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C_p - ESTIMATES FOR CERTAIN KERNELS ON LOCAL FIELDS

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1. The purpose of this paper is to extend to the case of n-dimensional vector spaces over a local field K the results proved in [8] in the case of $\mathbb{R}^n$. They concern necessary and sufficient conditions for certain kernels to give rise to operators (on $L^2(K^n)$) belonging to Schatten-von Neumann classes (for the theory of Schatten-von Neumann classes, see, for instance, [1]). Though the main ideas are the same as in [8], their actual application needs several adaptations to the new context.

In order to present the results, we have first to establish the notation and to remind some facts from the theory of local fields; the basic reference for this topic is [7].

Let K be a local field; that is, a locally compact, non-discrete, totally disconnected field with the valuation $|\cdot|$. We denote by $\mathfrak{O}=\{x\in K, |x|\leq 1\}$, $\mathfrak{O}^*=\{x\in K, |x| =1\}$, $\mathfrak{p}=\{x\in K, |x|<1\}$. It is known that there exists $\mathfrak{p}\in\mathfrak{P}$, such that $\mathfrak{P}=\mathfrak{p}\mathfrak{O}$ (this $\mathfrak{p}$ will be fixed in the sequel). The residue space $\mathfrak{O}/\mathfrak{p}$ is a finite field; let Q be a complete set of representatives for it. If card $Q=q$, then the image of K^* in $(0,\infty)$ under the valuation $|\cdot|$ is the multiplicative subgroup of $(0,\infty)$ generated by q; also $|\mathfrak{p}|=q^{-1}$.

Now, define on K^n, $|x|=\max_{1\leq i\leq n}|x_i|$. $x\mapsto|x|$ is an ultrametric valuation on K^n; that is, $x\mapsto|x|$ is a norm and $|x + y|\leq \max\{|x|,|y|\}$. We will denote by $\mathcal{B}_k=\{x\in K^n, |x|\leq q^{-k}\}$,

$S_k=\{x\in K^n, |x|=q^{-k}\}$; Φ_k will be the characteristic function of $\mathcal{B}_k$. $\mathcal{S}=\mathcal{S}(K^n)$ will denote the space of finite linear combinations of characteristic functions of balls.

The Fourier transform on K^n is defined as follows: let χ be a fixed character on K that is trivial on $\mathfrak{O}$ but is non trivial on $\mathfrak{p}^{-1}$. Then, for $f\in L^1(K^n)$

$$\hat{f}(x)=\int_{K^n} f(\xi)\,\overline{\chi(x\xi)}\,d\xi$$

The standard properties of the Fourier transform can be found in [7, chap.III].

Caution: the sign $|.|$ is used to denote the valuations on K, K^n, as well as the modulus of a complex number.

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2. In the sequel we shall recapture the main results of [8]. We shall consider operators given by kernels of the form

$$A(x,y)\hat{\varphi}(x-y) \tag{1}$$

where the main condition imposed on the continuous function A is its $\mathfrak{p}$-homogeneity:

$$A(\mathfrak{p}x,\ \mathfrak{p}y)=A(x,y),\quad \forall\ x,y\in K^n \tag{2}$$

Further restrictions on A will be stated when necessary.

Let us introduce also the equivalent of Besov spaces on K^n. If $R_k(x)=q^{-kn}\Phi_{-k}(x)$, then $\hat{R}_k=\Phi_k$; we define, for $p,q\geq 1$, $s\in\mathbb{R}$

$$\dot{B}^s_{pq}=\dot{B}^s_{pq}(K^n)=\{f\in\mathcal{S}'\mid \{q^{-sk}\|f*(R_k-R_{k-1})\|_p\}_{k\in\mathbb{Z}}\in \ell^q\}$$

These spaces appear in [7] (actually, we use their "homogeneous" version).

Now, let $T(A,\varphi)$ be the operator whose kernel is $A(x,y)\hat{\varphi}(x-y)$, and $T(A)$ the operator whose kernel is $A(x,y)$. The supplementary conditions on A which will appear below will tend towards establishing, for a fixed s, the equivalence "$T(A,\varphi)$ if and only if $\varphi\in\dot{B}^{n/p}_{pp}$".

<u>Lemma 1.</u> Let $E=\{(x,y)\in K^n \times K^n \ , \ |x-y|=1\}$, and $\chi(x,y)$ the characteristic function of E. Define

$$a_1(A) = \|T(\chi A)\|_{C_1}$$

Then

$$\| T(A,\varphi)\|_{C_1} \leq C . a_1 (A) \cdot \|\varphi\|_{\dot{B}^n_{11}}$$

The proof is similar to that of lemma 1 in [8]. The functions ψ_k are given simply by $\psi_k = \Phi_k - \Phi_{k-1}$.

Also, with a proof similar to that of lemma 2 in [8], we obtain

<u>Lemma 2</u>. Define

$$a_2(A)^2 = \sup_{|t|=1} \int_{K^n} |A(y+t,y)|^2 dy$$

Then

$$\| T(A,\varphi)\|_{C_2} \leqslant C . a_2 (A) \|\varphi\|_{\dot{B}^{n/2}_{22}}$$

In order to apply lemmas 1 and 2, we state the following result, which can be proved by methods analogous to those in [1 , XI, 9].

Proposition A. Suppose $\xi, \eta \in K^n$, $K(x,y)$ is a kernel defined on $(\xi+\mathcal{B}_1)\times(\eta+\mathcal{B}_1)$; and $T(K)$ is the operator corresponding to K; suppose also $\alpha > \frac{n}{2}$. Define

$$|||K||| = \max\left\{ \sup_{x,y} |K(x,y)| \, , \sup_{x,y,h} |h|^{-\alpha} |K(x,y+h)-K(x,y)| \right\}$$

Then, if $|||K|||<\infty$, $T(K) \in C_1$, and

$$\|T(K)\|_{C_1} \leq C \; |||K|||$$

We may now state

Theorem 1. Suppose A is continuous on $(K^n \times K^n)\setminus\{0\}$, satisfies (2) and, moreover, on $B=\{(x,y)\in K^n\times K^n, \max\{|x|,|y|\}=1\}$ we have

$$\text{(i)} \quad |A(x,y)| \leq C\,|x-y|^{\alpha}$$
$$\text{(ii)} \quad |A(x,y+h)-A(x,y)| \leq C\,|h|^{\alpha}$$

where $\alpha > \frac{n}{p}$, $1 \leq p \leq 2$.

Then there is a constant C_p (depending of course also on A), such that

$$\|T(A,\varphi)\|_{C_p} \leq C_p \|\varphi\|_{\dot{B}^{n/p}_{pp}}$$

Proof. We shall suppose $1<p<2$ (otherwise the proof is simpler). Consider the analytic family of kernels

$$\text{(3)} \qquad A_\lambda(x,y) = \left[\frac{|x-y|}{\max(|x|,|y|)}\right]^{\lambda-\alpha} A(x,y)$$

defined for $\frac{\alpha p}{2} \leq \operatorname{Re}\lambda \leq \alpha p$.

We will use interpolation between $\mathrm{Re}\lambda = \frac{\alpha p}{2}$ and $\mathrm{Re}\lambda = \alpha p$.

For $\mathrm{Re}\lambda = \frac{\alpha p}{2}$, the estimate is rather straightforward. If $|y| = q^k > 1$, and $|t| = 1$, we have, using condition (i) in the theorem:

$$|A_\lambda(y+t,y)| = \frac{1}{|y|^{\alpha(\frac{p}{2}-1)}}\,|A(y+t,y)| =$$

$$= q^{k\alpha(1-\frac{p}{2})}\cdot|A(p^k(y+t),p^k y)| \leq C\cdot q^{k\alpha(1-\frac{p}{2})}\;q^{-k\alpha} =$$

$$= C\, q^{-\frac{kn}{2}}\; q^{-\frac{k}{2}(\alpha p - n)}$$

Therefore, for $|t| = 1$,

$$\int_{K^n} |A(y+t,y)|^2 dy = C + \int_{|y|>1} |A(y+t,y)|^2 dy =$$

$$= C + \sum_{k\geq 1} \int_{|y|=q^k} |A(y+t,y)|^2 dy \leq C\,(1 + \sum_{k\geq 1} q^{nk}.q^{-nk}.q^{-k(\alpha p-n)}) < \infty$$

whence $a_2(A_\lambda) < \infty$.

Suppose now $\mathrm{Re}\lambda = \alpha p$. We have to estimate $a_1(A_\lambda) = \|T(\chi A_\lambda)\|_{C_1}$, where χ is the characteristic function of the set $E \subset K^n \times K^n$, $E = \{(x,y) \mid |x-y| = 1\}$.

We shall need some preliminary notations. Suppose $x = (x_1,\dots,x_n) \in K^n$. Then $x_i = \sum_{k=k_i}^{\infty} p^k x_{ik}$, where $x_{ik} \in \Omega$; that is, x_{ik} belongs to a set of q elements. Moreover, in this case $|x| = \max\{|x_i|\} = q^{-\min\{k_i\}}$. Define a function $\iota : K^n \to K^n$ by putting $\iota(x) = (\bar{x}_1,\dots,\bar{x}_n)$, where $\bar{x}_i = \sum_{K=k_i}^{-1} p^k x_{ik}$ (in case $k_i > -1$, we put $\bar{x}_i = 0$). Then $\iota(K^n)$ is a denumerable set of elements. Define $\mathcal{J}_0 = \{0\}$ and, for $k \geq 1$, $\mathcal{J}_k = \{x \in \iota(K^n) \mid |x| = q^k\}$.

Then $\iota(K^n) = \bigcup_{k=0}^{\infty} \mathcal{J}_k$, and the number of elements in $\mathcal{J}_k$ is less than q^{nk}. Also, $|x-y| = 1$ is equivalent to $\iota(x) = \iota(y)$ and $\iota(p^{-1}x) \neq \iota(p^{-1}y)$. If $B(x,\varepsilon)$ denotes the (closed) ball of center x

and radius ε, then, for $\xi \in \iota(K^n)$, we have $B(\xi,1)\,(=\xi+\mathcal{B}_1)=$ $= \bigcup_{\zeta\in Q} B(\xi+p\zeta, q^{-1})$, and we may write the set E as a disjoint union

$$E = \bigcup_{\xi\in\iota(K^n)} \bigcup_{\substack{\zeta,\eta\in Q\\ \zeta\neq\eta}} B(\xi+p\zeta,\, q^{-1})\times B(\xi+p\eta,\, q^{-1}) =$$

$$= \bigcup_{k=0}^{\infty} \bigcup_{\xi\in\mathcal{J}_k} \bigcup_{\substack{\zeta,\eta\in Q\\ \zeta\neq\eta}} B(\xi+p\zeta,\, q^{-1})\times B(\xi+p\eta,\, q^{-1})$$

Denote $E_{\xi;\zeta,\eta} = B(\xi+p\zeta, q^{-1})\times B(\xi+p\eta,\, q^{-1})$, $\chi_{\xi;\zeta,\eta} = \chi_{E_{\xi;\zeta,\eta}}$, and let us estimate $\|T(\chi_{\xi,\zeta,\eta} A_\lambda)\|_{C_1}$ (the estimate below is actually valid for $k\geq 1$; $\mathcal{J}_0$ can be treated similarly).

Recall that A_λ is given by formula (3); however, when $(x,y)\in E_{\xi,\zeta,\eta}$, we have $|x-y|=1$ and $\max(|x|,|y|)=q^k$.

Also, by homogeneity (relation (2)), we have

$$|A(x,y)| = |A(p^k x, p^k y)| \leq C\,|p^k x - p^k y| = C.q^{-k\alpha}$$

and, for $|h|<1$,

$$|h|^{-\alpha}|A(x,y+h)-A(x,y)| = |h|^{-\alpha}|A(p^k x, p^k y+p^k h)-A(p^k x, p^k y)| \leq$$
$$\leq C\,.\,|h|^{-\alpha}.q^{-k\alpha}.\,|h|^{\alpha} = C\,.\,q^{-k\alpha}$$

(since $(x,y)\in E_{\xi,\zeta,\eta}$ implies $|p^k x| = |p^k y| = 1$, we have applied (i) and (ii) in the hypothesis).

By Proposition A, we have

$$\|T(\chi_{\xi,\zeta,\eta})s_\lambda\|_{C_1} \leq C.q^{-k\alpha}$$

and therefore

$$\|T(A_\lambda)\|_{C_1} \leq C.\sum_{k=0}^{\infty} \sum_{\xi\in\mathcal{J}_k} \sum_{\substack{\zeta,\eta\in Q\\ \zeta\neq\eta}} q^{-k\alpha(p-1)}q^{-k\alpha} \leq$$
$$\leq C\sum^{\infty} q^{nk}q^{-k\alpha p} < \infty$$

(by the condition $\alpha > \frac{n}{p}$).

To end the proof, consider, for $\frac{\alpha p}{2} \leq \mathrm{Re}\,\lambda \leq \alpha p$, the analytic family of operators $\mathcal{T}_\lambda$, which associate to the function φ the operator $T(A_\lambda, \varphi)$. Then, for $\mathrm{Re}\lambda = \frac{\alpha p}{2}$, $\mathcal{T}_\lambda$ maps $\dot{B}_{22}^{n/2}$ into $\mathcal{C}_2$, while, for $\mathrm{Re}\lambda = \alpha p$, it maps $\dot{B}_{11}^{n}$ into $\mathcal{C}_1$. The desired conclusion follows by interpolation.

3. We shall now treat the reverse problem. We rely on the following lemma, whose proof is similar to that of corollary 1 in [8].

<u>Lemma</u> 3. Let A be some locally integrable kernel on $K^n \times K^n$; suppose $\alpha, \alpha' \in \mathcal{S}(K^n)$. Define the function a by its Fourier transform :

$$\hat{a}(u) = \alpha'(u) \int A(x+u, x)\,\alpha(x)\,dx$$

Suppose $\mathrm{supp}\,\alpha$, $\mathrm{supp}\,\alpha + \mathrm{supp}\,\alpha' \subset \{x \in K^n \mid |x| \leq R\}$, and denote by P, P' the projections onto $L^2(\mathrm{supp}\,\alpha)$, $L^2(\mathrm{supp}\,\alpha + \mathrm{supp}\,\alpha')$, respectively. Then, for $1 \leq p \leq \infty$,

$$\| a \|_{L^p} \leq C . R^{n(1-\frac{1}{p})} \| P' T(A) P \|_{\mathcal{C}_p}$$

where C depends on the multiplier norm (in L^1) of α' and on the uniform norm of α.

The following proposition, which we will use in the sequel, is an immediate consequence of [5, 3. Corollary 2].

<u>Proposition B.</u> If $m\in B^{n/2}_{21}(K^n)$, then $\hat{m}\in L^1(K^n)$; therefore m is a multiplier in all $L^p(K^n)$, $1\leq p\leq\infty$.

We may now state the theorem.

<u>Theorem 2.</u> Let A be continuous on $K^n x K^n\setminus\{0\}$, satisfying (2). Suppose that:

(i) there is $\alpha > \frac{n}{2}$, such that

$$|A(x+h,y)-A(x,y)|\leq C|y|^{\alpha} \quad \text{for } |x|=|y|=1,\ |h|<1$$

(ii) for any $u\in K^n\setminus\{0\}$, there exists $x\in K^n$, such that

$$A(x+u,x)\neq 0$$

Then

$$\|\varphi\|_{B^{n/p}_{pp}}\leq C\|T(A,\varphi\|_{C_p} \qquad \text{for } 1\leq p\leq\infty$$

<u>Proof.</u> Let $F=\{u\in K^n, |u|=1\}$. Let Ω_j be a finite open cover of F (we may choose it to be open also in K^n), D_j open sets in $K^n\setminus\{0\}$, $\varkappa_j\in\mathbb{C}$, $|\varkappa_j|=1$, such that $\mathrm{Re}(\varkappa_j A(x+u,x))>0$ for $u\in\Omega_j$, $x\in D_j$. Take $\Omega'_j\ll\Omega_j$, such that $\{\Omega'_j\}$ is still an open cover of F, and choose $\alpha_j, \alpha'_j\in\mathcal{S}(K^n)$ positive functions, such that

(i) $\alpha'_j(u)>0$ for $u\in\Omega'_j$

(ii) $\mathrm{supp}\,\alpha'_j\subset\Omega_j$, and $\mathrm{supp}\,\alpha_j\subset D_j$

(the possibility of this construction follows from (ii) in the statement of the theorem).

Define now functions b_{jk} by

$$\hat{b}_{jk}(u)=\hat{\varphi}(u)\alpha'_j(p^{-k}u)\int A(x+u,x)\alpha_j(p^{-k}x)\,dx$$

If P'_{jk}, P_{jk} are the corresponding projections, then we have, by lemma 3

$$(5) \qquad \| b_{jk} \|_{L^p} \leq C\, q^{-kn(1-\frac{1}{p})} \| P'_{jk} T(A,\varphi) P_{jk} \|_{C_p}$$

(Note that C depends on the multiplier norm of α'_j, and can be chosen therefore independently of k and j).

Denote

$$\theta_{jk}(u) = \alpha'_j(p^{-k}u) \int A(x+u,x)\, \alpha_j(p^{-k}x)\, dx$$

A change of variables yields

$$\theta_{jk}(u) = q^{-kn} \theta_{jo}(p^{-k}u)$$

Now, define ψ, ψ_k by $\hat{\psi} = \sum \kappa_j \theta_{jo}$, $\hat{\psi}_k(u) = \hat{\psi}(p^{-k}u)$
Note that supp θ_{jo}, supp $\hat{\psi} \subset S_1$. Also, for $|u| = 1$, $|v| < 1$, we have, by condition (i),

$$|\theta_{jo}(u+v) - \theta_{jo}(u)| \leq C |v|^{\alpha}$$

and therefore

$$|\hat{\psi}(u+v) - \hat{\psi}(u)| \leq C.\, |v|^{\alpha}$$

But we have Re $\hat{\psi} > 0$ on S_1, and therefore $\frac{1}{\hat{\psi}}$ satisfies a similar estimate on S_1:

$$\left| \frac{1}{\hat{\psi}}(u+v) - \frac{1}{\hat{\psi}}(u) \right| \leq C.\ v$$

It follows easily, since $\frac{1}{\hat{\psi}}$ is supported on S_1, that it belongs to $\dot{B}^{n/2}_{21}(K^n)$, and we may apply proposition B to conclude that it is a multiplier on any $L^p(K^n)$.

From (5) we obtain

$$q^{-kn}\|\varphi * \psi_k\|_{L^p} \leq \sum_j \|b_{jk}\|_{L^p} \leq C.q^{-kn(1-\frac{1}{p})}.\left(\sum_j \|P'_{jk}\ T(A,\varphi)P_{jk}\|_{C_p}\right)$$

whence

$$q^{-\frac{kn}{p}}\|\varphi*\psi_k\|_{L^p} \leq C\left(\sum_j \|P'_{jk}\ T(A,\varphi)P_{jk}\|^p_{C_p}\right)^{1/p}$$

Therefore

$$(6) \quad \sum_{k\in\mathbb{Z}} q^{-kd}\|\varphi*\psi_k\|^p_{L^p} \leq C.\sum_{k\in\mathbb{Z}}\sum_j \|P'_{jk}\ T(A,\varphi)P_{jk}\|^p_{C_p} \leq$$

$$\leq C\|T(A,\varphi)\|^p_{C_p}$$

since P'_{jk_1}, P'_{jk_2} and P_{jk_1}, P_{jk_2} are disjoint for $|k_1-k_2|$ sufficiently large (depending only on the sets D_j).

But, since $\frac{1}{\hat{\psi}_k}$ are multipliers of uniformly bounded norm in L^p, it follows that

$$\|\varphi * \psi_k\|_{L^p} \geq \|\varphi * (R_k - R_{k-1})\|_{L^p}$$

and, by (6),

$$\|\varphi\|_{B^{n/p}_{pp}} \leq C\|T(A,\varphi)\|_{C_p}$$

4. Final remarks. 1. The results presented apply in particular to commutators of multiplication operators with singular integral operators of the type considered in [7 ,VI.4] (this is the analogue for local fields of the commutators considered in [4],[6]).

2. In a recent work ([3]), Janson and Peetre extend the results of [2] and [8]; by combining their method