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A dual Galois theory of rank two for nonseparable extensions of fields *

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Angel POPESCU

Abstract In this work we extend the classical Galois-Krull theory for separable and normal extensions of fields, and the Jacobson theory for finite purely inseparable extensions of exponent 1, to general normal extensions of exponent 1 (the maximal purely inseparable subextension has exponent 1).

A. Distinguished algebraic extensions

Definition 1. An algebraic extension of fields K/k will be called distinguished if it is possible to find a purely inseparable subextension $L/k \subset K/k$ with K/L separable.

Proposition 1. Let K/k be a distinguished algebraic extension of fields, k_K/k the maximal separable subextension of K/k and K_o/k the maximal purely inseparable subextension of K/k. In this case $K = K_o \cdot k_K$, and K/K_o is separable. Conversely, for every purely inseparable extension N/k and every separable extension M/k the extension $K = N \cdot M/k$ is distinguished.

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<u>Proof</u> Let K/k be a distinguished extension of fields and L/k a purely inseparable extension with K/L separable. Every $x \in K$, pure over k ,is pure over L, so that $x \in L$ (K/L is separable). Hence $L = K_o$, the maximal purely inseparable subextension of K/k. Now we examine the diagram

$$\begin{tikzcd}
 & K & \\
 & K_o \cdot k_K \arrow[u, "\text{sep.}\ \text{pure}"] & \\
K_o \arrow[ur, "\text{sep.}"] \arrow[uur, bend left, "\text{sep.}"] & & k_K \arrow[ul, "\text{pure}"'] \arrow[uul, bend right, "\text{pure}"'] \\
 & k = K_o \cap k_K \arrow[ul, "\text{pure}"] \arrow[ur, "\text{sep.}"'] &
\end{tikzcd}$$

We conclude that $K/K_o \cdot k_K$ is pure and separable, and $K = K_o \cdot k_K$. The last part of the Prop.1 is a direct consequence of the Def.1.

<u>Corollary 1</u>. A separable , a purely inseparable,or a normale algebraic extension K/k is distinguished.

<u>Proof</u>. Nontrivial is the fact that a normal algebraic extension K/k is distinguished. But this appears in [3] as Prop. 12, §7,Cap. VII.

<u>Proposition 2</u>. Every algebraic extension K/k contains a maximal distinguished subextension K_d/k.

<u>Proof</u>. It is sufficient to take K_d/k as the maximal separable subextension of K/K_o, where K_o is the maximal purely inseparable subextension of K/k.

Remark 1. Generally speaking $K_d \neq K$, in other words, there exist algebraic extensions K/k that are not distinguished. This is the case in the following exemple, sent to us by the amiability of S. Iyanaga-it was constructed by a referee of one of my previous paper.

Exemple 1. Let p be an add prime, $F = F_p$ the prime field of characteristic p, t, s two independent variables over F. Put $k = F(t^p, s^p)$, $K = F(t, s, x)$, where x is a root of $x^2 - tx + s = 0$. K/k is normal of degree $2p^2$. Consider now $L = k(x)$, and $M = k(x^p)$. M/k is the maximal separable subextension of L/k. $F(t,s)$ is the maximal pure subextension of K/k, so that $k(x) \cap F(t,s) = k$ is the maximal pure extension of $L = k(x)$. If L/k were distinguished, we had $L = k \cdot M = M$, a contradiction. We conclude that L/k is a subextension of normal (distinguished) extension K/k, which is not distinguished.

B. A Galois type correspondence for distinguished subextensions

Let K/k be a normal (algebraic) extension of exponent 1, in other words the maximal pure subextension of K/k, K_o/k is of exponent 1, and k_K the maximal separable subextension of K/k. In this and the following section we consider, as a non-trivial case, all the fields having characteristic $p \neq 0$.

For an algebraic normal extension K/k we denote by $\mathcal{D}_{K/k}$, the K- liniar space of all k-derivations of K, and by $S = \mathrm{Aut}\,(K/k)$. It is clear that $K_o = K^S = \{x \in K, \sigma(x) = x$, for every $\sigma \in S\}$.

For a K- subspace of $\mathcal{D}_{K/k}$, $\mathcal{A}$, denote by $N(\mathcal{A}) = \bigcap\limits_{D \in \mathcal{A}} \mathrm{Ker} D$, the annulator of $\mathcal{A}$, and for a subextension $L/k \subset K/k$ denote by $\mathcal{A}(L) = \{ D \in \mathcal{D}_{K/k}, D(x) = 0$, for all $x \in L\}$.
A K-subspace of $\mathcal{D}_{K/k}$, $\mathcal{A}$, will be called <u>arithmetically maximal(A-maximal)</u> if for all other K-subspace $\mathcal{B}$ of $\mathcal{D}_{K/k}$ with $N(\mathcal{B}) = N(\mathcal{A})$, and $\mathcal{B} \supset \mathcal{A}$, we have $\mathcal{A} = \mathcal{B}$.

<u>Corollary 2</u>. If $\mathcal{A}$ is an A-maximal K-subspace of $\mathcal{D}_{K/k}$, we have $\mathcal{A}(N(\mathcal{A})) = \mathcal{A}$.

<u>Proof</u>. It is clear that $\mathcal{A} \subset \mathcal{A}(N(\mathcal{A}))$, and $N(\mathcal{A}(N(\mathcal{A}))) = N(\mathcal{A})$, because we always can find a derivation $D \in \mathcal{D}_{K/k}$ with $\mathrm{Ker} D = N(\mathcal{A})$ ([2], Exc.3, pag 185, and an extension of it using Zorn's Lemma for the infinite case).

For a derivation $D \in \mathcal{D}_{K_o/k}$ we denote by D^* the unique derivation in $\mathcal{D}_{K/k}$ which extend D ([3], Chap.X.Th.7 and consequences). Note that the application $D \dashrightarrow D^*$ is K_o-liniar and we can view $\mathcal{D}_{K_o/k}$ as a K_o-subspace in $\mathcal{D}_{K/k}$.

<u>Definition 2</u>. The set $G(K/k) = \{ (\sigma, \sigma E), \sigma \in S, E \in \mathcal{D}_{K/k}\}$, with the multiplication rule

(1) $\quad (\sigma_1, \sigma_1 E_1)(\sigma_2, \sigma_2 E_2) = (\sigma_1 \sigma_2, \sigma_1 \sigma_2 E_2 + \sigma_1 E_1 \sigma_2)$,

becomes a group, called <u>the Galois group of rank 2</u> of extension K/k (supposed normal).

Definition 3. The set $G'(K/k) = \{ (\sigma, \sigma E^*), \sigma \in S, E \in \mathcal{D}_{K_0/k} \}$, with the same kind of multiplication (1), becomes a subgroup of $G(K/k)$, called the dual Galois group of rank 2 associated with K/k.

Lemma 1. For $\sigma \in S$ and $E \in \mathcal{D}_{K_0/k}$, we have $\sigma E^* = E^*\sigma$. Moreover, $G'(K/k) \cong S \times \mathcal{D}_{K_0/k}$, where $\mathcal{D}_{K_0/k}$ is considered with the additive law of composition.

Proof. Let x be in K and $f(X) = a_0 + a_1 X + \ldots + X^t$, the minimal separable polynomial of x over K_0. (K/K_0 is separable). Denote by $f^E(X) = E(a_0) + E(a_1)X + \ldots + E(a_{t-1})X^{t-1}$. $E^*(x) = -f^E(x)/f'(x)$, and $\sigma E^*(x) = -f^E(\sigma(x))/f'(\sigma(x)) = E^*(\sigma(x))$, $a_i, E(a_i) \in K_0 = K^S$. The last part of the Lemma follows from the association: $(\sigma, \sigma E) \longrightarrow (\sigma, E)$ and the commutation of $\sigma \in S$ with E^*, where $E \in \mathcal{D}_{K_0/k}$.

Definition 4. A subgroup $M = (H, \mathcal{t})$ in $G'(K/k)$ is said to be closed in $G'(K/k)$ if H is closed in the Krull topology on $S = \mathrm{Aut}(k_K/k)$, and $\mathcal{t}$ is an A-maximal K-subspace of $\mathcal{D}_{K_0/k}$.

For a subextension T/k of K/k we denote by $\mathcal{t}_0(T) = \{ D \in \mathcal{D}_{K_0/k}, D^*(x) = 0, \text{for all } x \in T \}$, $\psi(T) = M_T = (H_T, \mathcal{t}_T)$, where $H_T = \{ \sigma \in S, \sigma(x) = x, \text{for all } x \in T \}$, $\mathcal{t}_T = \mathcal{t}_0(T \cap K_0)$, and by $\varphi(M) = L_M = (\mathrm{Fix}\, H \cap k_K).N_0(\mathcal{t})$, for $M = (H, \mathcal{t})$, closed in $G'(K/k)$, $\mathrm{Fix}\, H = \{ x \in K, \sigma(x) = x, \text{for all } \sigma \in H \}$, and $N_0(\mathcal{t}) = \{ x \in K_0, D(x) = 0, \text{for all } D \in \mathcal{t} \subset \mathcal{D}_{K_0/k} \}$.

THEOREM 1. Let K/k be a normal algebraic extension of exponent 1 (K_0/k is of exponent 1). With the above notations, the maps ψ and φ establish a one-to-one correspondence between the distinguished subextensions of K/k and the closed subgroups of G'(K/k).

<u>Proof</u>. Let L/k be a distinguished subextension in K/k. We want $\psi(L) = H_L \times \mathfrak{t}_L$ be closed in G'(K/k). H_L is identified with the subgroup of Aut (k_K/k) which leave unchanged the elements of $k_L = \{ x \in L, x$ is separable over $k \}$. From the classical Galois-Krull theory follows that H_L is closed in the Krull topology on S = Aut (k_K/k) = Aut (K/k). Moreover, $\mathfrak{t}_0(N_0(\mathfrak{t}_L)) = \mathfrak{t}_0(N_0(\mathfrak{t}_0(L \cap K_0))) = \mathfrak{t}_0(L \cap K_0) = \mathfrak{t}_L$, from the proof of Cor.2. We proved that $M_L = H_L \times \mathfrak{t}_L$ is closed indeed in G'(K/k).

Let $M = H \times \mathfrak{t}$, be a closed subgroup in G'(K/k). It is clear that L_M = (Fix H $\cap$ k_K).$N_0(\mathfrak{t})$ is distinguished in K/k (it is the compositum between a separable and a pure subextension of K/k).

Let us remark now that $L_M = \{ x \in$ Fix H, x is separable over $N_0(\mathfrak{t}) \subset K_0 \}$. For this, we have $N_0(\mathfrak{t}) \subset K_0 \subset$ Fix H, hence $L_M \subset \{ x \in$ Fix H, x separable over $N_0(\mathfrak{t}) \}$. Now let x be in Fix H, x separable over $N_0(\mathfrak{t})$. $x = \sum \xi_i \eta_i$, where $\xi_i \in k_K$, $\eta_i \in K_0$ (K/k is distinguished). $x = \sigma(x) = \sum \sigma(\xi_i).\eta_i = \sum \xi_i.\eta_i$, where $\sigma \in H$, hence $\xi_i \in$ Fix H $\cap$ k_K (we may consider η_i free over k). Now, if $D \in \mathfrak{t}$, D(x) = 0, x beeing separable over $N_0(\mathfrak{t})$ ($D(x) = -f^D(x)/f'(x)$, with $f(X) \in N_0(\mathfrak{t})[X]$, where f is the minimal polynomial of x over $N_0(\mathfrak{t})$). It follows that in a writing of $x = \sum \theta_i \tau_i$, $\theta_i \in k_K$, $\tau_i \in K_0$, θ_i free over k, we have $0 = D(x) = \sum \theta_i D(\tau_i)$, and $D(\tau_i) = 0$, so that $\tau_i \in N_0(\mathfrak{t})$. But $k_K \cap K_0 = k$, and we can combine the two writings of x to obtain $x \in L_M$.

Let L/k be a distinguished subextension in K/k. We shall prove that $\underline{L \subset \varphi\psi(L)}$. $L = k_L.(L \cap K_o)$, from Prop.1., and $\varphi\psi(L) = L_{H_L \times \mathcal{t}_L} = \{ x \in \mathrm{Fix}\, H_L, x$ separable over $N_o(\mathcal{t}_L) = N_o(\mathcal{t}(L \cap K_o)) = L \cap K_o \}$. If $x \in L$ and $\sigma \in H_L$, $\sigma(x) = x$, hence $x \in \mathrm{Fix}\, H_L$. L is distinguished, x is separable over $L \cap K_o$, so we have $x \in \varphi\psi(L)$.

Now we prove the converse part $\underline{\varphi\psi(L) \subset L}$. Let y be in $\varphi\psi(L)$, in other words $y \in \mathrm{Fix}\, H_L$ and y is separable over $N_o(\mathcal{t}_L) = L \cap K_o$. Write now $y = \sum \xi_i \eta_i$, with $\xi_i \in k_K$, $\eta_i \in K_o$ ($K = k_K.K_o$). In this writing we can consider η_i free over k. We have now $y = \sigma(y) = \sum \sigma(\xi_i).\eta_i$, so $\xi_i = \sigma(\xi_i)$ for all $\sigma \in H_L$, and we conclude that $\xi_i \in \mathrm{Fix}\, H_L \cap k_K = k_L$. In this way $y \in k_L.K_o$. But $y^p \in k_L.k = k_L$, so that y is pure over k_L and over L. y is also separable over $N_o(\mathcal{t}_L) = L \cap K_o$, so that y is separable over L. As a consequence $y \in L$.

Let $M = (H, \mathcal{t})$ be a closed subgroup in $G'(K/k)$. We shall prove now that $\underline{M = H \times \mathcal{t} \subset \psi\varphi(H, \mathcal{t})}$. Let σ be in H and $x \in \varphi(H, \mathcal{t})$. $\sigma(x) = x$, so $\sigma \in H_{\varphi(H, \mathcal{t})}$. Consider now $D \in \mathcal{t}$ and $x \in \varphi(H, \mathcal{t}) \cap K_o$. x is separable over $N_o(\mathcal{t})$. But $x \in K_o$ implies x is pure over $N_o(\mathcal{t})$, so that $x \in N_o(\mathcal{t})$ and $D(x) = 0$. We conclude that $D \in \mathcal{t}_{\varphi(H, \mathcal{t})}$ and the inclusion is proved.

Now we shall prove the converse part $\underline{\psi\varphi(H, \mathcal{t}) \subset (H, \mathcal{t})}$. For this, let σ be in $H_{\varphi(H, \mathcal{t})}$. σ leaves unchanged the elements of Fix H which are separable over $N_o(\mathcal{t})$. Let x be in $\mathrm{Fix}\, H \cap k_K$. H is closed in the Krull topology on S, hence it will be sufficient to prove that $\sigma(x) = x$. But $x \in$

Fix $H \cap k_K$ implies x separable over $N_o(\mathcal{t})$, so that $\sigma(x) = x$, and $H_{\varphi(H, \mathcal{t})} \subset H$. Let D be in $\mathcal{t}_{\varphi(H, \mathcal{t})} = \mathcal{t}_o($ Fix $H \cap N_o(\mathcal{t})) = \mathcal{t}_o(N_o(\mathcal{t})) = \mathcal{t}$ (Cor.2.). We used the fact that $N_o(\mathcal{t}) \subset$ Fix H, and the eqality $L_M \cap K_o =$ Fix $H \cap N_o(\mathcal{t})$. For the last eqality let x be in $L_M \cap K_o$, x is separable and pure over $N_o(\mathcal{t})$, so that $x \in N_o(\mathcal{t}) \cap$ Fix H. Conversely, for $y \in$ Fix $H \cap N_o(\mathcal{t})$, $y \in K_o$, and $y \in L_M$, since $y \in N_o(\mathcal{t})$. With that the proof of the Theorem 1. is over.

C. <u>A Galois type correspondence for arbitrary subextensions</u>

<u>Lemma 2.</u> For a subgroup H in S and a K-subspace $\mathcal{t}$ in $\mathcal{D}_{K/k}$, $M = \{ (\sigma, \sigma E) \in G(K/k), \sigma \in H, E \in \mathcal{t} \} \overset{not.}{=} (H, \mathcal{t})$ is a subgroup in G(K/k) if and only if $\sigma E \sigma^{-1} \in \mathcal{t}$, for all $\sigma \in H$, and $E \in \mathcal{t}$.

<u>Definition 5.</u> A subgroup $M = (H, \mathcal{t})$ is called <u>admissible</u> if H is closed in the Krull topology on S, $\mathcal{t}$ is an A-maximal K-subspace in $\mathcal{D}_{K/k}$, and if we can find a p-base $\{c_i\}$ of $N(\mathcal{t})$ over k_K such that $c_i \in$ Fix H, for all i.

<u>Remark 2.</u> $M = (H, \mathcal{t})$ is only a notation. Generally speaking $M = (H, \mathcal{t}) \not\cong H \times \mathcal{t}$ if we work with $\mathcal{D}_{K/k}$ instead of $\mathcal{D}_{K_o/k}$. It is not difficult to construct extensions K/k with $G(K/k) \not\cong S \times \mathcal{D}_{K/k}$.

In the following we denote by $\mathcal{t}_L = \mathcal{t}(L) = \{ D \in \mathcal{D}_{K/k}, D(x) = 0$, for all $x \in L \}$, where $L/k \subset K/k$, and by $N(\mathcal{t}) = \{ x \in K, D(x) = 0$, for all $D \in \mathcal{t} \}$, $\mathcal{t}$ being a K-subspace in $\mathcal{D}_{K/k}$.

<u>THEOREM 2.</u> Let K/k be a normal extension of exponent 1

(K_0/k has exponent 1). The maps $\overline{\Psi}(L) = (H_L, \mathcal{t}_L) \subset G(K/k)$, with $H_L = \{\sigma \in S,\ \sigma(x) = x, \text{for } x \in L\}$, $\mathcal{t}_L = \{D \in \mathcal{D}_{K/k},\ D(x) = 0, \text{for } x \in L\}$, and $\overline{\varphi}((H, \mathcal{t})) = \operatorname{Fix} H \cap N(\mathcal{t})$, establish a one-to-one correspondence between the arbitrary subextensions $L/k \subset K/k$ and the admissible subgroups $(H, \mathcal{t})$ in $G(K/k)$.

<u>Proof.</u> If $L/k \subset K/k$, $\overline{\Psi}(L) = (H_L, \mathcal{t}_L)$ is admissible in $G(K/k)$. For this we write $L = k_L[c_\alpha]_{\alpha \in \Lambda}$, where $\{c_\alpha\}_{\alpha \in \Lambda}$ is a p-base over k_K. H_L is closed in the Krull topology on $S = \operatorname{Aut}(K/k) = \operatorname{Aut}(k_K/k)$, and $\mathcal{t}_L$ is a A-maximal subspace in $\mathcal{D}_{K/k}$ (if $\mathcal{t}_L \subset \mathcal{B}$, with $N(\mathcal{B}) = L$, every $D \in \mathcal{B}$ is 0 on L, hence $D \in \mathcal{t}_L$). It is clear, using Lemma 2, that $(H_L, \mathcal{t}_L)$ is a subgroup in $G(K/k)$. Morever $c_\alpha \in L \subset \operatorname{Fix} H_L$, so that $\overline{\Psi}(L)$ is admissible in $G(K/k)$. But $L \subset \operatorname{Fix} H_L \cap N(\mathcal{t}(L)) = \overline{\varphi}\,\overline{\Psi}(L)$. Conversely, let x be in $\overline{\varphi}\,\overline{\Psi}(L) = \operatorname{Fix} H_L \cap N(\mathcal{t}(L)) = \operatorname{Fix} H_L \cap k_K[c_\alpha]_{\alpha \in \Lambda}$, since $k_K \subset N(\mathcal{t}(L)) \subset K$, and since we always can find a k_K-derivation on K, $D \in \mathcal{t}(L)$, with $\operatorname{Ker} D = k_K[c_\alpha]_{\alpha \in \Lambda}$. Write now $x = \sum \xi_{i_1,\dots,i_k} c_{\alpha_{i_1}}^{i_1} \cdots c_{\alpha_{i_k}}^{i_k}$ with $\xi_{i_1,\dots,i_k} \in k_K$. We have $x = \sigma(x)$ for $\sigma \in H_L$, hence $\xi_{i_1,\dots,i_k} \in k_L$ (c_α is a p-base over k_K and $c_\alpha \in L$); so that $x \in k_L[c_\alpha]_{\alpha \in \Lambda} = L$. Up to now we have proved that <u>$L = \overline{\varphi}\,\overline{\Psi}(L)$</u>.

We want now to prove that $(H, \mathcal{t}) \subset \overline{\Psi}\,\overline{\varphi}(H, \mathcal{t})$ for an admissible subgroup $(H, \mathcal{t})$ in $G(K/k)$. If $\sigma \in H$, and $D \in \mathcal{t}$, it is clear that $\sigma \in H_{\operatorname{Fix} H \cap N(\mathcal{t}(L))}$ and that $D \in \mathcal{t}(\operatorname{Fix} H \cap N(\mathcal{t}))$; let now σ be in $H_{\operatorname{Fix} H \cap N(\mathcal{t}(L))}$ and D be in $\mathcal{t}(\operatorname{Fix} H \cap N(\mathcal{t}))$. For $x \in \operatorname{Fix} H \cap k_K \subset \operatorname{Fix} H \cap N(\mathcal{t}(L))$ we have $\sigma(x) = x$. Since H is closed in $S = \operatorname{Aut}(k_K/k)$ we conclude that $\sigma \in H$. Write now $N(\mathcal{t}) = k_K[c_i]_{i \in I}$ with $c_i \in \operatorname{Fix} H$ ($(H, \mathcal{t})$ is admissible in $G(K/k)$). D is 0 on k_K and D is 0 on c_i, since $D \in \mathcal{t}(\operatorname{Fix} H \cap N(\mathcal{t}))$, so that

$D \in \mathcal{t}(N(\mathcal{t})) = \mathcal{t}$, $\mathcal{t}$ beeing A-maximal in $\mathcal{D}_{K/k}$, and we have proved that $(H, \mathcal{t}) = \overline{\Psi}\overline{\varphi}(H, \mathcal{t})$.

The last Remark For K/k purely inseparable, finte and of exponent 1, the A-maximal K-subspaces in $\mathcal{D}_{K/k}$ are exactely the restricted Lie algebras of Jacobson [1]. When K/k is infinite, purely inseparable, and of exponent 1, the A-maximale K-subspaces in $\mathcal{D}_{K/k}$ are exactely the closed (in the finite topology on $\mathcal{D}_{K/k}$) K-subspaces which are closed for taking p-powers ([G1], [G2], [OS]). It is not difficult to prove that when K/k is normale with K_0/k of exponent 1, every A-maximal subspace is closed in the finite topology and is closed for taking p-powers. The converse part of this affirmation is also true. All this facts we have proved independently from [G1], [G2], [OS], using another tools.

A C K N O W L E D G E M E N T

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R E F E R E N C E S

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