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ORU - COMPACT OPERATORS

by

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If E,F are Riesz spaces, $L_r(E,F)$ will be the ordered vector space of all differences of positive linear operators from E to F; the order relation on $L_r(E,F)$ is as usual: $U \geq 0$ if $U(E_+) \subset F_+$. $L_r^{\times}(E,F)$ will be the subspace of $L_r(E,F)$ consisting of all differences of positive order continuous operators. In case E and F are endowed with locally solid topologies, $L_r'(E,F)$ will be the subspace of $L_r(E,F)$ consisting of all differences of positive continuous operators. We shall write $E^{\sim}$ for $L_r(E,\mathbb{R})$, $E^{\times}$ for $L_r^{\times}(E,\mathbb{R})$ and E' for $L_r'(E;\mathbb{R})$.

For any order bounded linear operator $U:E \longrightarrow F$ we let $U':F^{\sim} \longrightarrow E^{\sim}$ be the transpose of U.

If $f \in E^{\sim}$ and $y \in F$ we shall denote by $f \otimes y$ the operator given by

$$(f\otimes y)(x) = f(x)y, \qquad x \in E.$$

For any order ideal G in $E^{\sim}$ we let $\mathcal{F}_G(E,F)$ be the collection of all operators of the form $\sum_{i=1}^{n} f_i \otimes y_i$ with $f_i \in G$ and $y_i \in F$; we also let $\mathcal{P}_G(E,F)$ be the collection of all operators of the form $\sum_{i=1}^{n} f_i \otimes y_i$ with $f_i \in G_+$ and $y_i \in F_+$. In particular we use the notations $\mathcal{F}(E,F)$ for $\mathcal{F}_{E^{\sim}}(E,F)$, $\mathcal{F}^{\times}(E,F)$ for $\mathcal{F}_{E^{\times}}(E,F)$, $\mathcal{F}'(E,F)$ for $\mathcal{F}_{E'}(E,F)$; a similar convention is applied to $\mathcal{P}_G(E,F)$.

For any Riesz subspace G of $E^{\sim}$, $|\sigma|(E,G)$ will be the topology on E given by the seminorms $x \longmapsto g(|x|)$ with $g \in G_+$.

A subset M of a Riesz space E endowed with a locally solid topology is called almost order bounded (shortly: ao-bounded) if for every neighborhood W of 0 there is $x \in E_+$ such that $(|u| - x)_+ \in W$ whenever $u \in M$.

Let E,F be Riesz spaces such that F is endowed with a locally solid topology, let $\mathcal{M}$ be an upwards directed collection of subsets of E_+ and let L be a directed subspace of $L_r(E,F)$ such that U(M) is topologically bounded whenever $U \in L$ and $M \in \mathcal{M}$. The solid $\mathcal{M}$-topology on L has $\{\mathcal{V}_{M,W} \mid M \in \mathcal{M},$ W neighborhood of 0 in F$\}$ as a basis for 0, $\mathcal{V}_{M,W}$ being the set of those $U \in L$ for which there is $V \in L$ such that $U \in [-V,V]$ and $V(M) \subset W$; it is a locally solid topology. In particular, taking $\mathcal{M}$ to be the collection of all finite subsets of E_+ we obtain the solid strong topology on $L_r(E,F)$; taking $\mathcal{M}$ to be the collection of all ao-bounded subsets of E_+ (E being endowed with a locally solid topology) we obtain the solid ao-bounded topology on $L_r'(E,F)$.

Let E,F be Banach lattices. The regular norm on $L_r(E,F)$ is given by

$$\|U\|_r = \inf\{\|V\| \mid U \in [-V,V]\},$$

$\| \ \|$ being the usual operator norm.

If X is a set, τ a topology on X and M a subset of X we shall denote by $\overline{M}^{\tau}$ the τ-closure of M; if τ is understood we shall omit it.

For any compact space X, C(X) will be the Banach lattice of all continuous real-valued functions on X (the norm being the usual sup norm); e will denote the function identically one on X.

1_E will be the identity map on a set E. The restriction of a map $U:E \to F$ to a subset $M \subset E$ will be denoted by $U|M$.

The group of lemmas below will be used in § 3.

LEMMA 1.1. Let E be a Riesz space endowed with a separated locally solid topology, $F \subset E$ be a subspace and $C \subset F \cap E_+$ be a subset. Suppose that the following hold:

i) For every $x \in F$ there is $y \in C$ such that $|x| \leq y$.

ii) Whenever $x \in E_+$, $y \in C$ and $x \leq y$ it follows that $x \in \overline{C}$.

Then $\overline{F}$ is an order ideal and $(\overline{F})_+ = \overline{C}$.

PROOF. Clearly $\overline{C} \subset \overline{F} \cap E_+$. To prove that $\overline{F}$ is an order ideal and that $\overline{F} \cap E_+ \subset \overline{C}$ it suffices to show that whenever $x \in E$, $y \in \overline{F}$ and $|x| \leq |y|$ it follows that $x_+ \in \overline{C}$ and $x_- \in \overline{C}$. As $y \in \overline{F}$ there is a net $(y_\delta) \subset F$ such that $y_\delta \to y$. Let $x_\delta = (x \wedge |y_\delta|) \vee (-|y_\delta|)$; as $|y_\delta| \to |y|$ and $|x| \leq |y|$ we have $x_\delta \to x$. It suffices therefore to show that $(x_\delta)_+ \in \overline{C}$ and $(x_\delta)_- \in \overline{C}$. But $|x_\delta| \leq |y_\delta|$ and by i), there is $z_\delta \in C$ such that $|y_\delta| \leq z_\delta$; hence $(x_\delta)_+ \leq z_\delta$ and $(x_\delta)_- \leq z_\delta$ which by ii) implies the conclusion.

If E is a Riesz space endowed with two locally solid topologies τ_1, τ_2 such that τ_1 is stronger than τ_2 we denote by $E_{\tau_1\tau_2}$ the set of those $x \in E$ with the property that τ_1 and τ_2 coincide on $[-|x|, |x|]$.

LEMMA 1.2. $E_{\tau_1\tau_2}$ is a τ_1-closed order ideal in E.

PROOF. Observe first that $x \in E_{\tau_1\tau_2}$ iff whenever $(y_\delta) \subset [0, |x|]$ and $y_\delta \to 0$ for τ_2 then $y_\delta \to 0$ for τ_1.

The fact that $E_{\tau_1\tau_2}$ is an order ideal is an easy consequence of the above remark and the Riesz decomposition property.

We prove that $E_{\tau_1\tau_2}$ is τ_1-closed. Let $x \in \overline{E}_{\tau_1\tau_2}^{\tau_1}$ and let $(y_\delta) \subset [0, |x|]$, $y_\delta \to 0$ for τ_2; we have to prove that $y_\delta \to 0$ for τ_1. So let W be a τ_1-neighborhood of 0. There is a solid τ_1-neighborhood W' of 0 such that $W'+W' \subset W$. As $x \in \overline{E}_{\tau_1\tau_2}^{\tau_1}$ there is $x_0 \in E_{\tau_1\tau_2}$ such that $x - x_0 \in W'$. As $0 \leq y_\delta \wedge |x_0| \leq |x_0|$ and $x_0 \in E_{\tau_1\tau_2}$ we have $y_\delta \wedge |x_0| \to 0$ for τ_1; consequently, there is δ_0 such that $y_\delta \wedge |x_0| \in W'$ whenever $\delta \geq \delta_0$. The relation

$$y_\delta \leq |x - x_0| + y_\delta \wedge |x_0|$$

implies that $y_\delta \in W$ whenever $\delta \geq \delta_0$. Hence $y_\delta \to 0$ for τ_1.

LEMMA 1.3. Let E be a Riesz space endowed with two locally solid topologies τ_1, τ_2 such that τ_1 is stronger than τ_2. Let $F \subset E_{\tau_1\tau_2}$ be a subset such that $x \in \overline{F}^{\tau_1}$ whenever $x \in E$, $y \in F$ and $|x| \leq |y|$.

Then $\overline{F}^{\tau_1} = \overline{F}^{\tau_2} \cap E_{\tau_1\tau_2}$.

PROOF. As τ_1 is stronger than τ_2, $\overline{F}^{\tau_1} \subset \overline{F}^{\tau_2}$; as $E_{\tau_1\tau_2}$ is τ_1-closed by lemma 1.2 and $F \subset E_{\tau_1\tau_2}$ it follows that $\overline{F}^{\tau_1} \subset E_{\tau_1\tau_2}$. Hence $\overline{F}^{\tau_1} \subset \overline{F}^{\tau_2} \cap E_{\tau_1\tau_2}$.

Now let $x \in \overline{F}^{\tau_2} \cap E_{\tau_1\tau_2}$. There is a net $(x_\delta) \subset F$ such that $x_\delta \longrightarrow x$ for τ_2. Let $y_\delta = (x_\delta \wedge |x|) \vee (-|x|)$. Obviously $y_\delta \longrightarrow x$ for τ_2 and $|y_\delta| \leq |x|$; as $x \in E_{\tau_1\tau_2}$ we have $y_\delta \longrightarrow x$ for τ_1, so in order to prove that $x \in \overline{F}^{\tau_1}$ it suffices to establish that $y_\delta \in \overline{F}^{\tau_1}$. But this follows from the relation $|y_\delta| \leq |x_\delta|$ and our hypothesis on F.

LEMMA 1.4. Let E,F be Riesz spaces such that F is endowed with a locally solid topology, let L be a subspace of $L_r(E,F)$ such that the modulus of any $U \in L$ exists and let $\mathcal{M}$ be an upwards directed collection of subsets of E_+ containing all the finite subsets. Denote by τ_1 the solid $\mathcal{M}$-topology on L and by τ_2 the solid strong topology on L. Let I be the set of those $U \in L$ with the property that for every neighborhood W of 0 in E and every $M \in \mathcal{M}$ there is $x \in E_+$ such that $|U|((u-x)_+) \in W$ whenever $u \in M$. Then I is a τ_1-closed order ideal in the Riesz space L contained in $L_{\tau_1\tau_2}$.

PROOF. Clearly I is an order ideal. To see that it is τ_1-closed let $U \in \overline{I}^{\tau_1}$, let W be a neighborhood of 0 in F and let $M \in \mathcal{M}$. There is a solid neighborhood W' of 0 such that $W'+W' \subset W$. As $U \in \overline{I}^{\tau_1}$ there is $U_0 \in I$ such that $|U-U_0|(M) \subset W'$. As $U_0 \in I$ there is $x \in E_+$ such that $|U_0|((u-x)_+) \in W'$ whenever $u \in M$. It follows then from the relation

$$|U|((u-x)_+) \leq |U-U_0|((u-x)_+) + |U_0|((u-x)_+) \leq$$
$$\leq |U-U_0|(u) + |U_0|((u-x)_+)$$

that $|U|((u-x)_+) \in W$ whenever $u \in M$. Hence $U \in I$.

To see that $I \subset L_{\tau_1\tau_2}$ let $U \in I$ and let $(U_\delta) \subset L$ be such that $0 \leq U_\delta \leq |U|$ and $U_\delta \longrightarrow 0$ for τ_2. Let W be a neighborhood of 0 in F and let $M \in \mathcal{M}$. There is a solid neighborhood W' of 0 such that $W'+W' \subset W$. As $U \in I$ there is $x \in E_+$ such that $|U|((u-x)_+) \in W'$ whenever $u \in M$; as $U_\delta \longrightarrow 0$ for τ_2 there is δ_0 such that $|U_\delta|(x) \in W'$ whenever $\delta \geq \delta_0$. Then the relation

$$|U_\delta|(u) \leq |U_\delta|(x) + |U_\delta|((u-x)_+) \leq |U_\delta|(x) + |U|((u-x)_+)$$

implies that $|U_\delta|(M) \subset W$ whenever $\delta \geq \delta_0$. Hence $(U_\delta) \longrightarrow 0$ for τ_1.

Lemmas 1.1 and 1.3 will be applied in the case when $F = \mathcal{F}_G(E_1,E_2)$ and $C = \mathcal{P}_G(E_1,E_2)$ (assuming that $\mathcal{F}_G(E_1,E_2)$ is contained in a Riesz space of operators); obviously, condition i) of lemma 1.1 is satisfied in this situation. To check condition ii) it suffices to prove it only for the operators in $\mathcal{P}_G(E_1,E_2)$ of the form $f \otimes y$ $(f \in G_+,\ y \in (E_2)_+)$.

2. oru-compact operators

A subset M of an Archimedean Riesz space E is called ru-totally bounded if M is contained in a principal order ideal E_x and is totally bounded for $\| \ \|_x$.

A linear operator U acting between the Riesz spaces E,F (F Archimedean) is called oru-compact if it takes order bounded subsets into ru-totally bounded subsets. The linear space of all oru-compact operators from E to F will be denoted by $L_{oru}(E,F)$; clearly $\mathcal{F}(E,F) \subset L_{oru}(E,F)$ and $RUS \in L_{oru}(E_0,F_0)$ whenever $U \in L_{oru}(E,F)$ and $S: E_0 \longrightarrow E$, $R: F \longrightarrow F_0$ are order bounded operators.

Using theorem 2.3 in [1] we see that $1_E \in L_{oru}(E,E)$ for each discrete Banach lattice E with order continuous norm; consequently, every operator $U: E_0 \longrightarrow E_1$ representable as a product $U_1 U_0$ of order bounded operators $U_0: E_0 \longrightarrow E$, $U_1: E \longrightarrow E_1$ with E as above is also oru-compact. In particular, every nuclear operator between two Banach lattices is oru-compact.

PROPOSITION 2.1. For any $x \in E_+$ and any $U \in L_{oru}(E,F)$, the set $M_x = \{ \bigvee_{i=1}^{n} U(y_i) \mid n \geq 1,\ y_i \in [-x,x] \}$ is ru-totally bounded. If F is ru-complete then $L_{oru}(E,F)$ is a Riesz space for the order induced by $L_r(E,F)$; the modulus of any $U \in L_{oru}(E,F)$ is given by

$$|U|(x) = \sup \{ U(y) \mid y \in [-x,x] \} \qquad (1)$$

for $x \in E_+$. In fact, $|U|(x)$ is the limit of the upwards directed set M_x for some norm $\| \ \|_y$.

PROOF. The inequality

$$\Big| \bigvee_{i=1}^{n} u_i - \bigvee_{i=1}^{n} v_i \Big| \leq \bigvee_{i=1}^{n} |u_i - v_i|$$

shows that $\{ \bigvee_{i=1}^{n} z_i \mid n \geq 1,\ z_i \in M \}$ is an ru-totally bounded set whenever M is. When F is ru-complete, the existence of the supremum in the right side of (1) follows from the fact that M_x is a totally bounded upwards directed subset in a principal ideal F_y complete for $\| \ \|_y$. Using the Riesz decomposition property one can see that (1) indeed gives the modulus of U.

COROLLARY 2.1. Let E,F,G be Riesz spaces with F,G ru-complete and let $T: F \longrightarrow G$ be a Riesz homomorphism. Then $|TU| = T|U|$ for every $U \in L_{oru}(E,F)$.

COROLLARY 2.2. If E,G are Riesz spaces, F is a Riesz subspace of G and F,G are ru-complete then $L_{oru}(E,F)$ is a Riesz subspace of $L_{oru}(E,G)$. If E, F are Riesz spaces with F order complete then $L_{oru}(E,F)$ is a Riesz subspace of $L_r(E,F)$.

We shall use the notations $L^{\times}_{oru}(E,F)$ for $L_{oru}(E,F) \cap L^{\times}_r(E,F)$, $L'_{oru}(E,F)$ for $L_{oru}(E,F) \cap L'_r(E,F)$, $L'^{\times}_{oru}(E,F)$ for $L'_{oru}(E,F) \cap L^{\times}_{oru}(E,F)$; clearly, these are order ideals in the Riesz space $L_{oru}(E,F)$ whenever F is ru-complete.

PROPOSITION 2.2. Let E,F be Riesz spaces endowed with complete metrisabe an order bounded operator such that $U|E_0 \in L_{oru}(E_0,F)$. Then $U \in L_{oru}(E,F)$.

PROOF. Let $x \in E_+$; we have to prove that $U([-x,x])$ is ru-totally bounded. There is a sequence $(x_n) \subset E_0$ such that $x_n \to x$; replacing, if necessary, x_n by $(x_n)_+ \wedge x$ we may assume that $0 \leq x_n \leq x$. We may also assume that $\sum_{n=1}^{\infty} \rho(2^n(x - x_n)) < \infty$ where ρ is a Riesz pseudonorm defining the topology of E; hence $x - x_n \leq 2^{-n}u$ where $u = \sum_{n=1}^{\infty} 2^n(x - x_n)$. For each n there is $y_n \in F_+$ such that $U([-x_n,x_n])$ is totally bounded in F_{y_n}. Choose $\alpha_n > 0$ such that $\sum_{n=1}^{\infty} \sigma(\alpha_n y_n) < \infty$ (σ being a Riesz pseudonorm defining the topology of F) and let $y = v \vee \sum_{n=1}^{\infty} \alpha_n y_n$ where v is an upper bound for $U([-x \vee u, x \vee u])$. Obviously, $U([-x,x]) \subset F_y$ and $U([-x_n,x_n])$ is totally bounded in F_y for every n. Take any $z \in [-x,x]$; then $z_n = (z \wedge x_n) \vee (-x_n) \in [-x_n,x_n]$ and $|z - z_n| \leq x - x_n \leq 2^{-n}u$ which implies $|U(z) - U(z_n)| \leq 2^{-n}y$. This shows that any finite ε-net for $U([-x_n,x_n])$ will be an $(\varepsilon + 2^{-n})$-net for $U([-x,x])$; hence $U([-x,x])$ is totally bounded in F_y.

<u>PROPOSITION 2.3.</u> Let E,F be Riesz spaces such that F is ru-complete and endowed with a locally solid topology. Let also F_0 be a closed ideal in F. Then any $U \in L_{oru}(E,F)$ with the property that $U(E) \subset F_0$ belongs to $L_{oru}(E,F_0)$.

PROOF. Let $x \in E_+$; we have to prove that $U([-x,x])$ is ru-totally bounded in F_0. There is $y \in F_+$ such that $U([-x,x])$ is totally bounded in F_y; clearly $U([-x,x]) \subset F_y \cap F_0$ and $F_y \cap F_0$ is a closed ideal in F_y. By proposition 1.9.2 in [3] there is a sequence $(x_n) \subset F_y \cap F_0$ such that $\|x_n\|_y \to 0$ and $U([-x,x]) \subset \overline{co}\{x_n \mid n \geq 1\}$ (the closure being taken with respect to $\| \ \|_y$). Let $z_n = \|x_n\|_y^{-1/2} x_n$. As $\|z_n\|_y \to 0$, the relation

$$0 \leq \bigvee_{i=1}^{n+p} |z_i| - \bigvee_{i=1}^{n} |z_i| \leq \bigvee_{i=n+1}^{n+p} |z_i|$$

shows that $(\bigvee_{i=1}^{n} |z_i|)_{n \geq 1}$ is a Cauchy sequence for $\| \ \|_y$; hence it is convergent to $z \in F_y \cap F_0$. Consequently, we have $|x_n| = \|x_n\|_y^{1/2} |z_n| \leq \|x_n\|_y^{1/2} z$ which implies that $x_n \in F_z$ and $\|x_n\|_z \to 0$. Consider now the space $\mathbb{R}^{\mathbb{N}}$ with the product topology and let $K = \{(\alpha_n)_{n \geq 1} \mid \alpha_n \geq 0, \sum_{n=1}^{\infty} \alpha_n \leq 1\}$; K is a compact convex subset of $\mathbb{R}^{\mathbb{N}}$. As $\|x_n\|_y \to 0$, the map $(\alpha_n) \mapsto \sum_{n=1}^{\infty} \alpha_n x_n$ from K into F_y is continuous; this implies that $\overline{co}\{x_n \mid n \geq 1\} \subset \{\sum_{n=1}^{\infty} \alpha_n x_n \mid (\alpha_n) \in K\}$. Therefore, for every $u \in [-x,x]$ we have $U(u) = \sum_{n=1}^{\infty} \alpha_n x_n$ (convergence in F_y) for some $(\alpha_n) \in K$. As $\|x_n\|_z \to 0$, the series $\sum_{n=1}^{\infty} \alpha_n x_n$ is also convergent is F_z; as the inclusion $F_z \to F_y$ is continuous we still have $U(u) = \sum_{n=1}^{\infty} \alpha_n x_n$ in F_z. Consequently, $U([-x,x])$ is contained in the compact subset $\overline{co}\{x_n \mid n \geq 1\}$ of F_z, hence it is ru-totally bounded in F_0.

The next results consider the completeness of $L_{oru}(E,F)$ for certain topologies.

THEOREM 2.1. Let E,F be Riesz spaces such that F is endowed with a complete metrizable locally solid topology. Then $L_{oru}(E,F)$ is complete for the solid strong topology.

PROOF. Let ρ be a Riesz pseudonorm defining the topology of F and let $(U_\delta)\subset L_{oru}(E,F)$ be a Cauchy net. The completeness of F implies the existence of a linear map $U:E\longrightarrow F$ such that $U(x)=\lim_\delta U_\delta(x)$ for every $x\in E$. To see that $U\in L_{oru}(E,F)$ consider an $x\in E_+$. There is an increasing sequence (δ_n) of indices such that $\rho(|U_\delta-U_{\delta_n}|(x))\leq 2^{-2n}$ whenever $n\geq 1$ and $\delta\geq\delta_n$. Put $V_n=U_{\delta_{n+1}}-U_{\delta_n}$. As $V_n\in L_{oru}(E,F)$ there is $y_n\in F_+$ such that $V_n([-x,x])$ is totally bounded in F_{y_n}. Choose $\alpha_n>0$ such that $\sum_{n=1}^{\infty}\rho(\alpha_n y_n)<\infty$ and let

$$y=\Big(\sum_{n=1}^{\infty}\alpha_n y_n\Big)\vee\Big(\sum_{n=1}^{\infty}2^n|V_n|(x)\Big).$$

As $V_n([-x,x])$ is totally bounded in F_y and $\sup_{z\in[-x,x]}\|V_n(z)\|_y\leq 2^{-n}$ it follows that the series $U_{\delta_1}|E_x+\sum_{n=1}^{\infty}V_n|E_x$ converges uniformly on order intervals to an oru-compact operator $V:E_x\longrightarrow F_y$. As $V(z)=\lim_{n\to\infty}U_{\delta_n}(z)$ for every $z\in E_x$, it follows by the choice of the sequence (δ_n) that $U(z)=V(z)$ for $z\in E_x$. This shows that $U([-x,x])$ is ru-totally bounded; as x is arbitrary, $U\in L_{oru}(E,F)$.

To see that $U_\delta\longrightarrow U$ for the solid strong topology, consider an $x\in E_+$ and use the same notations as above. From the relation

$$U(z)-U_{\delta_n}(z)=\sum_{i=n}^{\infty}V_i(z)\leq\sum_{i=n}^{\infty}|V_i|(x)\ ,\ z\in[-x,x]$$

we infer that (using proposition 2.1)

$$|U-U_{\delta_n}|(x)\leq\sum_{i=n}^{\infty}|V_i|(x)\ ;$$

hence

$$\rho(|U-U_{\delta_n}|(x))\leq\sum_{i=n}^{\infty}\rho(|V_i|(x)).$$

As (U_δ) is a Cauchy net, this is sufficient for the convergence.

COROLLARY 2.3. Let E,F be Riesz spaces such that F is endowed with a complete metrizable locally solid topology. Then $L_{oru}(E,F)$ is closed in $L_r(E,F)$ for the solid strong topology.

THEOREM 2.2. If E,F are Banach lattices then $L_{oru}(E,F)$ is a Banach lattice for the regular norm.

PROOF. Follows from the above theorem and the well known-fact that $L_r(E,F)$ is complete for the regular norm.

THEOREM 2.3. Let E,F be Riesz spaces such that F is endowed with a complete metrizable locally solid topology. Then $L^{\times}_{oru}(E,F)$ is closed in $L_{oru}(E,F)$ for the solid strong topology.

PROOF. Let $U \in \overline{L^{\times}_{oru}(E,F)}$, let (x_α) be a net decreasing to 0 and let y be a lower bound for $(|U|(x_\alpha))$. We may assume that $x_\alpha \leq x$ for some $x \in E$. There is a sequence $(U_n) \subset L^{\times}_{oru}(E,F)$ such that $\sum_{n=1}^{\infty} \rho(2^n|U - U_n|(x)) < \infty$ (ρ being a Riesz pseudonorm defining the topology of F). Let $z = \sum_{n=1}^{\infty} 2^n|U - U_n|(x)$. We have

$$y \leq |U|(x_\alpha) \leq |U - U_n|(x) + |U_n|(x_\alpha) \leq 2^{-n}z + |U_n|(x_\alpha).$$

As $|U_n|(x_\alpha) \downarrow 0$ it follows that $y \leq 2^{-n}z$; as F is Archimedean we find that $y \leq 0$.

3. The approximation of oru-compact operators by finite-rank operators

Our plan is the following: first we give the main technical tool (theorem 3.1) which provides the basis for all approximation results below.

We characterize them (theorem 3.3) the band generated by $\mathcal{F}_G(E,F)$ in $L_{oru}(E,F)$, G being a band in $E^{\sim}$; as a corollary we obtain that, under quite general hypothesis, the band generated by $\mathcal{F}(E,F)$ in $L_{oru}(E,F)$ is the whole $L_{oru}(E,F)$.

The leading idea of the remaining results in the section is as follows: consider an order ideal L in $L_{oru}(E,F)$, a subspace of L of the form $\mathcal{F}_G(E,F)$ where G is an order ideal in $E^{\sim}$ and a topology τ on L. Our task will be to prove that $\overline{\mathcal{F}_G(E,F)}^{\tau}$ is an order ideal in L and that $(\overline{\mathcal{F}_G(E,F)}^{\tau})_+ = \overline{\mathcal{P}_G(E,F)}^{\tau}$ for certain L and τ. We shall also encounter situations when we are given two topologies τ_1, τ_2 on L such that τ_1 is stronger than τ_2; in such a case either we prove that $\overline{\mathcal{F}_G(E,F)}^{\tau_1} = \overline{\mathcal{F}_G(E,F)}^{\tau_2}$ or we characterizr those $U \in \overline{\mathcal{F}_G(E,F)}^{\tau_2}$ which also belong to $\overline{\mathcal{F}_G(E,F)}^{\tau_1}$.

As corollaries to all these results, we determine the closure of the subspace of finite-rank operators in various spaces of oru-compact operators with respect to various topologies.

<u>THEOREM 3.1.</u> Let E,F be Riesz spaces with F ru-complete and let $U \in L_{oru}(E,F)$, $x \in E_+$, $y \in F_+$ be such that $U([-x,x])$ is totally bounded in F_y. Suppose also that at least one of the following conditions holds:

i) E admits x as a strong order unit.

ii) $F^{\sim}$ separates F.

iii) There are $f \in E_+$ and $z \in (F_y)_+$ such that $|U| \leq f \otimes z$.

Then for any $\varepsilon > 0$ there is $V \in \mathcal{F}(E,F)$ such that $|U - V|(x) \leq \varepsilon y$; in case iii) we also have $V \in \mathcal{F}_{E^{\sim}_f}(E,F)$ and $|V| \leq f \otimes (z + \varepsilon y)$.

If in addition we have in cases i) and ii) $U \in L^{\times}_{oru}(E,F)$ then $V \in \mathcal{F}^{\times}(E,F)$; if moreover E and F are endowed with locally solid topologies, F' separates F and $U \in L'_{oru}(E,F)$ than $V \in \mathcal{F}'(E,F)$.

In all the statements above about V one may replace the letter $\mathcal{F}$ by

$\mathcal{P}$ whenever $U\in L_{oru}(E,F)_+$.

PROOF. As F is ru-complete there is a compact space K and an order isomorphism $T:C(K)\longrightarrow F_y$ such that $T(e)=y$. Let $U_x\in L_{oru}(E_x,C(K))$ be given by $U_x=T^{-1}UJ_x$ where $J_x:E_x\longrightarrow E$ is the inclusion map. Consider the topology on $E_x^{\sim}$ given by the norm $f\longmapsto |f|(x)$. The map $\Phi:K\longrightarrow E_x^{\sim}$ given by $\Phi(s)=$ $=U_x'(\delta_s)$ is continuous (δ_s being the unit mass supported by $s\in K$). Hence there is an open covering $(G_i)_{1\leq i\leq m}$ of K such that $|\Phi(s)-\Phi(t)|(x)\leq\varepsilon$ whenever $s,t\in G_i$ ($1\leq i\leq m$); in case iii), we also assume that $|\varphi(s)-\varphi(t)|\leq\varepsilon$ whenever $s,t\in G_i$, where $\varphi=T^{-1}(z)$. Let $(\varphi_i)_{1\leq i\leq m}$ be a continuous partition of unit subordonated to the covering (G_i) (that is, $\varphi_i\in C(K)_+$, supp $\varphi_i\subset G_i$ and $\sum_{i=1}^{m}\varphi_i=e$). In case i), define $h_i\in E^{\sim}$ by $h_i=\Phi(s_i)$ where s_i is any point in G_i. In case ii), if $\varphi_i=0$ put $h_i=0$; if not, there is $f_i\in F_+^{\sim}$ such that $f_i(T(\varphi_i))>0$ (as $F^{\sim}$ separates F). Let $g_i\in F_+^{\sim}$ be given by

$$g_i(u)=\alpha_i\sup_{m\geq 1} f_i(u\wedge mT(\varphi_i)),\quad u\in F_+$$

where $\alpha_i>0$ is choosen so that $g_i(y)=1$. Define $h_i\in E^{\sim}$ by $h_i=U'(g_i)$.

In case iii), observe that the hypothesis implies that $|\Phi(s)|\leq$ $\leq\varphi(s)J_x'(f)$ for every $s\in K$. This is equivalent to the relation

$$\langle\Phi(s),u\rangle\leq\varphi(s)f(|u|)$$

for every $u\in E_x$. Let f_s be a Hahn-Banach extension of $\Phi(s)$ such that $f_s(u)\leq$ $\leq\varphi(s)f(|u|)$ for every $u\in E$. This implies that $f_s\in E^{\sim}$ and $|f_s|\leq\varphi(s)f$. In the case $U\geq 0$ one may choose f_s in $E_+^{\sim}$ by replacing it, if necessary, with $(f_s)_+$. Define h_i to be f_{s_i} where s_i is any point in G_i.

Now in all three cases define $V\in\mathcal{F}(E,F)$ by $V=\sum_{i=1}^{m}h_i\otimes T(\varphi_i)$. We prove that $|U-V|(x)\leq\varepsilon y$. To this purpose, let $V_x\in L_{oru}(E_x,C(K))$ be given by $V_x=T^{-1}VJ_x$. Obviously,

$$\| |U-V|(x)\|_y=\sup_{z\in[-x,x]}\|(U_x-V_x)(z)\|.$$

For any $z\in[-x,x]$ we have

$$\|(U_x-V_x)(z)\|=\sup_{s\in K}|\langle\delta_s,U_x(z)-V_x(z)\rangle|=$$

$$=\sup_{s\in K}\left|\left\langle\Phi(s)-\sum_{i=1}^{m}\varphi_i(s)U_x'(k_i),\ z\right\rangle\right|$$

where $k_i=\delta_{s_i}$ in cases i), iii) and $k_i=(J_yT)'(g_i)$ in case ii) ($J_y:F_y\longrightarrow F$ being the inclusion map). To show that the later expression is $\leq\varepsilon$, it suffices, by the properties of the partition of the unit, to prove that $|\langle\Phi(s)-U_x'(k_i),\ z\rangle|\leq\varepsilon$ whenever $s\in G_i$ and $\varphi_i\neq 0$. This is obvious in cases i) and iii); in case ii) we have

$$\langle\Phi(s)-U_x'(k_i),\ z\rangle=\langle k_i,\langle\Phi(s),z\rangle e-U_x(z)\rangle$$

as $\langle k_i, e\rangle = g_i(y) = 1$. Let $\psi_i \in C(K)$ be such that $0 \leq \psi_i \leq e$, supp $\psi_i \subset G_i$ and $\psi_i(s) = 1$ for $s \in$ supp φ_i. As $(e - \psi_i)\wedge \varphi_i = 0$ it follows that $\langle k_i, e - \psi_i\rangle = 0$ be the definition of g_i. Hence

$$\langle k_i, \psi\rangle = \langle k_i, \psi_i\psi\rangle + \langle k_i, (e - \psi_i)\psi\rangle = \langle k_i, \psi_i\psi\rangle$$

for any $\psi \in C(K)$. In particular,

$$\langle \Phi(s) - U_x'(k_i), z\rangle = \langle k_i, \psi_i(\langle \Phi(s), z\rangle e - U_x(z))\rangle.$$

Now observe that

$$\begin{aligned}\|\psi_i(\langle\Phi(s), z\rangle e - U_x(z))\| &\leq \sup_{t\in G_i} |\langle \delta_t, \langle \Phi(s), z\rangle e - U_x(z)\rangle| = \\ &= \sup_{t\in G_i} |\langle \Phi(s), z\rangle - \langle \Phi(t), z\rangle| \leq \\ &\leq \sup_{t\in G_i} |\Phi(s) - \Phi(t)|(x) \leq \varepsilon.\end{aligned}$$

Recalling that $\|k_i\| = 1$ we obtain $|\langle \Phi(s) - U_x'(k_i), z\rangle| \leq \varepsilon$ which is the desired conclusion.

In the case iii), remark that

$$(1) \qquad |h_i| \leq (\varphi(s) + \varepsilon) f$$

whenever $s \in G_i$. Indeed,

$$|h_i| = |f_{s_i}| \leq \varphi(s_i) f \leq (\varphi(s) + \varepsilon) f.$$

From (1) we infer that

$$\varphi_i(s)|h_i| \leq \varphi_i(s)(\varphi(s) + \varepsilon) f$$

for any $s \in K$, which is equivalent to

$$|h_i| \otimes \varphi_i \leq f \otimes \varphi_i(\varphi + \varepsilon e).$$

Applying T we obtain

$$|h_i| \otimes T(\varphi_i) \leq f \otimes T(\varphi_i(\varphi + \varepsilon e)).$$

Hence

$$\begin{aligned}|V| &\leq \sum_{i=1}^{m} |h_i| \otimes T(\varphi_i) \leq f \otimes \sum_{i=1}^{m} T(\varphi_i(\varphi + \varepsilon e)) = f \otimes T(\varphi + \varepsilon e) = \\ &= f \otimes (x + \varepsilon y).\end{aligned}$$

Finally, suppose that $U \in L^{\times}_{cru}(E,F)$ (in cases i) and ii)). That $V \in \mathcal{F}^{\times}(E,F)$ follows from the fact that whenever $U \in L^{\times}_{cru}(E,F)$ and $g \in F^{\sim}$ then $U'(g) \in E^{\times}$; indeed, $|U'(g)| \leq |U|'(|g|)$ and $|U|$ takes decreasing to 0 nets into ru-convergent to 0 nets.

The remaining assertions in the theorem are easily obtained.

THEOREM 3.2. Let E,F be Riesz spaces such that F

= 0; as we also have $|V'| \wedge |V''| = 0$, the relation

$$|U - V'| \leq |U - V'| + |V''| = |U - V' - V''| = |U - V|$$

implies that $|U - V'|(x) \leq \varepsilon y$, which concludes the proof.

COROLLARY 3.2. Let E,F be Riesz spaces such that F is ru-complete and at least one of the conditions in the statement of theorem 3.2 holds. Then the band generated by $\mathcal{F}(E,F)$ in $L_{oru}(E,F)$ is equal to $L_{oru}(E,F)$ and the band generated by $\mathcal{F}^{\times}(E,F)$ in $L_{oru}(E,F)$ is equal to $L^{\times}_{oru}(E,F)$.

PROOF. Follows from theorems 3.1, 3.3, 2.3 (the later will be applied to a principal ideal of F, endowed with its norm $\| \|_y$).

For E,F Riesz spaces with F order complete we shall denote by $\mathcal{J}(E,F)$ the band generated by $\mathcal{F}(E,F)$ in $L_r(E,F)$ and by $\mathcal{J}^{\times}(E,F)$, the band generated by $\mathcal{F}^{\times}(E,F)$ in $L_r(E,F)$.

THEOREM 3.4. Let E,F be Riesz spaces such that F is order complete and at least one of the conditions in the statement of theorem 3.2 holds. Then $L_{oru}(E,F) \subset \mathcal{J}(E,F)$ and $L^{\times}_{oru}(E,F) \subset \mathcal{J}^{\times}(E,F)$.

PROOF. It suffices to show that $\mathcal{J}(E,F)^{\perp} \subset L_{oru}(E,F)^{\perp}$ and $\mathcal{J}^{\times}(E,F)^{\perp} \subset L^{\times}_{oru}(E,F)^{\perp}$. So let $U \in \mathcal{J}(E,F)^{\perp}$ (respectively $U \in \mathcal{J}^{\times}(E,F)^{\perp}$) and let $V \in L_{oru}(E,F)$ (respectively $V \in L^{\times}_{oru}(E,F)$), $x \in E_+$ and $\varepsilon > 0$. There is $y \in F_+$ such that $V([-x,x])$ is totally bounded in F_y. By theorem 3.1 there is $W \in \mathcal{F}(E,F)$ (respectively $W \in \mathcal{F}^{\times}(E,F)$) such that $|V - W|(x) \leq \varepsilon y$. Using the relations $|U| \wedge |W| = 0$ and

$$|U| \wedge |V| = |U| \wedge |W| + |U| \wedge |V - W| \leq |V - W|$$

we obtain $(|U| \wedge |V|)(x) \leq \varepsilon y$; as ε and x are arbitrary, $|U| \wedge |V| = 0$.

THEOREM 3.5. Let E,F be Riesz spaces such that F is ru-complete and endowed with a separated locally solid topology. Consider the solid strong topology on $L_{oru}(E,F)$. Then $\overline{\mathcal{F}_G(E,F)}$ is an order ideal in $L_{oru}(E,F)$ and $(\overline{\mathcal{F}_G(E,F)})_+ = \mathcal{P}_G(E,F)$ whenever G is an order ideal in $E^{\sim}$.

PROOF. By lemma 1.1 it suffices to show that whenever $U \in L_{oru}(E,F)_+$, $f \in G_+$ and $y \in F_+$ are such that $U \leq f \otimes y$ then $U \in \overline{\mathcal{P}_G(E,F)}$. So let $x \in E_+$ and let W be a solid neighborhood of 0 in F. There is $z \in F_+$ such that $y \in F_z$ and $U([-x,x])$ is totally bounded in F_z. There is also $\varepsilon > 0$ such that $\varepsilon z \in W$. By theorem 3.1 we can find $V \in \mathcal{P}_G(E,F)$ verifying $|U - V|(x) \leq \varepsilon z$. Hence $|U - V|(x) \in W$, which concludes the proof.

For the next results we shall need the following lemma.

LEMMA 3.1. Let E,F be Riesz spaces such that F is ru-complete and endowed with a locally solid topology. Let $U \in L_{oru}(E,F)_+$, $f \in E^{\sim}_+$ and $y \in F_+$ be such that $U \leq f \otimes y$. Consider the solid ao-bounded topology on $L'_{oru}(E,F)$, the topology on E being $|\sigma|(E,E^{\sim}_f)$. Then $U \in \overline{\mathcal{P}_{E^{\sim}_f}(E,F)}$.

PROOF. Let W be a neighborhood of 0 in F and let M be an ao-bounded subset of E_+. There is an open solid neighborhood W' of 0 such that W' + W' + W' ⊂

$\subset W$. We can find $\varepsilon > 0$ such that $\varepsilon y \in W'$ and $x \in E_+$ such that $f((u - x)_+) \leq \varepsilon$ whenever $u \in M$. There is also $z \in F_+$ with the property that $y \in F_z$ and $U([-x,x])$ is totally bounded in F_z. Let $\eta > 0$ be such that $\eta z \in W'$ and $\varepsilon(y + \eta z) \in W'$. By theorem 3.1 there is $V \in \mathcal{P}_{E^\sim_f}(E,F)$ verifying $|U - V|(x) \leq \eta z$ and $V \leq f \otimes (y + \eta z)$. Then we have for any $u \in M$

$$\begin{aligned}|U - V|(u) &\leq |U - V|(x) + |U - V|((u - x)_+) \leq \\ &\leq \eta z + |U|((u - x)_+) + |V|((u - x)_+) \leq \\ &\leq \eta z + f((u - x)_+)y + f((u - x)_+)(y + \eta z) \leq \\ &\leq \eta z + \varepsilon y + \varepsilon(y + \eta z) \in W' + W' + W'.\end{aligned}$$

Hence $|U - V|(M) \subset W$ and the proof is complete.

THEOREM 3.6. Let E,F be Riesz spaces endowed with separated locally solid topologies such that F is ru-complete. Let τ_1 be the solid ao-bounded topology on $L'_{oru}(E,F)$ and let τ_2 be the solid strong topology on $L'_{oru}(E,F)$. Then $\overline{\mathcal{F}_G(E,F)}^{\tau_1} = \overline{\mathcal{F}_G(E,F)}^{\tau_2}$ and $\overline{\mathcal{P}_G(E,F)}^{\tau_1} = \overline{\mathcal{P}_G(E,F)}^{\tau_2}$ whenever G is an order ideal in E'.

PROOF. An application of lemma 3.1 shows that $U \in \overline{\mathcal{P}_G(E,F)}^{\tau_1}$ whenever $U \in L'_{oru}(E,F)_+$, $f \in G_+$ and $y \in F_+$ are such that $U \leq f \otimes y$. By lemma 1.4, $L'_{oru}(E,F) = L'_{oru}(E,F)_{\tau_1 \tau_2}$; therefore, $\overline{\mathcal{F}_G(E,F)}^{\tau_1} = \overline{\mathcal{F}_G(E,F)}^{\tau_2}$ by lemma 1.3. Finally, lemma 1.1 gives

$$\overline{\mathcal{P}_G(E,F)}^{\tau_1} = (\overline{\mathcal{F}_G(E,F)}^{\tau_1})_+ = (\overline{\mathcal{F}_G(E,F)}^{\tau_2})_+ = \overline{\mathcal{P}_G(E,F)}^{\tau_2}.$$

THEOREM 3.7. Let E,F be Riesz spaces endowed with separated locally solid topologies such that F is ru-complete and F' separates F. Consider the solid ao-bounded topology on $L'_{oru}(E,F)$. Then

$$L'_{oru}(E,F) = \overline{\mathcal{F}'(E,F)},$$
$$L'_{oru}(E,F)_+ = \overline{\mathcal{P}'(E,F)}.$$

A similar statement holds for $L'^{\times}_{oru}(E,F)$.

PROOF. Let τ_1 be the solid ao-bounded topology on $L'_{oru}(E,F)$ and let τ_2 be the solid strong topology on $L'_{oru}(E,F)$. By theorem 3.6,

$$\overline{\mathcal{F}'(E,F)}^{\tau_1} = \overline{\mathcal{F}'(E,F)}^{\tau_2},$$
$$\overline{\mathcal{P}'(E,F)}^{\tau_1} = \overline{\mathcal{P}'(E,F)}^{\tau_2}.$$

Now theorem 3.1 shows that $L'_{oru}(E,F) = \overline{\mathcal{F}'(E,F)}^{\tau_2}$ and $L'_{oru}(E,F)_+ = \overline{\mathcal{P}'(E,F)}^{\tau_2}$. A similar reasoning is employed for $L'^{\times}_{oru}(E,F)$.

COROLLARY 3.3. Let E,F be Riesz spaces endowed with complete metrizable locally solid topologies. Suppose that at least one of the conditions in the statement of theorem 3.2 holds. Let τ_1 be the solid ao-bounded topology on $L_r(E,F)$ and let τ_2 be the solid strong topology on $L_r(E,F)$. Then

$$L_{oru}(E,F) = \overline{\mathcal{F}(E,F)}^{\tau_1} = \overline{\mathcal{F}(E,F)}^{\tau_2},$$

$$L_{oru}(E,F)_+ = \overline{\mathcal{P}(E,F)}^{\tau_1} = \overline{\mathcal{P}(E,F)}^{\tau_2}$$

(the closures being taken in $L_r(E,F)$). A similar statement holds for $L^{\times}_{oru}(E,F)$.

PROOF. Observe that $L_{oru}(E,F) = L'_{oru}(E,F)$, $E' = E^{\sim}$, $F' = F^{\sim}$ and apply the preceding results.

One of the main results in [5] (theorem 4.4) asserts that whenever E,F are Banach lattices such that F has order continuous norm, then

$$\mathcal{J}(E,F) = \overline{\mathcal{F}(E,F)}^{\tau_1} = \overline{\mathcal{F}(E,F)}^{\tau_2},$$

$$\mathcal{J}(E,F)_+ = \overline{\mathcal{P}(E,F)}^{\tau_1} = \overline{\mathcal{P}(E,F)}^{\tau_2},$$

τ_1 and τ_2 being defined above; in particular, it is seen that $L_{oru}(E,F) = \mathcal{J}(E,F)$ in this case. Corollary 3.3 is an extension of this result in the sense that it states that the equalities

$$\overline{\mathcal{F}(E,F)}^{\tau_1} = \overline{\mathcal{F}(E,F)}^{\tau_2},$$

$$\overline{\mathcal{P}(E,F)}^{\tau_1} = \overline{\mathcal{P}(E,F)}^{\tau_2} = (\overline{\mathcal{F}(E,F)}^{\tau_1})_+$$

still hold evem if F has not order continuous norm; however, if F is order complete but has not order continuous norm, the equality $\mathcal{J}(E,F) = \overline{\mathcal{F}(E,F)}^{\tau_1}$ is no longer true. Indeed, there is in this case an order bounded disjoint sequence $(x_n) \subset F_+$ such that $\liminf_{n\to\infty} \|x_n\| > 0$. Take $E = l_\infty$ and define $U \in L_r(E,F)$ by

$$U((\alpha_n)_{n\geq 1}) = \sup_{n\geq 1} \alpha_n x_n$$

for $(\alpha_n)_{n\geq 1} \in (l_\infty)_+$. Clearly $U \in \mathcal{J}(E,F)$ but $U \notin \overline{\mathcal{F}(E,F)}^{\tau_1}$ as $\overline{\mathcal{F}(E,F)}^{\tau_1} = L_{oru}(E,F)$ and $U \notin L_{oru}(E,F)$.

<u>THEOREM 3.8.</u> Let E,F be Riesz spaces endowed with separated locally solid topologies (F ru-complete) and let G be an order ideal in E'. Let τ_1 be the solid $\mathcal{M}$-topology on $L'_{oru}(E,F)$, $\mathcal{M}$ being the collection of those subsets of E_+ which are bounded for the topology of E and ao-bounded for $|\sigma|(E,G)$; let also τ_2 be the solid strong topology on $L'_{oru}(E,F)$. Then $\overline{\mathcal{F}_G(E,F)}^{\tau_1}$ is the order ideal of those $U \in \overline{\mathcal{F}_G(E,F)}^{\tau_2}$ with the following property: for every neighborhood W of 0 in F and every $M \in \mathcal{M}$ there is $x \in E_+$ such that $|U|((u-x)_+) \in W$ whenever $u \in M$; we also have $(\overline{\mathcal{F}_G(E,F)}^{\tau_1})_+ = \overline{\mathcal{P}_G(E,F)}^{\tau_1}$.

PROOF. By lemma 3.1 we have $U \in \overline{\mathcal{P}_G(E,F)}^{\tau_1}$ whenever $U \in L'_{oru}(E,F)_+$, $f \in G_+$ and $y \in F_+$ are such that $U \leq f \otimes y$. Therefore, lemma 1.1 implies that $\overline{\mathcal{F}_G(E,F)}^{\tau_1}$ is an order ideal and $(\overline{\mathcal{F}_G(E,F)}^{\tau_1})_+ = \overline{\mathcal{P}_G(E,F)}^{\tau_1}$. Let I be the order ideal of those $U \in L'_{oru}(E,F)$ with the property in the statement of the theorem. By lemma 1.4, I is a τ_1-closed ideal contained in $L'_{oru}(E,F)_{\tau_2}$. As $\mathcal{F}_G(E,F) \subset$

$\subset I$ we have $\overline{\mathcal{F}_G(E,F)}^{\tau_1} \subset \overline{\mathcal{F}_G(E,F)}^{\tau_2} \cap I$. To prove the reverse inclusion, we have by lemma 1.3

$$\overline{\mathcal{F}(E,F)}^{\tau_2} \cap I \subset \overline{\mathcal{F}(E,F)}^{\tau_2} \cap L'_{oru}(E,F)_{\tau_1 \tau_2} = \overline{\mathcal{F}(E,F)}^{\tau_1}.$$

<u>COROLLARY 3.4.</u> Let E,F be Banach lattices such that E' has order continuous norm. Let τ_1 be the topology given by the regular norm on $L_{oru}(E,F)$ and let τ_2 be the solid strong topology on $L_{oru}(E,F)$. Denote by I the order ideal of those $U \in L_{oru}(E,F)$ with the following property: for every $\varepsilon > 0$ there is $x \in E_+$ such that $\| |U|((u - x)_+)\| \leq \varepsilon$ whenever $\|u\| \leq 1$. Then I consists precisely of those $U \in L_{oru}(E,F)$ with compact modulus; we have

$$\overline{\mathcal{F}(E,F)}^{\tau_1} = I = L_{oru}(E,F)_{\tau_1 \tau_2},$$

$$(\overline{\mathcal{F}(E,F)}^{\tau_1})_+ = \overline{\mathcal{P}(E,F)}^{\tau_1}.$$

PROOF. As E' has order continuous norm, a subset of E is ao-bounded for $|\sigma|(E,E')$ iff it is norm bounded (see [2]); consequently, $\overline{\mathcal{F}(E,F)}^{\tau_1} = I$ by theorem 3.8 (as theorem 3.2 implies that $L_{oru}(E,F) = \overline{\mathcal{F}(E,F)}^{\tau_2}$). On the other side, by lemma 3.1 and lemma 1.3 we have

$$\overline{\mathcal{F}(E,F)}^{\tau_1} = \overline{\mathcal{F}(E,F)}^{\tau_2} \cap L_{oru}(E,F)_{\tau_1 \tau_2} = L_{oru}(E,F)_{\tau_1 \tau_2}.$$

As $I = \overline{\mathcal{F}(E,F)}^{\tau_1}$, every U in I has compact modulus. Conversely, let $|U|$ be compact. To prove that U satisfies the condition in the definition of I, it will suffice by theorem 1E in [2] to show that $\| |U|(x_n)\| \longrightarrow 0$ for every norm bounded disjoint sequence (x_n); but this follows from the compacity of U and from the fact that every norm bounded disjoint sequence in a Banach lattice with order continuous duak is weakly convergent to 0 (see [2]).

The above result is an extension of the corresponding result in [5] (theorem 4.5) which is proved in the case when F has also order continuous norm.

4. A situation when $L_{oru}(E,F) = \{0\}$

For every Riesz space E we shall denote by $\hat{E}$ the set of all Riesz homomorphisms $h: E \longrightarrow \mathbb{R}$.

A σ-unñversally complete Riesz space is a Riesz space E with the property that every disjoint sequence in E_+ admits a supremum.

<u>THEOREM 4.1.</u> Let E be a σ-order complete σ-universally complete Riesz space. The following are equivalent:

i) $\hat{E} = \{0\}$.

ii) $L_{oru}(E,F) = \{0\}$ for every Archimedean Riesz space F.

PROOF.

ii) $\Longrightarrow$ i) Obvious.

i) $\Longrightarrow$ ii) Let F be an Archimedean Riesz space and let $U \in L_{oru}(E,F)$.

Replacing F by its Dedekind extension we may assume that F is order complete; then $U = U_+ - U_-$, hence we may confine ourselves to the case $U \geq 0$. Let $x \in E_+ \setminus \{0\}$; there is $y \in F_+ \setminus \{0\}$ such that $U([0,x])$ is totally bounded in F_y. There are a σ-stonian space X, a stonian space Y and order isomorphisms $S: E_x \to C(X)$, $T: F_y \to C(Y)$ such that $S(x)$ and $T(y)$ are the functions identically equal to one. Let $V = TUS^{-1}$ and, for each $t \in Y$, let μ_t be the Radon measure associated with $V'(\delta_t)$. The proof will now be divided into several steps.

STEP 1). For every sequence (M_n) of closed-open subsets of X such that $\chi_{M_n} \downarrow 0$ in $C(X)$ we have $\| V(\chi_{M_n}) \| \to 0$ (χ_M being the characteristic function of $M \subset X$). PROOF. As V is a compact operator, the sequence $(V(\chi_{M_n}))$ is norm convergent to its infimum $u \in C(Y)$. The sequence $(nS^{-1}(\chi_{M_n} - \chi_{M_{n+1}}))$ is a disjoint sequence in E_+; as E is σ-universally complete, there is an upper bound $z \in E$ for this sequence. It follows that $mS^{-1}(\chi_{M_m} - \chi_{M_{m+p}}) \leq z$ for $p \geq 1$; as $S^{-1}(\chi_{M_{m+p}}) \downarrow 0$ when $p \to \infty$, $mS^{-1}(\chi_{M_m}) \leq z$. Consequently, $T^{-1}(u) \leq$ $\leq US^{-1}(\chi_{M_m}) \leq m^{-1}U(z)$ for $m \geq 1$, which implies that $u = 0$.

STEP 2). μ_t is σ-normal for every $t \in Y$ (that is , every σ-meagre subset (a countable union of closed nowhere dense Baire subsets) has μ_t-measure 0).

PROOF. Let $M \subset X$ be a closed nowhere dense Baire subset; we have $M = \bigcap_{n=1}^{\infty} M_n$ where M_n is closed-open and $\chi_{M_n} \downarrow 0$ in $C(X)$. By step 1), $\| V(\chi_{M_n}) \| \to 0$, which implies that M has measure 0 for every μ_t.

STEP 3). $\mu_t(\{s\}) = 0$ for every $s \in X$ and $t \in Y$.

PROOF. Let $\mathcal{F}$ be the set of all closed-open subsets of X containing s; $\mathcal{F}$ is a downwards directed set. The net $(V(\chi_M))_{M \in \mathcal{F}}$ is norm convergent to its infimum $u \in C(Y)$; hence there is a sequence $(M_n) \subset \mathcal{F}$ such that $\| V(\chi_{M_n}) - u \| \to 0$. There is a closed-open subset M of X such that $\chi_{M_n} \downarrow \chi_M$ in $C(X)$. We observe that $V(\chi_N) = 0$ for any closed-open subset $N \subset M \setminus \{s\}$. Indeed, $M_n \setminus N \in \mathcal{F}$ which implies $V(\chi_{M_n}) - V(\chi_N) \geq u$ for $n \geq 1$; consequently, $u - V(\chi_N) \geq$ $\geq u$, that is, $V(\chi_N) = 0$. This shows in particular that if $s \notin M$ then $V(\chi_M) = 0$. As $\chi_{M_n \setminus M} = \chi_{M_n} - \chi_M \downarrow 0$ in $C(X)$, step 1) implies $\| V(\chi_{M_n}) - V(\chi_M) \| \to 0$; consequently, if we have $V(\chi_M) = 0$ then $\| V(\chi_{M_n}) \| \to 0$ and we conclude in this situation that $\mu_t(\{s\}) = 0$ for every $t \in Y$. It will therefore suffice, in order to finish the proof, to show that the assumption $V(\chi_M) \neq 0$ leads to a contradiction. Indeed, in this case we have by the above remark $s \in M$ and $\mu_t(N) = 0$ for any closed-open $N \subset M \setminus \{s\}$; as μ_t is regular, this implies $\mu_t(M \setminus \{s\}) = 0$. Consequently,

$$\chi_M \mu_t = g(t) \delta_s \tag{1}$$

with $g(t) \in \mathbb{R}_+$. The above relation implies that $V(\chi_M f) = \delta_s(f) g$ for any $f \in C(X)$;

in particular, $g = V(\chi_M) \in C(X) \setminus \{0\}$. For any $z \in E_+$ we have

$$\delta_s(S(z \wedge nx))g = V(\chi_M S(z \wedge nx)) \leq VS(z \wedge nx)) = TU(z \wedge nx)$$

hence

$$\delta_s(S(z \wedge nx))T^{-1}(g) \leq U(z \wedge nx) \leq U(z).$$

As $T^{-1}(g) \neq 0$, this implies that $\sup_{n \geq 1} \delta_s(S(z \wedge nx)) < \infty$; consequently, the relation

$$h(z) = \sup_{n \geq 1} \delta_s(S(z \wedge nx)) \ , \quad z \in E_+$$

defines a nonzero element of $\hat{E}$, which establishes the desired contradiction.

STEP 4). Let X be a compact space and let $\mu_1, \ldots, \mu_m$ be positive Radon measures on X such that $\mu_i(\{s\}) = 0$ for $1 \leq i \leq m$ and $s \in X$. Then for every $\varepsilon > 0$ there are disjoint Baire subsets M_1, M_2 such that $M_1 \cup M_2 = X$ and $\mu_i(M_j) \geq 2^{-1} \mu_i(X) - \varepsilon$ for $1 \leq i \leq m$ and $j = 1,2$.

PROOF. Let $\mu = \sum_{i=1}^{m} \mu_i$. By the Radon-Nikodym theorem there are Baire functions f_i such that $\mu_i = f_i \mu$. For each i there is a simple Baire function g_i such that $\int_X |f_i - g_i| d\mu \leq \varepsilon/2$. There is a partition of X into disjoint Baire subsets $N_1, \ldots, N_m$ such that each g_i has the form

$$g_i = \sum_{k=1}^{m} c_{ik} \chi_{N_k} .$$

As μ is regular and $\mu(\{s\}) = 0$ for any $s \in X$, it follows that for every Baire subset M with $\mu(M) > 0$ there are disjoint Baire subsets M', M" such that $M = M' \cup M''$ and $\mu(M') > 0$, $\mu(M'') > 0$. Consequently, a well-known measure-theoretic result implies that there are disjoint Baire subsets N_{1k}, N_{2k} such that $N_k = N_{1k} \cup N_{2k}$ and $\mu(N_{jk}) = 2^{-1}\mu(N_k)$ $(1 \leq k \leq m)$. Put $M_j = \bigcup_{k=1}^{m} N_{jk}$. We have

$$\begin{aligned}\mu_i(M_j) &= \int_{M_j} f_i d\mu \geq \int_{M_j} g_i d\mu - \varepsilon/2 = \\ &= \sum_{k=1}^{m} c_{ik}\mu(N_{jk}) - \varepsilon/2 = 2^{-1}\sum_{k=1}^{m} c_{ik}\mu(N_k) - \varepsilon/2 = \\ &= 2^{-1}\int_X g_i d\mu - \varepsilon/2 \geq 2^{-1}\int_X f_i d\mu - \varepsilon = 2^{-1}\mu_i(X) - \varepsilon .\end{aligned}$$

STEP 5). Let $M \subset X$ and $N \subset Y$ be closed-open subsets such that $\inf_{t \in N} \mu_t(M) > 0$. Then there are disjoint closed-open subsets M_1, M_2 such that $M = M_1 \cup M_2$ and $\inf_{t \in N} \mu_t(M_j) > 0$, $j = 1,2$.

PROOF. Let $\varepsilon = 3^{-1} \inf_{t \in N} \mu_t(M)$. As V is compact, there is an open covering $(G_i)_{1 \leq i \leq m}$ of N such that $|V(f)(t) - V(f)(t')| \leq \varepsilon$ whenever $t, t' \in G_i$ and $\|f\| \leq 1$. Let t_i be any point in G_i. By steps 3) and 4), there are disjoint Baire subsets P_1, P_2 such that $M = P_1 \cup P_2$ and $\mu_{t_i}(P_j) \geq 2^{-1}\mu_{t_i}(M) - \varepsilon$. As X is σ-stonian, there is a closed-open subset M_j such that $(M_j \setminus P_j) \cup (P_j \setminus M_j)$ is σ-meagre (see [4]);

of course, $M_1 \cap M_2 = \emptyset$ and $M = M_1 \cup M_2$. By step 2), each μ_t is σ-normal; therefore, $\mu_{t_i}(P_j) = \mu_{t_i}(M_j)$. Consequently, we have for any $t \in G_i$

$$\mu_t(M_j) \geqslant \mu_{t_i}(M_j) - \varepsilon \geqslant 2^{-1}\mu_{t_i}(M) - \varepsilon \geqslant 2^{-1}\inf_{t\in N} \mu_t(M) - \varepsilon > 0.$$

As $N \subset \bigcup_{i=1}^{n} G_i$, the proof is complete.

STEP 6). We may now conclude the proof of the theorem. Suppose that $U(x) \neq 0$; then there is a nonvoid closed-open subset $N \subset Y$ such that $\inf_{t\in N} \mu_t(X) > 0$. Using step 5), we construct inductively a sequence (M_n) of disjoint closed-open subsets such that $\alpha_n = \inf_{t\in N} \mu_t(M_n) > 0$ for $n \geqslant 1$. The sequence $(n\alpha_n^{-1}S^{-1}(\chi_{M_n}))$ is a disjoint sequence in E_+; as E is σ-universally complete, there is an upper bound $z \in E$ for this sequence. We have

$$\alpha_n \chi_N \leqslant V(\chi_{M_n})$$

whence

$$nT^{-1}(\chi_N) \leqslant U(n\alpha_n^{-1}S^{-1}(\chi_{M_n})) \leqslant U(z)$$

for $n \geqslant 1$. Consequently, $T^{-1}(\chi_N) = 0$ which leads to a contradiction. Therefore, $U(x) = 0$ for any $x \in E_+$ and the proof is complete.

The next result offers a more convenient form for the condition i) in the above theorem.

THEOREM 4.2. Consider the following conditions imposed to a Riesz space E:

i) $\hat{E} = \{0\}$.

ii) E contains no nonzero atomic elements.

Then i) $\Longrightarrow$ ii); the converse is true whenever E is σ-order complete, order separable and σ-universally complete.

Recall that an atomic element is an element x such that $[0, |x|]$ contains scalar multiples of $|x|$ only.

PROOF.

i) $\Longrightarrow$ ii). If $x \in E_+ \setminus \{0\}$ is atomic, it is easily seen that the relation

$$h(y) = \sup\{\alpha \mid \alpha \in \mathbb{R}, \alpha x \leqslant y\}, \quad y \in E_+$$

defines an element of $\hat{E} \setminus \{0\}$.

Suppose now that E is σ-order complete, order separable and σ-universally complete and let $h \in \hat{E}$. If $h \neq 0$ there is $x \in E_+$ such that $h(x) = 1$. Denote by $C(x)$ the set $\{y \mid y \in E, y \wedge (x - y) = 0\}$. Let M be a maximal subset in $C(x) \cap h^{-1}(\{0\}) \setminus \{0\}$ consisting of pairwise disjoint elements. As E is order separable, M is at most countable. We have $x = \sup M$; indeed, if $z = \sup M$ and $z \neq x$ then $x - z \in C(x) \cap M^{\perp} \setminus \{0\}$; as E is σ-order complete and contains no nonzero atomic elements, we may write $x - z = u_1 + u_2$ with $u_i \in C(x) \setminus \{0\}$ and $u_1 \wedge u_2 = 0$. Therefore either $h(u_1) = 0$ or $h(u_2) = 0$ which contradicts the maximality of M.

Dispose the elements of M in a sequence (y_m). Then (ny_m) is a disjoint sequence in E_+; as E is σ-universally complete, this sequence has an upper bound $z \in E$. We have

$$x - \sum_{i=1}^{n} y_i = \bigvee_{i=n+1}^{\infty} y_i \leq (n+1)^{-1} z;$$

consequently, $1 = h(x) \leq (n+1)^{-1} h(z)$ for every $n \geq 1$, which leads to a contradiction.

As examples of σ-order complete, order separable, σ-universally complete Riesz spaces containing no nonzero atomic elements we mention the space $L_0(\mu)$ of equivalence classes of measurable functions on a σ-finite measure space (X, Σ, μ) without atoms; also, the maximal extension of $C([0,1])$. Recall that a Riesz space $\widetilde{E}$ is called the maximal extension of a Riesz space E if it is order complete, E is order dense in $\widetilde{E}$ and every subset of $\widetilde{E}_+$ consisting of pairwise disjoint elements is order bounded; every Archimedean Riesz space possesses a unique (up to isomorphism) maximal extension. It is known that the maximal extension of $C([0,1])$ is not isomorphic to any $L_0(\mu)$.

The following example shows that the order separability condition in the implication ii) $\Longrightarrow$ i) cannot be dropped.

<u>EXAMPLE.</u> A σ-order complete, σ-universally complete Riesz space containing no nonzero atomic elements such that $\hat{E} \neq \{0\}$.

Let F be any σ-order complete σ-universally complete Riesz space containing no nonzero atomic elements and let x be a fixed element in $F_+ \setminus \{0\}$. Let Ω be the set of all countable ordinals and let F^{Ω} be the power Riesz space (with the product order). Consider the subset $E \subset F^{\Omega}$ consisting of those $f = (x_\alpha)_{\alpha \in \Omega}$ for which there are $\lambda_f \in \mathbb{R}$ and $\alpha_f \in \Omega$ such that $x_\alpha = \lambda_f x$ whenever $\alpha \geq \alpha_f$. Then E is a Riesz space satisfying our requirements; the map $f \longmapsto \lambda_f$ belongs to $\hat{E} \setminus \{0\}$.

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