $\mathcal{F}\subseteq\mathcal{F}'$) if $F\subseteq F'$, $\Gamma\subseteq\Gamma'$ and v is given by restriction from v'. Let k,k' be the residue fields of the valuation rings R resp. R' of $\mathcal{F}$ resp. $\mathcal{F}'$. The extension $\mathcal{F}\subseteq\mathcal{F}'$ (or $R\subseteq R'$) is called immediate if $\Gamma=\Gamma'$ and $k=k'$. Moreover if for every $x\in F'$ and every $\gamma\in\Gamma$ there exists an element y such that $v(x-y)\geq\gamma$ then $\mathcal{F}\subseteq\mathcal{F}'$ (or $R\subset R'$) is dense.

(2.2) A well ordered sequence $a=(a_\nu)_{\nu<\theta}$ of elements from F is called fundamental (shortly we write a is f.s.) if

i) a has not a last element, i.e. θ is a limit ordinal,

ii) $v(a_\nu-a_\rho)<v(a_\tau-a_\nu)$ for all $\rho<\nu<\tau$,

iii) for every $\gamma\in\Gamma$ there exists a $\rho<\theta$ such that $v(a_\tau-a_\nu)>\gamma$ for all $\nu<\tau$ with $\nu>\rho$.

Let ε, $0<\varepsilon<1$ be a real number. If $\Gamma\subset\mathbb{R}$ and θ is the ordinal of $\mathbb{N}$ then a is f.s exactly when the sequence $(\varepsilon^{v(a_\nu)})_{\nu<\theta}$ of real numbers is fundamental.

(2.2.1) An element $y\in F'$ is called the limit of a f.s. $a=(a_\nu)_{\nu<\theta}$ from F if

$$v(y-a_\rho)=v(a_{\rho+1}-a_\rho) \text{ for all } \rho<\theta.$$

If there exists the limit must be unique (see iii)).

(2.2.2) The extension $\mathcal{F}\subseteq\mathcal{F}'$ is dense iff every element from F' is the limit of a f.s. from F.

(2.2.3) $\mathcal{F}$ is called complete if every dense extension of it is trivial, or equivalently if every f.s. from F has a limit in F. A complete valued field of rank one (i.e. $\Gamma\subseteq\mathbb{R}$) is henselian, i.e. its valuation ring satisfies Hensel's Lemma. $\mathcal{F}'$ is the completion of $\mathcal{F}$ if $\mathcal{F}\subseteq\mathcal{F}'$ is dense and $\mathcal{F}'$ is complete (every valued field has an unique completion). $\mathcal{F}$ is called complete relatively to $\mathcal{F}'$ if no element from $F'\setminus F$ is a limit of a f.s. from F. A valued field $\mathcal{E}=(E,w,\Gamma)$, $\mathcal{F}\subset\mathcal{E}\subset\mathcal{F}'$ is called the complete closure of $\mathcal{F}$ relatively to $\mathcal{F}'$ if $\mathcal{F}\subseteq\mathcal{E}$ is dense and $\mathcal{E}$ is complete relatively to $\mathcal{F}'$. Clearly E is the subfield of all elements from F' which are limits of some f.s. from F.

(2.3) A well ordered sequence $a=(a_\sigma)_{\sigma<\theta}$ of elements from F is called pseudo convergent (shortly we write a is a p.c.s.) if it satisfies i) and ii) from (2.2).

(2.3.1) An element $y\in F'$ is called a pseudo limit (shortly we write a p.ℓ.) of a p.c.s. $a=(a_\sigma)_{\sigma<\theta}$ from F if

(1) $\quad v(y-a_\rho) = v(a_{\rho+1} - a_\rho)$

for all $\rho<\theta$. Note that this happens if (1) holds just for $\rho\gg 0$ i.e., for ρ sufficiently large. When a is not a f.s. then there exists a $\delta\in\Gamma$ such that $v(a_\rho-a_\sigma)<\delta$ for all $\rho<\sigma<\theta$. If y is a p.ℓ. of a in F' and $b\in F'$ is an element such that $v(b)\geq\delta$ then $y+b$ is another p.ℓ. of a. Thus a p.ℓ. is not unique in general.

(2.3.2) If the extension $\mathcal{F}\subseteq\mathcal{F}'$ is immediate then every element from $F'\setminus F$ is a p.ℓ. of a p.c.s. from F having no p.ℓ. in F (see [K]).

(2.3.3) $\mathcal{F}$ is called maximally complete if every immediate extension of it is trivial or equivalent if every p.c.s. from F has a p.ℓ. in F. A maximally complete valued field is complete and so henselian if $\Gamma\subseteq\mathbb{R}$. $\mathcal{F}'$ is a maximally complete immediate extension of $\mathcal{F}$ if $\mathcal{F}\subseteq\mathcal{F}'$ is immediate and $\mathcal{F}'$ is maximally complete. Every valued field has a maximally complete immediate extension which can be not unique if $p=\text{char } F>0$ (see [K]).

$_{\rho<\theta}$ is a p.c.s. from F then the following statements hold (see [K] or [S] ch.II for proofs):

(2.4.1) $v(a_\sigma - a_\rho) = v(a_{\rho+1} - a_\rho)$ for all $\rho<\sigma<\theta$,

(2.4.2) either

i) $v(a_\rho) < v(a_\sigma)$ for all $\rho<\sigma<\theta$, or

ii) $v(a_\rho) = v(a_\sigma)$ for $\sigma > \rho \gg 0$.

(2.4.3) if y is a p.ℓ. of a then either

i) $v(y) > v(a_\rho)$, for all ρ in the case (2.4.2) i), or

ii) $v(y) = v(a_\rho)$ for $\rho \gg 0$ in the case (2.4.2) ii).

(2.4.4) for every polynomial f in Y over F the sequence $(f(a_\rho))_{\rho<\theta}$ is ultimately pseudo-convergent (thus by (2.4.2) it follows either $v(f(a_\rho)) < v(f(a_\sigma))$ or $v(f(a_\rho)) = v(f(a_\sigma))$ for $\sigma > \rho \gg 0$).

(2.4.5) if $v(f(a_\tau)) < v(a_{\tau+1} - a_\tau)$ for a certain polynomial $f \in F[Y]$ and a certain $\tau < \theta$ then $v(f(a_\sigma)) = v(f(a_\tau))$ for all $\sigma \geq \tau$.

(2.5) <u>Lemma</u>. Suppose that $\mathcal{F} \subseteq \mathcal{F}'$ is dense and let $a = (a_\rho)_{\rho<\theta}$ be a p.c.s. from F which is not f.s. Then a has a p.ℓ. in F' iff it has one in F.

<u>Proof</u>. Let $b \in F'$ be a p.ℓ. of a. By (2.2.2) b is a limit of a f.s. $c = (c_\lambda)_{\lambda<\omega}$ from F. Since a is not a f.s. there exists an element $\delta \in \Gamma$ such that $v(a_\sigma - a_\rho) < \delta$ for all $\rho<\sigma<\theta$. Choose a λ sufficiently large in order to have $v(b - c_\lambda) > \delta$. Then $v(c_\lambda - a_\rho) = v(b - a_\rho) = v(a_{\rho+1} - a_\rho)$ for all $\rho<\theta$ and so c_λ is a p.ℓ. of a in F (see (2.4.1)).□

(2.6) <u>Lemma</u>. Let y be an element from F' which is not in F. Then one and only one of the following statements holds:

i) y is a p.ℓ. of a p.c.s. from F having no p.ℓ.-es in F,

ii) the set $\Lambda := \{v'(y-b) \mid b \in F\}$ has a largest element.

<u>Proof</u>. Suppose ii) does not hold. Then we show i) by adapting the proof of [S] Ch.II, Lemma 20 to our case. Select in Λ a cofinal well ordered subset $\Lambda' = \{v(y - a_\rho) \mid \rho<\theta\}$, $a_\rho \in F$ having no last element because ii) does not hold. We have

$$v(a_\sigma - a_\rho) = v(y - a_\rho)$$

if $\rho<\sigma<\theta$ because $v(y-a_\rho) < v(y-a_\sigma)$ by construction. Since $(v(y-a_\rho))_\rho$

increases monotonically we conclude that $a=(a_\rho)_{\rho<\theta}$ is a p.c.s. from F and y is a p.ℓ. of it. If $z\in F$ is another p.ℓ. of a then

$$v(y-z)>v(a_\sigma-a_\rho)=v(y-a_\rho)$$

for all $\rho<\sigma<\theta$, i.e. Λ' is not cofinal in Λ (Contradiction!). Thus has no p.ℓ. in F. □

(2.7) A p.c.s. $a=(a_\rho)_{\rho<\theta}$ from F is called

i) transcendental if $v(f(a_\rho))=v(f(a_\sigma))$ for all nonzero polynomia $f\in F[Y]$ and all $\sigma>\rho\gg 0$,

ii) algebraic if $v(f(a_\rho))<v(f(a_\sigma))$ for at least a nonzero polyno-mial $f\in F[Y]$ and all $\sigma>\rho\gg 0$.

(2.7.1) A p.c.s. $a=(a_\rho)_{\rho<\theta}$ from F has a p.ℓ. in F iff there exist a polynomial $f\in F[Y]$ of degree one such that

(1) $v(f(a_\rho))<v(f(a_\sigma))$

for all $\sigma>\rho\gg 0$. Indeed if $b\in F$ is a p.ℓ. of a then $v(b-a_\rho)=v(a_\sigma-a_\rho)$ fo all $\rho<\sigma<\theta$ and taking $f=Y-b$ we get (1) fulfilled for all $\sigma>\rho$. Conver-sely, if (1) holds for $f=c(Y-b)$, $c\neq 0$ then $(v(b-a_\rho))_\rho$ increases monoto nically for $\rho\gg 0$ and so we get

$$v(b-a_\rho)=v((b-a_\sigma)+(a_\sigma-a_\rho))=v(a_\sigma-a_\rho),\ \sigma>\rho\gg 0$$

Thus b is a p.ℓ. of a (see (2.3.1)).

(2.7.2) A transcendental p.c.s. from F has no p.ℓ. in F (see(2.7.1 A p.ℓ. of a transcendental p.c.s from F is transcendental over F.

(2.7.3) If a is a transcendental p.c.s. from F then there exists an immediate transcendental extension $\mathcal{F}(z)=(F(z),\bar{v},\Gamma)$ of $\mathcal{F}$ in which z a p.ℓ. of a. Conversely, if $\mathcal{F}(u)=(F(u),w,\Gamma)$ is a transcendental exten-sion of $\mathcal{F}$ in which u is a p.ℓ. of a then $\mathcal{F}(z)$ and $\mathcal{F}(u)$ are analytical-ly equivalent over $\mathcal{F}$, the equivalence being given by $z\to u$. This result is in fact [K] Theorem 2 when a has no p.ℓ. in F; but this is certainly true by (2.7.2).

(2.7.4) If a is algebraic having no p.ℓ. in F then there exists an immediate algebraic extension $\mathcal{F}(z)=F(z),\bar{v},\Gamma)$ of $\mathcal{F}$ in which z is a p.ℓ. of a. The defining equation is $f(z)=0$, where f is a nonzero polynomial of least degree for which (2.7) ii) holds (such polynomial f is irreducible of degree ≥ 2). Conversely, if u is a root of f and if $\mathcal{F}(u)=(F(u),w,\Gamma)$ is an immediate extension of $\mathcal{F}$ in which u is a p.ℓ. of a then $\mathcal{F}(u)$ and $\mathcal{F}(z)$ are analytically equivalent over $\mathcal{F}$, the equivalence being given by $u \rightsquigarrow z$. This result is in fact [K] Theorem 3.

(2.8) Lemma. Let $a=(a_\rho)_{\rho<\theta}$ be an algebraic p.c.s. from F which is not f.s. and $g \in F[Y]$ a nonzero polynomial satisfying (2.7)ii). Then the sequence $v(g(a_\rho))_{\rho<\theta}$ is bounded in Γ.

Proof. Let $\mathcal{F}'$ be an extension of $\mathcal{F}$ containing a p.ℓ. y of a (apply (2.7.4)). By (2.4.4), (2.4.2) i) and (2.4.3) i) we have

$$v(g(y))>v(g(a_\rho)) \quad \text{for } \rho \gg 0 .$$

Since a is not a f.s., y is not unique and a has an infinite set of p.ℓ. in $\mathcal{F}'$. Thus changing y (if necessary) we can suppose $g(y)\neq 0$. Then $v(g(y))$ is the wanted bound. □

(2.9) Lemma. Let $a=(a_\rho)_{\rho<\theta}$ be a transcendental p.c.s. from F which is not f.s. If $\mathcal{F} \subseteq \mathcal{F}'$ is dense then a is also transcendental over $\mathcal{F}'$.

Proof. Suppose that a is algebraic over F'. Then there exists a nonzero polynomial $f \in F'$ satisfying (2.7) ii). By Lemma (2.8) the sequence $v(f(a_\rho))_{\rho<\theta}$ is bounded in Γ. Choose a polynomial $g \in F[Y]$ such that the valuation of all coefficients of g-f is bigger than all $v(f(a_\rho))$, $\rho<\theta$. Then $v(g(a_\rho))=v(f(a_\rho))$ for all $\rho<\theta$ and so a is algebraic over F. Contradiction! □

§ 3. When immediate algebraic extensions are dense?

Let $\mathcal{F}=(F,v,\Gamma)$ be a valued field, p=char F, $a=(a_\rho)_{\rho<\theta}$ a p.c.s from F and f a nonzero polynomial from F[Y]. The notation $f^{(j)}=\frac{1}{j!}\frac{\partial^j f}{\partial Y^j}$, $f^{(0)}=f$ has sense also when p>0 because the coefficients of $\frac{\partial^j f}{\partial Y^j}$ are "multiple" of j!. By Taylor's formula we have

$$(*) \quad f(a_\rho)=f(a_\tau)+\sum_{j\geq 1} f^{(j)}(a_\tau)(a_\rho-a_\tau)^j, \quad \rho<\tau<\theta .$$

Taylor's formula holds also when p>0; to see this substitute all constants by variables and apply the usual formula in a ring of polynomials over $\mathbb{Z}$, then change back variables by constants. Let $\bar{p}$ be the residue field characteristic of the valuation ring of $\mathcal{F}$.

(3.1) Lemma. Suppose that

(1) $v(f(a_\rho))=v(f(a_\tau))$ for $\tau>\rho\gg 0$

Then

(2) $v(f(a_\tau))<v(f^{(j)}(a_\tau)(a_\rho-a_\tau)^j)$

for every $j\geq 1$ providing $\tau>\rho\gg 0$.

Proof. If ρ increases, $\rho<\tau$ then $v(a_\rho-a_\tau)$ increases too. Thus we can suppose that the nonzero elements from

$$\left\{f^{(j)}(a_\tau)(a_\rho-a_\tau)^j\right\}_{j\geq 0}$$

have their valuation different. Then (2) follows from (*) and (1).□

(3.1.1) Remark. When f is a monic polynomial of degree e satisfying (1) then the above Lemma gives $v(f(a_\tau))<ev(a_\rho-a_\tau)<ev(a_{\tau+1}-a_\tau)$ for $\tau>\rho\gg 0$.

(3.2). Lemma. Let e be a positive integer such that $f^{(e)}\neq 0$ and

(1) $v(f^{(e)}(a_\rho))=v(f^{(e)}(a_\tau))$ for $\tau>\rho\gg 0$.

Then

(2) $v(f^{(e)}(a_\tau)(a_\rho-a_\tau)^e)<v(f^{(u)}(a_\tau)(a_\rho-a_\tau)^u)$

for $\tau>\rho\gg 0$ if $u>e$ and either

i) $\bar{p}=0$, or

ii) $\bar{p}>0$ and $\vartheta_{\bar{p}}(\binom{u}{e})=0$, where $\binom{u}{e}=\frac{u!}{e!(u-e)!}$ and $\vartheta_{\bar{p}}:\mathbb{Q}^*\longrightarrow\mathbb{Z}$ denotes the $\bar{p}$-adic valuation.

Proof. Applying Lemma (3.1) to $h=f^{(e)}$ (by (1) h satisfies the hypothesis) we obtain

$$v(h(a_\tau))<v(h^{(u-e)}(a_\tau)(a_\rho-a_\tau)^{u-e})$$

for $\tau>\rho\gg 0$. Since $h^{(j)}=\binom{j+e}{e}f^{(j+e)}$ for all j we get (2).□

(3.2.1) Remark. Suppose $\bar{p}>0$ and let $u=\sum_{i\geq 0}\alpha_i\bar{p}^i$, $e=\sum_{i\geq 0}\beta_i\bar{p}^i$, $0\leq\alpha_i,\beta_i<\bar{p}$ be the $\bar{p}$-adic expansion of u resp. e. Then $\vartheta_{\bar{p}}(\binom{u}{e})=0$ iff $\alpha_i\geq\beta_i$ for eve -

ry $i\geq 0$. For the proof note first that

$$\vartheta_{\bar{p}}(u!)=\sum_{i\geq 0}\alpha_i\vartheta_{\bar{p}}((\bar{p}^i)!) \text{ and } \vartheta_{\bar{p}}((\bar{p}^i)!)=\frac{\bar{p}^i-1}{\bar{p}-1}.$$

In particular if $e=\bar{p}^i$ and $\vartheta_{\bar{p}}(u)=i$ then $\vartheta_{\bar{p}}(\binom{u}{e})=0$.

(3.3) Lemma. Suppose that

(1) $\quad v(f(a_\rho))<v(f(a_\tau)) \quad$ for $\tau>\rho\gg 0$.

Then there exists a positive integer j such that

(2) $v(f(a_\tau))>v(f^{(j)}(a_\tau)(a_\rho-a_\tau)^j)$

For the proof apply (*) like in Lemma (3.1).

(3.4) Corollary. Suppose that f is a nonzero polynomial of least degree for which (2.7) ii) holds. Then the valuation of the nonzero elements from

$$\left\{f^{(j)}(a_\tau)(a_\rho-a_\tau)^j\right\}_{j\geq 0},$$

$\tau>\rho\gg 0$ is different and reaches the minimum for either

i) $j=1$ if $\bar{p}=0$, or

ii) one j of the form $\bar{p}^i$, $i\in\mathbb{N}$ if $\bar{p}>0$.

Proof. By hypothesis $f^{(1)}$ satisfies (2.7) i). If $\bar{p}=0$ then it is enough to apply Lemma (3.3) and Lemma (3.2) for e=1.

Now suppose $\bar{p}>0$. Let u be a positive integer such that $f^{(u)}\neq 0$ and $t=\bar{p}^{\vartheta_{\bar{p}}(u)}$. By Remark (3.2.1) we have $\vartheta_{\bar{p}}(\binom{u}{t})=0$. Taking $h=f^{(t)}$ like in Lemma (3.2) we get

$$h^{(u-t)}=\binom{u}{t}f^{(u)}.$$

Then $h\neq 0$ because otherwise $\vartheta_{\bar{p}}(\binom{u}{t})>0$ and $\bar{p}=p$. Contradiction! Thus $f^{(t)}\neq 0$. By hypothesis $f^{(t)}$ satisfies (2.7) i). Applying Lemma (3.2) we get

$$v(f^{(t)}(a_\tau)(a_\rho-a_\tau)^t)<v(f^{(u)}(a_\tau)(a_\rho-a_\tau)^u),$$

for $\tau>\rho\gg 0$. Then the valuation of the nonzero elements from

$$\left\{f^{(j)}(a_\tau)(a_\rho-a_\tau)^j\right\}_{j\geq 1}, \quad \tau>\rho\gg 0$$

reaches the minimum for $j=\bar{p}^i$, $i\in\mathbb{N}$. By Lemma (3.3) we are ready. □

(3.5) Lemma. Let t be a positive integer and y, z $(z\neq 0)$ two elements from $\mathbb{F}$ such that

(1) $f^{(t)}(y)\neq 0$ and $v(z^t f^{(t)}(y)) < v(z^j f^{(j)}(y))$ for all $j\neq t$.

Suppose that F is henselian. Then f is in $F[Y]$ a multiple of a polynomial h of degree t.

Proof. Using Taylor's formula we have

$$f(y + zX)=f(y) + \sum_{j=1}^{e} f^{(j)}(y)z^j X^j, \quad e:=\deg f$$

Put $c_j:=z^{j-t}\frac{f^{(j)}(y)}{f^{(t)}(y)}$, $j\geq 0$ and $g:=\sum_{j=0}^{e} c_j X^j$. By hypothesis $v(c_j)>0$ for all $j\neq t$ and $c_t=1$. Let $(R,\underline{m})$ be the valuation ring of $\mathfrak{F}$. We have $g\in R[X]$ and $g\equiv X^t \bmod \underline{m}$. Applying Hensel's Lemma we get $g=h'h''$ for some polynomials h', $h''\in R[X]$ such that $\deg h'=t$ and $h'\equiv X^t \bmod \underline{m}$. Take $h:=z^t h'(\frac{Y-y}{z})$

(3.6) Proposition. Suppose that F is henselian and f is a nonzero polynomial of least degree for which (2.7) ii) holds. Then f is linear if $\bar{p}=0$. Otherwise $\deg f=\bar{p}^s$ for a certain $s\in\mathbb{N}$ and

$$v(f^{(\bar{p}^s)}(a_\tau)(a_\rho-a_\tau)^{\bar{p}^s}) < v(f^{(j)}(a_\tau)(a_\rho-a_\tau)^j), \quad \tau>\rho>>0$$

for all j, $0\leq j<\bar{p}^s$.

Proof. By Corollary (3.4) the valuation of the nonzero elements from

$$\left\{ f^{(j)}(a_\tau)(a_\rho-a_\tau)^j\right\}_{j\geq 0}, \quad \tau>\rho>>0$$

is different and reaches the minimum for either

i) $j=1$ if $\bar{p}=0$, or

ii) one j of the form $\bar{p}^s$, $s\in\mathbb{N}$ if $\bar{p}>0$.

Choose $\tau>\rho>>0$ and apply Lemma (3.5) for $y=a_\tau$, $z=a_\rho-a_\tau$. Then f is a multiple in $F[Y]$ of a linear polynomial h if $\bar{p}=0$ or of a polynomial h of degre $\bar{p}^s$. By (2.7.4) f is irreducible in $F[Y]$ and so f,h are associated in the divisibility. In particular $\deg f=\bar{p}^s$. □

(3.7) Lemma. Let $\hat{\mathfrak{F}}=(\hat{F},\hat{v},\Gamma)$ be the completion of $\mathfrak{F}$. Suppose that a is not f.s. and f is a nonzero polynomial of least degree for which (2.7) ii) holds. Then every nonzero polynomial g from $\hat{F}[Y]$ with $\deg g<\deg f$ satisfies (2.7) i).

Proof. It is enough to take g monic. Since a is not f.s there exists $\delta\in\Gamma$ such that $v(a_{\rho+1}-a_\rho)<\delta$ for all $\rho<\theta$. Choose a monic polynomial $g \in F[T]$ with $\deg g = e := \deg g$ such that the coefficients of $g-g$ have their valuation bigger than $e\gamma$. By assumption $\tilde{g}$ satisfies (2.7) i) ($e \leq \deg f$) and so

$$v(\tilde{g}(a_\rho)) < e\gamma \quad \text{for } \rho \gg 0$$

(see Remark (3.1.1)). Then we get

$$v(\tilde{g}(a_\rho)) = \hat{v}(g(a_\rho)) \quad \text{for } \rho \gg 0$$

Thus g satisfies (2.7) i). □

(3.8) <u>Proposition</u>. Suppose that $\Gamma \leq \mathbb{R}$, a is not f.s. and f is a nonzero polynomial of least degree for which (2.3) ii) holds. Then f is linear if $\bar{p}=0$. Otherwise $\deg f = \bar{p}^s$ for a certain $s \in \mathbb{N}$ and

$$v(f^{(\bar{p}^s)}(a_\tau)(a_\rho - a_\tau)^{\bar{p}^s}) < v(f^{(j)}(a_\tau)(a_\rho - a_\tau)^j), \quad \tau > \rho \gg 0$$

for all j, $0 \leq j < \bar{p}^s$.

<u>Proof</u>. Let $\hat{F}$ be the completion of F. By Lemma (3.7) we can apply Proposition (3.6) to f over F because $\hat{F}$ is henselian ($\Gamma \leq \mathbb{R}$). □

(3.9) <u>Theorem</u>. An algebraic p.c.s from F which is not f.s. has a p.ℓ. in F if $\bar{p}=0$ and $\Gamma \leq \mathbb{R}$.

<u>Proof</u>. Let a be any algebraic p.c.s. from F and f a nonzero polynomial of least degree for which (2.7) ii) holds. By Proposition (3.8) f is linear and so a has a p.ℓ. in F (see (2.7.1)). □

(3.10) <u>Corollary</u>. If $\bar{p}=0$ and $\Gamma \leq \mathbb{R}$ then every algebraic immediate valued field extension of $\mathcal{F}$ is dense.

<u>Proof</u>. Let $\mathcal{F}' = (F', v', \Gamma)$ be an algebraic immediate valued field extension of $\mathcal{F}$ and y an element from $F' \setminus F$. By (2.3.2) y is a p.ℓ. of a p.c.s. a from F having no p.ℓ. in F. Using (2.7.2) we note that a is algebraic. If a is not f.s. then a has a p.ℓ. in F by Theorem (3.9). Contradiction! Thus a is f.s. and so y belongs to the complete closure of $\mathcal{F}$ relatively to $\mathcal{F}'$. As y was arbitrarily chosen we get $\mathcal{F} \subseteq \mathcal{F}'$ dense.

(3.10.1) <u>Remark</u>. Actually Theorem 3.9 and the above Corollary are also consequences of "Der Defektsatz" from [O] §9 no.55 (see e.g. [P_1] Corollary (4.2)).

(3.11) Corollary. Let R, R' be the valuation rings of $\mathcal{F}$ resp. $\mathcal{F}'$ and $q \subset R$ a prime ideal of height one from R. Suppose that

(i) $R/q \cong R'/qR'$,

(ii) char $R/q = 0$

(iii) $\mathcal{F} \subseteq \mathcal{F}'$ is immediate and algebraic.

Then $\mathcal{F} \subseteq \mathcal{F}'$ is dense.

Proof. By (i) qR' is a prime ideal and $R_q \subseteq R'_{qR'}$ is an immediate valuation ring extension (see (iii)). Applying Corollary (3.10) we get $R_q \subseteq R'_{qR'}$ dense. Thus for every $y \in F'$ and every $t \in q$ there exists $a \in F$ such that $v(y-a) \geq v(t)$. Since the elements from $\Gamma \setminus v(q \setminus \{0\})$ are smaller than the elements from $v(q \setminus \{0\})$ we get $R \subseteq R'$ dense too. □

(3.12) Theorem. Let $\mathcal{F}' = (F', v', \Gamma)$ be an algebraic immediate valued field extension of $\mathcal{F}$. Suppose that $\bar{p} > 0$, $\Gamma \subseteq \mathbb{R}$ and for every $y \in F'$, $[F(y):F] < \bar{p}$. Then $\mathcal{F} \subseteq \mathcal{F}'$ is dense.

Proof. Let y be an element from $F' \setminus F$. Like above y is a p.ℓ. of an algebraic p.c.s. a from F having no p.ℓ. in F. Let f be a nonzero polynomial of least degree for which (2.7) ii) holds. By hypothesis deg $f < \bar{p}$. Using Proposition (3.8) we get deg $f=1$ if a is not f.s. But this is not possible because a has no p.ℓ. in F. Thus a is f.s. and so $\mathcal{F} \subseteq \mathcal{F}'$ is dense. □

(3.12.1) Remark. The above Theorem cannot be improved too much because there exist algebraic (even separable) immediate valued field extensions which are not dense as shows the following example inspired from [O] § 9, no.57.

(3.13) Example. Let k be a field, X a variable, $\Gamma = \mathbb{Q}$ and F the fraction field of the group algebra $k[\Gamma]$, i.e. the elements of F are rational functions in $(X^{\gamma})_{\gamma \in \mathbb{Q}}$. Let $\tilde{F}$ be the field of all formal sums $\sum_{n \in \mathbb{N}} \alpha_n X^{\gamma_n}$, where $\alpha_n \in k$ and $\gamma = (\gamma_n)_{n \in \mathbb{N}}$ is a monotonically increasing sequence from Γ. The correspondence

$$\sum_{n \in \mathbb{N}} \alpha_n X^{\gamma_n} \rightsquigarrow \gamma_s, \quad s = \min\{n \in \mathbb{N} \mid \alpha_n \neq 0\}$$

defines a valuation $\tilde{v}: \tilde{F}^* \to \Gamma$. Clearly $\tilde{\mathcal{F}} = (\tilde{F}, \tilde{v}, \Gamma)$ is an immediate extension of $\mathcal{F} = (F, v, \Gamma)$ (in fact a maximally complete immediate extension of $\mathcal{F}$). Suppose $p = $ char $k > 0$ and denote $\rho_n = \frac{p^{n+1} - 1}{p^{n+1}}$. Clearly

$\rho=(\rho_n)_{n\in\mathbb{N}}$ is a monotonically increasing sequence bounded by $\frac{1}{p-1}$. Take

$$y=-1+\sum_{n\geq 0}(-1)^n X^{\rho_n},$$

$F':=f(y)$, $v'=\tilde{v}|_{F'}$, $\mathcal{F}'=(F',v',\Gamma)$. Clearly there exists no $z\in F$ such that $v(y-z)>\frac{1}{p-1}$, i.e. $\mathcal{F}\subseteq\mathcal{F}'$ is not dense. On the other hand $\mathcal{F}\subseteq\mathcal{F}'$ is immediate ($\mathcal{F}'\subseteq\tilde{\mathcal{F}}$) and algebraic because y is a solution of

$$f:=Y^p+XY+1$$

To see this note that $1+\rho_n=p\rho_{n+1}$, $n\geq 0$ and $p\rho_0=1$.

Denote $a_s=-1+\sum_{0\leq n\leq s}(-1)^n X^{\rho_n}$, $a=(a_s)_{s\in\mathbb{N}}$. Then a is a p.c.s. from F and y is a p.l. of it. A small computation give us

$$v(f(a_s))=p\rho_{s+1}$$

$$v(f^{(1)}(a_s)(a_{s-1}-a_s))=1+v(a_{s-1}-a_s)=1+\rho_s=p\rho_{s+1}$$

$$v(f^{(p)}(a_s)(a_{s-1}-a_s)^p)=p\rho_s\quad,\ s\geq 1.$$

Since $\rho_{s+1}>\rho_s$ we have

$$v(f^{(p)}(a_s)(a_t-a_s)^p)<v(f^{(j)}(a_s)(a_t-a_s)^j),$$

$s>t\geq 0$ for all $j, 0\leq j<p$. This illustrates our Proposition (3.8).

(3.14) Corollary. Let R, R' be the valuation rings of $\mathcal{F}$ resp. $\mathcal{F}'$ and $q\subseteq R$ a prime ideal of height one from R. Suppose that

(i) $R/qR\cong R'/qR'$,

(ii) $\tilde{p}=\operatorname{char} R/q>0$ and for every $y\in F'$, $[F(y):F]<\tilde{p}$,

(iii) $\mathcal{F}\subseteq\mathcal{F}'$ is immediate.

Then $\mathcal{F}\subseteq\mathcal{F}'$ is dense.

The proof goes like in Corollary (3.11) using Theorem (3.12).

§ 4. The structure of finite dense valued field extensions

In this section we shall use a theorem of Néron desingularization on dense valued field extensions. The result belongs in fact to N.Schappacher [Scha]; our proof follows the proof of $[P_1]$ Lemma (4.3).

(4.1) Theorem (Néron-Schappacher). Let $\mathcal{F}\subseteq\mathcal{F}'$ be a dense valued field extension and $R\subseteq R'$ their valuation ring extension. Suppose that $F\subseteq F'$ is separable. Then R' is a filtered inductive union of its smooth sub-R-algebras of finite presentation.

Proof. Let $B\subset R'$ be a sub-R-algebra of finite type, let us say $B:=R[y]$ for some elements $y=(y_1,\dots,y_n)$ from B. It is enough to embed B in a smooth sub-R-algebra $B'\subseteq R'$ of finite presentation. Since the extension $F\subseteq F'$ is separable there exists a system of polynomials $f=(f_1,\dots,f_r)$, $r:=n-\operatorname{trdeg}_F F(y)$ and a rxr-minor M of Jacobian matrix $J:=(\frac{\partial f}{\partial Y})$ such that $M(y)\neq 0$. Choose M in J such that $v(M(y))$ is minimum. Since $\mathcal{F}\subseteq\mathcal{F}'$ is immediate there exists an element $d\in R$ such that $v(d)=$ $=v'(M(y))$ and so $dR'=M(y)R'$. If $v(d)=0$ then $B':=B_{M(y)}$ is a smooth R-algebra of finite presentation (see e.g. Lemma (7.2)). Suppose now $v(d)>0$. Since $\mathcal{F}\subseteq\mathcal{F}'$ is dense there exist a system of elements $\tilde{y}\in R^n$ such that $v'(y-\tilde{y})\geq 2v(d)$ and so $y\equiv\tilde{y} \bmod d^2R'$. Changing y by $y-\tilde{y}$ and Y by $Y+\tilde{y}$ we may suppose from now on that $y\in d^2R'$, let us say $y=dz$ for an element $z\in dR'^n$. We have

$$\frac{\partial f}{\partial Y}(0)\equiv\frac{\partial f}{\partial Y}(y) \bmod d^2R'$$

Thus $v(M(0))=v(d)$ and every rxr-minor of $(\frac{\partial f}{\partial Y}(0))$ is divisible by d. By theory there exist two invertible matrices U,W such that $U(\frac{\partial f}{\partial Y}(0))W$ has a diagonal form

$$\begin{Vmatrix} d_1 & 0 & 0 \\ & \ddots & \vdots \\ 0 & d_r & 0 \end{Vmatrix}$$

where $d_i\in R$, $v(d_1)\leq v(d_2)\leq\dots\leq v(d_r)$ and $\prod_{i=1}^{r} d_i=d$. Applying on f,z the invertible transformations given by U resp. W^{-1} we can suppose that $(\frac{\partial f_i}{\partial Y_j}(0))=d_i\delta_{ij}$, δ_{ij} being Kronecker' symbol.

We have

$$0=f_i(y)=f_i(0)+d\,d_i(z_i+Q_i(z));\quad 1\leq i\leq r$$

where $Q=(Q_i)_{1\leq i\leq r}$ are polynomials in $Z=(Z_1,\dots,Z_n)$ over R containing only monomials of degree ≥ 2. Then $f_i(0)\in dd_iR'\cap R=dd_iR$ and so $f_i(0)=$

$=aa_i c_i$ for some suitable $c_i \in R$, $1 \leq i \leq r$. Denote $B'':=R[z]$,

$$h_i := c_i + Z_i + Q_i \quad , \qquad 1 \leq i \leq r$$

Since $F(z)=F(y)$ we get $\mathrm{trdeg}_F F(z)=n-r$. Note that $h(z)=0$ and $\frac{\partial h}{\partial Z}(z)$ contains a $r\times r$-minor u from $1+zR'$. Then $B':=B''_u$ is a smooth R-algebra of finite presentation (apply Lemma (7.2) like above). □

(4.2) <u>Theorem</u>. Let $\mathcal{F}\subseteq\mathcal{F}'$ be a dense valued field extension and $R\subseteq R'$ their valuation ring extension. Suppose that $F\subseteq F'$ is finite separable. Then R' is etale and essentially finite over R.

<u>Proof</u>. Let $B\subseteq R'$ be a finite sub-R-algebra of R' whose fraction field is F'. By Theorem (4.1) there exists a smooth sub-R-algebra $B'\subseteq R'$ of finite presentation containing B. Clearly B' is normal because R is so. Then B' contains the integral closure C of R in F'. By [B] VI, §8, no.3 R' is a localization of C and so of B' too. In particular R' is smooth over R. Since B' is of finite type there exists a finite sub-R-algebra A of R' such that B' is contained in one localization of A. Thus R' is a localization of A, i.e. R' is essentially finite over R. Now it is enough to note that a smooth, essentially finite algebra is etale. □

(4.3) <u>Corollary.</u> Let $\mathcal{F}\subseteq\mathcal{F}'$ be an immediate valued field extension and $R\subseteq R'$ their valuation ring extension. Suppose that $F\subseteq F'$ is finite separable, $\Gamma\subseteq\mathbb{R}$ and either

i) $\bar{p}=0$ ($\bar{p}$ being the residue field characteristic of R) or,

ii) $\bar{p}>0$ and for every $y\in F'$, $[F(y):F]<\bar{p}$.

Then R' is etale and essentially finite over R.

For the proof apply Corollary (3.10) and Theorem (3.12).

(4.4) <u>Corollary</u>. Let $\mathcal{F}\subseteq\mathcal{F}'$ be an immediate valued field extension, $R\subseteq R'$ their valuation ring extension, and $q\subset R$ a prime ideal of height one. Suppose that $R/q\cong R'/qR'$, $[F':F]<\infty$, $F\subseteq F'$ separable and either

(i) $p:=\mathrm{char}\, R/q=0$ or

(ii) $p>0$ and $[F(y):F]<p$ for every $y\in F'$.

Then R' is etale and essentially finite over R.

For the proof apply Corollaries (3.11) and (3.14).

§ 5. Algebraic valued field extensions

Let $\mathcal{F}\subseteq\mathcal{F}'$ be a valued field extension with the same value group Γ, $R\subseteq R'$ their valuation rings, $\underline{m}$ the maximal ideal from R, $q\subset R$ a prime idea[l], $\bar{z}$ an element from $\bar{R}':=R'/qR'$ and $\bar{f}$ a monic polynomial from $\bar{R}[Z]$, $\bar{R}:=R/q$. Suppose that $\bar{f}(\bar{z})=0$ and $\bar{w}:=(\partial\bar{f}/\partial Z)(\bar{z})\notin\underline{m}\bar{R}'$. Let f,u be some liftings of $\bar{f}$ resp. $\bar{z}$ to $R[Z]$ resp. R′ and denote $\tilde{R}':=(R'[Z]/(f))_{(\underline{m}R',Z-u)}$, $\tilde{R}:=R[Z]_{\underline{m}\tilde{R}'\cap R[Z]}$, where $z\in\tilde{R}'$ is given by Z.

(5.1) Lemma. Then

(i) $\tilde{R}$, $\tilde{R}'$ are valuation rings with the same value group Γ - etale extensions of R resp. R′,

(ii) $\tilde{R}/q\tilde{R}\cong\bar{R}[Z]_{\underline{m}\bar{R}'\cap\bar{R}[Z]}$

(iii) $\bar{R}'\cong\tilde{R}'/q\tilde{R}'$

(iv) the inclusion $\tilde{R}\subseteq\tilde{R}'$ is a valuation ring extension compatible with $R\subseteq R'$.

Proof. Using some facts from the henselization theory (see [EGA] or [R]) we get etale the inclusions $R\subseteq\tilde{R}$, $R'\subseteq\tilde{R}'$ and we see that (ii)-(iv) hold. Remains to note that $\tilde{R},\tilde{R}'$ are valuation rings. Since R,R′ are normal rings we get $\tilde{R},\tilde{R}'$ normal too by the etality. In particular $\tilde{R},\tilde{R}'$ are domains and let $\tilde{F},\tilde{F}'$ be their fraction fields. Then $\tilde{R},\tilde{R}'$ are localizations of the integral closure of R resp. R′ in $\tilde{F}$ resp. $\tilde{F}'$. Thus R,R′ are valuation rings (see a Remark from [B] VI §8 no.3). □

(5.2) Proposition. Let $\mathcal{F}\subseteq\mathcal{F}'$ be a valued field extension with the same value group of finite rank $t\in\mathbb{N}$, and $R\subseteq R'$ their valuation ring extension. Suppose that

(i) $[F':F]<\infty$,

(ii) for every factor domain $\bar{R}$ of R with char $\bar{R}>0$, $\bar{F}=\mathrm{Fr}\,\bar{R}\subset\bar{F}\otimes_R R'$ is a separable field extension and it holds $[\bar{F}(\bar{y}):\bar{F}]<\mathrm{char}\,\bar{R}$ for every $\bar{y}\in\bar{R}\otimes_R R'$.

Then $R\subseteq R'$ is etale and essentially finite.

Proof. Apply induction on t. If t=0 then the valuation is trivia[l] and we have R=k, R′=k′, $k\subseteq k'$ being the residue field extension of $R\subseteq R'$. Then $k\subseteq k'$

is a finite separable field extension which is clearly etale.

Suppose $t \geq 1$. Let $q \subset R$ be a prime ideal of height one and $\bar{R} := R/q$, $\bar{R}' := R'/qR'$. Let $\bar{F}$, $\bar{F}'$ be the fraction fields of $\bar{R}$ resp. $\bar{R}'$ ($qR' \subset R'$ is a prime ideal because R, R' have the same value group). By (i) we get $[\bar{F}':\bar{F}] < \infty$. Thus $\bar{R} \subseteq \bar{R}'$ is etale and essentially finite by induction hypothesis. Using Jacobian criterion [R] V Theorem 1 there exist $\bar{z} \in \bar{R}'$, $\bar{f} \in \bar{R}[Z]$ such that

(1) $\bar{R}' = \bar{R}[\bar{z}]_{m\bar{R}' \cap \bar{R}[\bar{z}]}$,

(2) $\bar{f}(\bar{z}) = 0$ and $\bar{w} := (\partial \bar{f}/\partial Z)(\bar{z}) \notin m\bar{R}' \cap \bar{R}[\bar{z}]$

Let $f, u\ \tilde{R}', z, \tilde{R}$ be like in Lemma (5.1). Then

(3) $\bar{R}' \cong \tilde{R}'/q\tilde{R}' \cong \tilde{R}/q\tilde{R}$

and by Corollary (4.4) $\tilde{R} \subseteq \tilde{R}'$ is etale and essentially finite. But $R \subseteq \tilde{R}$, $R' \subseteq \tilde{R}'$ are also etale and essentially finite by construction. Then it is enough to apply the following Lemma which follows from Lemma (7.6).

(5.2.1) Lemma. Let $R \subseteq R' \subseteq R''$ be two valuation ring extensions. Suppose that $R \subseteq R''$, $R' \subseteq R''$ are etale and essentially finite. Then $R \subseteq R'$ is etale and essentially finite.

(5.3) Theorem. Let $\mathfrak{F} \subseteq \mathfrak{F}'$ be a valued field extension with the same value group Γ of finite rank, $R \subseteq R'$ their valuation ring extension and k the residue field of R. Suppose that

(i) char $k = 0$,

(ii) $F \subseteq F'$ is algebraic,

Then R' is a filtered inductive union of its etale, essentially finite sub-R-algebras of finite presentation. Moreover if $[F':F] < \infty$ then $R \subseteq R'$ is etale, essentially finite and essentially of finite presentation.

Proof. If $[F':F] < \infty$ then $R \subseteq R'$ is etale and essentially finite by Proposition (5.2). Then $R \subseteq R'$ is essentially of finite presentation by the following Lemma which follows from Corollary (7.4).

(5.3.1) Lemma. Let $A \subseteq B$ be two domains and $q \subset B$ a prime ideal. Suppose that A is normal and B_q is smooth and essentially of finite type over A. Then B_q is essentially of finite presentation over A. Now suppose that $[F':F]=\infty$. Then express F' as a filtered inductive union of its subfields which are finite extensions of F, let us say $F'=\bigcup_{i\in I} F_i$. As above $R\subseteq R_i:=R'\cap F_i$ is etale, essentially finite and essentially of finite presentation. Thus R_i is a filtered inductive union of its etale, essentially finite sub-R-algebras of finite presentation. Since R' is a filtered inductive union of $(R_i)_{i\in I}$ we are ready. □

(5.4) Corollary. Conserving the notations and hypothesis of Theorem (5.3) suppose that R is henselian and $[F':F]<\infty$. Then R' is a finite free R-module and $[k':k]=[F':F]$.

Proof. By Theorem (5.3) $R\subseteq R'$ is essentially finite, i.e. $R'\cong C_{\underline{m}R'\cap C}$ for a finite sub-R-algebra $C\subseteq R'$. Then C is quasi-local because R is henselian and so $R'\cong C$. Thus R' is finite over R. Since R' is torsionless as an R-module we get also R' free over R. The second statement follows from [B] VI, §8 Theorem 2 because R' is the unique valuation ring from F' dominating R. □

(5.4.1) Remark. When R is a valuation ring containing a field of characteristic zero then its integral closure in every finite field extension F′ of F=FrR is a finite free R-module by [B] VI § 8 Theorem 2 and [O] § 9 no.55 "Der Defektsatz" (see also [Ri] G Theorem 2). Thus our Corollary (5.4) is in particular a consequence of "Der Defektsatz".

§ 6. Obstructions for desingularization.

The Theorems (4.1) and (5.3) suggest the introduction of the following definition.

(6.1) A valued field extension $\mathcal{F}\subseteq\mathcal{F}'$, or their valuation ring extensions $R\subseteq R'$ is called

(i) a desingularization extension (shortly a d-extension) if R′ is a filtered inductive union of its smooth sub-R-algebras of finite presentation;

(ii) a weak desingularization extension (shortly a w. d-extension) if R′ is a filtered inductive limit of smooth R-algebras of finite presentation;

(iii) formally a desingularization extension (shortly a f.d-extension if for every nonzero element $x\in R$, R'/xR' is a filtered inductive limit of smooth R/xR-algebras of finite presentation.

(6.1.1) Remark (i) A w.d-extension is separable.

(ii) Composite extensions and filtered inductive unions of d-extensions (resp. w.d. or f.d.) are also d-extensions (resp.w.d. or f.d.). Thus Theorem (4.1) says in fact that a separable dense valued field extension is a d-extension.

(6.2) Lemma. Let $\mathcal{F}\subseteq\mathcal{F}'$ be a f.d. - extension of rank one immediate valued fields of characteristic $p>0$ and $a=(a_\rho)_{\rho<\theta}$ a p.c.s. from F such that

(1) a^{p^s} has a p.ℓ. in F for a certain $s\in\mathbb{N}$

(2) a has a p.ℓ. in F′,

(3) a is not a f.s.

Then a has a p.ℓ. in F.

Proof. Let y be a p.ℓ. of a in F′ and b a p.ℓ. of a^{p^s} in F. Then y^{p^s} is a p.ℓ. of a^{p^s} in F′ and so we have

$$v(y^{p^s}-b)>v((a_{\rho+1}-a_\rho)^{p^s}) \text{ for all } \rho<\theta$$

If c is a nonzero element from F then $(ca_\rho)_{\rho<\theta}$ is a p.c.s. having a p.ℓ. in F iff a has one. Thus multiplying a by a suitable c we can suppose $v(y)\geq 0$, $v(b)\geq 0$, $v(a_\rho)\geq 0$ and so we reduce to the case when y,b and a are from R′,where $R\subseteq R'$ is the valuation ring extension of $\mathcal{F}\subseteq\mathcal{F}'$.

Denote $f:=Y^{p^s}-b\in R[Y]$. Since $\mathcal{F}\subseteq\mathcal{F}'$ is immediate there exists $0\neq d\in R$ such that $v(d)=v(f(y))$ if $f(y)\neq 0$; otherwise take for d a superior bound of $(p^s v(a_{\rho+1}-a_\rho))_{\rho<\theta}$, a being not a f.s. Now, let $\bar{R}':=R'/dR'$ be a filtered inductive limit of some smooth $\bar{R}:=R/dR$-algebras $(\bar{B}_i)_{i\in I}$ of finite presentation and $\bar{\varphi}_i:\bar{B}_i\to\bar{R}'$, $i\in I$ the limit maps. Since $\bar{y}:=y+dR'$ is a solution of f in $\bar{R}'$ there exist a $j\in I$ and an element $\bar{y}_j\in\bar{B}_j$ such that $f(\bar{y}_j)=0$ and $\bar{\varphi}_j(\bar{y}_j)=\bar{y}$. As $\mathcal{F}$ has rank one dim R=1 and so $\bar{R}$ is henselian (dim $\bar{R}=0$). Thus the map $\bar{R}\to\bar{B}_j$ has a retraction $\bar{\Psi}$. Then the element $\bar{z}:=\bar{\Psi}(\bar{y}_j)$ is a solution of f in $\bar{R}$.

Let z be a lifting of $\bar{z}$. It follows $f(z)\in dR$ and so

$$v((z-y)^{p^s})=v(f(z)-f(y))\geq v(d)>p^s v(a_{\rho+1}-a_\rho) \text{ for all } \rho<\theta .$$

Thus $v(z-y)>v(a_{\rho+1}-a_\rho)$, $\rho<\theta$, i.e. z is a p.ℓ. of a in R (see (2.3.1)).

(6.2.1) Remark. The only reason for which we introduced this Lemma was to illustrate our next Theorem (6.6) on an easy example. This Theorem will allow us to substitute (1) in the above Lemma by asking for a to be algebraic over F.

(6.3) Lemma. Let $a=(a_\rho)_{\rho<\theta}$ be an algebraic p.c.s. from $\mathcal{F}=(F,v,\Gamma$ $f\in F[Y]$ a nonzero polynomial of least degree for which (2.7) ii) holds and $I=\{j\geq 1 \mid f^{(j)}\neq 0\}$. Then there exists an ordinal $\tau<\theta$ such that

(i) $v(f^{(j)}(a_\rho))=v(f^{(j)}(a_\sigma))$ for $j\in I$, $\sigma>\rho\geq\tau$,

(ii) $v(f(a_\rho))<v(f(a_\sigma))$ for $\sigma>\rho\geq\tau$,

(iii) there exists an $i\in I$ such that

$v(f^{(i)}(a_\tau))+i\delta < v(f^{(j)}(a_\tau))+j\delta$,

for every $j\in I$, $j\neq i$ and for every $\delta \geq v(a_{\tau+1}-a_\tau)$ such that there exists $\rho, \tau<\rho<\theta$ with $\delta < v(a_{\rho+1}-a_\rho)$.

Proof. Using (2.7) we can find a τ satisfying (i), (ii). For (iii) it is enough to apply the following Lemma which is a slight variation of [K] Lemma 4.

(6.3.1) Lemma. Let $\beta_1,\ldots,\beta_m$ be any elements of an ordered abelian group Γ, $t_1,\ldots,t_m$ some distinct integers and $(\alpha_\rho)_{\rho<\theta}$ a well ordered monotone increasing set of elements from Γ. Then there exist an ordinal $\tau<\theta$ and an integer i, $1\leq i\leq m$ such that

$$\beta_j + t_j\delta > \beta_i + t_i\delta$$

for all $j\neq i$ and $\delta\geq\alpha_\tau$ with $\delta<\alpha_\rho$ for a certain $\rho<\theta$ (depending on δ).

(6.4) Lemma. Let $\mathcal{F}$, a, f, I, τ be like in Lemma (6.3), $\mathcal{F}'=(F',v',\Gamma')$ an extension of $\mathcal{F}$ and y a p.ℓ. of a in F'. Then

(i) $v'(f^{(j)}(y))=v(f^{(j)}(a_\tau))$ for all $j\in I$

(ii) $v'(f(y))>v(f(a_\rho))$ for all $\rho\geq\tau$.

Proof. By (2.4.4) $(f^{(j)}(a_\rho))_{\rho<\theta}$, $j\geq 0$ are ultimately p.c.s. : either of the type

- (2.4.2) i) when j=0, or
- (2.4.2) ii) when $j\geq 1$

Using (2.4.3) we have

$$v'(f^{(j)}(y))=v(f^{(j)}(a_\rho)),\quad j\in I$$
$$v'(f(y))>v(f(a_\rho))$$

for $\rho >> 0$. Now we are ready by Lemma (6.3).□

(6.5) Proposition. Let $\mathcal{F}$, $\mathcal{F}'$, a, f, I, τ be like in the above Lemma and y an element from F'. Then y is a p.ℓ. of a iff it satisfies the following conditions:

(i) $v(f^{(j)}(y))=v(f^{(j)}(a_\tau))$ for $j\in I$,

(ii) $v(f(y))>v(f(a_\rho))$ for all $\rho\geq\tau$.

Proof. The necessity follows from the above Lemma. Suppose that y is not a p.ℓ. but (i)-(iii) hold. Then there exists $\rho \gg 0$ (see (2.3.1 let us say $\rho \gg \tau$, such that $\gamma := v(y-a_\rho) \neq v(a_{\rho+1}-a_\rho)$. Since $(y-a_\sigma)_{\sigma<\theta}$ is a p.c.s. we have

(1) $v(y-a_{\sigma+1}) \geq v(y-a_\sigma)$, $\sigma \gg 0$

by (2.4.2). If $\gamma > v(a_{\rho+1}-a_\rho)$ then we get

$$v(y-a_{\rho+1}) = v(a_{\rho+1}-a_\rho) < v(y-a_\rho) = \gamma$$

which contradicts (1) by taking ρ sufficiently large.

Now we may assume $\gamma < v(a_{\rho+1}-a_\rho)$. Then using (2.4.5) it follows

(2) $v(y-a_\sigma) = \gamma$ for all $\sigma \geq \rho$.

By Taylor's formula we get

(3) $f(a_\sigma) = f(y) + \sum_{j \geq 1} f^{(j)}(y)(a_\sigma - y)^j$

Using (2), (3), (i) and Lemma (6.3) (see the choice of τ) we have

(4) $v(f(a_\sigma)-f(y)) = v(f^{(i)}(y)) + i\gamma$

for a certain $i \geq 1$ and all $\sigma \geq \rho$. Then $v(f(a_\sigma)-f(y))$ does not depend of $\sigma \geq \rho$. Since $(v(f(a_\sigma)))_{\sigma \geq \rho}$ increases monotonically we get $v(f(y)) \leq v(f(a_\rho))$ which contradicts (ii). □

(6.6) Theorem. Let $\mathcal{F} \subseteq \mathcal{F}'$ be a w.d-extension of rank one immediate valued fields. Then

(π) every algebraic p.c.s. from F which is not f.s. has a p.ℓ. in F if it has one in F'.

Proof. Let $a = (a_\rho)_{\rho<\theta}$ be an algebraic p.c.s. from F which is not a f.s. and y a p.ℓ. of a in F'. Multiplying a by a suitable element from F^* we can suppose $v(y)>0$ and $v(a_\rho)>0$, $\rho<\theta$ like in Lemma (6.2). Let $R \subsetneq R'$ be the valuation ring extension of $\mathcal{F} \subsetneq \mathcal{F}'$ and choose a polynomial $f \in R[Y]$ of least degree for which (2.7) ii) holds. Like in the proof of Lemma (2.8) we can change y (if necessary) for to have $f(y) \neq 0$.

By Lemma (2.5) it is enough to show that a has a p.ℓ. in the completion $\hat{R}$ of R (the valuation ring of the completion $\hat{\mathcal{F}}$ of $\mathcal{F}$). Using Lemma (3.7) f is still over $\hat{R}$ of "least degree for which (2.7) ii) holds . Let Σ, σ be like in Lemma (6.5). By Proposition (6.5) a has a p.ℓ. in $\hat{R}$ iff there exists in $\hat{R}$ a solution of the following inequations:

$$(1)\qquad \begin{cases} v(f(Y)) > v(f(a_\rho)) & \text{for all } \rho \geq \sigma \\ v(f^{(j)}(Y)) = v(f^{(j)}(a_\sigma)) & \text{for all } j \in I \end{cases}$$

But (1) has in R' the solution y and so a solution in $\hat{R}$ of the following system of equations

$$(2)\quad v(f^{(j)}(Z)) = v(f^{(j)}(y)), \quad j \in I \cup \{0\}$$

it is still a solution to (1).

Let $(d_j)_{j \in I \cup \{0\}}$ be some elements from R such that $v(d_j) = v(f^{(j)}(y))$ ($\mathcal{F} \subset \mathcal{F}'$ is immediate). Then (2) has solutions in $\hat{R}$ iff the following system of equations in (Z, U_j, U'_j):

$$(3)\qquad \begin{cases} f^{(j)}(Z) = d_j U_j & , j \in I \cup \{0\} \\ U_j U'_j = 1 \end{cases}$$

has solutions in $\hat{R}$. Since (3) has in R' a solution induced by y we are ready by the following

(6.6.1) Lemma. Let $g = (g_1, \ldots, g_s)$ be a system of polynomials in $T = (T_1, \ldots, T_n)$ over R and $\tilde{R}$ a henselian local R-algebra. Suppose that $R \subseteq R'$ is a w. d-extension and g has a solution $t = (t_1, \ldots, t_n)$ in R'. Then g has also a solution in $\tilde{R}$.

Proof. Let R' be a filtered inductive limit of some smooth R-algebras $(B_i)_{i \in I}$ of finite presentation and $\varphi_i : B_i \longrightarrow R'$, $i \in I$ the limit maps. Then there exists a $j \in I$ and an element $t' \in B_j^n$ such that $g(t') = 0$ and $\varphi_j(t') = t$. By henselianity the map $\tilde{R} \longrightarrow \tilde{R} \otimes_R B_j$ has a retraction ψ_j. Then the element $t = \psi_j(1 \otimes t')$ is a solution of g in R. □

(6.6.2) Remark. Lemma (6.6.1) is a variation of [P_1] Theorem (6.1)

(6.7) Corollary. Let $\mathcal{F} \subseteq \mathcal{F}'$ be a w.d-extension of immediate valued fields of rank one and let $\mathcal{F}'' = (F'', v'', \Gamma)$ be the complete closure of $\mathcal{F}$ relatively to $\mathcal{F}'$. Then every element from $F' \setminus F''$ is a p.ℓ. of a p.c.s. from F which is transcendental over F'. In particular F'' is algebraically closed in F.

Proof. Let y be an element from $F' \setminus F''$. By (2.3.2) y is a p.ℓ. of a p.c.s. a from F having no p.ℓ. in F. As $y \notin F''$, a is not f.s. Thus a is transcendental over F by Theorem (6.6). Moreover a is also transcendental over F" as shows Lemma (2.9). □

(6.8) Corollary. Let $\mathcal{F} \subseteq \mathcal{F}'$ be an immediate algebraic w. d-extension of valued fields of rank one.

Then $\mathcal{F} \subseteq \mathcal{F}'$ is dense and separable.

For proof apply Corollary (6.7) and (6.1.1) i).

(6.9) Theorem. Let $\mathcal{F} \subseteq \mathcal{F}'$ be an algebraic immediate extension of valued fields of rank one. Then the following statements are equivalent:

(i) $\mathcal{F} \subseteq \mathcal{F}'$ is a d-extension

(ii) $\mathcal{F} \subseteq \mathcal{F}'$ is a w. d-extension,

(iii) $\mathcal{F} \subseteq \mathcal{F}'$ is a separable f. d-extension,

(iv) $\mathcal{F} \subseteq \mathcal{F}'$ is dense and separable.

Proof. Applying Theorem (4.1) and the above Corollary we get (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iv). The implication (iii)$\Rightarrow$(ii) is a technical consequence of [P_2] Lemma (9.1) (we do not include details because the methods are completely different from those used here). Clearly (ii)$\Rightarrow$(iii) is trivial (see (6.1.1) i)). □

(6.10) Remark. Theorem (6.6) and Corollary (6.9) provide us a lot of examples of immediate extensions which are not d-extensions. For instance the exemple given in (3.13) is not even a f.d-extension. In particular there exist flat morphisms $u: A \to A'$ of quasi-local rings $(A, \underline{m}), (A', \underline{m}')$ such that dim A=dim A'=0, $\underline{m}A' = \underline{m}'$, $A/\underline{m} \cong A'/\underline{m}'$ but A' is not a filtered inductive limit of smooth A-algebras of finite presentation (note that this does not happen if A' is noetherian (see [P_2] Corollary (3.3)).

§ 7. Smooth algebras over normal rings

The aim of this section is to give the proofs to Lemmas (5.3.1) and (5.2.1) (see (7.4) and (7.6)).

(7.1) Lemma. Let $A \subseteq A'$ be two domains, F the fraction field of A, B a smooth A-algebra of finite presentation and $w: B \to A'$ a morphism of

A-algebras. Suppose that

(i) A is normal,

(ii) $\dim(F \otimes_A B) = \mathrm{trdeg}_F F(\mathrm{Im}\, w)$.

Then there exists an element $b \in B$ such that

(1) $w(b)=1$,

(2) the map $B_b \to A'$ induced by w is injective.

This Lemma is a particular form of [P1] Lemma (2.4).

(7.2) Lemma. Let A be a normal domain, C a finite type A-algebra, $y=(y_1,\ldots,y_n)$ a system of generators of C over A, $r:=n-\mathrm{trdeg}_A C$ and $f=(f_1,\ldots,f_r)$ a system of polynomials from $A[Y]$, $Y=(Y_1,\ldots,Y_n)$. Suppose that

(i) C is a domain,

(ii) $f(y)=0$ and there exists a $r \times r$-minor M of the Jacobian matrix $(\frac{\partial f}{\partial Y})$ such that $M(y)$ is invertible in C.

Then C is a smooth A-algebra of finite presentation.

Proof. Clearly $B:=(A[Y]/f)_M$ is a smooth A-algebra of finite presentation. Let $w: B \to C$ be the map given by $Y \to y$ and F the fraction field of A. We have

$$\dim F \otimes_A B = n-r = \mathrm{trdeg}_A C$$

Applying Lemma (7.1) we find an element $b \in B$ such that $w(b)=1$ and $B_b \cong C$. Thus C is a smooth A-algebra of finite presentation. □

(7.3) Proposition. Let A be a normal domain, C a finite type A-algebra and $q \subset C$ a prime ideal. Suppose that

(i) C is a domain,

(ii) $\Omega_{C_q/A}$ is a free C_q-module.

Then C_q is smooth and essentially of finite presentation over A. If $\Omega_{C_q/A}=0$ then C_q is etale over A.

Proof. Let $y=(y_1,\ldots,y_n)$ be a system of generators of C over A. We have

$$\Omega_{C/A} = \bigoplus_{j=1}^{n} C dY_j / \{dg \mid g \in A[Y],\ g(y)=0\}$$

where $Y=(Y_1,\dots,Y_n)$. Since $\Omega_{C/A} \otimes C_q \cong \Omega_{C_q/A}$ is free over C_q there exists a system of polynomials $f=(f_1,\dots,f_r)$, $r := n - \mathrm{rank}_{C_q} \Omega_{C_q/A}$ in $A[Y]$ such that

(1) $f(y)=0$,

(2) there exists a $r\times r$-minor M of $(\frac{\partial f}{\partial Y})$ such that $M(y) \notin q$.

But $\mathrm{rank}_{C_q} \Omega_{C_q/A} = \mathrm{trdeg}_A C$ and applying Lemma (7.2) we get $C_{M(y)}$ smooth and of finite presentation over A. Thus C_q is smooth and essentially of finite presentation over A. □

(7.4) Corollary. Let A be a normal domain, C a finite type A-algebra and $q \subset C$ a prime ideal. Suppose that

(i) C is a domain,

(ii) C_q is smooth over A.

Then C_q is essentially of finite presentation over A.

(7.5) Lemma. Let $A \subseteq A' \subseteq A''$ be two extensions of normal domains and $q \subset A''$ a prime ideal. Suppose that A''_q is etale over A' and essentially of finite type over A. Then $A'_{q\cap A}$ is essentially of finite type over A.

Proof. By Jacobian criterion for etality [R] V, Theorem 1 there exists an element $z \in A''$ such that

(1) $A'[z]_{q\cap A[z]} = A''_q$,

(2) z is a root of a monic polynomial f from $A'[Z]$ such that $w := (\partial f/\partial Z)(z) \notin q$.

Thus z is integral over the normal ring A' and so $\tilde{f} := \mathrm{Irr}(z, F') \in A'[Z]$, F' being the fraction field of A'. Changing f by $\tilde{f}$ we can also suppose f irreducible over F' (we could avoid this change applyin from the beginning [I] III (8.1)). Since A''_q is essentially of finite type over A there exists a polynomial $f' \in A'[Z]$ such that $w' = f'(z) \notin q$ and $T := A'[z]_{ww'}$ is of finite type over A. Express A' as a filtered inductive union of its sub-A-algebras of finite type containing the coefficients of f and f', let us say $A' = \bigcup_{i\in I} A'_i$. Then T is the filtered inductive union of $T_i := A'_i[z]_{ww'}$. Since T is of finite type over A we get

$T=T_i$ for a certain $i\in I$. In particular

(3) $A''_q = A'_i[z]_{q\cap A'_i[z]}$

Now note that $A'_i[z]_{q\cap A'_i[z]}$ is a localization of a finite free $(A'_i)_{q\cap A'_i}$ - algebra, f being irreducible over F'. By faithfully flatness we get

(4) $A'_{q\cap A'} = (A'_i)_{q\cap A'_i}$

from (3). □

(7.6) Lemma. Let $R\subseteq R'\subseteq R''$ be two valuation ring extensions. Suppose that $R\subseteq R''$, $R'\subseteq R''$ are etale and essentially of finite type. Then $R\subseteq R'$ is etale and essentially of finite type too.

Proof. By Lemma (7.5) we get $R\subseteq R'$ essentially of finite type. Since $R'\subseteq R''$ is etale we get $\Omega_{R''/R}=0$ (see [R] V Theorem 2). Thus the following sequence

$$0 \to \Omega_{R'/R} \otimes R'' \to \Omega_{R''/R} \to \Omega_{R''/R'} = 0$$

is exact. But $\Omega_{R''/R} = 0$ because $R\subseteq R''$ is etale. Then $\Omega_{R'/R} \otimes_{R'} R''=0$ and so $\Omega_{R'/R}=0$ by faithfully flatness.

Applying Proposition (7.3) we get $R\subseteq R'$ etale and essentially of finite presentation. □

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