INSTITUTUL
DE
MATEMATICA

INSTITUTUL NATIONAL
PENTRU CREATIE
STIINTIFICA SI TEHNICA

ISSN 0250 3638

COHEN-MACAULAY NILPOTENT STRUCTURES

by

Nicolae MANOLACHE

PREPRINT SERIES IN MATHEMATICS
No.68/1985

BUCURESTI

COHEN-MACAULAY NILPOTENT STRUCTURES

by

Nicolae MANOLACHE*)

November 1985

*) Department of Mathematics, The National Institute for Scientific and Technical Creation, Bd.Pacii 220, 79622 Bucharest, Romania.

COHEN-MACAULAY NILPOTENT STRUCTURES

by

Nicolae MANOLACHE

This paper deals with multiple structures on smooth varieties as support, embedded in a smooth variety. It was inspired by [2], where several constructions of nilpotent structures are presented, which led to the classification of nilpotent curves of small multiplicity in threefolds (e.g. curves of degree 4 in $\mathbb{P}^3(\mathbb{C})$). These constructions were already used in several works (e.g. [3], [4], [11]).

The "new" idea here is to use the frame of algebraic linkage (cf. [12]), like in [6], where a method to double a structure is described. For the case of curves, the main observation is that a kind of algebraic linkage can be considered, such that the curve which plays the role of the "union" of two structures need not be locally complete intersection, but Cohen-Macaulay.

If Y is a Cohen-Macaulay structure on a smooth variety X, embedded in a smooth variety P, and E is a vector bundle on X, then the kernel of a surjective homomorphism $I_Y \twoheadrightarrow E$ (where I_Y is the ideal of Y in P) is the ideal of a "thicker" Cohen-Macaulay structure Z on X. Lemma 2 shows that all Cohen-Macaulay structures on a union X of smooth curves are obtained applying succesively this method, taking E's vector bundles on the irreducible components of X. The same fact, proved here by linkage, follows also by what is called in [2] Cohen-Macaulay stratification. We mention that we found it independently.

When the dimension is greater than 1, not all Cohen-Macaulay nilpotent structures can be obtained with this method, the first example being already with multiplicity 3. The main result of this paper is that all locally complete intersection structures up to multiplicity 4 on a smooth X are obtainable like above, so that they are given by the same (or similar) constructions as those described by Banica and Forster for curves (cf. [2]). We describe also all multiplicity 3 Cohen-Macaulay structures on smooth support. However, we note that the existence of certain multiple structures requires rather strong topological conditions, and it is not an easy task to give "interesting particular cases" (as for instance surfaces in $\mathbb{P}^4$ - cf. [8], [9]).

The results of this paper were obtained some years ago in paralel with working on multiple 5 and 6 structures on a smooth curve as support, and they were presented at the INCREST Seminar on Algebraic Geometry.

I express my thanks to C.Banica with whom I had useful discussions on the subject and who stimulated me to study further multiple structures.

In the following P will be a smooth algebraic variety over an algebraically closed field k. If Y is a closed subscheme of P we shall denote by I_Y the ideal of Y in P. Closed subschemes of P of dimension 1 (resp. 2) will be called curves (resp. surfaces).

In this section, using an adaptation of the definitions from [12], we present some general lemmas.

DEFINITION 1. Let Y_1, Y_2 two closed subschemes of P. We say that Y_1, Y_2 are locally algebraically linked (shortly l.a.l.) if:

(1) Y_1, Y_2 are equidimensional, without embedded components;

(2) there exist an equidimensional l.c.i. (i.e. locally complete intersection) subscheme Y of P such that:

$$\text{(A)}\quad I_{Y_2}/I_Y = \mathrm{Hom}_{O_P}(O_{Y_1}, O_Y)$$

$$\text{(B)}\quad I_{Y_1}/I_Y = \mathrm{Hom}_{O_P}(O_{Y_2}, O_Y)\ .$$

REMARKS: 1) It follows from the definition that dim Y=dim Y_1= =dim Y_2=d.

2) If d=1 then the condition (1) is equivalent to "Y_1, Y_2 C.M. curves" (i.e. Y_1, Y_2 are locally Cohen-Macaulay).

3) If we take Y an equidimensional l.c.i. subscheme of P and Y_1 a C.M. proper subscheme of Y with dim Y_1=dim Y, then the subscheme Y_2 defined by (A) is C.M. and Y_1, Y_2 are l.a.l.

4) In the assumptions of the definition, with Y_1, Y_2 C.M., we have the exact sequences (dual to each other):

$$(1)\qquad 0 \to \omega_{Y_i} \otimes \omega_Y^{-1} \to O_Y \longrightarrow O_{Y_j} \longrightarrow 0$$

which can be written:

$$(2)\qquad 0 \longrightarrow I_Y/I_{Y_i}I_{Y_j} \longrightarrow I_{Y_i}/I_{Y_i}I_{Y_j} \xrightarrow{p_i} \omega_{Y_j} \otimes L_j \longrightarrow 0$$

where $L_j = \omega_Y^{-1}|Y_j$.

4) Ferrand method of doubling: taking Y_1=Y_2 and a surjection p like p_i in (2), then Y given by (2) is locally Gorenstein (by a lemma of Fossum), hence l.c.i. if the codimension is 2 (cf. [6]).

5) From the exact sequences (1) one obtains the exact sequences of sheaves of multiplicative groups:

(1') $$0 \longrightarrow J_j \longrightarrow \mathcal{O}^*_Y \longrightarrow \mathcal{O}^*_{Y_j} \longrightarrow 0$$

and then:

(3) $$H^1(J_j) \longrightarrow \mathrm{Pic}\, Y \longrightarrow \mathrm{Pic}\, Y_j \longrightarrow H^2(J_j)$$

When supp Y_j=supp Y, then $I_j=I_{Y_j}/I_Y$ is a nilpotent ideal in $\mathcal{O}_Y$ and, like in [1], we have the following

LEMMA. If Z is a subscheme of nilpotent ideal I in Y, dim Y=1 and $\mathcal{J}$ is the kernel of the map of the sheaves of multiplicative groups $\mathcal{O}^*_Y$ $\mathcal{O}^*_Z$, then $H^1(\mathcal{J})=H^1(I)$.

As it is very well known, if char k=0 (or if char k=p and $I^p=0$), then $I \cong \mathcal{J}$ by exponential (respectively by truncated exponential), for any dimension of Y. In particular, if $I^2=0$ then $x \to 1+x$ gives $I \approx \mathcal{J}$.

6) If we take $Y \subset P$ only C.M. instead of l.c.i. and $Y_1 \subset Y$ like at Remark 3 then Y_2 defined by (A) (we shall call in the following Y_2 "the residue of Y_1 in Y") is no longer C.M. in general (but locally it has depth 1).

LEMMA 1. Let Y, Y_1 be Cohen-Macaulay curves embedded in a smooth variety P, Y_1 closed proper subscheme of Y. Then the residue Y_2 of Y_1 in Y is a Cohen-Macaulay curve.

Proof. The question is local, so that we can take A, A_1 C.M. local rings of dimension 1, A_1 quotient ring of A by an ideal I_1 and consider A_2 defined by $(0: I_1)_A=I_2$. Then it is an easy excercise to show that a nonzero divisor in A is nonzero divisor in A/I_2.

COROLLARY. If Y is a Cohen-Macaulay curve, nonreduced, then there are two proper subschemes Y_1, Y_2 of Y of dimension 1 and a Cohen-Macaulay $\mathcal{O}_{Y_1}$-module, F, such that we have an exact sequence:

$$0 \to \frac{I_Y}{I_{Y_1} I_{Y_2}} \to \frac{I_{Y_2}}{I_{Y_1} I_{Y_2}} \to F \to 0$$

DEFINITION. If we have the situation from the above corollary with $F = (0 : I_1)_{\mathcal{O}_Y}$ we say that Y is a linking curve of Y_1 and Y_2 (in this order!).

REMARKS: 1) Given Y_1, Y_2 it is not true that a linking curve does exist or that it is unique.

2) If Y is a linking curve of Y_1 and Y_2 and Y is l.c.i. then it is also a linking curve of Y_2 and Y_1.

LEMMA 2. If Y is a Cohen-Macaulay curve and X is a smooth irreducible component of Y_{red}, $Y \neq X$, then there is a Cohen-Macaulay curve Y' ("less nilpotent" along X) and a vector bundle E on X, such that I_Y is given by an exact sequence:

$$0 \to I_Y / I_X I_{Y'} \to I_{Y'} / I_X I_{Y'} \to E \to 0$$

or, equivalently:

$$0 \to E \to \mathcal{O}_Y \to \mathcal{O}_{Y'} \to 0$$

Proof. This is lemma 1 with $Y_1 = X$ smooth, because $(0 : I_X)_{\mathcal{O}_Y} = \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{O}_X ; \mathcal{O}_Y)$ is a C.M. $\mathcal{O}_X$-module of dimension 1, hence locally free.

REMARK. Lemma 2 gives inductively, in principle, all C.M. structures on curves whose support is a union of smooth curves. (It can be used directly to produce, for instance, triple structures on a line in $\mathbb{P}^3$, described firstly in [7]). Namely, if Y' is a Cohen-Macaulay curve, E is a locally free sheaf on a smooth curve X and $p: I_{Y'}/I_X I_{Y'} \longrightarrow E$ a surjection, then the kernel of $I_{Y'} \to I_{Y'}/I_X I_{Y'} \to E$ defines a C.M. curve Y.

Applying this succesively one reobtains the known fact that the multiplicity $m_x(Y)=m(\mathcal{O}_{Y,m})$:=the multiplicity of the local ring $\mathcal{O}_{Y,x}$ is constant with respect to $x \in Y$, for any C.M. structure Y on a smooth X. We call $m_x(Y)=m(Y)$ "the mutliplicity of Y" and when m(Y)=2, 3 etc. we call Y a double, triple (and so on) structure on X

DEFINITION. If Y, Y_1 are C.M. equidimensional subschemes of P and the residue Y_2 of Y_1 in Y is C.M. we say that Y is a linking scheme of Y_1 on Y_2 (in this order).

LEMMA 3. Let Y, Y_1, Y_2 be C.M. equidimensional subschemes of P, such that Y_1, Y_2 are closed proper subschemes of Y, of dimension dim Y. Then Y is a linking scheme of Y_1 and Y_2 iff the following conditions are fulfilled:

(a) the residue of Y_1 in Y is C.M.

(b) $I_{Y_1} I_{Y_2} \subset I_Y$

(c) $m_p(Y)=m_p(y_2)+m_p(A_1)$, where $A_1=(0: I_{Y_1})_{\mathcal{O}_Y}$.

Proof. Let Y_2' be the curve defined by the ideal $A_1 \subset \mathcal{O}_Y$. From (a) Y_2' is C.M., from (c) $m(\mathcal{O}_{Y_2',x})=m(\mathcal{O}_{Y_2,x})$ for any x and from (a) Y_2' is a closed subscheme of Y_2. It follows $Y_2=Y_2'$.

COROLLARY. If Y_1, Y_2 are Cohen-Macaulay equidimensional subschemes of P such that they are closed proper subschemes of a l.c.i. (more generally Gorenstein) scheme $Y \subset P$ and:

(1) $I_{Y_1} I_{Y_2} \subset I_Y$

(2) $m_p(Y)=m_p'(Y_1)+m_p(Y_2)$

then Y is a linking scheme of Y_1 and Y_2.

Proof. In this case $A_1 \cong \omega_{Y_1} \otimes L$, where L is invertible on Y_1 and then $m_p(A_1)=m_p(Y_1)$.

REMARKS: 1) In dimensions greater than 1, even when the support is smooth, it is no longer true that all C.M. nilpotent structures Y are obtained in a process described by exact sequences:

$$0 \to I_{Y_1}/I_X^2 \to I_X/I_X^2 \to E_0 \to 0$$

$$0 \to \frac{I_{Y_{r+1}}}{I_X I_{Y_r}} \to I_{Y_r}/I_X I_{Y_r} \to E_r \to 0$$

where E_i are vector bundles on X , r=1,...,t and $Y_{t+1}=Y$.

For instance, if X=plane, $P=\mathbb{P}^4$ take $I_X=(x,y)$ and u,v,w the homogenous coordinates on X, then $I_Y=(w^2x^2+v^2(ux+vy),\ w^2xy-uv(ux+vy),\ w^2y^2+u^2(ux+vy),\ x^3,x^2y,xy^2,y^3)$ defines a triple structure on X, which cannot be obtained like above.

We call those C.M. nilpotent structures on a smooth X which can be obtained like above, nilpotent structures of type I.

2) As we are interested mainly in l.c.i. structures Y and for them we want to know also the dualizing sheaf, to the exact sequence (3) and the lemma under it we must add the computation of $\omega_Y|_X$.

LEMMA 4. Let Y_o be a linking scheme of equidimensional C.M. schemes Y_1 and X, where X is regular. If Y is a l.c.i. scheme containing Y_o (locally) and dim Y=dim Y_i , then $\omega_{Y_i}|_X \cong (I_Y:I_{Y_i})/ /I_Y+I_X(I_Y:I_{Y_i})$, locally, (i=0,1). The linking exact sequence:

$$0 \quad \omega_{Y_i} \to \omega_{Y_o} \to F^\vee \otimes \omega_X \to 0$$

gives, by restriction to X, locally, the exact sequence:

$$(4) \qquad 0 \to \frac{(I_Y : I_{Y_1})}{I_Y + I_X(I_Y : I_{Y_o})} \to \omega_{Y_o}\big|X \to F^V \otimes \omega_X \longrightarrow 0$$

Proof. Straightforward computation.

LEMMA 5. If $Y \subset P$ is a nilpotent structure on a smooth variety X, then $m_x(Y)$ is constant with respect to $x \in X$ (this constant will be called the multiplicity of Y, denoted m(Y)).

Proof. Let ξ be the generic point of X. We show that $m_x(Y) = m_\xi(Y)$ for any $x \in X$. We have surjections $\mathcal{O}_{x,P} \to \mathcal{O}_{x,Y}$ and $\mathcal{O}_{x,Y} \to \mathcal{O}_{x,X}$. Denote respectively by R, B, A the completions of the above rings in the topologies of their maximal ideals; thus we obtain surjections $q: R \to B$, $p: B \to A$, where R and A are regular complete rings. Then $R \simeq k[[x_1,\dots,x_n;u_1,\dots,u_r]]$, $A \simeq k[[u_1,\dots,u_r]]$ and the map pq can be considered to be the natural projection. Take $s: A \to R$ the section of pq which sends $u_i \in A$ in $u_i \in R$. Then, B being C.M. and $B/_{(u_1,\dots,u_2)}$ having finite length, it follows that $(u_1,\dots,u_r)$ is an A-regular sequence in B, hence B is Cohen--Macaulay as an A-module. But A is regular, so that B is A free, $B \simeq A^d$. Then $B - A^d = K(X)^d$ and so $m(B) = d = m(B_\xi)$.

THEOREM 1. If X is a smooth sub variety of the smooth variety P and Y is a l.c.i. structure on X with $m(Y) \leq 4$, then Y is of type I.

Proof. All is done if we show that the residue of X in Y is C.M. of type I. This is a local property: for any $x \in Y$ we have to show that $Hom_{\mathcal{O}_{x,P}}(\mathcal{O}_{x,X}, \mathcal{O}_{x,Y}) = (0:I_X) \subset \mathcal{O}_{x,Y} = B$ defines a C.M. quotient of B of type I (here $I = I_X/I_Y$). With the notations from the proof of lemma 5, we have surjections $k=k[[x_1,\dots,x_n;u_1,\dots,u_r]] \to$ $\to B$, $B \to k[[u_1,\dots,u_r]] = A$ such that $R \to A$ is the natural surjection and take $s:A \to R$ the natural section. Then $\varphi : \mathcal{Y}=\mathrm{Spec}\, B \to$ $\to \mathrm{Spec}\, A=X$ is a flat family of μ-multiple points in $\mathrm{Spec}\, k[[x_1,\dots,x_n]]$ $\mu=m(Y)$. We shall compute all possible ideals $I_B=\ker(R \to B)$ and show directly that $(0:I)_B=\frac{(I_B:I_A)}{I_B}$ defines a C.M. quotient of B, of type I.

For this, one method can be to compute the Hilbert scheme $\mathcal{H}_\mu$ of μ-multiple points of Spec $S=k[[x_1,\dots,x_n]]$ (or, equivalently, of μ-colength ideals of $k[[x_1,\dots,x_n]]$), using the fact that any ideal $J\subset S$ of colength μ contains m^μ, where m is the maximal ideal of S (this can be proved by induction on μ, cf. [10]); the exact sequence

$$0 \to \frac{J}{m^\mu} \to \frac{S}{m^\mu} \to \frac{S}{J} \to 0$$

shows that $\mathcal{H}_\mu$ can be realized as a subscheme in the Grassmannian of codimension μ linear subspaces of A/m^μ , given by annulation of certain determinants. Then we can use the universality of $\mathcal{H}_\mu$ to compute I_B.

Instead of that we shall do the computations directly for the family $\mathcal{Y} \to X$, taking succesively $\mu=2, 3, 4$.

For this we need firstly a classification of ideals of colengths 2, 3, 4 in $S=k[[x_1,\dots,x_n]]$. If $k=\mathbb{C}$ and $n=2$ these are classified in [5], but they are not hard to do over any k, using the observation that, in the terminology of this paper, any C.M. structure on a closed point is of type I. As any k-vector space contains one of dimension 1, for any colength μ ideal Y in S there is one of colength $p-1$, J' and an exact sequence:

$$0 \to \frac{J}{mJ'} \to \frac{J'}{mJ'} \to k \to 0$$

Thus, ideals of colength 2 are given by exact sequences:

$$0 \to \frac{I_2}{m^2} \to \frac{m}{m^2} \to k \to 0$$

hence in convenient coordinates, $J_2=(x_1^2,x_2,\dots,x_n)$.

The ideals of colength 3 are given by exact sequences

$$0 \to \frac{J_3}{mJ_2} \to \frac{J_2}{mJ_2} \to k \to 0$$

hence $J_3=(x_1^3,x_2,\dots,x_n)$, or $J_3=(x_1^2,x_1x_2,x_2^2,x_3,\dots,x_n)$, in convenient coordinates.

In the same way one obtains changing conveniently the coordinates, the ideals of colength 4:

$J_4=(x_1^4,x_2,\dots,x_n)$, $J_4=(x_1^3,x_1x_2,x_2^2,x_3,\dots,x_n)$ or

$J_4=(x_1^2,x_2^2,x_3,\dots,x_n)$; if char $k=2$ we have also

$J_4=(x_1x_2,\ x_1^2+x_2^2,\ x_3,\dots,x_n)$.

Coming back to our family $\mathcal{Y}\to X$, take firstly $\mu=2$. Then the fiber $Y_o\subset \mathrm{Spec}\ S$ of φ over the closed point is given by an ideal of the form $(x_1^2,x_2,\dots,x_n)$, eventually changing the coordinates. This shows that around Y_o, $\mathcal{H}_2$ contains ideals of the form $(x_1^2,\ x_2+a_2x_1,\dots,x_n+a_nx_1)$, where a_i are local coordinates on $\mathcal{H}_2$; this shows that $I_B=(x_1^2,\ x_2,\dots,x_u)$, in convenient coordinates. It follows $(I_B:I_A)=I_A$, hence the residue is trivially of type I.

If $\mu=3$, then Y_o is given by an ideal of the form $(x_1^3,x_2,\dots,x_u)$ (recall that we consider only l.c.i.) and the same procedure as above shows $I_B\simeq(x_1^3,x_2,\dots,x_n)$, in suitable x_i.

Then $(I_B:I_A)=(x_1^2,x_2,\dots,x_n)$ defines a C.M. structure of type I.

If $\mu=4$, then the nontrivial case is $I_{Y_o}=(x_1^2,x_2^2,x_3,\dots,x_n)$ and for char k=2 also $I_{Y_o}=(x_1x_2,\ x_1^2+x_2^2,\ x_3,\dots,x_n)$.

Consider $I_{Y_o}=(x_1^2,x_2^2,\ x_3,\dots,x_u)$. Then $I_B\ (x_1^2+ax_1+bx_2+cx_1^2+ +dx_1x_2+ex_2^2+fx_1^3+qx_1^2x_2+hx_1x_2^2+ix_2^3,\ x_2^2+Ax_1+Bx_2+Cx_1^2+Dx_1x_2+Ex_2^2+Fx_1^3+Gx_1^2x_2+ +Hx_1x_2^2+Ix_2^3,\ (x_1,x_2)^4,\ x_3+a_3x_1+b_3x_2,\dots,x_n+a_nx_1+b_nx_2)$, where $a,b,\dots,A,B,\dots,a_i,b_i$ are nonunits in $A=k[[u_1,\dots,u_r]]$ and satisfy the equations of $\mathcal{H}_4$ around I_{Y_o}, or equivalently, are such that B/pB_p is given by an ideal of colength 4 in $R_p/pR_p\simeq A_p/pA_p[[x_1,\dots,x_n$ for any $p\in$ Spec A. Making the change of coordinates $X_i=x_i+a_ix_1+ +b_ix_2$ $(i\geq 3)$ we have $I_B\simeq(x_1^2+ax_1+,\dots,x_2^2+Ax_1+,\dots,X_3,\dots,X_n)$ which show that taking n=2 is not reducing the generality. If we consider the matrix of the coefficients of $x_1^3,\ x_1^2x_2,\ x_1x_2^2,\ x_2^3$ in $x_1F_1,x_2F_1,x_1F_2,x_2F_2\in I_B=(F_1,F_2)$, we see that it is invertible, hence $x_1^3,\ x_1^2x_2,\ x_1x_2^2,\ x_2^3$ can be expressed in $x_1,\ x_2,\ x_1^2,\ x_1x_2,\ x_2^2$ mod I_B. Then I_B can be written

$$I_B=(x_1^2+ax_1+bx_2+cx_1^2+dx_1x_2+ex_2^2,\ x_2^2+Ax_1+Bx_2+Cx_1^2+Dx_1x_2+Ex_2^2\ ,\ (x_1,x_2)^4),$$

with $a,b,\dots,A,B,\dots$ new nonunits in A. It is easy to see that I_B can be written in the form: $I_B=(x_1^2+ax_1+bx_2+cx_1x_2,\ x_2^2+Ax_1+Bx_2+ +Cx_1x_2,\ (x_1,x_2)^4)$. We show now that $aB-bA=0$. Indeed, if $aB-bA\neq 0$, then there is a $p\in$ Spec A such that $aB-bA\notin p$, hence the fiber of φ over p is given by an ideal $(x_1+\alpha x_1^2+\beta x_1x_2+\gamma x_2^2,\ x_2+\alpha' x_1^2+\beta' x_1x_2+ +\gamma' x_2^2)$, not of colength 4. We may then take $F_1,\ F_2$ of the form: $F_1=x_1^2+\ell(ax_1+bx_2)+cx_1x_2,\ F_2=x_2^2+m(ax_1+bx_2)+dx_1x_2$, where a,b have no common factor. If both l,m are zero, it is easy to see that $(I_B:I_A)=(x_1^2,\ x_1x_2,\ x_2^2)$ defines a C.M. structure of type I. Assume, to make a choice, $\ell\neq 0$. We show $a=0$. Indeed, if $a\neq 0$, taking the fiber over $p\in$ Spec A such that $a,\ell\notin p$, the ideal of the fiber over p will be of the form $(x_1^2+ax_1+bx_1x_2+cx_2^2,\ dx_2^2+ex_1+fx_1x_2,\ (x_1,x_2)^4)$, where a is invertible and d or f are not zero. It is easy to see that an ideal of this kind cannot have colength 4. Thus, $I_B=(x_1^2+lbx_2+ +cx_1x_2,\ x_2^2+mbx_2+dx_1x_2,\ (x_1,x_2)^4)$ and $l,b\neq 0$. Then we must have $d=0$, $m=0$, because otherwise localizing conveniently we should have l, b, d, m invertible and the two conics $x_1^2+lbx_2+cx_1x_2=0$, $x_2^2+mbx_2+ +dx_1x_2=0$ will not intersect 4 times in origin.

Thus, we have showed $I_B=(x_1^2+ax_2+bx_1x_2,x_2^2)$ and then $I_B'=(I_B:I_A)=(x_1^2+ax_2,\ x_1x_2,x_2^2)$ is of type I, because $(I_{B'}:I_A)= =(x_1^2,x_2)$ which is of type I. (In fact, if char $k\neq 2$, $I_B\simeq (x_1^2+ax_2,x_2^2)$, with a new x_1).

Assume now char $k=2$ and $I_{Y_o}=(x_1x_2,\ x_1^2+x_2^2,\ x_3,\ldots,x_n)$. Like above, we reduce our computations to the case $I_B=(x_1x_2+ax_1+bx_2+cx_1^2+ +dx_1x_2+lx_2^2,\ x_1^2+x_2^2+Ax_1+Bx_2+Cx_1^2+Dx_1x_2+Ex_2^2,\ (x_1,x_2)^4)$ where $a,b,\ldots,A,B,\ldots$ are nonunits in A. With new coordinates and coefficients, $I_B=(x_1x_2+ax_1+bx_2+cx_1^2+dx_2^2,\ x_1^2+x_2^2+Ax_1+Bx_2+Cx_2^2+Dx_2^2,\ (x_1,x_2)^4)$. Like above we must have $aB-bA=0$, hence

$$F_1=(x_1x_2+l(ax_1+bx_2)+cx_1^2+dx_2^2\ ,\quad F_2=x_1^2+x_2^2+m(ax_1+bx_2)+ex_1^2+fx_2^2)$$

where a,b, if not zero, have no common factor.

Dividing by $(1+e)$, F_2 can be written, with new coefficients $F_2=x_1^2+m(ax_1+bx_2)+ex_2^2$, where e is invertible. Substituting x_1^2 from $F_2\in I_B$ into F_1 and changing notations we may assume $F_1=x_1x_2+ +l(ax_1+bx_2)+cx_2^2$, $F_2=x_1^2+m(ax_1+bx_1)+dx_2^2$, where d is invertible and the other coefficients are nonunits. Then it is not hard to show that I_B must be of the form $I_B=(x_1x_2,\ x_1^2+ax_2^2,\ x_3,\ldots,x_n)$, where a is an invertible element. Then $I_B:I_A=(x_1^2,\ x_1x_2,\ x^2,\ x_3,\ldots,x_n)$ defines a C.M. structure of type I.

§2. Constructions of nilpotent structures

In this section we give constructions of C.M. nilpotent structures of type I on smooth support X, embedded is a smooth P, mainly up to multiplicity 4 (this includes all l.c.i. structures up multiplicity 4, by theorem 1) and a construction for a C.M. structure of multiplicity 3, not of type I. Many of the descriptions given here are essentially those from C. Bănica and O Forster's paper [2], but, some are new, for example 1.2., 1.1.2., 1.2.1. in char=2. We shall present them briefly, without insisting on aspects, as the dimensions of the families of nilpotent structures we consider.

1. Structures Y such that the residue of X in Y is X. These are obtained from exact sequences:

$$0 \longrightarrow \frac{I_Y}{I_X^2} \longrightarrow \frac{I_X}{I_X^2} \longrightarrow Q \longrightarrow 0$$

where Q is a vector bundle on X.

1.1. Q=L is a line bundle (This is Ferrand's doubling). The exact sequence by which $X_2=Y$ is obtained can be written also:

$$0 \to L \longrightarrow \mathcal{O}_{X_2} \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

We have $\omega_{X_2}|_X = \omega_X \otimes L^{-1}$ and the exact sequence:

$$H^1(L) \to \operatorname{Pic} X_2 \longrightarrow \operatorname{Pic} X \longrightarrow H^2(L) .$$

If $x \in X$ then the ideal I_{X,X_2} is of the form $(x_1^2, x_2, \dots, x_n)$, where x_i are a convenient system of coordinates of X in $x \in X$ (i.e. $I_{X,x} = (x_1, \dots, x_n)$, $n = \operatorname{codim}_x X$).

1.2. Q=E is a vector bundle of rank 2. One obtains triple structures X_3. The defining exact sequence can be written:

$$0 \longrightarrow E \longrightarrow \mathcal{O}_{X_3} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

<u>Local structure</u>. $I_{X,X_3}=(x_1^2, x_1x_2, x_2^2, x_3,\dots,x_n)$, where x_i are convenient local coordinates around $x\in X$.

By lemma 4, $\omega_{X_3}|X \simeq \omega_X \otimes E^v$.

REMARK. When $\mathrm{codim}_P X=2$, then X_3 is simply the first infinitesimal neighbourhood, $X^{(1)}$.

1.q. Q is a vector bundle of rank q. Then the codimension of X must be at least q, the local structure of X_{q+1} is of the form $((x_1,\dots,x_q)^2, x_{q+1},\dots,x_n)$ and we have $\omega_{X_{q+1}}|X \simeq \omega_X \otimes Q^v$.

1.1.0. Here we consider C.M. structures Y such that the residue of X in Y is of the type 1.1 above. They are among those which can be obtained from exact sequences

$$0 \longrightarrow I_Y/I_XI_{X_2} \longrightarrow I_{X_2}/I_XI_{X_2} \longrightarrow T \longrightarrow 0$$

where T is a vector bundle on X.

In order to separate "nonlinear" equation of X_2 we consider the kernel of the natural map $I_{X_2}/I_XI_{X_2} \longrightarrow I_X/I_X^2$ which is $I_X^2/I_XI_{X_2}$. Like in 2 , one sees easily that the multiplication map $L^2 \simeq I_X/I_{X_2} \otimes I_X/I_{X_2} \longrightarrow I_X^2/I_XI_{X_2}$ is locally (hence globally) an isomorphism. In the notations from 1.1, L^2 is generated by $x_1^2 (\mathrm{mod}\ I_XI_{X_2})$. We shall make our discussion upon the map $L^2 \longrightarrow I_{X_2}/I_XI_{X_2} \longrightarrow T$, which is not zero, because otherwise in the structure Y thus obtained the residue of X would be X and not X_2.

1.1.1. T is a line bundle. Then $T=L^2(D)$. Denote by Y_3 the structure of multiplicity 3 thus obtained. It is given by an exact sequence:

$$0 \rightarrow I_{Y_3}/I_X I_{X_2} \rightarrow I_{X_2}/I_X I_{X_2} \xrightarrow{p} L^2(D) \rightarrow 0$$

where p is a surjection such that $L^2 \rightarrow I_{X_2}/I_X I_{X_2} \xrightarrow{p} L^2(D)$ is the natural inclusion.

Local structure. In convenient local coordinates, we have $I_{Y_3} \simeq (x_1^3, x_2, \dots, x_n)$ in points $x \in X \setminus D$ and I_{Y_3} $(x_1^2 - fx_2,\ x_1x_2,\ x_2^2,\ x_3, \dots, x_n)$ for $x \in \mathrm{Supp}\, D$, where f is the local equation of D in x. It follows that Y_3 is l.c.i. only when D=0. Lemma 4 gives $\omega_{Y_3}|X \quad L^{-2} \otimes \omega_X$.

1.1.2. T is a vector bundle of rank 2. Structures Y_4 of multiplicity 4, given by exact sequences:

$$0 \rightarrow I_{Y_4}/I_X I_{X_2} \rightarrow I_{X_2}/I_X I_{X_2} \xrightarrow{p} T \rightarrow 0$$

where p is a surjection.

Local structure. Consider firstly that $L^2 \rightarrow I_{X_2}/I_X I_{X_2} \xrightarrow{p} T$ is the composition $L^2 \hookrightarrow L^2(D) \subset T$, D being a divizor on X. (If X is a curve this happens always). In a point $x \notin \mathrm{supp}\, D$, in convenient coordinates, $I_{Y_4} \simeq (x_1^3,\ x_1x_2,\ x_2^2,\ x_3, \dots, x_n)$ and in a point $x \in \mathrm{supp}\, D$ $I_{Y_4} \simeq (x_1^2 - fx_3,\ x_1x_2,\ x_1x_3,\ x_2^2,\ x_2x_3,\ x_3^2,\ x_4, \dots, x_n)$ where f is the equation of D in x. It is clear that, in codimension 2, only the first case can occur.

We note that $\omega_{Y_4}|X \simeq \omega_X \otimes T^\vee$ only if D=0 (for $D \neq 0$ we have that the kernel of the canonical map $\omega_{Y_4}|X \twoheadrightarrow \omega_X \otimes T^\vee$ is concentrated in the points of D.

In general, the homomorphism L^2 T is of rank 1 on an open subset of X. Let V be the complement of it. In points $x \notin V$ the structure is of the form $I_{T_4} \simeq (x_1^3, x_1x_2, x_2^2, x_3, \dots, x_n)$ and, in a point $x \in V$, I_{T_4} $(x_1^2 - fx_2 - gx_3, x_1x_2, x_1x_3, x_2^2, x_2x_3, x_3^2, x_4, \dots, x_n)$.

1.1.3. T is a vector bundle of rank 3 (this is possible only if $\mathrm{codim}_P X$ 3.

One obtains structures Y_5 of multiplicity 5, given by exact sequences:

$$0 \to I_{Y_5}/I_X I_{X_2} \longrightarrow I_{X_2}/I_X I_{X_2} \xrightarrow{p} T \longrightarrow 0$$

where p is a surjection.

<u>Local structure</u>. Consider firstly that p, when restricted to L^2, is the composition $L^2 \hookrightarrow L^2(D) \subset T$, D being a divizor on X. In a point $x \notin \mathrm{Supp}\, D$, in convenient coordinates $I_{Y_5} \simeq (x_1^3, x_1x_2, x_1x_3, x_2^2, x_2x_3, x_3^2, x_4, \dots, x_n)$ and in a point $x \in \mathrm{supp}\, D$: $I_{Y_5} \simeq$ $\simeq (x_1^2 - fx_2, x_1x_2, x_1x_3, x_1x_4, (x_2, x_3, x_4)^2, x_5, \dots, x_n)$, f being the local equation of D. ⊗ Like above, consider secondly that p is of rank 1 outside a smaller V. The structure is outside V of the form $I_{Y_5} = (x_1^3, x_1x_2, x_1x_3, x_2^2, x_2x_3, x_3^2, x_4, \dots, x_n)$ and in V of the form I_{Y_5} $(x_1^2 - fx_2 - gx_3 - hx_4, x_1x_2, x_1x_3, x_1x_4, (x_2, x_3, x_4)^2, x_5, \dots, x_n)$.

⊗ [Similar to 1.1.2, the canonical surjection $\omega_{Y_5}|_X \to \omega_X \otimes T^{\vee}$ has the kernel concentrated in the points of D.]

Of course, one can continue this types of structures, taking greater ranks for T.

1.2.0. In this subsection we deal with structures Z such that the residue of X in Z is of the type 1.2. above. They are among those which can be obtained from an exact sequence:

$$0 \to I_Z/I_XI_{X_3} \to I_{X_3}/I_XI_{X_3} \to M \to 0$$

where M is a vector bundle on X. Like at 1.1.0, we want "to separate" the local equations "of degree 2": for this one takes the kernel $I_{X_3}/I_XI_{X_3}$ I_X/I_X^2, which is $I_X^2/I_XI_{X_3}$. To compute this, observe that the multiplication $E \otimes E \simeq I_X/I_{X_3} \otimes I_X/I_{X_3} \to I_X^2/I_XI_{X_3}$ factors through S^2E, it is surjective and rank S^2E=3=rank $(I_X^2/I_XI_{X_3})$=3, hence $I_X^2/I_XI_{X_3} \simeq S^2E$. Thus, we have a canonical map $\varphi : S^2E \to M$, which is not zero (by the argument used already above).

REMARKS: 1) If the codimension is 2, E reduces always the conormal bundle $\nu = I_X/I_X^2$, because $I_{X_3} = I_X^2$.

2) In S^2E we have the (geometric) subfibration given locally by $\Delta = \{ae_1^2 + be_1e_2 + ce_2^2 \mid b^2 - 4ac = 0\}$, if e_1, e_2 is any basis of E (Δ is the subfibration of "perfect squares"). We shall organize our discussion upon Δ, like in [2], where there is treated the case codimension 2 and M line bundle.

1.2.1. M is a line bundle on X. (A) Consider firstly that the image of $\varphi : S^2E \to M$ is of the form M(-D), where D is an effective divizor on X. Assume char $k \neq 2$. Then $K = \ker \varphi$ is a subbundle of S^2E and above a point $x \in X$ it intersects Δ in two lines or in one line (i.e. it is tangent to Δ). The locus of $x \in X$ above which $\Delta \cap K$ = two lines is open in X and $\Delta_X = \{x \in X \mid \Delta_x \cap K_x = \text{one line}\}$ is closed, given locally, if $\varphi : S^2E \to M' = M(-D)$ is the application $\varphi(e_1^2) = \ell$,

$\varphi(e_1e_2)=m$, $\varphi(e_2^2)=n$, by the equation $m^2-4\ell n=0$.

Local structure. Observe that we can assume $\Delta_X\neq X$, because if $\Delta_X=X$ then in points $x\notin D$ we should have I_{Z_4} of the form $(x_1^3,x_1x_2,x_2^2,x_3,\dots,x_n)$ and the rezidue of X in Z_4 would n… be of the type 1.2. Then Δ_X is a divizor on X and the local equations of Δ_X show that we have $\Delta_X=2(c_1(M')-c_1(E))$, or $\deg\Delta_X=2(\deg M'-\deg\Lambda^2E)$. In the points $x\in X$, $x\notin D$, $x\notin\Delta_X$ the ideal I_{Z_4} is of the form $(x_1^2-ax_1x_2,\ x_2^2-bx_1x_2,\ x_3,\dots,x_n)$ where a,b are noninvertible elements. If we pass to the completion, then $\hat{I}_{Z,x}\simeq(x_1^2,x_2^2,x_3,\dots,x_n)$, in convenient coordinates.

In points $x\notin D$, $x\in\Delta_X$, in convenient local coordinates, $I_{Z_4}\simeq(x_1^3,x_1x_2,x_2^2-fx_1^2,x_3,\dots,x_n)$, where f is the equation of Δ_X in x.

In points $x\in D$, $x\notin\Delta_X$, in convenient local coordinates $I_{Z_4}\simeq(x_1x_2-gx_3,\ x_1^2-ax_1x_2,\ x_2^2-bx_1x_2,\ x_1x_3,\ x_2x_3,\ x_3^2,\ x_4,\dots,x_n)$, where g is the equation of D in x and a,b are noninvertible elements. If we pass to the completion, $\hat{I}_{x,Z_4}$ is with new coordinates, like above, but a=b=0.

In points $x\in\Delta_X$, $x\notin D$ we have $I_{Z_4}=(x_1^2-gx_3,\ x_1x_2,\ x_2^2-fx_1^2,\ x_1x_3,\ x_2x_3,\ x_3^2,\ x_4,\dots,x_n)$, where g is the equation of D in x and f that of Δ_X.

Thus, we have shown that Z_4 is l.c.i. iff D=0, $D_X=0$. This shows that a necessary condition for the existence of l.c.i. Z_4 is $2(c_1(M)-L_1(E))=0$ and then $\omega_{Z_4}|X\simeq\omega_X\otimes M^{-1}$.

Assume now char k=2. If locally $\varphi:S^2E\to M(-D)=M'$ is given by $\varphi(e_1^2)=a$, $\varphi(e_1e_2)=b$, $\varphi(e_2^2)=c$ then the locus of the points $x\in X$ where $b^2+ac=0$ is invariant and denoted in the following Δ_X. Like above, we have $\Delta_X=2(c_1(M')-c_1(E))$.

<u>Local structure</u>. In the points $x \notin D$ the ideal I_{Z_4} is of the form a) $I_{Z_4} \simeq (x_1x_2, x_1^2 + \lambda x_2^2, x_2^3, x_3, \ldots, x_n)$ or of the form b) $I_{Z_4} \simeq (x_1^2 + \lambda x_1x_2, x_2^2 + \mu x_1x_2, x_1^2x_2, x_1x_2^2, x_3, \ldots, x_n)$. In the points $x \notin \Delta_X$ in the first case λ is invertible and in the second $1+\lambda\mu$ is invertible, so that Z_4 is l.c.i. in points $x \notin D$, $x \notin \Delta_X$.

If $x \in \Delta_X$, then in the form a) λ is the local equation of Δ_X in x and in the form b) this equation is $1+\lambda\mu$. Consider now $x \in D$. Then I_{Z_4} is of one of the forms:

a') $(x_1^2+gx_3, x_1x_2, x_1^2+\lambda x_2^2, x_2^3, x_1x_3, x_2x_3, x_3^2, x_4, \ldots, x_n)$, or

b') $(x_1x_2+gx_3, x_1^2+\lambda x_1x_2, x_2^2+\mu x_1x_2, x_1^2x_2, x_1x_2^2, x_1x_3, x_2x_3, x_3^2, x_4, \ldots, x_n)$

where λ, respectively λ, μ, have interpretations related to Δ_X like above and g is the equation in x of D. Thus we see that Z_4 is l.c.i. iff $D=0$, $\Delta_X=0$. A necessary condition for the existence of a l.c.i. Z_4 is $2(c_1(M)-c_1(E))=0$. By lemma 4 we have also $\omega_{Z_4}|X \simeq \omega_X \otimes M^{-1}$, for a l.c.i. Z_4.

(B) Consider now that the image of $\varphi: S^2E \to M$ is $I_C \otimes M$, where C is closed subscheme of X. Then the local structure of Z_4 is like above for points $x \notin C$ and in the points of C it is of the form

$$(x_1^2-ax_3, x_1x_2-bx_3, x_2^2-cx_3, x_1x_3, x_2x_3, x_3^2, x_4, \ldots, x_n)$$

Note that also now, if locally $\varphi(e_1^2)=\ell$, $\varphi(e_1e_2)=m$, $\varphi(e_2^2)=n$, then the locus where m^2-4ln is zero does not depend upon the choice of e_1, e_2.

It is not difficult to describe other C.M. structures taking M vector bundle of greater rank, but we shall not give such details here.

1.1.1.0. We consider here C.M. structures T given by kernels of surjections: $I_{Y_3} \to I_{Y_3}/I_X I_{Y_3} \xrightarrow{p} N \to 0$ where N is a vector bundle on X, p is a surjection. In fact we shall give here only the case:

1.1.1.1. (N a line bundle)-construction described in [2] for complex curves embedded in threefolds, but the "general" case is essentially the same. Like until now we discuss upon the restriction of p to $K = \ker\left(I_{Y_3}/I_X I_{Y_3} \to I_X/I_X^2\right) = I_X^2 \cap I_{Y_3}/I_X I_{Y_3} \simeq$ $\simeq L^3(D)$ (the last isomorphism follows, for instance, observing that $L^3 \simeq \frac{I_X^3}{I_X^2 I_{X_3}} \to K$ is an isomorphism outside D and then that $L^3(D)=K$).

We shall not consider $p|K=0$, because in this case the residue of X in T_4 would not be of type 1.1.1.

It follows that N is of the form $L^3(D+\Gamma)$.

<u>Local structure</u>. In points $x \notin D$, $x \notin \Gamma$, the ideal I_{T_4} is of the form $(x_1^4, x_2, \ldots, x_n)$. In points $x \notin D$, $x \in \Gamma$, I_{T_4} is of the form $(x_1^3 - gx_2, x_1x_2, x_2^2, x_3, \ldots, x_n)$, where g is the equation of Γ in x. In points $x \in D$, $x \notin \Gamma$ we have $I_{T_4} = (x_1^2 - fx_2 - ax_1x_2, x_2^2, x_3, \ldots, x_n)$ with convenient coordinates. If char $k \neq 2$, then with new x_1: $I_{T_4} = (x_1^2 - fx_2, x_2^2, x_3, \ldots, x_n)$. In points $x \in D$, $x \in \Gamma$ the ideal of T_4 is of one of the forms:

$$I_{T_4} = (x_1x_2 - g(x_1^2 - fx_2),\ x_2^2, x_1(x_1^2 - fx_2),\ x_3, \ldots, x_n)$$

(where f is the equation of D and g that of Γ in x) or:

$$I_{T_4}=(x_1x_2-gx_3,x_1^2-fx_2-hx_3,x_2^2,x_1x_2,x_1x_3,x_3^2,x_4,\dots,x_n)$$

It follows that T_4 is l.c.i. iff $\Gamma=0$ and, by lemma 4, $\omega_{T_4}|X \simeq \omega_X \otimes L^{-3}(-D)$ in this case.

2. We analize here C.M. structures of multiplicty 3 on a amooth X, not of type I. In the proof of Theorem 1, with the notations from there, there was omitted the study of $\mathcal{H}_3$ around colength 3 ideals of the form $I_o=(x_1^2,x_1x_2,x_2^2,x_3,\dots,x_n)$, such points not being complete intersections. It is clear that, around such an ideal, $\mathcal{H}_3$ contains ideals of the form:

$$(x_1^2+a_1x_1+b_1x_2,x_1x_2+a_2x_1+b_2x_2,x_2^2+a_3x_1+b_3x_2,(x_1,x_2)^3,x_3,\dots,x_n)\ ,$$

where a_i,b_i are such that the ideal has colength 3, i.e. the matrix:

$$\begin{pmatrix} 0 & 0 & 1 & a_1 & 0 & a_2 & 0 & a_3 & 0 \\ 0 & 1 & 0 & b_1 & a_1 & b_2 & a_2 & b_3 & a_3 \\ 1 & 0 & 0 & 0 & b_1 & 0 & b_2 & 0 & b_3 \\ a_1 & a_2 & a_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_1 & b_2 & b_3 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

has rank 3 (see [5], VI.2). The minors of rank 4 of this matrix are the local equations of $\mathcal{H}_3$. It follows that $\mathcal{Y}$ contains around I_o ideals like above with $a_i,b_i\in k[[u_1,\dots,u_r]]=A$ satisfying the same equations. Using the factoriality of A one shows that around I_o, $\mathcal{Y}$ contains ideals of the form

$$(x_1^2+b^2d\ (ax_1+bx_2),\ x_1x_2-abd(ax_1+bx_2),\ x_2^2+a^2d(ax_1+bx_2),\ (x_1,x_2)^3,\ x_3,\dots,x_n),$$ where $a,b,d\in A$, a and b coprime.

Take Y be a C.M. structure on a smooth X, not of type I. Then, around some points $x\in X$, $I_{Y,x}$ is like above, with $a,b,d\neq 0$, ad, bd noninvertible, a,b coprime. Let Y' be the residue of X in Y. Then, generically, $I_{Y'}$ is of the form $(x_1^2,x_2,\ldots,x_n)$ and in a point $x\in X$ like above, $I_{Y'}=(ax_1+bx_1, x_1^2,x_1x_2,x_2^2,x_3,\ldots,x_n)$. It follows that $I_{Y'}/I_X^2$ is a vector bundle of rank n-1 and the canonical inclusion map $j:I_{Y'}/I_X^2 \rightarrow I_X/I_X^2$ has rank n-2 and generically rank (j)=n-1. The locus C where rank j=n-1 is l.c.i. in X, of codimension 2.

It follows that I_Y is the kernel of a composition $I_{Y'} \xrightarrow{p} I_{Y'}/I_XI_{Y'} \xrightarrow{u} L \rightarrow 0$, where p is the canonical surjection, L is an invertible sheaf on X and u is a surjection such that restrcited to $K=\ker(I_{Y'}/I_XI_{Y'} \rightarrow I_X/I_XI_C)=I_{Y'}\cap I_XI_C/I_XI_{Y'}$ is also a surjection (if $\mathrm{codim}_P X=2$ then this condition is always fulfilled)

Example. If $X=\mathbb{P}^2$ is given by x=y=0 in $P=\mathbb{P}^4$ and u,v,w are the homogeneous coordinates on X, then a straightforward computation following the above description shows that any C.M. triple structure on X not of type I is of the form:

$$I\cong(Px^2+B^2(Ax+By),\ Pxy-AB(Ax+By),\ Py^2+A^2(Ax+By),\ (x,y)^3),$$

where A, B, P are forms of degrees deg A=deg B=d, deg P=3d-1 in u, v, w such that A, B, P have no common zero on X.

REFERENCES

1. M.Artin, Some numerical criteria for contractibility of curves on an algebraic surface, Amer. J. Math., 84, 485-496 (1962).
2. C.Bănică; O.Forster, Sur les structures multiples (manuscript).
3. C.Bănică: M.Putinar, On complex vector bundles on rational threefolds, Math. Proc. Camb. Phil. Soc., 97, 279-288, (1985).
4. C.Bănică; N.Manolcahe, Rank 2 stable vector bundles on $\mathbb{P}^3(\mathbb{C})$ with Chern classes $c_1=1$, $c_2=4$, Math. Z., 190, 315-339 (1985).
5. J.Briançon, Description de $Hilb^n\mathbb{C}\{x,y\}$, Inventiones math., 41, 45-89, (1977).
6. D.Ferrand, Courbes gauches et fibres de rang 2, C.R. Acad. Sci. Paris, 281, 345-347, (1977).
7. R.Harthorne, Stable vector bundles of rank 2 on $\mathbb{P}^3$, Math. Ann., 238, 229-280, (1978).
8. K.Hulek,; C.Okonek: A.Van de Ven, Multiplicity 2 structure on Castelnuovo surfaces.
9. K.Hulek: A.Van de Ven, The Horrocks-Mumford bundle and the Ferrand construction, manuscripta math., 50, 313-335, (1985).
10. A.Iarrobino, Punctual Hilbert Schemes, Mem.Amer.Math.Soc., 188, (1977).
11. N.Manolache, Rank 2 stable vector bundles on $\mathbb{P}^3$ with Chern classes $c_1=-1$, $c_2=2$, Rev. Roum. Math. Pures et Appl., 26, No.9, 1203-1209, (1981).
12. C.Peskine; L.Szpiro, Liaison des variétés algèbriques, Inventiones math., 26, 271-302, (1974).