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I. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces on which there are fixed two symmetry operators J_1 and, respectively, J_2 (recall that an operator $J \in \mathcal{L}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space, is called symmetry if $J=J^*=J^{-1}$). Let also $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be an operator such that the number of the negative squares of the selfadjoint operator $J_1 - T^* J_2 T$ is a cardinal $\varkappa$. We denote this fact by $\varkappa^-(J_1 - T^* J_2 T) = \varkappa$.

Given two others Hilbert spaces $\mathcal{K}$ and $\mathcal{K}'$ on which we fix two symmetry operators J and, respectively, J' let us consider the Hilbert spaces $\tilde{\mathcal{K}}_1 = \mathcal{H}_1 \oplus \mathcal{K}$ and $\tilde{\mathcal{K}}_2 = \mathcal{H}_2 \oplus \mathcal{K}'$. Then the operators $\tilde{J}_1 = J_1 \oplus J$ and $\tilde{J}_2 = J_2 \oplus J'$ are symmetry operators on $\tilde{\mathcal{K}}_1$ and, respectively $\tilde{\mathcal{K}}_2$. We are interested in searching for description of all operators $\tilde{T} \in \mathcal{L}(\tilde{\mathcal{K}}_1, \tilde{\mathcal{K}}_2)$ such that:

$$\tilde{T} = \begin{pmatrix} T, & A \\ B, & X \end{pmatrix} : \begin{matrix} \mathcal{H}_1 \\ \oplus \\ \mathcal{K} \end{matrix} \longrightarrow \begin{matrix} \mathcal{H}_2 \\ \oplus \\ \mathcal{K}' \end{matrix} ,$$

and $\varkappa^-(\tilde{J}_1 - \tilde{T}^* \tilde{J}_2 \tilde{T}) = \varkappa'$, where $\varkappa'$ is another given cardinal number.

This problem that we stated above in full generality is related in dfferent, more or less particular, cases to other problems as in [2], [3], [6], [11], and so on.

It turns out that linear operators on indefinite inner product spaces are involved and in the language of these spaces, one advantage is that the problem becomes in a certain way independent on the symmetry operators and the notation and formulations can be considerably simplified. However these facts do not go too far due to the lack of an adequate substitute of the square root of operators in these spaces.

Our approach is to adapt the idea of [1] to this indefinite situation. In order to do that we consider in the second section the problem of indefinite factorizations of selfadjoint operators.

Corollary 2.5 exhibits the best controllable situation that we will use here. It turns out that certain relations exist between the numbers $\kappa^-(J_1-T^*J_2T)$ and $\kappa^-(J_2-TJ_1T^*)$. These facts are presented in the third section.

One important point in our approach is the problem of finding a good substitute for the defect relations which in the indefinite case are no longer true. Thus we are led to prove the existence of certain link operators between the defect subspaces.

In the last section we present in our main result the solution to the above problem when $\kappa^-(J_1)$, $\kappa^-(J_2)<\infty$ and κ' has the least admissible value. We have to note that the existence problem when κ' is greater than this least admissible value can be easily deduced from here , while the problem of description (as it is suggested by the results from the first sections) involves more parameters and, on the other hand, some of the old parameters may be unbounded.

We have used without specification in this paper only elementary material from Krein space theory (see 4); more complicated things are precisely quoted. Let us also describe our context: we think about a Krein space $\mathcal{K}$ as a Hilbert space with a symmetry operator $J\in\mathcal{L}(\mathcal{K})$ and the indefinite inner product $[\cdot,\cdot]$ is introduced as follows:

$$[x,y]=(Jx,y) \quad , \ x,y\in\mathcal{K}.$$

There are many such choices of J, for a fixed inner product $[\cdot,\cdot]$; each of it is called a fundamental symmetry(f.s) . For a fixed f.s. J on $\mathcal{K}$, denoting by $\#$ the involution we get $T^{\#}=JT^*J$, $T\in\mathcal{L}(\mathcal{K})$.

II. Let $\mathcal{H}$ be Hilbert space and $A \in \mathcal{L}(\mathcal{H})$, $A=A^*$. Considering the function signum defined as follows

$$\mathrm{sgn}: \mathbb{R} \longrightarrow \{-1,0,1\} \quad , \quad \mathrm{sgn}(t)= \begin{cases} -1 , & t<0 \\ 0 , & t=0 \\ 1 , & t>0 , \end{cases}$$

let S_A denote the selfadjoint partial isometric operator $\mathrm{sgn}(A)$.

Clearly we have $\ker(S_A)=\ker A$, $S_A\mathcal{H}=\overline{\mathcal{R}(A)}$ and $A=S_A|A|$. Then the signature numbers of A can be defined as follows:

$$\varkappa^-(A)=\dim\ker(I+S_A) , \quad \varkappa^+(A)=\dim\ker(I-S_A) , \quad \varkappa^0(A)=\dim\ker S_A .$$

Let us recall also that $\varkappa^-(A)$($\varkappa^+(A)$) is the number of negative(positive) squares of the quadratic form (Ax,x) , $x \in \mathcal{H}$ (see e.g. [7]). On the other hand, since S_A is a symmetry operator on $\overline{\mathcal{R}(A)}$ one can consider the indefinite inner product

$$[x,y]=(S_A x,y) , \quad x,y \in \overline{\mathcal{R}(A)}.$$

Let us denote by $\mathcal{H}_A$ the Krein space obtained in this way. Also since $S_A\mathcal{R}(|A|^{\frac{1}{2}}) \subset \mathcal{R}(|A|^{\frac{1}{2}})$ and $\mathcal{R}(|A|^{\frac{1}{2}})$ is dense in $\mathcal{H}_A$ it follows that $(\mathcal{R}(|A|^{\frac{1}{2}}) ,[\cdot,\cdot])$ is a space with indefinite inner product, which is decomposable and non-degenerate, hence can be completed to a Krein space which coincides with $(\mathcal{H}_A,[\cdot,\cdot])$.

The first statement from the next result belongs to Potapov[11] and Bognár-Krámli [5] . We sketch the proof for the reader's convenience.

2.1. PROPOSITION. *Let $\mathcal{H}$ be a Hilbert space, $A \in \mathcal{L}(\mathcal{H})$, $A=A^*$, $\mathcal{K}$ a Krein space and J a fundamental symmetry of $\mathcal{K}$. Then*

(i) There exists $B \in \mathcal{L}(\mathcal{H},\mathcal{K})$ such that $A=B^*JB$ if and only if $\varkappa^+(J) \geq \varkappa^+(A)$ and $\varkappa^-(J) \geq \varkappa^-(A)$.

(ii) Assuming $\varkappa^+(J) \geq \varkappa^+(A)$ and $\varkappa^-(J) \geq \varkappa^-(A)$ let $B \in \mathcal{L}(\mathcal{H},\mathcal{K})$. Then $A=B^*JB$ if and only if $B=C|A|^{\frac{1}{2}}+X$ where C is a linear mapping from $\mathcal{R}(|A|^{\frac{1}{2}})$ in $\mathcal{K}$, isometric with respect to the indefinite inner products, such that $C|A|^{\frac{1}{2}} \in \mathcal{L}(\mathcal{H},\mathcal{K})$ and

$X\in\mathcal{L}(\ker A,\mathcal{K})$ with the properties $X^{\#}X=0$ and $\mathcal{R}(X)\subset(C\mathcal{R}(|A|^{\frac12}))^{[\perp]}$.

PROOF. (i) Let $B\in\mathcal{L}(\mathcal{H},\mathcal{K})$ and $A=B^*JB$. Then

$$0<(Ax,x)=(B^*JBx,x)=(JBx,Bx)=[Bx,Bx],$$

for every $x\in\ker(I-S_A)-\{0\}$, particularly it follows $B|\ker(I-S_A)$ is one-to-one and $\overline{B\ker(I-S_A)}$ is a non-negative subspace of $\mathcal{K}$, hence $\varkappa^+(A)\le\varkappa^+(\mathcal{K})=\varkappa^+(J)$. Similarly we have $\varkappa^-(A)\le\varkappa^-(J)$.

Conversely, assuming $\varkappa^+(A)\le\varkappa^+(J)$ and $\varkappa^-(A)\le\varkappa^-(J)$ it follows that there exists $C\in\mathcal{L}(\mathcal{H}_A,\mathcal{K})$, a Krein space isometry, i.e.

$$C^*JC=S_A|\mathcal{H}_A .$$

Taking $B=C|A|^{\frac12}\in\mathcal{L}(\mathcal{H},\mathcal{K})$ we get

$$B^*JB=|A|^{\frac12}C^*JC|A|^{\frac12}=|A|^{\frac12}S_A|A|^{\frac12}=A .$$

(ii) If $B\in\mathcal{L}(\mathcal{H},\mathcal{K})$ such that $A=B^*JB$ then we can define the linear mapping $C:\mathcal{R}(|A|^{\frac12})\longrightarrow\mathcal{K}$ by

$$C|A|^{\frac12}x=Bx\quad,\qquad x\in\mathcal{H}_A.$$

It follows that C is well defined, since $|A|^{\frac12}$ is one-to-one on $\mathcal{H}_A$. Let us also define $X\in\mathcal{L}(\ker A,\mathcal{K})$ by

$$Xx=Bx\quad,\ x\in\ker A.$$

Let $x,y\in\mathcal{H}$. Then

$$[C|A|^{\frac12}x,C|A|^{\frac12}y]=(JBx,By)=(B^*JBx,y)=(Ax,y)=(S_A|A|^{\frac12}x,|A|^{\frac12}y)=$$
$$=[|A|^{\frac12}x,|A|^{\frac12}y]\quad,$$

i.e. C is isometric with respect to the indefinite inner products.

Since

$$[C|A|^{\frac12}x,Xy]=(JBx,By)=(B^*JBx,y)=(Ax,y)=0$$

for every $x\in\mathcal{H}$, $y\in\ker A$, it follows $\mathcal{R}(X)\subset(C\mathcal{R}(|A|^{\frac12}))^{[\perp]}$, and from

$$(X^{\#}Xx,y)=[Xx,Xy]=(JBx,By)=(Ax,y)=0\quad,\ x,y\in\ker A$$

we get $X^{\#}X=0$.

Conversely, let $B=C|A|^{\frac12}+X\in\mathcal{L}(\mathcal{H},\mathcal{K})$ with C and X as above (i.e. C isometric, $X^{\#}X=0$ and $\mathcal{R}(X)\subset(C\mathcal{R}(|A|^{\frac12}))^{[\perp]}$). Then for every $x,y\in\mathcal{H}$ we have $x=x_1+x_2$, $y=y_1+y_2$ according to the decomposition of

$\mathcal{H}=\mathcal{H}_A\oplus\ker A$ and

$$(B^*JBx,y)=(JBx,By)=(JC|A|^{\frac{1}{2}}x_1+JXx_2,C|A|^{\frac{1}{2}}y_1+Xy_2)=$$
$$=[C|A|^{\frac{1}{2}}x_1,C|A|^{\frac{1}{2}}y_1]+[C|A|^{\frac{1}{2}}x_1,Xy_2]+[Xx_2,C|A|^{\frac{1}{2}}y_1]+[Xx_2,Xy_2].$$

Since the second and the third terms are null from $\mathcal{R}(X)\subset (C\mathcal{R}(|A|^{\frac{1}{2}}))^{[\perp]}$ and also the last term is null from $X^{\#}X=0$ we get

$$(B^*JBx,y)=[C|A|^{\frac{1}{2}}x_1,C|A|^{\frac{1}{2}}y_1]=[|A|^{\frac{1}{2}}x_1,|A|^{\frac{1}{2}}y_1]=(|A|^{\frac{1}{2}}S_A|A|^{\frac{1}{2}}x_1,y_1)$$
$$=(Ax,y)$$

i.e. $B^*JB=A$. ∎

In the next corollaries the notation are as in the above proposition.

2.2. COROLLARY. If $\overline{B\mathcal{H}_A}$ is a non-degenerate subspace of $\mathcal{K}$ then the corresponding C is closable.

PROOF. Indeed, $\overline{B\mathcal{H}_A}=\overline{C|A|^{\frac{1}{2}}\mathcal{H}}$ and due to its non-degeneracy it follows that $C|A|^{\frac{1}{2}}\mathcal{H}+((C|A|^{\frac{1}{2}})\mathcal{H})^{[\perp]}$ is dense in $\mathcal{K}$. Let $x\in\mathcal{H}$, $y\in C|A|^{\frac{1}{2}}\mathcal{H}$ and $z\in((C|A|^{\frac{1}{2}})\mathcal{H})^{[\perp]}$, $y=C|A|^{\frac{1}{2}}h$ with $h\in\mathcal{H}$. Then

$$[C|A|^{\frac{1}{2}}x,y+z]=[C|A|^{\frac{1}{2}}x,C|A|^{\frac{1}{2}}h]=[|A|^{\frac{1}{2}}x,|A|^{\frac{1}{2}}h] ,$$

hence $y+z\in\mathcal{D}(C^{\#})$ and $C^{\#}(y+z)=|A|^{\frac{1}{2}}h$. Particularly $C^{\#}$ is densely defined, hence C is closable. ∎

Since the closure of every closable isometric operator (between Krein spaces) is also isometric, the preceding corollary means that everytime when $\overline{B\mathcal{H}_A}$ is non-degenerate the corresponding C can be chosen closed.

2.3. COROLLARY. Assume that $\kappa(\mathcal{H}_A)=\min(\kappa^-(A),\kappa^+(A))<\infty$. Then $\overline{B\mathcal{H}_A}$ is a regular subspace of $\mathcal{K}$ if and only if the corresponding C is bounded.

PROOF. Let us assume that $\kappa^-(A)<\infty$ and consider the fundamental decomposition (f.d.) $\mathcal{H}_A=\mathcal{H}_A^+[+]\mathcal{H}_A^-$ where $\mathcal{H}_A^+=\ker(I-S_A)$, $\mathcal{H}_A^-=\ker(I+S_A)$. Then

$$\mathcal{D}(C)=|A|^{\frac{1}{2}}\mathcal{H}_A=|A|^{\frac{1}{2}}\mathcal{H}_A^-+|A|^{\frac{1}{2}}\mathcal{H}_A^+=\mathcal{H}_A^-+|A|^{\frac{1}{2}}\mathcal{H}_A^+$$

where $|A|^{\frac{1}{2}}\mathcal{H}_A^+\subset\mathcal{H}_A^+$, hence a positive linear submanifold of $\mathcal{H}_A$.

Also,

$$\mathcal{R}(C)=C\mathcal{H}_A^- + C|A|^{\frac{1}{2}}\mathcal{H}_A^+ ,$$

and since C is isometric let us note that $C\mathcal{H}_A^-$ is a negative finite dimensional subspace, $C|A|^{\frac{1}{2}}\mathcal{H}_A^+$ is a positive linear manifold and $[C\mathcal{H}_A^-, C|A|^{\frac{1}{2}}\mathcal{H}_A^+]=0$. Therefore

$$\overline{B\mathcal{H}_A}=\overline{\mathcal{R}(C)}=C\mathcal{H}_A^- + \overline{C|A|^{\frac{1}{2}}\mathcal{H}_A^+},$$

and the conclusion follows from [4 , Theorem VI.3.5]. ■

2.4. COROLLARY. Assume that A has closed range. Then the formula

$$B=C|A|^{\frac{1}{2}}+X$$

establishes a one-to-one correspondence between all the operators $B\in\mathcal{L}(\mathcal{H},\mathcal{K})$ such that $B^*JB=A$ and all the pairs (C,X) such that $C\in\mathcal{L}(\mathcal{H}_A,\mathcal{K})$ is a Krein space isometry and $X\in\mathcal{L}(\ker A,\mathcal{K})$ satisfies $X^{\#}X=0$ and $\mathcal{R}(X)\subset\mathcal{R}(C)^{[\perp]}$.

PROOF. The boundedness of C follows from the fact that $|A|^{\frac{1}{2}}$ is invertible on $\mathcal{H}_A$ and $C|A|^{\frac{1}{2}}\in\mathcal{L}(\mathcal{H}_A,\mathcal{K})$. The rest of the statement is contained in the proof of Proposition 2.1. ■

2.5. COROLLARY. Assume that $\varkappa^-(A)=\varkappa^-(J)<\infty$. The the formula

$$B=C|A|^{\frac{1}{2}}$$

establishes a one-to-one correspondence between all the operators $B\in\mathcal{L}(\mathcal{H},\mathcal{K})$ such that $B^*JB=A$ and all the Krein space isometric operators $C\in\mathcal{L}(\mathcal{H}_A,\mathcal{K})$.

PROOF. Let $B\in\mathcal{L}(\mathcal{H},\mathcal{K})$ be such that $B^*JB=A$. Then $B=C|A|^{\frac{1}{2}}+X$ as in Proposition 2.1. Consider the f.d. $\mathcal{H}_A=\mathcal{H}_A^-[+]\mathcal{H}_A^+$ as in the proof of Corollary 2.3. Since $C\mathcal{H}_A^-$ is a negative subspace of dimension $\varkappa^-(A)=\varkappa^-(J)$ it follows that it is maximal uniformly negative in $\mathcal{K}$ and $(C\mathcal{H}_A^-)^{[\perp]}$ is a maximal uniformly positive subspace. Taking account of $C|A|^{\frac{1}{2}}\mathcal{H}_A^+\subset(C\mathcal{H}_A^-)^{[\perp]}$ the boundedness of C follows as in the proof of Corollary 2.3. On the other hand,

$$\mathcal{R}(X)\subset\mathcal{R}(C)^{[\perp]}=(C\mathcal{H}_A^-)^{[\perp]},$$

therefore $\mathcal{R}(X)$ is positive. But, from $X^{\#}X=0$ it follows $\mathcal{R}(X)$ neutral, hence $\mathcal{R}(X)=0$, i.e. $X=0$. ■

III. Let $\mathcal{K}$ be a Krein space and $A\in\mathcal{L}(\mathcal{K})$, $A=A^{\#}$. In this section we present several facts regarding the cardinal numbers $\varkappa^{0}[A]$, $\varkappa^{-}[A]$ and $\varkappa^{+}[A]$ which are called respectively the rank of isotropy, the rank of negativity and the rank of positivity of A , associated to the quadratic form

$$[Ax,x] \qquad , \quad x\in\mathcal{K}$$

in the sense of [7] . If J is a fundamental symmetry(f.s) of $\mathcal{K}$ then, considering the notation from the previous section with respect to the J-inner product which turns $\mathcal{K}$ into a Hilbert space,

$$\varkappa^{0}[A]=\varkappa^{0}(JA) \ , \ \varkappa^{-}[A]=\varkappa^{-}(JA) \ , \ \varkappa^{+}[A]=\varkappa^{+}(JA).$$

<u>3.1. PROPOSITION</u>. <u>Let</u> $\mathcal{K}_1$ <u>and</u> $\mathcal{K}_2$ <u>be Krein spaces and</u> $T\in\mathcal{L}(\mathcal{K}_1,\mathcal{K}_2)$

<u>Then</u>:

$$\varkappa^{0}[I_1-T^{\#}T]=\varkappa^{0}[I_2-TT^{\#}] , \tag{3.1}$$

$$\varkappa^{-}[I_1-T^{\#}T]+\varkappa^{-}(\mathcal{K}_2)=\varkappa^{-}[I_2-TT^{\#}]+\varkappa^{-}(\mathcal{K}_1) \tag{3.2}$$

$$\varkappa^{+}[I_1-T^{\#}T]+\varkappa^{+}(\mathcal{K}_2)=\varkappa^{+}[I_2-TT^{\#}]+\varkappa^{+}(\mathcal{K}_1). \tag{3.3}$$

<u>PROOF</u>. Let $\mathcal{K}$ denote the Krein space $\mathcal{K}_1[+]\mathcal{K}_2$ and $\hat{T}\in\mathcal{L}(\mathcal{K})$ defined by

$$\hat{T}=\begin{bmatrix} I_1 & , & T^{\#} \\ T & , & I_2 \end{bmatrix}$$

and note that it can be factorized in two dual ways:

$$\hat{T}=\begin{bmatrix} I_1 & , & 0 \\ T & , & I_2 \end{bmatrix}\begin{bmatrix} I_1 & , & 0 \\ 0 & , & I_2-TT^{\#} \end{bmatrix}\begin{bmatrix} I_1 & , & T^{\#} \\ 0 & , & I_2 \end{bmatrix} ,$$

and

$$\hat{T}=\begin{bmatrix} I_1 & , & T^{\#} \\ 0 & , & I_2 \end{bmatrix}\begin{bmatrix} I_1-T^{\#}T & , & 0 \\ 0 & , & I_2 \end{bmatrix}\begin{bmatrix} I_1 & , & 0 \\ T & , & I_2 \end{bmatrix}$$

whence by counting $\varkappa^{0}[\hat{T}]$, $\varkappa^{-}[\hat{T}]$ and $\varkappa^{+}[\hat{T}]$ these ways the desired formulae follow. ∎

<u>3.2. REMARK</u>. The previous Proposition can be equivalently formulated

as follows as follows:

Let $\mathcal{H}_1$, $\mathcal{H}_2$ be Hilbert spaces, J_1 and J_2 symmetries on $\mathcal{H}_1$ and, respectively, $\mathcal{H}_2$, and $T\in\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)$. Then:

$$(3.1)' \qquad \varkappa^0(J_1-T^*J_2T)=\varkappa^0(J_2-TJ_1T^*)$$

$$(3.2)'+(3.3)' \qquad \varkappa^{\mp}(J_1-T^*J_2T)+\varkappa^{\mp}(J_2)=\varkappa^{\mp}(J_2-TJ_1T^*)+\varkappa^{\mp}(J_1)\ . \blacksquare$$

<u>3.3. COROLLARY</u>. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be Pontrjagin spaces ($\varkappa^-(\mathcal{K}_1)$, $\varkappa^-(\mathcal{K}_2)<\infty$) and $T\in\mathcal{L}(\mathcal{K}_1,\mathcal{K}_2)$. Then $\varkappa^-[I_1-T^\#T]<\infty$ if and only if $\varkappa^-[I_2-TT^\#]<\infty$ and in this case we have

$$(3.4) \qquad \varkappa^-[I_1-T^\#T]-\varkappa^-[I_2-TT^\#]=\varkappa^-(\mathcal{K}_1)-\varkappa^-(\mathcal{K}_2). \blacksquare$$

<u>3.4. REMARK</u>. With the notation from the previous Remark and additionally $\varkappa^-(J_1)$, $\varkappa^-(J_2)<\infty$, the above Corollary is equivalent to

$$\varkappa^-(J_1-T^*J_2T)<\infty \quad \text{if and only if} \quad \varkappa^-(J_2-TJ_1T^*)<\infty$$

and in this case we have:

$$(3.4)' \qquad \varkappa^-(J_1-T^*J_2T)-\varkappa^-(J_2-TJ_1T^*)=\varkappa^-(J_1)-\varkappa^-(J_2).$$

For obvious reasons we shall refer to (3.4) or (3.4)' as "index formula". ■

We record also as consequence of Proposition 3.1 an well-known fact [11], with the advantage of an elementary proof (see also [6]).

<u>3.5. COROLLARY</u>. Let $\mathcal{K}$ be a Pontrjagin space and $T\in\mathcal{L}(\mathcal{K})$. Then T is contraction (strict contraction, uniform contraction) if and only if $T^\#$ is contraction (strict contraction, uniform contraction).

<u>PROOF</u>. We have only to note that T contraction means $\varkappa^-[I-T^\#T]=0$, T strict contraction means $\varkappa^-[I-T^\#T]=0$ and $\varkappa^0[I-T^\#T]=0$ and T uniform contraction is equivalent to $\varkappa^-[I-T^\#T]=0$ and $I-T^\#T$ is invertible. ■

IV. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Krein spaces and $T\in\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)$. By fixing two f.s.'s J_1 and J_2 on $\mathcal{H}_1$ and respectively $\mathcal{H}_2$ we define the following bounded operators:

$$J_T=\operatorname{sgn}(J_1-T^*J_2T) \quad , \quad J_{T^*}=\operatorname{sgn}(J_2-TJ_1T^*)$$
$$D_T=|J_1-T^*J_2T|^{\frac{1}{2}} \quad , \quad D_{T^*}=|J_2-TJ_1T^*|^{\frac{1}{2}}.$$

It follows that the following equalities hold

(4.1) $\quad J_TD_T=D_TJ_T \quad , \quad J_{T^*}D_{T^*}=D_{T^*}J_{T^*}$

(4.2) $\quad J_TD_T^2=J_1-T^*J_2T \quad , J_{T^*}D_{T^*}^2=J_2-TJ_1T^*$

(4.3) $\quad TJ_1(J_1-T^*J_2T)=(J_2-TJ_1T^*)J_2T \quad ,T^*J_2(J_2-TJ_1T^*)=(J_1-T^*J_2T)J_1T^*$

Defining the subspaces

$$\mathcal{D}_T=\overline{\mathcal{R}(D_T)}\subset\mathcal{H}_1 \quad , \quad \mathcal{D}_{T^*}=\overline{\mathcal{R}(D_{T^*})}\subset\mathcal{H}_2 ,$$

from now on we always will consider $\mathcal{D}_T$ as a Krein space with J_T as f.s., i.e.

$$[x,y]=(J_Tx,y) \quad , \quad x,y\in\mathcal{D}_T ,$$

and correspondingly, we consider $\mathcal{D}_{T^*}$ as a Krein space with J_{T^*} as f.s., i.e.

$$[u,v]=(J_{T^*}u,v) \quad , \quad u,v\in\mathcal{D}_{T^*} .$$

With the notation from the previous sections we have

$$\varkappa^{\mp}(\mathcal{D}_T)=\varkappa^{\mp}(J_T)=\varkappa^{\mp}(J_1-T^*J_2T)=\varkappa^{\mp}[I-T^{\#}T] \quad ,$$
$$\varkappa^{\mp}(\mathcal{D}_{T^*})=\varkappa^{\mp}(J_{T^*})=\varkappa^{\mp}(J_2-TJ_1T^*)=\varkappa^{\mp}[I-TT^{\#}]$$

particularly, by changing the f.s.'s J_1 and J_2, the above construction leads to isometric isomorphic Krein space $\mathcal{D}_T$, respectively $\mathcal{D}_{T^*}$. This is one more reason to adopt this simplified notation somehow independent of J_1 and J_2.

4.1. PROPOSITION. For any Krein spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, $T\in\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)$ and fixed f.s.'s J_1 and J_2 on $\mathcal{H}_1$ and respectively $\mathcal{H}_2$, there exist uniquely determined operators $L_T\in\mathcal{L}(\mathcal{D}_T,\mathcal{D}_{T^*})$ and $L_{T^*}\in\mathcal{L}(\mathcal{D}_{T^*},\mathcal{D}_T)$ such that

(4.4) $\quad D_{T^*}L_T=TJ_1D_T \quad ,D_TL_{T^*}=T^*J_2D_{T^*}$.

PROOF. Let us first note that the existence of L_T is equivalent to the existence of a non-negative constant μ such that

(4.5) $\quad TJ_1D_T^2J_1T^*\leq \mu\cdot D_{T^*}^2$.

Then, considering the non-negative inner product $(\cdot,\cdot)_{D^2_{T^*}}$ on $\mathcal{D}_{T^*}$, defined by

$$(x,y)_{D^2_{T^*}} = (D^2_{T^*}x,y) \qquad , \; x,y\in \mathcal{D}_{T^*} ,$$

since we have

$$TJ_1D^2_TJ_1T^* = TJ_1(J_1-T^*J_2T)J_TJ_1T^* = (J_2-TJ_1T^*)J_2TJ_TJ_1T^* =$$
$$= D^2_{T^*}J_{T^*}J_2TJ_TJ_1T^*$$

and analogously

$$TJ_1D^2_TJ_1T^* = TJ_1J_TT^*J_2J_{T^*}D^2_{T^*} ,$$

it follows that the operator $TJ_1J_TT^*J_2J_{T^*}$ is $(\cdot,\cdot)_{D^2_{T^*}}$-symmetric, hence by a result from [9], [12] and [10] it follows that there exists $\mu \geq 0$ such that

$$TJ_1J_TT^*J_2J_{T^*}D^2_{T^*} \leq \mu D^2_{T^*} ,$$

i.e. we have (4.5) and the existence of the operator $L_T\in\mathcal{L}(\mathcal{D}_T,\mathcal{D}_{T^*})$ is proved. The fact that it is uniquely determined by the first equality of (4.4) follows from the fact that on $\mathcal{D}_{T^*}$ the operator D_{T^*} is one-to-one.

The statement concerning L_{T^*} follows by interchanging the roles of T and T^*, respectively J_1 and J_2. ■

4.2. REMARK. Let us assume that the operator $I-T^{\#}T$ has closed range, equivalently by Proposition 3.1 that $I-TT^{\#}$ has closed range. It follows that D_T is invertible on $\mathcal{D}_T$ and D_{T^*} is invertible on $\mathcal{D}_{T^*}$.

On the other hand, from (4.3) we have

$$TJ_1\mathcal{D}_T \subset \mathcal{D}_{T^*} \qquad , \; T^*J_2\mathcal{D}_{T^*} \subset \mathcal{D}_T$$

hence, in this case, the operators L_T and L_{T^*} can be explicitly written

$$L_T = D^{-1}_{T^*}TJ_1D_T|\mathcal{D}_T \qquad , L_{T^*} = D^{-1}_TT^*J_2D_{T^*}|\mathcal{D}_{T^*} .$$

Particularly this is the case if $\mathcal{H}_1$ and $\mathcal{H}_2$ are of finite dimensions (see [3] for applications of this situation). ■

4.3. REMARK. Let us assume that one can choose J_1 and J_2 such that $TJ_1=J_2T$. Then , it is easy to prove that , with respect to this f.s.'s the operators L_T and L_{T^*} can be explicitely written

$$L_T=J_2T \quad \text{and} \quad L_{T^*}=J_1T^* .$$

Particularly, when $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces, i.e. J_1 and J_2 are identity operators, the relations from (4.4) are exactly the " defect relations" (see e.g. [13]). ■

4.4. COROLLARY. With the notation from the above proposition we have

$$L_{T^*}=J_TL_T^*J_{T^*}|\mathcal{D}_{T^*} . \tag{4.6}$$

PROOF By multiplying on right the first equality from (4.4) with J_TD_T we get

$$D_{T^*}L_TJ_TD_T=TJ_1(J_1-T^*J_2T)=(J_2-TJ_1T^*)J_2T=D_{T^*}J_{T^*}D_{T^*}J_2T^*.$$

Since D_{T^*} is one-to-one on $\mathcal{D}_{T^*}$ it follows

$$L_TJ_TD_T=J_{T^*}D_{T^*}J_2T^* ,$$

whence we get

$$J_{T^*}L_TJ_TD_T=D_{T^*}J_2T^* ,$$

and then passing to adjoints

$$D_TJ_TL_T^*J_{T^*}=TJ_2D_{T^*}.$$

Taking account on the uniqueness of the operator L_{T^*} the equality (4.6) now follows. ■

4.5. REMARK. Recalling the conventions from the beginning of this section the above Corollary means that $L_{T^*}=L_T^{\#}$. ■

The following result will be of technical importance during the next section.

4.6. COROLLARY. The following equalities hold

$$J_T-D_TJ_1D_T=L_T^*J_{T^*}L_T \tag{4.7}$$

and

$$J_{T^*}-D_{T^*}J_2D_{T^*}=L_{T^*}^*J_TL_{T^*} . \tag{4.8}$$

PROOF We first make the convention that during this proof the inverse operators that appear have to be understood in the generalized sense.

Let us consider the obvious equality

$$x=J_1(J_1-T^*J_2T)x+J_1T^*J_2Tx \qquad ,x\in\mathcal{H}_1,$$

whence it follows that for any $y \in \mathcal{R}(J_1 - T^* J_2 T)$ we have

$$(J_1 - T^* J_2 T)^{-1} y = J_1 y + J_1 T^* J_2 T (J_1 - T^* J_2 T)^{-1} y \quad . \tag{4.9}$$

It is easy to see, from (4.1) and (4.2) that

$$(J_1 - T^* J_2 T)^{-1} \subset D_T^{-1} J_T D_T^{-1} \tag{4.10}$$

and also from (4.3), that

$$(J_2 - T J_1 T^*)^{-1} T J_1 y = J_2 T (J_1 - T^* J_2 T)^{-1} y \quad , \tag{4.11}$$

where y runs in $\mathcal{R}(J_1 - T^* J_2 T)$.

It follows from (4.9) , by means of (4.10) and (4.11), that

$$D_T^{-1} J_T D_T^{-1} y = J_1 y + J_1 T^* (J_2 - T J_1 T^*)^{-1} T J_1 y \quad , \tag{4.12}$$

for any $y \in \mathcal{R}(J_1 - T^* J_2 T)$. Since, analogously with (4.10), we have

$$(J_2 - T J_1 T^*)^{-1} \subset D_{T^*}^{-1} J_{T^*} D_{T^*}^{-1} \quad ,$$

(4.12) can be written

$$D_T^{-1} J_T D_T^{-1} y = J_1 y + J_1 T^* D_{T^*}^{-1} J_{T^*} D_{T^*}^{-1} T J_1 y,$$

or equivalently,

$$J_T z = D_T J_1 D_T z + D_T J_1 T^* D_{T^*}^{-1} J_{T^*} D_{T^*}^{-1} T J_1 D_T z \quad , \tag{4.13}$$

for any $z \in D_T^{-1} \mathcal{R}(J_1 - T^* J_2 T) = \mathcal{R}(D_T)$. Using now (4.4) in (4.13) it follows

$$J_T z = D_T J_1 D_T z + L_T^* J_T L_T z \qquad , z \in \mathcal{R}(D_T) \ ,$$

i.e. (4.7) holds on a dense linear manifold of $\mathcal{D}_T$, hence everywhere. (4.8) follows by duality from (4.7). ∎

<u>4.7 REMARK</u> Let us consider the Krein space $\mathcal{H}_1 [+] \mathcal{D}_{T^*}$ organized in the following way: first take the Hilbert space $\mathcal{H}_1 \oplus \mathcal{D}_{T^*}$ on which consider the symmetry $J_1 \oplus J_{T^*}$ and then define the indefinite inner product by means of this symmetry. Analogously one can consider the Krein space $\mathcal{H}_2 [+] \mathcal{D}_T$. From the above results it is straightforward to prove the following equalities:

$$\begin{bmatrix} T & , D_{T^*} \\ D_T, & -J_T L_{T^*} \end{bmatrix} \begin{bmatrix} J_1 & , 0 \\ 0 & , J_{T^*} \end{bmatrix} \begin{bmatrix} T^* & , D_T \\ D_{T^*}, & -L_{T^*}^* J_T \end{bmatrix} = \begin{bmatrix} J_2 & , 0 \\ 0 & , J_T \end{bmatrix}$$

and

$$\begin{bmatrix} T^* & , D_T \\ D_{T^*}, & -L_{T^*} J_T \end{bmatrix} \begin{bmatrix} J_2 & , 0 \\ 0 & , J_T \end{bmatrix} \begin{bmatrix} T & , D_{T^*} \\ D_T, & -J_T L_{T^*} \end{bmatrix} = \begin{bmatrix} J_1 & , 0 \\ 0 & , J_{T^*} \end{bmatrix}$$

i.e. the elementary rotation operator $R(T)$ defined by

$$(4.14) \qquad R(T)=\begin{bmatrix} T & , D_{T^*} \\ D_T & , -J_T L_{T^*} \end{bmatrix}$$

is a Krein space unitary operator from $\mathcal{H}_1 [+] \mathcal{D}_{T^*}$ onto $\mathcal{H}_2 [+] \mathcal{D}_T$.

The operator $R(T)$ is the indefinite variant of the elementary rotation (e.g. see [13]) from the positive-definite situation. The case $J_T \geq 0$ of the considerations in this section was done in [6] ; there $R(T)$ also played the role of a fundamental cell for the analysis of a matrix having $\varkappa$ negative squares in a perfect analogy with the positive-definite case. ∎

V. Let us consider now two Pontrjagin spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$. If we denote $\kappa = \kappa^-[I-T^\# T]$ then it is necessary the following inequality

(5.1) $$\kappa - \kappa^-(\mathcal{H}_1) + \kappa^-(\mathcal{H}_2) \geq 0$$

(by Corollary 3.3). Let also $\widetilde{\mathcal{H}}_1$ and $\widetilde{\mathcal{H}}_2$ be two Pontrjagin spaces such that $\mathcal{H}_1 \subset \widetilde{\mathcal{H}}_1$ and $\mathcal{H}_2 \subset \widetilde{\mathcal{H}}_2$. It follows that it is necessary to have $\kappa^-(\mathcal{H}_1) \leq \kappa^-(\widetilde{\mathcal{H}}_1)$, $\kappa^-(\mathcal{H}_2) \leq \kappa^-(\widetilde{\mathcal{H}}_2)$ and that there exist two Pontrjagin subspaces $\mathcal{K} \subset \widetilde{\mathcal{H}}_1$ and $\mathcal{K}' \subset \widetilde{\mathcal{H}}_2$ such that

(5.2) $$\widetilde{\mathcal{H}}_1 = \mathcal{H}_1 [+] \mathcal{K} \quad , \quad \widetilde{\mathcal{H}}_2 = \mathcal{H}_2 [+] \mathcal{K}' .$$

Considering $\kappa' \in \mathbb{N}$, we are interested in the following problem of lifting with prescribed number of negative squares:

(5.3) $$\left[\begin{array}{l} \text{Determine all the operators } \widetilde{T} \in \mathcal{L}(\widetilde{\mathcal{H}}_1, \widetilde{\mathcal{H}}_2) \text{ such that} \\ T = P_2 \widetilde{T} | \mathcal{H}_1 \quad \text{and} \quad \kappa^-[I - \widetilde{T}^\# \widetilde{T}] = \kappa' , \end{array} \right.$$

where P_2 denotes the projection of $\widetilde{\mathcal{H}}_2$ onto $\mathcal{H}_2$ along $\mathcal{K}'$(i.e. the projection of $\widetilde{\mathcal{H}}_2$ onto $\mathcal{H}_2$ which is orthogonal with respect to the indefinite inner product).

Let us fix from now on the f.s.'s J_1, J_2, J and J' on the appropriate spaces $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{K}$ and $\mathcal{K}'$. Then the operators

$$\widetilde{J}_1 = \begin{pmatrix} J_1 & , 0 \\ 0 & , J \end{pmatrix} \quad , \quad \widetilde{J}_2 = \begin{pmatrix} J_2 & , 0 \\ 0 & , J' \end{pmatrix}$$

(where the block-matrices are understood with respect to the decompositions (5.2)) are also f.s.'s on $\widetilde{\mathcal{H}}_1$ and respectively $\widetilde{\mathcal{H}}_2$. With respect to all these f.s.'s we will consider the notation from the beginning of the preceding section. On the other hand, since every operator $\widetilde{T} \in \mathcal{L}(\widetilde{\mathcal{H}}_1, \widetilde{\mathcal{H}}_2)$ such that $T = P_2 \widetilde{T} | \mathcal{H}_1$ is represented by

(5.4) $$\widetilde{T} = \begin{pmatrix} T & , A \\ B & , X \end{pmatrix}$$

with respect to (5.2), it turns out that the problem (5.3) is equivalent to the following one:

(5.5) $$\left[\begin{array}{l} \text{Determine all the operators } \widetilde{T} \in \mathcal{L}(\widetilde{\mathcal{H}}_1, \widetilde{\mathcal{H}}_2) \text{ with the} \\ \text{matrix representation (5.4) and } \kappa^-(\widetilde{J}_1 - \widetilde{T}^* \widetilde{J}_2 \widetilde{T}) = \kappa' . \end{array} \right.$$

Let us denote by $T_r \in \mathcal{L}(\tilde{\mathcal{H}}_1, \mathcal{H}_2)$ the upper row, respectively $T_c \in \mathcal{L}(\mathcal{H}_1, \tilde{\mathcal{H}}_2)$ the left column, of the operator matrix from (5.4),

(5.6) $$T_r = (T \quad , A)$$

and

(5.7) $$T_c = \begin{pmatrix} T \\ B \end{pmatrix}.$$

In order to approach the problem (5.5) we need to consider two particular cases of it, namely the following problems:

(5.8) $\left[\begin{array}{l}\text{Determine all the operators } T_r \in \mathcal{L}(\tilde{\mathcal{H}}_1, \mathcal{H}_2) \text{ with the} \\ \text{matrix (5.6) and such that } \varkappa^-(\tilde{J}_1 - T_r^* J_2 T_r) = \varkappa'.\end{array}\right.$

and

(5.9) $\left[\begin{array}{l}\text{Determine all the operators } T_c \in \mathcal{L}(\mathcal{H}_1, \tilde{\mathcal{H}}_2) \text{ with the} \\ \text{matrix (5.7) and such that } \varkappa^-(J_1 - T_c^* \tilde{J}_2 T_c) = \varkappa'.\end{array}\right.$

5.1. LEMMA. (i) *If the problem (5.8) has solutions then it is necessary that*

$$\varkappa' \geqslant \max(\varkappa, \varkappa^-(\tilde{\mathcal{H}}_1) - \varkappa^-(\mathcal{H}_2)).$$

(ii) *If* $\varkappa' = \varkappa \geqslant \varkappa^-(\tilde{\mathcal{H}}_1) - \varkappa^-(\mathcal{H}_2)$ *holds then the formula*

(5.10) $$T_r = (T \quad , D_{T^*}\Gamma)$$

establishes a bijective correspondence between the set of all solutions of the problem (5.8) and the set of all operators $\Gamma \in \mathcal{L}(\mathcal{H}, \mathcal{D}_{T^*})$ *with the property* $\varkappa^-(J - \Gamma^* J_{T^*} \Gamma) = 0$.

Moreover, with the above notation, the operators U and V defined by

(5.11) $$UD_{T_r^*} = D_{\Gamma^*} D_{T^*},$$

and

(5.12) $$VD_{T_r} = \begin{pmatrix} D_T & , -J_T L_{T^*}\Gamma \\ 0 & , D_\Gamma \end{pmatrix}$$

are unitary operators between Pontrjagin spaces $\mathcal{D}_{T_r^*}$ *and* $\mathcal{D}_{\Gamma^*}$, *respectively* $\mathcal{D}_{T_r}$ *and* $-\mathcal{D}_T[+]\mathcal{D}_\Gamma$.

PROOF. (i) Let T_r be a solution of the problem (5.8). It follows by

Remark 3.4 that

$$\kappa^-(J_2-T_r\tilde{J}_1T_r^*)=\kappa'-\kappa^-(\tilde{\mathcal{H}}_1)+\kappa^-(\mathcal{H}_2),$$

hence $\kappa'\geq\kappa^-(\tilde{\mathcal{H}}_1)-\kappa^-(\mathcal{H}_2)$ is necessary. On the other hand, by Proposition 2.1 there exist a Krein space $\mathcal{K}$, a f.s. S of $\mathcal{K}$ and an operator $Y\in\mathcal{L}(\mathcal{H}_2,\mathcal{K})$ such that $\kappa^-(S)=\kappa^-(J_2-T_r\tilde{J}_1T_r^*)$, $\kappa^+(S)\geq\kappa^+(J_2-T_r\tilde{J}_1T_r^*)$ and $J_2-T_r\tilde{J}_1T_r^*=Y^*SY$, equivalently

$$J_2-TJ_1T^*=(A\ ,Y^*)\begin{pmatrix}J & , & 0\\ 0 & , & S\end{pmatrix}\begin{pmatrix}A^*\\ Y\end{pmatrix}\tag{5.13}$$

whence, also by Proposition 2.1, it follows

$$\kappa^-(J_2-TJ_1T^*)\leq\kappa^-(J)+\kappa^-(S)$$

which turns out to be equivalent to $\kappa'\geq\kappa$.

(ii) Assuming $\kappa'=\kappa\geq\kappa^-(\tilde{\mathcal{H}}_1)-\kappa^-(\mathcal{H}_2)$ and considering T_r a solution of the problem (5.8) then (5.13) holds and in addition

$$\kappa^-(J_2-TJ_1T^*)=\kappa^-(J)+\kappa^-(S)$$

hence by Corollary 2.5 it follows that there exists uniquely determined operator $\Lambda\in\mathcal{L}(\mathcal{D}_{T^*},\mathcal{H}[+]\mathcal{K})$ such that

$$\begin{pmatrix}A^*\\ Y\end{pmatrix}=\Lambda D_{T^*}\tag{5.14}$$

and

$$\Lambda^*\begin{pmatrix}J & , & 0\\ 0 & , & S\end{pmatrix}\Lambda=J_{T^*}.\tag{5.15}$$

By representing the operator Λ as $\Lambda=\begin{pmatrix}\Lambda_1\\ \Lambda_2\end{pmatrix}$ with respect to $\mathcal{H}[+]\mathcal{K}$, from (5.14) we get $A=D_{T^*}\Lambda_1^*$ and from (5.15) we get

$$J_{T^*}-\Lambda_1^*J\Lambda_1=\Lambda_2^*S\cdot\Lambda_2.\tag{5.16}$$

By taking $\Gamma=\Lambda_1^*$ we get the formula (5.10) and noting that $\kappa^-(J_{T^*}-\Lambda_1^*J\Lambda_1)\geq\kappa-\kappa^-(\tilde{\mathcal{H}}_1)+\kappa^-(\mathcal{H}_2)$, $\kappa^-(\Lambda_2^*S\Lambda_2)\leq\kappa^-(S)=$ $=\kappa-\kappa^-(\tilde{\mathcal{H}}_1)+\kappa^-(\mathcal{H}_2)$ and using also (5.15), it follows

$$\kappa^-(J_{T^*}-\Gamma J\Gamma^*)=\kappa-\kappa^-(\tilde{\mathcal{H}}_1)+\kappa^-(\mathcal{H}_2),$$

which by Remark 3.4 is equivalent to $\kappa^-(J-\Gamma^*J_{T^*}\Gamma)=0$.

It is easy to prove that

$$D_{T_r^*}J_{T_r^*}D_{T_r^*}=D_{T^*}D_{\Gamma^*}J_{\Gamma^*}D_{\Gamma^*}D_{T^*}\tag{5.17}$$

whence, defining the operator $U: \mathcal{R}(D_{T_r^*}) \longrightarrow \mathcal{R}(D_{\Gamma^*})$ by (5.11), it follows that it is isometric with respect to the indefinite inner products and by taking account of $\kappa^-(\mathcal{D}_{T_r^*}) = \kappa^-(\mathcal{D}_{\Gamma^*}) < \infty$, the densities of the domain and the range of U, and [4, Theorem VI.3.5], it follows that U is unitary between the Pontrjagin spaces $\mathcal{D}_{T_r^*}$ and $\mathcal{D}_{\Gamma^*}$.

We claim now that

$$(5.18) \qquad D_{T_r} J_{T_r} D_{T_r} = \begin{pmatrix} D_T & , & 0 \\ -\Gamma^* L_{T^*} J_T, & D_\Gamma \end{pmatrix} \begin{pmatrix} J_T & , & 0 \\ 0 & , & J \end{pmatrix} \begin{pmatrix} D_T & , -J_T L_{T^*}\Gamma \\ 0 & , \ D_\Gamma \end{pmatrix}.$$

Indeed

$$D_{T_r} J_{T_r} D_{T_r} = \begin{pmatrix} J_1 - T^* J_2 T & , & -T^* J_2 D_{T^*} \Gamma \\ -\Gamma^* D_{T^*} J_2 T & , & J - \Gamma^* D_{T^*} J_2 D_{T^*} \Gamma \end{pmatrix}$$

and the right side of (5.18) equals

$$\begin{pmatrix} J_1 - T^* J_2 T & , & -D_T L_{T^*} \Gamma \\ -\Gamma^* L_{T^*}^* D_T & , & J - \Gamma^* (J_{T^*} - D_{T^*} J_2 D_{T^*}) \Gamma \end{pmatrix}$$

hence the claim follows by taking account of the definition of L_{T^*} (Proposition 4.1) and Corollary 4.6.

Defining the operator $V: \mathcal{R}(D_{T_r}) \longrightarrow \mathcal{D}_T [+] \mathcal{D}_\Gamma$ by (5.12), it follows from (5.18) that V is isometric with respect to the indefinite inner products and, similarly as for U, we can argue that it is bounded and it uniquely extends to a unitary operator between the Pontrjagin spaces $\mathcal{D}_{T_r}$ and $\mathcal{D}_T [+] \mathcal{D}_\Gamma$.

Conversely, if $\Gamma \in \mathcal{L}(\mathcal{H}, \mathcal{D}_{T^*})$ such that $\kappa^-(J_{T^*} - \Gamma J \Gamma^*) = \kappa - \kappa^-(\tilde{\mathcal{H}}_1) + \kappa^-(\mathcal{H}_2)$, by taking T_r as in (5.10), it turns out that (5.17) holds, particularly $\kappa^-(J_{T^*} - \Gamma J \Gamma^*) = \kappa^-(J_2 - T_r \tilde{J}_1 T_r^*)$ hence $\kappa^-(\tilde{J}_1 - T_r^* J_2 T_r) = \kappa$.

The fact that the correspondence from (5.10) is one-to-one is clear. ■

5.2. LEMMA. (i) If the problem (5.9) has solution then it is necessary that the following inequality holds

$$(5.19) \qquad \kappa' \geq \kappa - \kappa^{-}(\tilde{\mathcal{H}}_2) + \kappa^{-}(\mathcal{H}_2).$$

(ii) If $\kappa' = \kappa - \kappa^{-}(\tilde{\mathcal{H}}_2) + \kappa^{-}(\mathcal{H}_2)$ then the formula

$$(5.20) \qquad T_c = \begin{pmatrix} T \\ \Gamma D_T \end{pmatrix}$$

establishes a bijective correspondence between the set of all solutions of the problem (5.9) and the set of all operators $\Gamma \in \mathcal{L}(\mathcal{D}_T, \mathcal{H}')$ such that $\kappa^{-}(J' - \Gamma J_T \Gamma^{*}) = 0$. Moreover, the operators U_* and V_* defined by

$$(5.21) \qquad U_* D_{T_c} = D_\Gamma D_T$$

$$(5.22) \qquad V_* D_{T_c^*} = \begin{pmatrix} D_{T^*} & , -J_{T^*} L_T \Gamma^{*} \\ 0 & , D_{\Gamma^*} \end{pmatrix}$$

are unitary operators between the Pontrjagin spaces $\mathcal{D}_{T_c}$ and $\mathcal{D}_\Gamma$, respectively $\mathcal{D}_{T_c^*}$ and $\mathcal{D}_{T^*} [+] \mathcal{D}_{\Gamma^*}$. ∎

The proof follows from Lemma 5.1 reasoning by duality.

Now, we prove the main result of this paper.

5.3. THEOREM. (i) If the problem (5.5) has solution then the following inequalities are necessary

$$(5.23) \qquad \kappa' \geq \max(\kappa - \kappa^{-}(\tilde{\mathcal{H}}_2) + \kappa^{-}(\mathcal{H}_2),\ \kappa^{-}(\tilde{\mathcal{H}}_1) - \kappa^{-}(\tilde{\mathcal{H}}_2)).$$

(ii) If $\kappa' = \kappa - \kappa^{-}(\tilde{\mathcal{H}}_2) + \kappa^{-}(\mathcal{H}_2) \geq \kappa^{-}(\tilde{\mathcal{H}}_1) - \kappa^{-}(\tilde{\mathcal{H}}_2)$ hold, then the formula

$$(5.24) \qquad \tilde{T} = \begin{pmatrix} T & , & D_{T^*}\Gamma_1 \\ \Gamma_2 D_T, & -\Gamma_2 L_T^{*} J_{T^*}\Gamma_1 & + D_{\Gamma_2^*} \Gamma D_{\Gamma_1} \end{pmatrix}$$

establishes a bijection between the set of all solutions of the problem (5.5) and the set of all triplets $(\Gamma_1, \Gamma_2, \Gamma)$ characterized as follows:

$$\Gamma_1 \in \mathcal{L}(\mathcal{H}, \mathcal{D}_{T^*}), \quad \kappa^{-}(J - \Gamma_1^{*} J_{T^*} \Gamma_1) = 0,$$
$$\Gamma_2 \in \mathcal{L}(\mathcal{D}_T, \mathcal{H}'), \quad \kappa^{-}(J' - \Gamma_2 J_T \Gamma_2^{*}) = 0,$$
$$\Gamma \in \mathcal{L}(\mathcal{D}_{\Gamma_1}, \mathcal{D}_{\Gamma_2^*}), \quad \kappa^{-}(I - \Gamma^{*}\Gamma) = 0.$$

Moreover, with the above notation, the Pontrjagin spaces $\mathcal{D}_{\tilde{T}}$

and $\mathcal{D}_{\Gamma_2}[+]\mathcal{D}_{\Gamma}$, respectively $\mathcal{D}_{T^*}$ and $\mathcal{D}_{\Gamma_1^*}[+]\mathcal{D}_{\Gamma^*}$, are unitary equivalent.

PROOF. The first statement follows easily by combining the first statements in Lemma 5.1 and 5.2.

Let assume now that $k' = k - \kappa^-(\tilde{\mathcal{H}}_2) + \kappa^-(\mathcal{H}_2) \geq k^-(\tilde{\mathcal{H}}_1) - \kappa^-(\tilde{\mathcal{H}}_{\ldots}$ hold and let also $\tilde{T}$ be a solution of the problem (5.5) in this case.

Considering the row operator T_r and the column operator T_c associated to $\tilde{T}$ by (5.6) and (5.7) it follows immediately that

$$k^-(\tilde{J}_1 - T_c^* J_2 T_c) = k' \quad , \quad k^-(J_1 - T_r^* \tilde{J}_2 T_r) = k \ ,$$

whence by Lemma 5.1 there exists $\Gamma_1 \in \mathcal{L}(\mathcal{H}, \mathcal{D}_{T^*})$ such that

$$T_r = (T \ , \ D_{T^*}\Gamma_1) \quad , \quad k^-(J - \Gamma_1^* J_{T^*}\Gamma_1) = 0$$

and by Lemma 5.2 there exists $\Gamma_2 \in \mathcal{L}(\mathcal{D}_T, \mathcal{H}')$ such that

$$T_c = \begin{pmatrix} T \\ \Gamma_2 D_T \end{pmatrix} \quad , \quad k^-(J' - \Gamma_2 J_T \Gamma_2^*) = 0$$

consequently, $\tilde{T}$ has the following matrix representation

$$\tilde{T} = \begin{pmatrix} T & , & D_{T^*}\Gamma_1 \\ \Gamma_2 D_T & , & X \end{pmatrix} . \tag{5.25}$$

Considering now $\tilde{T}$ as a row extension of T_c it follows by Lemma 5.1 the existence of $\Delta \in \mathcal{L}(\mathcal{H}, \mathcal{D}_{T_c^*})$ such that

$$\begin{pmatrix} D_{T^*}\Gamma_1 \\ X \end{pmatrix} = D_{T_c^*}\Delta \quad , \quad k^-(J - \Delta^* J_{T_c^*}\Delta) = 0 \ . \tag{5.26}$$

On the other hand, by the last statement of Lemma 5.2, there exists a Pontrjagin space unitary operator $V_* : \mathcal{D}_{T^*}[+]\mathcal{D}_{\Gamma_2^*} \longrightarrow \mathcal{D}_{T_c^*}$ such that

$$V_* D_{T_c^*} = \begin{pmatrix} D_{T^*} & , & -J_{T^*} L_T \Gamma_2^* \\ 0 & , & D_{\Gamma_2^*} \end{pmatrix}$$

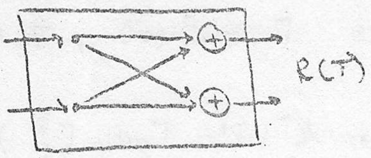
which from (5.26) yields the existence of $\Lambda \in \mathcal{L}(\mathcal{H}', \mathcal{D}_{T^*}[+]\mathcal{D}_{\Gamma_2^*})$ such that

$$\begin{pmatrix} D_{T^*}\Gamma_1 \\ X \end{pmatrix} = \begin{pmatrix} D_{T^*} & , & 0 \\ -\Gamma_2 L_T^* J_{T^*} , & & D_{\Gamma_2^*} \end{pmatrix} \Lambda$$

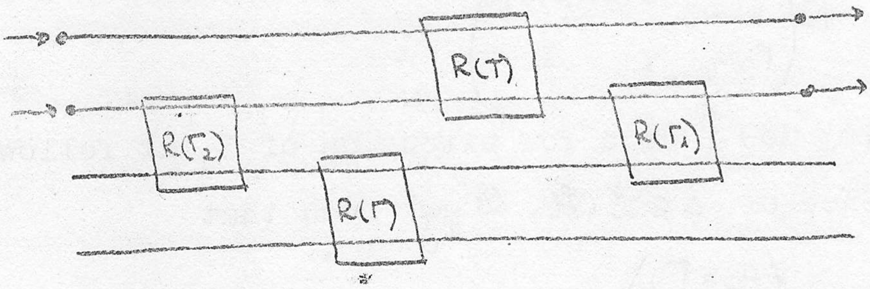
and $\varkappa^-(J' - \Lambda^*(J_{T^*}\oplus I)\Lambda)=0$. By representing $\Lambda = \begin{pmatrix}\Lambda_1\\ \Lambda_2\end{pmatrix}$ with respect to $\mathcal{D}_{T^*}[+]\mathcal{D}_{\Gamma_2^*}$, it follows from (5.28) that $\Lambda_1 = \Gamma_1$ and $\varkappa^-(J' - \Gamma_1^* J_{T^*}\Gamma_1 - \Lambda_2^*\Lambda_2)=0$, hence we can consider Λ as a column extension of Γ_1 and applying once more Lemma 5.2 we get $\Gamma \in \mathcal{L}(\mathcal{D}_{\Gamma_1}, \mathcal{D}_{\Gamma_2^*})$, Hilbert space contraction, such that $\Lambda_2 = \Gamma D_{\Gamma_1}$. By using this known operator Λ in (5.27) we get the desired formula of X.

The unitary equivalence of the Pontrjagin spaces $\mathcal{D}_{\tilde{T}}$ and $\mathcal{D}_{\Gamma_2}[+]\mathcal{D}_{\Gamma}$, respectively $\mathcal{D}_{\tilde{T}^*}$ and $\mathcal{D}_{\Gamma_1^*}[+]\mathcal{D}_{\Gamma^*}$, follow by combining the last assertions of Lemma 5.1 and 5.2. ■

5.4 REMARK Considering the elementary lattice section



then the formula (5.24) has the following lattice filter representation (in transfer representation)



■

5.5 REMARK. 1) Using the results of [6] and those of this paper, one can try an inspection of the concluding problems in [8].

2) Theorem 5.3 contains, as particular cases, the structure of two by two Hilbert space block-matrices whose defect spaces have a finite number of negative squares and of two by two Krein space J-contractions (see also [6]).

3) Applications of Theorem 5.3 (mimicing those of [1]) will be presented elsewhere.

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