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STABLE RANK OF C^*-ALGEBRAS OF TOEPLITZ

OPERATORS ON POLYDISKS

by

Gabriel NAGY

In [3] M.A.Rieffel introduced the notion of topological stable rank for Banach algebras and obtained some results concerning this invariant for C^*-algebras.

R.H.Herman and L.N.Vaserstein show in [2] that the stable rank introduced by Rieffel coincides, in the case of C^*-algebras, with the usual algebraic one.

Since for commutative C^*-algebras which have a manifold as spectrum the stable rank can be recovered from the dimension of the manifold, some of Rieffel's results formulated in terms of ideals, quotients or crossed-products (which we shall use here) are generalisations of well-known facts from classical dimension theory.

In one example from his paper Rieffel shows that the stable rank for the C^*-algebra $\mathcal{T}$ of 1-dimensional Toeplitz operators (i.e. on the disk) is equal to 2. This result seems to suggest that the algebra $\mathcal{T}$, viewed as an extension of the algebra $C(\mathbb{T})$ of continuous function on the 1-dimensional torus, is a sort of a "noncommutative disk".

Here we show by studying the stable rank for C^*-algebras $\mathcal{T}_n$ of n-dimensional Toeplitz operators (i.e. on polydisks) that this image of a noncommutative disk fails at the test of tensor products, in the sense that the stable rank for the algebra $\mathcal{T}_n$ (which is isomorphic to the tensor product of n copies of $\mathcal{T}$) is not equal to the stable rank of the algebra of continuous functions on the n-polydisk , but is close to the stable rank of the algebra of continuous funcions on the n-torus.

The paper has two sections.

The first one contains the notations we use and some preliminary results which are due to Rieffel.

The second sections contains the estimates for the stable rank of the algebras $\mathcal{T}_n$. Our determinations are precise for even numbers n and with approximation of 1 in the odd case. We obtain more than this, namely some evaluations for the stable rank and connected stable rank (in the sense of [3]) of the tensor product of the algebras $\mathcal{T}_n$ with a class of commutative C^*-algebras (of continuous functions on multidimensional torus).

I would like to express my gratitude to Professor Dan Voiculescu for the useful discussions in writing this paper.

I

We begin by recalling some definitions and results from [3] .

For a unital C^*-algebra A and a natural number $n \geq 1$ we consider

$$Lg_n(A)=\left\{(a_1,\ldots,a_n)\in A^n \,\middle|\, \exists b_1,\ldots,b_n\in A \text{ s.t. } b_1a_1+\ldots+b_na_n=1\right\}$$

For any C^*-algebra A we denote by $\tilde{A}$ the C^*-algebra with adjoined unit.

We denote by $\mathcal{T}_n$ the C^*-algebra generated by the Toeplitz operators T_φ , with continuous symbol φ on the n-torus $\mathbb{T}^n = \{(z_1,\ldots,z_n) \in \mathbb{C}^n \mid |z_1|=\ldots=|z_n|=1\}$ acting on the Hardy space $H^2(\mathbb{T}^n)$. It is well-known that we have an isomorphism $\mathcal{T}_n \simeq \underbrace{\mathcal{T}\otimes\ldots\otimes\mathcal{T}}_{n\text{-times}}$ where $\mathcal{T}$ stands for $\mathcal{T}_1$.

We denoty by $\mathcal{K}$ the C^*-algebra of compact operators on a separable Hilbert space. Then we have $\mathcal{T}=\{T_\varphi + K \mid \varphi\in C(\mathbb{T}), K\in\mathcal{K}\}$ and defining $p:\mathcal{T}\longrightarrow C(\mathbb{T})$ by $p(T_\varphi + K)=\varphi$ one obtains the Toeplitz extension (see [1]), namely the exact sequence

$$(1) \qquad 0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \xrightarrow{\;p\;} C(\mathbb{T}) \longrightarrow 0$$

For a C^*-algebra A we consider the topological stable rank of A, defined in [3], as the least integer $n\geq 1$ (in case it does not exist we put ∞) for which $Lg_n(\tilde{A})$ is dense in $\tilde{A}^n$ (if A is unital we can replace $\tilde{A}$ by A). According to [2] this number coincides with the usual stable rank of A, denoted by sr(A) which is the least integer $n\geq 1$ (if it does not exist it is again taken to be ∞) such that for any $(a_1,\ldots,a_n,a_{n+1})\in Lg_{n+1}(\tilde{A})$ there exist $b_1,\ldots,b_n\in\tilde{A}$ such that $(a_1+b_1a_{n+1},\ldots,a_n+b_na_{n+1})\in$ $\in Lg_n(\tilde{A})$ (if A is unital we can replace $\tilde{A}$ by A). For a unital C^*-algebra A we denote by GL(n,A) (respectively $GL^o(n,A)$) the group of invertible elements of $M_n(A)$ (respectively the connected component of the unit in GL(n,A)), for any $n\geq 1$. For a C^*-algebra A we denote by csr(A) the connected stable rank of A, defined in [3] as the least integer $n\geq 1$ (if it does not exist put $csr(A)=\infty$) for which the action of $GL^{o}(m,\tilde{A})$ on $Lg_m(\tilde{A})$ is transitive for any $m \geq n$. This action is by left multiplication, considering $Lg_m(\tilde{A})$ as consisting of column vectors. According to Corollary 8.5 of [3] $csr(A)$ is the least number n such that $Lg_m(\tilde{A})$ is connected for any $m \geq n$. (Again if A is unital we can replace $\tilde{A}$ by A).

We introduce the notation $msr(A)=\max(csr(A),sr(A))$ for any C^*-algebra A.

Corollary 4.10 of [3] shows that

(2) $$msr(A) \leq sr(A)+1 \quad \text{for any } C^*\text{-algebra } A$$

By Theorem 6.4 of [3] we have

(3) $$sr(A \otimes \mathbb{K}) = \begin{cases} 1 & \text{if } sr(A)=1 \\ 2 & \text{if } sr(A) \neq 1 \end{cases}$$

REMARK. From inequality (2) it follows that for any stable C^*-algebra A we have $msr(A) \leq 3$. An obvious consequence of Theorem 7.1 and Corollary 8.6. of [3] is that, for a unital C^*-algebra A, we have

(4) $$msr(A \otimes C(\mathbb{T})) \leq sr(A)+1$$

If J is a closed bilateral ideal of the C^*-algebra A, with our notations, by Theorems 4.3, 4.4, 4.11 of [3] we have

(5) $$\max(sr(J),sr(A/J)) \leq sr(A) \leq \max(sr(J),msr(A/J))$$

An easy consequence of the definition is that for any two C^*-algebras A and B we have

(6) $sr(A \oplus B) = \max(sr(A), sr(B))$

From Proposition 3.5 of [3] we have

(7) $sr(\mathcal{K}) = 1$

Example 4.13 of [3] shows that

(8) $sr(\mathcal{T}) = 2$

1. PROPOSITION. Let n be an integer, $n \geq 1$. Then

(9) $sr(C(\mathbb{T}^{2n})) = msr(C(\mathbb{T}^{2n})) = n+1$

(10) $n = sr(C(\mathbb{T}^{2n-1})) \leq msr(C(\mathbb{T}^{2n-1})) \leq n+1$.

PROOF: We shall apply Proposition 1.7 of [3] which states that for a compact manifold X we have $sr(C(X)) = \left[\frac{\dim X}{2}\right] + 1$ where by [ ] we denote the integer part. This formula leads to the equalities

$$sr(C(\mathbb{T}^{2n})) = n+1 \quad \text{and} \quad sr(C(\mathbb{T}^{2n-1})) = n$$

The second part of (10) follows from (2). To prove the second part of (9) use $C(\mathbb{T}^{2n}) \simeq C(\mathbb{T}^{2n-1}) \otimes C(\mathbb{T})$ which by (4) gives $msr(C(\mathbb{T}^{2n})) \leq sr(C(\mathbb{T}^{2n-1})) + 1 = n+1$.

Q.E.D.

II

We begin this section with some formulas similar to (5) and (6), but concerning the connected stable rank.

2. LEMMA. Let A, B be C^*-algebras and J a closed bilateral ideal of A. Then

(11) $\quad \mathrm{csr}(A \oplus B) = \max(\mathrm{csr}(A), \mathrm{csr}(B))$

(12) $\quad \mathrm{csr}(A) \leq \max(\mathrm{csr}(J), \mathrm{csr}(A/J))$

PROOF: (11) is obvious. The proof of (12) will be along the same lines as Theorem 4.11 of [3]. With no loss of generality we may suppose that A is unital (this discussion is made in Rieffel's paper). If max $(\mathrm{csr}(J), \mathrm{csr}(A/J)) = \infty$ inequality (12) is trivial. Let us suppose that $\max(\mathrm{csr}(J), \mathrm{csr}(A/J)) = m < \infty$. Take $n \geq m$. The natural surjection $A \to A/J$ will be denoted by $A \ni x \mapsto \hat{x} \in A/J$ and the same notation will be kept for the surjection $M_n(A) \to M_n(A/J)$. Choose $(a_1, \ldots, a_n) \in Lg_n(A)$. Because $n \geq \mathrm{csr}(A/J)$, the exists $G_o \in GL^o(n, A/J)$ such that

$$G_o \cdot \begin{pmatrix} \hat{a}_1 \\ \vdots \\ \hat{a}_n \end{pmatrix} = \begin{pmatrix} \hat{1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

G_o, being in $GL^o(n, A/J)$, has a lifting $G \in GL^o(n, A)$) (i.e. $\hat{G} = G_o$).

The property of G_o can be rewritten as

$$G \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in J^n ,$$

so there exist $j_1, \ldots, j_n \in J$ such that

$$G.\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} j_1+1 \\ j_2 \\ \vdots \\ j_n \end{pmatrix}$$

The left action of $GL(n,A)$ on $Lg_n(A)$ takes $Lg_n(A)$ into itself as shows Proposition 4.1 of [3] , and so

$$(j_1+1,\ j_2,\dots,j_n) \in Lg_n(A)\ .$$

Now if C is an arbitrary unital C^*-subalgebra of A we have $Lg_n(C)=Lg_n(A)\cap C^n$. (Indeed, the only thing to prove is that $Lg_n(A)\cap C^n \subset Lg_n(C)$ and this follows from the fact that $(c_1,\dots,c_n)\in Lg_n(C)$ if and only if $c_1^*c_1+\dots+c_n^*c_n$ is invertible in C or, equivalently, in A).

By this remark we conclude that

$$(j_1+1,j_2,\dots,j_n)\in Lg_n(\tilde{J})$$

Because $n \geqslant csr(J)$ we can find $H\in GL^o(n,\tilde{J})$ such that

$$H\cdot\begin{pmatrix} j_1+1 \\ j_2 \\ \vdots \\ j_n \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and so

$$HG\cdot\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and because $HG \in GL^{o}(n,A)$ it follows that $GL^{o}(n,A)$ acts trasitively on $Lg_n(A)$, this property having place for any $n \geq m$. This shows that $m \geq csr(A)$.

Q.E.D.

We shall work later with "msr", so we combine this lemma with (5) and (6) of Section I and write shortly

(13) $$msr(A \oplus B) = \max\ (msr(A),\ msr(B))$$

(14) $$msr(A) \leq \max\ (msr(J),\ msr(A/J)).$$

To obtain estimates of the stable rank for the algebras of type $\mathcal{T}_n \otimes C(\mathbb{T}^k)$ we shall consider some exact sequences similar to (1).

Let us define inductively the following sequence of ideals of the algebras $\mathcal{T}_n$. Put $J_1 = \mathcal{K}$ and $J_{n+1} = J_n \otimes \mathcal{T} + \mathcal{T}_n \otimes \mathcal{K}$ for any $n \geq 1$.

Assuming that J_n is a closed two-sided ideal of $\mathcal{T}_n$, J_{n+1} will be a sum of two closed two-sided ideals of $\mathcal{T}_{n+1}$ and so will be itself such an ideal.

3. <u>LEMMA</u>. For each $n \geq 1$ we have an exact sequence

(15) $$0 \longrightarrow J_n \longrightarrow \mathcal{T}_n \xrightarrow{p_n} C(\mathbb{T}^n) \longrightarrow 0$$

PROOF: We proceed by induction. The case n=1 is mentioned in Section I (1). Suppose that for n we have the exact sequence (15). Tensoring "term by term" the exact sequences (1) and (15) with suitable C^*-algebras (all of which are nuclear) we obtain the following three exact sequences

$$0 \longrightarrow J_n \otimes \mathcal{T} \longrightarrow \mathcal{T}_{n+1} \xrightarrow{p_n \otimes \mathrm{Id}_{\mathcal{T}}} C(\mathbb{T}^n) \otimes \mathcal{T} \longrightarrow 0$$

$$0 \longrightarrow J_n \otimes \mathcal{K} \longrightarrow \mathcal{T}_n \otimes \mathcal{K} \xrightarrow{p_n \otimes \mathrm{Id}_{\mathcal{K}}} C(\mathbb{T}^n) \otimes \mathcal{K} \longrightarrow 0$$

$$0 \longrightarrow C(\mathbb{T}^n) \otimes \mathcal{K} \longrightarrow C(\mathbb{T}^n) \otimes \mathcal{T} \xrightarrow{\mathrm{Id}_{C(\mathbb{T}^n)} \otimes p} C(\mathbb{T}^n) \otimes C(\mathbb{T}) \longrightarrow 0$$

Define $\tilde{P}_{n+1} : \mathcal{T}_{n+1} \longrightarrow C(\mathbb{T}^n) \otimes C(\mathbb{T})$ by

$$\tilde{P}_{n+1} = (\mathrm{Id}_{C(\mathbb{T}^n)} \otimes p) \circ (p_n \otimes \mathrm{Id}_{\mathcal{T}}) = p_n \otimes p$$

$\tilde{P}_{n+1}$ is obviously a surjective $*$-morphism and because the second of the three exact sequences is actually a "restriction" of the first, we have

$$\mathrm{Ker}\, \tilde{P}_{n+1} = (P_n \otimes \mathrm{Id}_{\mathcal{T}})^{-1} (\mathrm{Ker}(\mathrm{Id}_{C(\mathbb{T}^n)} \otimes P)) =$$

$$= (P_n \otimes \mathrm{Id}_{\mathcal{T}})^{-1} (C(\mathbb{T}^n) \otimes \mathcal{K}) = J_n \otimes \mathcal{T} + \mathcal{T}_n \otimes \mathcal{K} = J_{n+1}$$

and so we obtain the exact sequence

$$0 \longrightarrow J_{n+1} \longrightarrow \mathcal{T}_{n+1} \xrightarrow{\tilde{p}_{n+1}} C(\mathbb{T}^n) \otimes C(\mathbb{T}) \longrightarrow 0$$

Finally we compose $\tilde{P}_{n+1}$ with the isomorphism $C(\mathbb{T}^n) \otimes C(\mathbb{T}) \simeq C(\mathbb{T}^{n+1})$. Q.E.D.

The following lemma will be useful for computing the stable ranks for the ideals J_n.

4. LEMMA. For any $n \geq 1$ we have an exact sequence

(16) $0 \longrightarrow J_n \otimes \mathcal{K} \longrightarrow J_{n+1} \longrightarrow (J_n \otimes C(\mathbb{T})) \oplus (C(\mathbb{T}^n) \otimes \mathcal{K}) \longrightarrow 0$

PROOF: Again we tensor "term by term" the exact sequence (15), from lemma 3, with suitable C^*-algebras and obtain the following two isomorphisms

$$J_n \otimes \mathcal{T} / J_n \otimes \mathcal{K} \simeq J_n \otimes C(\mathbb{T})$$

$$\mathcal{T}_n \otimes \mathcal{K} / J_n \otimes \mathcal{K} \simeq C(\mathbb{T}^n) \otimes \mathcal{K}$$

Since $J_{n+1} = J_n \otimes \mathcal{T} + \mathcal{T}_n \otimes \mathcal{K}$, the only thing we have to remark is that $(J_n \otimes \mathcal{T}) \cap (\mathcal{T}_n \otimes \mathcal{K}) = J_n \otimes \mathcal{K}$.

Q.E.D.

5. <u>PROPOSITION</u>. The ideals J_n have the following properties:

(17) $\quad \mathrm{msr}(J_n) \leq 3$ for any $n \geq 1$

(18) $\quad \mathrm{msr}(J_n \otimes C(\mathbb{T}^k)) \leq 3$ for any $n, k \geq 1$.

Furthermore

(19) $\quad \mathrm{msr}(J_1) \leq 2$,

(20) $\quad \mathrm{msr}(J_1 \otimes C(\mathbb{T})) \leq 2$

(21) $\quad \mathrm{msr}(J_2) \leq 2$.

PROOF: We prove (17) and (18) simultaneously by induction of n. For n=1 (17) is trivial since $J_1 = \mathcal{K}$ and by the remark preceding Proposition 1 we will have $\mathrm{msr}(J_1) \leq 3$.

(18) is also trivial since $J_1 \otimes C(\mathbb{T}^k)$ is also a stable C^*-algebra. Suppose (17) and (18) hold for n. Using Lemma 4 and (13), (14) from Lemma 2 we get the inequality

$$\operatorname{msr}(J_{n+1}) \leqslant \max(\operatorname{msr}(J_n \otimes \mathbb{K}),\ \operatorname{msr}(J_n \otimes C(\mathbb{T})),\ \operatorname{msr}(C(\mathbb{T}^n) \otimes \mathbb{K}))$$

Since the C^*-algebras $J_n \otimes \mathbb{K}$ and $C(\mathbb{T}^n) \otimes \mathbb{K}$ are stable, their msr's are not greater than 3 (by the Remark from Section I). The only thing to apply is now the inductive hypothesis, namely (18) for n, and what follows is inequality (17) for n+1.

To get (18) for n+1 we tensor the exact sequence given by Lemma 4 "term by term" with $C(\mathbb{T}^k)$ and obtain the following exact sequence (modulo some obvious isomorphisms)

$$0 \to J_n \otimes C(\mathbb{T}^k) \otimes \mathbb{K} \to J_{n+1} \otimes C(\mathbb{T}^k) \to (J_n \otimes C(\mathbb{T}^{k+1})) \oplus$$
$$\oplus (C(\mathbb{T}^{n+k}) \otimes \mathbb{K}) \to 0$$

and by (13) (14) from Lemma 2 we have

$$\operatorname{msr}(J_{n+1} \otimes C(\mathbb{T}^k)) \leqslant \max(\operatorname{msr}(J_n \otimes C(\mathbb{T}^k) \otimes \mathbb{K}),\ \operatorname{msr}(J_n \otimes C(\mathbb{T}^{k+1})),$$
$$\operatorname{msr}(C(\mathbb{T}^{n+k}) \otimes \mathbb{K}))$$

Using exactly the same arguments as before we get (18) for n+1.

To prove the additional properties (19) (20) (21) let us remark first that by (7) of Section I $\operatorname{sr}(\mathbb{K})=1$ and so by (2) of Section I $\operatorname{msr}(\mathbb{K}) \leqslant \operatorname{sr}(\mathbb{K})+1=2$, which proves (19).

The proof of (20) uses the same arguments because by (10) of Section I we have $\operatorname{sr}(C(\mathbb{T}))=1$ and by (3) of Section I we have $\operatorname{sr}(\mathbb{K} \otimes C(\mathbb{T}))=1$. Finally, to obtain (21) we shall write down the exact sequence given by Lemma 4, modulo the isomorphism $\mathbb{K} \otimes \mathbb{K} \simeq \mathbb{K}$, that is

(22) $0 \to \mathbb{K} \to J_2 \to (\mathbb{K} \otimes C(\mathbb{T})) \oplus (C(\mathbb{T}) \otimes \mathbb{K}) \to 0$

By Lemma 2 (13), (14) it follows that

$msr(J_2) \leq \max(msr(\mathbb{K}), msr(\mathbb{K} \otimes C(\mathbb{T}))) \leq 2$.

Q.E.D.

6. THEOREM. The sequence of C^*-algebras $(\mathcal{T}_n)_{n \geq 1}$ has the following properties

(23) $sr(\mathcal{T}_{2p}) = msr(\mathcal{T}_{2p}) = p+1$ for any $p \geq 1$

(24) $p+1 \leq sr(\mathcal{T}_{2p+1}) \leq msr(\mathcal{T}_{2p+1}) \leq p+2$ for any $p \geq 1$

(25) $sr(\mathcal{T}_n \otimes C(\mathbb{T}^k)) = msr(\mathcal{T}_n \otimes C(\mathbb{T}^k)) = \frac{n+k}{2}+1$ for any $n, k \geq 1$ such that $n \equiv k \pmod 2$

(26) $\frac{n+k+1}{2} \leq sr(\mathcal{T}_n \otimes C(\mathbb{T}^k)) \leq msr(\mathcal{T}_n \otimes C(\mathbb{T}^k)) \leq \frac{n+k+3}{2}$ for any $n, k \geq 1$ such that $n \not\equiv k \pmod 2$.

PROOF: Using Lemma 3, one inequality from (5) of Section I, and (14) from Lemma 2 we get

$$sr(C(\mathbb{T}^n)) \leq sr(\mathcal{T}_n) \leq msr(\mathcal{T}_n) \leq \max(msr(J_n), msr(C(\mathbb{T}^n)))$$

For $n=2p$ using Proposition 1 of Section I we obtain

$$p+1 \leq sr(\mathcal{T}_{2p}) \leq msr(\mathcal{T}_{2p}) \leq \max(msr(J_{2p}), p+1) \quad \text{for any } p \geq 1$$

Since $msr(J_{2p}) \leq p+1$ (this follows from (21) for $p=1$ and (17) for $p \geq 2$) the preceding inequalities imply (23).

For odd numbers n, $n \geq 3$ we obtain by Proposition 1 of Section I $p+1 \leq sr(\mathcal{T}_{2p+1}) \leq msr(\mathcal{T}_{2p+1}) \leq \max(msr(J_{2p+1}), msr(C(\mathbb{T}^{2p+1})))$.

By Proposition 1 of Section I and by Proposition 5 we have

$$\max(\mathrm{msr}(J_{2p+1}), \mathrm{msr}(C(\mathbb{T}^{2p+1}))) \leq \max(3, p+2) = p+2$$

for any $p \geq 1$ which gives the inequalities (24).

Tensoring "term by term" the exact sequence from Lemma 3 with $C(\mathbb{T}^k)$ we get the exact sequence

$$0 \longrightarrow J_n \otimes C(\mathbb{T}^k) \longrightarrow \mathcal{J}_n \otimes C(\mathbb{T}^k) \longrightarrow C(\mathbb{T}^{n+k}) \longrightarrow 0$$

and as before we infer

$$\mathrm{sr}(C(\mathbb{T}^{n+k})) \leq \mathrm{sr}(\mathcal{J}_n \otimes C(\mathbb{T}^k)) \leq \mathrm{msr}(\mathcal{J}_n \otimes C(\mathbb{T}^k)) \leq$$

$$\leq \max(\mathrm{msr}(J_n \otimes C(\mathbb{T}^k)), \mathrm{msr}(C(\mathbb{T}^{n+k})))$$

For $n \equiv k \pmod 2$, or equivalently if $n+k$ is even applying Proposition 1 of Section I we have

$$\frac{n+k}{2}+1 \leq \mathrm{sr}(\mathcal{J}_n \otimes C(\mathbb{T}^k)) \leq \mathrm{msr}(\mathcal{J}_n \otimes C(\mathbb{T}^k)) \leq$$

$$\leq \max(\mathrm{msr}(J_n \otimes C(\mathbb{T}^k)), \frac{n+k}{2}+1)$$

Now by Proposition 5 (if $n=k=1$ we use (20)) we have $\max(\mathrm{msr}(J_n \otimes ((\mathbb{T}^k)), \frac{n+k}{2}+1) = \frac{n+k}{2}+1$ and so from these inequalites we get (25).

For $n \not\equiv k \pmod 2$ arguing as before we have

$$\frac{n+k+1}{2} \leq \mathrm{sr}(\mathcal{J}_n \otimes C(\mathbb{T}^k)) \leq \mathrm{msr}(\mathcal{J}_n \otimes C(\mathbb{T}^k)) \leq$$

$$\leq \max(\mathrm{msr}(J_n \otimes C(\mathbb{T}^k)), \mathrm{msr}(C(\mathbb{T}^{n+k})) \leq$$

$$\leq \max(3, \frac{n+k+3}{2}) = \frac{n+k+3}{2}$$

and so we obtain (26).

Q.E.D.

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