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KK-groups of Crossed Products by Groups Acting on Trees

by Mihai V.PIMSNER

Let G be a locally compact, second countable group that acts continuously on some tree X. The aim of this paper is to express the KK-groups of the crossed products by G (both full and reduced) in terms of the KK-groups of the crossed products by the stabilizer subgroups corresponding to the action of G on X. This will be done by exhibiting six terms cyclic exact sequences for the KK-groups of the crossed products, which for G discrete, are analoguous to those of J.P.Serre [27] for the homology and cohomology of G with values in a G-module M.

The impetus to the present results came from discussions with J.Anderson on his joint work with W.Paschke on the K-theory of HNN extensions [1], presented at the X^{th} Operator Theory Conference in Bucureşti. I realized that the methods used in [23] for the free groups, extend naturally to this general setting.

Since the computation of the K-groups of reduced crossed products by free groups in [23], important progress in the understanding of the K-theory of discrete groups has been made.

On one hand there is the approach of G.G.Kasparov and A.Connes for certain discrete subgroups of Lie groups. They emphasize the role played by the Lie group, by relating the KK-theory of the subgroup to the KK-theory of the former.

This method is effective for subgroups of solvable Lie groups [4], [9] , for subgroups of Lorentz groups [15] and depends generally on the positive solution of the Connes-Kasparov conjecture.

On the other hand, there are partial results on the K-groups of free and amalgamated products and of HNN-extensions of groups.

Thus E.C.Lance introduced condition "Λ" , in order to use the methods of [23] to compute the K-groups of the reduced C^*-algebra of certain free products of groups. This has been extended by T.Natsume [20] to certain amalgamated products and recently, J.Anderson and W.Paschke [1] combined the above results with those of [22] to get results for the K-groups of the reduced C^*-algebra of certain HNN-extensions.

J.Cuntz introduced the important notion of K theoretic amenability to relate the KK-groups of the reduced crossed products to those of the full ones, in the hope that the latter are easier to compute. This is indeed the case for the full C^*-algebra of free product groups [7] , but this does not seem to work for crossed products by such groups. The class of K-amenable groups contains the amenable groups and is stable under the operations of taking subgroups or direct and free products [7] . Moreover G.G.Kasparov [15] proved that the Lorentz groups SO(n,1) are KK-amenable and recently P.Julg and A.Valette [11] proved the striking result that groups acting on trees with amenable stabilizers are KK-amenable.

The results of the present paper show that the methods for computing the KK-groups of the full and of the reduced crossed products by groups acting on trees are in fact the same, and closely related to the original methods of [23]. The link with amalgamated products and with HNN-extensions of groups is provided by the results of [27], which identify a discrete group acting on a tree with the fundamental group of a graph of groups. (The graphs of groups with one edge and one vertex, respectively with one edge and two vertices, yield HNN-extensions and respectively amalgamated products). Let us also mention that due to the work of G.G.Kasparov ([16] 8.4 page 36) the above mentioned results of this paper have consequences for the strong Novikov conjecture.

The case when G is arbitrary (locally compact, second countable) seems to be completely new, except for the computation of K_* of the reduced C^*-algebra of $SL_2(Q_p)$ done in [24] by using the explicit knowledge of the representation theory of this group. The most interesting examples seem to be the reductive groups over local fields with one dimensional Bruhat-Tits building [30]. This is another confirmation of Tits' philosophy that the building is the analogue of the symetric space G/K, and suggests the existence of a theory, parallel to that of Kasparov and Connes, with the building playing the role of G/K.

The paper is divided into four sections. The first one contains a brief analysis of actions of groups on trees and the necessary definitions and notations. It is mainly an adaption of the methods and results of P.Julg and A.Valette [11] to our purposes.

The second section shows how generalisations of the

Toeplitz extensions of [23] can be naturally constructed out of the tree X . We show that each subset of edges of the fundamental domain of X determines two Toeplitz extensions (one for the full and one for the reduced crossed product). As in [23] the exact sequences determined in KK-theory by these Toeplitz extensions yield the final results. This is done in the fourth section.

The third section is devoted to the computation of the KK-groups of the Toeplitz algebras. It will be clear in the fourth section that the total Toeplitz extensions(correspon-ding to all edges of the fundamental domain of X) play a special role, so it is enough to treat only this case. The use of the equivariant Kasparov groups KK^G makes the proof more natural and more general than that in [23] . Moreover one of the needed homotopies is the one exhibited by Julg and Valette in [11] .

Let us also mention that we get as a corollary of the main results, a generalisation of the theorem of Julg and Valette of [11] : namely the condition that the stabilizers be amenable is weakened to the condition that they be only K-amenable.

We have havily used Kasparov's equivariant KK^G-theory (for trivially graded C^*-algebras). The general references for the notations and results used in the paper are [12] , [13] and [14] . However for the exact sequences determined in KK by the Toeplitz extensions one needs the results as they appear in [29] .

I am grateful to J.Anderson for sharing his insight on this subject and for providing me with a copy of J.P.Serre's book [27] .

§ 1.

We shall denote by X^0 (respectively by X^1) the set of vertices (resp. of edges) of a tree. An orientation of the tree will be defined by specifying the origin and the terminus of each edge $y \in X^1$, i.e. by a map

$$X^1 \longrightarrow X^0 \times X^0 \quad , \qquad y \longmapsto (o(y) , t(y))$$

An oriented tree will be denoted simply by X . The opposite oriented tree will be denoted by X_{op} . It has the same vertices and edges as X , the origin and terminus of an edge being reversed. The disjoint union $Y = X^1 \amalg X^1_{op}$ is called the set of oriented edges of the tree X . Since X^1_{op} is another copy of X^1 , the identity map $X^1 \longrightarrow X^1_{op}$ determines a map

$$Y \longrightarrow Y \quad , \qquad y \longmapsto \bar{y}$$

such that $y \neq \bar{y}$ and $\bar{\bar{y}} = y$. The oriented edge $\bar{y}$ is called the inverse edge of y . Moreover the origin and terminus of an oriented edge is well defined, leading to a map

$$Y \longrightarrow X^0 \times X^0 \quad , \quad y \longmapsto (o(y) , t(y))$$

satisfying $o(y) = t(\bar{y})$. Finally, the modulus of an oriented edge $y \in Y$ is defined by

$$|y| = \begin{cases} y & \text{if } y \in X^1 \\ \bar{y} & \text{if } y \in X^1_{op} \end{cases}$$

The locally compact group G is said to act on the oriented tree X, if G acts continuously on both X^0 and X^1 (endowed with the discrete topology) and preserves the orientation. This means that if we denote by

$$G \times X^0 \ni (g,P) \longmapsto gP \in X^0$$
$$G \times X^1 \ni (g,y) \longmapsto gy \in X^1$$

the (left) actions of G on X^0 and respectively on X^1, then

$$o(gy) = g\, o(y) \qquad \text{and} \qquad t(gy) = g\, t(y) \quad .$$

<u>Remark</u>: It is well known that if G acts on a tree, then it acts also on an oriented tree. This is done by adding midpoints to the edges of the tree (barycentric subdivision) so that the group acts without inversion, in which case one can always find a G invariant orientation.

If G acts on the oriented tree X, then it acts also on X_{op}. We thus get an action of G on the set Y of oriented edges, denoted

$$G \times Y \ni (g,y) \longmapsto gy \in Y$$

which satisfies

$$g\bar{y} = \overline{gy},$$
$$o(gy) = g\, o(y), \quad t(gy) = g\, t(y),$$
$$g|y| = |gy|,$$

for every $g \in G$ and $y \in Y$. In particular the maps $y \longmapsto \bar{y}$ and $y \longmapsto |y|$ are G equivariant.

Definition 1. Every vertex $P \in X^0$ determines a map, still denoted by P,

$$P : X^0 \setminus \{P\} \longrightarrow Y$$

that sends each vertex $Q \in X^0$, $Q \neq P$, to the unique oriented edge $P(Q)$ that satisfies

i) $o(P(Q)) = Q$ and

ii) $t(P(Q))$ belongs to the unique geodesic joining Q and P.

We shall denote by $\bar{P} : X^0 \setminus \{P\} \longrightarrow Y$ the map $Q \mapsto \overline{P(Q)}$ and by $|P| : X^0 \setminus \{P\} \longrightarrow X^1$ the map $Q \mapsto |P(Q)|$.

Remark The map $|P|$ was introduced by P.Julg and A.Valette in [11]. The next lemma is essentially due to them.

Lemma 1. i) The map $X^0 \setminus \{P\} \ni Q \longmapsto |P|(Q) \in X^1$ is a bijection. Moreover Y is the disjoint union of the immage of P and of the immage of $\bar{P}$.

ii) The maps P and P' agree on the vertices that do not lie on the geodesic $[P,P']$.

iii) If g belongs to G, then

$g(P(g^{-1}Q)) = (gP)(Q)$; $g(|P|(g^{-1}Q)) = |gP|(Q)$ and

$g(\bar{P}(g^{-1}Q)) = (\overline{gP})(Q)$ for every $Q \neq gP$.

Proof: Part i) of this lemma is obvious, while part ii) and iii) follow from the fact that G maps geodesics into geodesics.

Definition 2. For every $P \in X^0$ we shall denote by X^1_P the intersection of the immage of the map P with X^1.

Thus X^1_P consists of all edges of X^1 that "point" to P, i.e. of those edges of X^1 whose terminus is closer to P than their origin (with respect to the natural metric on X^0).

Lemma 2. i) The sets X^1_P and $X^1_{P'}$ differ by a finite number of edges (lying on the geodesic $[P,P']$).

ii) If g belongs to G, then $g(X^1_P) = X^1_{gP}$.

Proof: Is a straightforward consequence of the definition and of lemma 1.

We shall denote by $\overline{X}^0$ the one point compactification of X^0. For X^1 it will be more convenient to consider the following two points compactification determined by the orientation: $\overline{X}^1$ denotes the set $X^1 \cup \{-\infty, +\infty\}$. A fundamental system of neighborhoods of $+\infty$ is given by finite intersections of sets $X^1_P \cup \{\infty\}$, while a fundamental system of neighborhoods of $-\infty$ is given by the complement of finite unions of sets $X^1_P \cup \{\infty\}$, $P \in X^0$. Note that if the tree is finite, then $-\infty$ and $+\infty$ are isolated points.

<u>Definition 3.</u> We shall denote by $C_+(X^1)$ the set of continuous functions $f \in C(\overline{X}^1)$ that vanish at $-\infty$. Similarly if E is a Banach space, $C_+(X^1,E)$ will have the obvious meaning. It is easy to see that $C_+(X^1)$ is generated by $C_o(X^1)$ and by the characteristic functions of the sets $X^1_P \cup \{\infty\}$, $P \in X^o$. These latter functions will be denoted by χ_P.

In view of lemma 2. it is easy to see that the action of G extends continuously to $\overline{X}^1$ by defining $g(+\infty) = +\infty$ $g(-\infty) = -\infty$ for every $g \in G$.

<u>Definition 4.</u> For each oriented edge $y \in Y$ we shall denote by X^o_y the set of those vertices P such that y belongs to the immage of the map P.

Thus X^o_y consists of those vertices P with the property that y "points" to P.

<u>Lemma 3.</u> i) X^o is the disjoint union of X^o_y and $X^o_{\overline{y}}$.

ii) If g belongs to G then $g(X^o_y) = X^o_{gy}$.

<u>Proof</u> : Straightforward.

We turn now to the known connection (see [27]) between groups acting on trees and graphs of groups. In order to fix the notation, let us recall the graph of groups associated to the action of G on X. ([27] I. 5.4)

We shall denote by Σ the oriented graph $G\backslash X$, i.e. :

$$\Sigma^o = G\backslash X^o \quad , \quad \Sigma^1 = G\backslash X^1$$

with origin and terminus maps given by :

$$\hat{o}(\hat{y}) = \widehat{o(y)} \qquad \text{and} \qquad \hat{t}(\hat{y}) = \widehat{t(y)} \quad .$$

We shall also fix a lifting of Σ . By this we shall mean the following :

1) We identify Σ^o and Σ^1 with subsets of X^o and respectively X^1 . (In particular we get maps $X^o \ni P \mapsto \hat{P} \in \Sigma^o \subset X^o$ and $X^1 \ni y \mapsto \hat{y} \in \Sigma^1 \subset X^1$ with the property that P and $\hat{P}$ (resp. y and $\hat{y}$) are conjugate by an element of G .

2) We fix for each $y \in X^1$ an element $\gamma_y \in G$, such that $\gamma_y\, y = \hat{y}$.

3) We fix for each $y \in \Sigma^1$ the edges $y^t, y^o \in X^1$ such that $\hat{y}^t = y = \hat{y}^o$ and $t(y^t), o(y^o) \in \Sigma^o$.

<u>Remark</u> Usually one takes a particular lifting of Σ by requiring that $o(y) \in \Sigma^o$ for every $y \in \Sigma^1$, and that Σ^o together with those $y \in \Sigma^1$ such that both $o(y)$ and $t(y)$ belong to Σ^o be a subtree of X .

Let G_P (resp. G_y) denote the stabilizer of the vertex P (resp. of the edge y). The graph of groups is then defined by G_P for $P \in \Sigma^o$ and G_y for $y \in \Sigma^1$, with homomorphisms

$$\sigma_y \; : G_y \longrightarrow G_{\hat{t}(y)} = G_{t(y^t)}$$
$$\sigma_{\bar{y}} \; : G_y \longrightarrow G_{\hat{o}(y)} = G_{o(y^o)}$$

defined by

$$\sigma_y(g) = \gamma_{yt}^{-1}\, g\, \gamma_{yt}$$

$$\sigma_{\bar{y}}(g) = \gamma_{y^o}^{-1}\, g\, \gamma_{y^o}$$

for every $g \in G$.

We conclude this section with a final definition.

<u>Definition 5.</u> For every $P \in \Sigma^o$ (resp. $y \in \Sigma^1$) we shall denote by X_P^o (resp. X_y^1) the orbit of P (resp. of y). Moreover, for every subset S of X^o (resp. of X^1) we shall denote by X_S^o (resp. X_S^1) the sets $\bigcup_{P \in \Sigma^o} X_P^o$ (resp. $\bigcup_{y \in \Sigma^1} X_y^1$).

By $\vec{X}_S^1 \subset \vec{X}^1$ we shall denote the space $X_S^1 \cup \{-\infty, +\infty\}$ with the induced totology.

By $C_+(X_S^1)$ we shall denote those functions $f \in C(\vec{X}_S^1)$ that vanish at $-\infty$, and $\chi_{P,S}$ will be the characteristic function of the set $\vec{X}_S^1 \cap (X_P^1 \cup \{+\infty\})$. (see definition 2.)

§ 2.

In this section we introduce the Toeplitz extensions for crossed products (both full and reduced) by groups acting on trees. First we shall prove some general facts about crossed products of short exact sequences.

Throughout this paper we shall denote by dg a fixed left Haar measure on G, and by $\Delta : G \longrightarrow \mathbb{R}^+$ the corresponding modular function. If A is a C^*-algebra on which G acts continuously by automorphisms, we shall use the notation of [14] and denote this action by

$$G \times A \ni (g,a) \longmapsto g(a) \in A .$$

We shall denote by $G \ltimes A$ (resp. $G \ltimes_r A$) the full (resp. the reduced) crossed product of A by this action of G [21] . Recall that if $C_c(G,A)$ denotes the set of continuous functions $k : G \longrightarrow A$ with compact support, then one defines the involution and respectively the convolution by

$$k^*(t) = \Delta(t)^{-1}\, t(k(t^{-1})^*)$$

$$k_1 * k_2(t) = \int_G k_1(s)\, s(k_2(s^{-1}t))\, ds$$

for every $t \in G$. If (π , U) is a covariant representation on a (right) Hilbert B-module $\mathcal{H}$, i.e. :

$\pi : A \longrightarrow \mathcal{L}(\mathcal{H})$ - is a representation of A ,

$G \ni g \longmapsto U_g \in \mathcal{L}(\mathcal{H})$ is a unitary representation of G , continuous in the strict topology, such that

$$U_g \pi(a) U_{g^{-1}} = \pi(g(a))$$

for every $g \in G$ and $a \in A$, then we shall denote by $\pi_G(k) \in \mathcal{L}(\mathcal{H})$ the operator defined by

$$\pi_G(k)\xi = \int_G \pi(k(s))\, U_s(\xi)\, ds \quad .$$

π_G is a $*$-representation of $C_c(G,A)$ on $\mathcal{L}(\mathcal{H})$, that extends to a $*$-representation (still denoted by π_G) of the full crossed product. If this representation factors through the reduced crossed product, we shall denote the corresponding representation by $\pi_{G,r}$. Similarly if $f : A \longrightarrow B$ is a G equivariant $*$-homomorphism, we shall denote by f_G (resp. $f_{G,r}$) the induced map on the full (resp. reduced) crossed products.

We shall also identify the multiplier algebra $\mathcal{M}(A)$ with $\mathcal{L}(A)$, where A is regarded as a right Hilbert A-module [12]. Following [14] we shall denote the induced action of G on $\mathcal{M}(A)$ also by $g \longmapsto g(x)$, and we shall say that the element $x \in \mathcal{M}(A)$ is G continuous, if the map $g \longmapsto g(x)$ is norm continuous.

Recall also from [[21] 7.6.2], that every $x \in \mathcal{M}(A)$ determines a multiplier $L(x)$ of both $G \ltimes A$ and $G \ltimes_r A$ by the formula :

$$L(x)k\,(t) = x\,k(t) \;,$$

i.e. as an element of $\mathcal{L}(G \ltimes A)$ (resp. $\mathcal{L}(G \ltimes_r A)$), $L(x)$ acts on the dense subset $C_c(G,A)$ by the above formula. (In fact, the above formula gives a multiplier on each C^*-

completion of $C_c(G,A)$). Moreover, the map $x \mapsto L(x)$ is a $*$-homomorphism.

If $I \subset A$ is a G invariant (i.e. $g(I) \subset I$ for every $g \in G$) closed two sided ideal, we shall identify $G \ltimes I$ (resp. $G \ltimes_r I$) with a closed two sided ideal in $G \ltimes A$ (resp. $G \ltimes_r A$).

Lemma 4. Let

$$0 \longrightarrow I \xrightarrow{i} A \xrightarrow{q} A/_I \longrightarrow 0$$

be an exact sequence of C^*-algebras. Suppose that G acts on I , A and $A/_I$, and that both i and q are G equivariant. Suppose moreover that q has a completely positive cross section φ with the following properties :

There exists a G equivariant $*$-representation $\rho : A/I \longrightarrow \mathcal{M}(A)$ and a G continuous projection $p \in \mathcal{M}(A)$ satisfying $a(p - g(p)) \in I$ for every $a \in A$, $g \in G$,such that

$$\varphi(x) = p\,\rho(x)p \ .$$

Then both sequences

$$0 \longrightarrow G \ltimes I \xrightarrow{i_G} G \ltimes A \xrightarrow{q_G} G \ltimes A/I \longrightarrow 0$$

and

$$0 \longrightarrow G \ltimes_r I \xrightarrow{i_{G,r}} G \ltimes_r A \xrightarrow{q_{G,r}} G \ltimes_r A/I \longrightarrow 0$$

are exact and both quotient maps admit a completely positive cross section of norm one.

Proof : It is straightforward that the maps q_G and $q_{G,r}$ are onto, so let us first prove the existence of the completely positive cross sections. The proofs for the full and reduced crossed products being the same, let us prove for simplicity only the second case.

It is easy to see that the map $\rho_G : C_c(G,A/I) \longrightarrow C_c(G,\mathcal{M}(A))$ defined by

$$\rho_G(k)(t) = \rho(k(t))$$

extends to a $*$-homomorphism $\rho_{G,r} : G \ltimes_r A/I \longrightarrow \mathcal{M}(G \ltimes_r A)$. It follows that the map $\varphi_{G,r} : G \ltimes_r A/I \longrightarrow \mathcal{M}(G \ltimes_r A)$ defined by

$$\varphi_{G,r}(x) = L(p)\,\rho_{G,r}(x)L(p)$$

is completely positive of norm one. Moreover if $k \in C_c(G,A/I)$ then

$$\varphi_{G,r}(k)(t) = L(p)\,\rho_{G,r}(k)L(p)\,(t) = p\,\rho(k(t))\,t(p) =$$

$$= \varphi(k(t)) + p\,\rho(k(t))\,(\,t(p) - p\,)\,.$$

Since $p\,\rho(x)\,\rho(x^*)p \in A$ for every $x \in A/I$, it follows that $p\,\rho(x) \in A$ for every $x \in A/I$, so that the above formula implies, on one hand that

$$\varphi_{G,r}(k) \in C_c(G,A) \quad \text{for every } k \in C_c(G,A/I)$$

and on the other hand that

$$q_{G,r} \circ \varphi_{G,r}(k) = k \quad \text{for every} \quad k \in C_c(G, A/I).$$

This shows that $\varphi_{G,r}$ takes values in $G \ltimes_r A$ and that $q_{G,r} \circ \varphi_{G,r}(x) = x$ for every $x \in G \ltimes_r A/I$.

The exactness of the sequences is now easily established : we have to prove exactness only at $G \ltimes A$ (resp. at $G \ltimes_r A$), and this in turn is equivalent to $x - \varphi_G \circ q_G(x) \in G \ltimes I$ for every $x \in G \ltimes A$ (resp. $x - \varphi_{G,r} \circ q_{G,r}(x) \in G \ltimes_r I$ for every $x \in G \ltimes_r A$). But the above relations have to be checked only on a dense subset, so let $k \in C_c(G,A)$. Then

$$k(t) - \varphi_G \circ q_G(k)(t) = k(t) - \varphi_{G,r} \circ q_{G,r}(k)(t) =$$
$$k(t) - \varphi(q(k(t))) - p\,\rho(q(k(t)))\,(\,t(p) - p\,)$$

Since both $k(.) - \varphi(q(k(.)))$ and $p\,\rho(q(k(.)))\,(\,.(p) - p\,)$ belong to $C_c(G,I)$, we get the desired result.

<u>Remark</u> The exactness of the sequence corresponding to full crossed products is true in general [28] . However we shall need the completely positive cross section in the sequel.

Suppose now that $G \times Z \ni (g,z) \longmapsto gz \in Z$ is a continuous action (on the left) of G on the discrete set Z . Since the stabilizers G_z of every point $z \in Z$ are open subgroups, the restriction of dg to G_z (still denoted by dg) is a left Haar measure on G_z . We shall denote by S the orbit space $G \backslash Z$, by $z \longmapsto \hat{z}$ the quotient map and we shall moreover identify S with a fixed transversal $S \subset Z$ for the action of G . For each $z \in Z$ we shall fix an element $\gamma_z \in G$ such that $\gamma_z z = \hat{z}$, and we shall denote by Z_s

$s \in S$, the orbit of s .

We shall record for further use the following proposition originally due to P.Green [10].

Proposition 5. i) The following isomorphisms hold :

$$G \ltimes C_o(Z,A) \simeq \bigoplus_{s \in S} (G_s \ltimes A) \otimes \mathcal{K}(l^2(Z_s))$$

$$G \ltimes_r C_o(Z,A) \simeq \bigoplus_{s \in S} (G_s \ltimes_r A) \otimes \mathcal{K}(l^2(Z_s))$$

where $\mathcal{K}(\mathcal{H})$ denotes the algebra of compact operators on $\mathcal{H}$, and where the action of G on $C_o(Z,A)$ is defined by $g(f)(z) = g(f(g^{-1}z))$ for every $f \in C_o(Z,A)$, $g \in G$ and $z \in Z$.

ii) If we regard $C_c(G \times Z,A)$ as a subalgebra of $C_c(G,C_o(Z,A))$, then one may describe the above isomorphisms explicitely by

$$\phi(k) = \sum_{s \in S} k^s_{z',z''} \otimes e_{z',z''}$$

for $k \in C_c(G \times S,A)$ and $z',z'' \in Z$ satisfying $\hat{z}' = \hat{z}'' = s$, where $e_{z',z''}$ is the canonical matrix unit in $\mathcal{K}(l^2(Z_s))$, and where $k^s_{z',z''} \in C_c(G_s,A)$ is defined by

$$k^s_{z',z''}(g) = \gamma_{z'}(k(\gamma^{-1}_{z'}\, g\, \gamma_{z''}\, , z'))\, \Delta(\gamma_{z''})$$

for every $g \in G_s$.

We shall construct for every subset $S \in \Sigma^1$, $S \neq \emptyset$, two Toeplitz extensions, one for the full and one for the reduced crossed product of A by the action of G . We start with the exact sequence

$$0 \longrightarrow C_o(X_S^1,A) \xrightarrow{i} C_+(X_S^1,A) \xrightarrow{q} A \longrightarrow 0$$

where i is the natural inclusion and q the evaluation map at $+\infty$. (see also definition 5.) . Note that q has a completely positive cross section φ , defined by

$$\varphi(a) = \chi_{P,S}\, \rho(a)\, \chi_{P,S}$$

where $\rho(a)$ is the constant function $\rho(a)(y) = a$ for every $y \in \bar{X}_S^1$, and $\chi_{P,S}$ appears in definition 5. Lemma 2. shows that $\chi_{P,S}$ is G continuous and that $f(\chi_{P,S} - g(\chi_{P,S})) = f(\chi_{P,S} - \chi_{gP,S})$ belongs to $C_o(X_S^1,A)$ for every $f \in C_+(X_S^1,A)$, so that we may apply lemma 4. to get the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & G \ltimes C_o(X_S^1,A) & \xrightarrow{i_G} & G \ltimes C_+(X_S^1,A) & \xrightarrow{q_G} & G \ltimes A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G \ltimes_r C_o(X_S^1,A) & \xrightarrow{i_{G,r}} & G \ltimes_r C_+(X_S^1,A) & \xrightarrow{q_{G,r}} & G \ltimes_r A & \longrightarrow & 0
\end{array}$$

with exact top and bottom sequences. Applying proposition 5. for the set X_S^1 with transversal S and elements γ_y , $y \in X_S^1$, provided by the construction of the graph of groups determined by G (see section 1.) we get the following Toeplitz extensions.

Proposition 6. For every nonvoid subset $S \in \Sigma^1$, there is a commutative diagram with exact horizontal sequences :

$$\begin{array}{ccccccccc}
0 \to & \bigoplus_{y\in S}((G_y \ltimes A)\otimes \mathcal{K}(l^2(X^1_y))) & \xrightarrow{\hat{i}_G} & G\ltimes C_+(X^1_S,A) & \xrightarrow{q_G} & G\ltimes A & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \to & \bigoplus_{y\in S}((G_y \ltimes_r A)\otimes \mathcal{K}(l^2(X^1_y))) & \xrightarrow{\hat{i}_{G,r}} & G\ltimes_r C_+(X^1_S,A) & \xrightarrow{q_{G,r}} & G\ltimes_r A & \to 0
\end{array}$$

where :

- The vertical homorphisms are determined by the projection from the full crossed product onto the reduced one.

- The maps $\hat{i}_G$ and $\hat{i}_{G,r}$ act in the following way : if $k \in C_c(G_y,A)$, then $\hat{i}_G(k\otimes e_{y',y''}) = \hat{i}_{G,r}(k\otimes e_{y',y''}) \in$ $\in C_c(G \times \bar{X}^1_S, A)$ ($\hat{y}' = \hat{y}'' = y$) is the function

$$\hat{i}_G(k\otimes e_{y',y''})(g,z) = \hat{i}_{G,r}(k\otimes e_{y',y''})(g,z) =$$

$$= \begin{cases} \gamma^{-1}_{y'}(k(\gamma_{y'}\, g\gamma^{-1}_{y''}))\,\Delta(\gamma^{-1}_{y''}) & \text{if } g\in \gamma^{-1}_{y'} G_y\, \gamma_{y''} \text{ and } z = y' \\ 0 & \text{otherwise} \end{cases}$$

Moreover, the maps q_G and $q_{G,r}$ have both a completely positive cross section of norm one.

<u>Example</u> In the case of the free group on n generators, the graph Σ has one vertex and n edges $y_1,\dots,y_n$. The extension determined by $S = \{y_k\}$ was called in [23] the k^{th}-Toeplitz extension, while the extension corresponding to $S = \Sigma^1$ appears in the last section of [23].

<u>Remark</u> In order to get precisely the extensions of [23],

one should compress the above extension with the projection $\chi_{0,S}$, for some fixed origin $0 \in X^0$. The above form for the Toeplitz extension has been suggested in the case $G = \mathbb{Z}$ by M.A.Rieffel [25].

It is easy to pass from the above Toeplitz extension to the compressed one, since the latter is a full corner in the former.

§ 3.

In this section we show that the K-theory of the Toeplitz algebra $G \ltimes C_+(X^1, A)$ (resp. $G \ltimes_r C_+(X^1, A)$) is isomorphic to that of the algebra $\bigoplus_{P \in \Sigma^0} (G_P \ltimes A)$ (resp. $\bigoplus_{P \in \Sigma^0} (G_P \ltimes_r A)$). (note that $X^1_{\Sigma^1} = X^1$) . To this end we shall construct the elements $\alpha \in KK^G(C_0(X^0), C_+(X^1))$ and $\beta \in KK^G(C_+(X^1), C_0(X^0))$ [14] and eventually prove that one is almost the inverse of the other. Since we are working with KK^G , we shall from now on assume that the group G is second countable and that the tree X is countable.

We start with some remarks concerning KK^G theory.

Remark 1. i) It is well known that every G equivariant $*$-homomorphism $f : A \to B$, or more generally $f : A \to \mathcal{K}(\mathcal{E})$, where $\mathcal{E}$ is a right Hilbert B-module, determines a class $[f]$ in $KK^G(A,B)$. The class $[f]$ may be defined by the triplet $(\mathcal{E}, f, 0)$, where the grading of $\mathcal{E}$ is trivial (i.e. the whole Hilbert module $\mathcal{E}$ is considered positive).

If $g : B \to \mathcal{K}(\mathcal{F})$ is another G equivariant $*$-homomorphism, then $[f] \otimes_B [g]$ is equal to the class of the $*$-homomorphism $f \otimes 1 : A \to \mathcal{K}(\mathcal{E} \otimes_B \mathcal{F})$. (Note that we follow the convention of [13], so that if $f : A \to B$ and $g : B \to C$, then the class of $g \circ f$ corresponds to $[f] \otimes_B [g]$).

ii) It is obvious from the definitions, that if the triplet $(\mathcal{E}, \varphi, T)$ representing an element in $KK^G(A,B)$ has the property that $\varphi(A) \subset \mathcal{K}(\mathcal{E})$, then $(\mathcal{E}, \varphi, 0)$ represents the same element. This shows that if we decompose $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1$, $\varphi = \varphi_0 \oplus \varphi_1$ (according to the grading) then

$$[(\mathcal{E}, \varphi, T)] = [\varphi_0] - [\varphi_1] \quad .$$

Let us first define the element $\alpha \in KK^G(C_0(X^0), C_+(X^1))$. To this end consider the right Hilbert $C_+(X^1)$-module

$$l^2(X^0) \otimes C_+(X^1)$$

endowed with the product G-action. (In accordance with the definition of right Hilbert modules, the scalar product in a Hilbert space will be linear in the second variable.) Let $\{e_P\}_{P \in X^0}$ denote the canonical basis of $l^2(X^0)$, and define $d : C_0(X^0) \longrightarrow \mathcal{K}(l^2(X^0) \otimes C_+(X^1))$ by the formula

$$d(f)\, e_P \otimes \xi \quad = \quad f(P)\, e_P \otimes \chi_P \cdot \xi$$

for every $f \in C_0(X^0)$, $\xi \in C_+(X^1)$ and $P \in X^0$.

<u>Definition 6.</u> The class of d in $KK^G(C_0(X^0), C_+(X^1))$ (see remark 1.) will be denoted by α.

We turn now to the definition of the element β . For each $P \in X^0$ denote by $E_P \subset X^1$ the set of edges with one extremity P . Thus :

$$E_P = \{y \in X^1 \; ; \; o(y) = P\} \amalg \{y \in X^1 \; ; \; t(y) = P\} \quad .$$

If y belongs to E_P, we shall denote by $e_{y,P}$ the corresponding unit vector in $l^2(E_P)$. (Again the scalar product is linear in the second variable). The sections of the field of Hilbert spaces $\{l^2(E_P)\}_{P\in X^o}$, that tend to zero at infinity, is in a natural way a right Hilbert $C_o(X^o)$-module which will be denoted by E. It is generated by the sections $e_{y,P}$ for $P\in X^o$ and $y\in E_P$, where $e_{y,P}$ is identified with the section

$$e_{y,P}(Q) = \begin{cases} e_{y,P} & \text{if } P = Q \\ 0 & \text{otherwise} . \end{cases}$$

The action of the group G on X, gives the following natural action of G on E:

$$g\, e_{y,P} = e_{gy,gP} .$$

We shall denote by $\psi_+ , \psi_- : C(\bar{X}^1) \longrightarrow \mathcal{L}(E)$ the $*$-homomorphisms defined by :

$$\psi_+(f)\, e_{y,P} = \begin{cases} f(y)\, e_{y,P} & \text{if } o(y) = P \\ f(\infty)\, e_{y,P} & \text{if } t(y) = P \end{cases}$$

$$\psi_-(f)\, e_{y,P} = \begin{cases} f(-\infty)\, e_{y,P} & \text{if } o(y) = P \\ f(y)\, e_{y,P} & \text{if } t(y) = P . \end{cases}$$

Since G preserves the orientation of the tree, it is easy to see that both ψ_+ and ψ_- are G equivariant.

Let now $O \in X^0$ be a vertex and denote by $T_O \in \mathcal{L}(C_o(X^0), E)$ the operator

$$T_O(\xi)(P) = \begin{cases} \xi(P)\, e_{]O[(P),P} & \text{if } P \neq O \\ o & \text{if } P = O\, ; \end{cases}$$

for every $\xi \in C_o(X^0)$ (see definition 1.). Here $C_o(X^0)$ is regarded as a Hilbert $C_o(X^0)$-module in the usual way [12]. Note that in this case, $C_o(X^0)$ is naturally isomorphic to $\mathcal{K}(C_o(X^0))$, so that we may consider $C_o(X^0) \subset \mathcal{L}(C_o(X^0))$.

<u>Lemma 7.</u> The following formulae hold :

i) $1 - T_O^* T_O = \delta_O$, where $\delta_O \in C_o(X^0)$ is the usual delta function.

ii) $\psi_+(\chi_O) = \psi_-(\chi_O) + T_O T_O^*$; in particular $\psi_+(\chi_O)\, T_O = T_O$ and $\psi_-(\chi_O)\, T_O = o$.

iii) $T_O - T_{O'} \in \mathcal{K}(C_o(X^0), E)$

iv) $g(T_O) = T_{gO}$ for every $g \in G$.

<u>Proof</u> : i) The definition of the $C_o(X^0)$-valued inner product of E shows that

$$\langle T_O(\xi) \mid T_O(\eta) \rangle (P) = \begin{cases} \bar{\xi}(P)\, \eta(P) & \text{if } P \neq O \\ o & \text{if } P = O \end{cases}$$

so that $\langle T_O^* T_O(\xi) \mid \eta \rangle = \langle T_O(\xi) \mid T_O(\eta) \rangle = \langle (1 - \delta_O)\xi \mid \eta \rangle$ for every $\xi, \eta \in C_o(X^0)$.

ii) It is easy to see from the definition of T_O, that

$$T_O T_O^* e_{y,P} = \begin{cases} e_{]O[(P),P} & \text{if } P \neq O \text{ and } y = \,]O[(P) \end{cases}$$

The definition of χ_0 (and of course the fact that X is a tree) implies that if $o(y) = P$, then

$$\chi_0(y) = \begin{cases} 1 & \text{if } y = O(P) = |O|(P) \\ 0 & \text{otherwise} \end{cases}$$

while if $t(y) = P$, then

$$\chi_0(y) = \begin{cases} 0 & \text{if } y = \overline{O}(P) = |O|(P) \\ 1 & \text{otherwise} \end{cases} .$$

Point ii) follows now easily from the definition of ψ_- and ψ_+.

iii) This is a straightforward consequence of lemma 1.

iv) Recall from [14], that $g(T_0)(\xi)$ is by definition equal to $g(T_0(g^{-1}\xi))$, for every $\xi \in C_0(X^0)$. It is sufficient to verify the above relation for $\xi = \delta_P$, $P \in X^0$. In this case $g^{-1}\delta_P = \delta_{g^{-1}P}$, and

$$T_0(\delta_{g^{-1}P}) = \begin{cases} e_{|O|(g^{-1}P),g^{-1}P} & \text{if } g^{-1}P \neq 0 \\ 0 & \text{if } g^{-1}P = 0 \end{cases}$$

so that

$$g(T_0(g^{-1}\delta_P)) = \begin{cases} e_{g(|O|(g^{-1}P)),P} & \text{if } P \neq g0 \\ 0 & \text{if } P = g0 \end{cases} .$$

The result now follows from lemma 1. .

The triplet $(\mathcal{E}, \psi, \tilde{T})$ defining $\beta \in KK^G(C_+(X^1), C_o(X^o))$ is now obtained as follows : the graded Hilbert $C_o(X^o)$-module

$\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1$ consists of two copies of $E \oplus C_0(X^0)$, $\mathcal{E}_0$ being positive and $\mathcal{E}_1$ negative. The graded G equivariant $*$-homomorphism $\psi : C(\bar{X}^1) \longrightarrow \mathcal{L}(\mathcal{E})$ is given by $\psi = \psi_0 \oplus \psi_1$, where $\psi_0 : C(X^1) \longrightarrow \mathcal{L}(\mathcal{E}_0)$, is defined by

$$\psi_0(f) = \psi_-(f) \oplus f(+\infty)$$

and $\psi_1 : C(\bar{X}^1) \longrightarrow \mathcal{L}(\mathcal{E}_1)$ by

$$\psi_1(f) = \psi_+(f) \oplus f(-\infty) .$$

Finally to describe the operator $\tilde{T} \in \mathcal{L}(\mathcal{E})$ of degree one, it will be sufficient to define $T \in \mathcal{L}(\mathcal{E}_0, \mathcal{E}_1)$, $\tilde{T}$ being then determined by the matrix

$$\begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix} .$$

For T we shall fix some origin $0 \in X^0$ and take the operator

$$T = T_0 + T_0^* + 1 - (T_0 T_0^* + T_0^* T_0) ,$$

i.e. $T(e \oplus \xi) = (T_0 \xi + (1 - T_0 T_0^*) e) \oplus (T_0^* e + (1 - T_0^* T_0)\xi)$

for every $e \in E$ and $\xi \in C_0(X^0)$.

Lemma 8. The triplet $(\mathcal{E}, \psi, \tilde{T})$ defines an element in $KK^G(C(\overline{X}^1), C_0(X^0))$.

Proof : The preceding lemma shows that T (and hence $\tilde{T}$) is G continuous and that $g(\tilde{T}) - \tilde{T} \in \mathcal{K}(\mathcal{E})$, for every $g \in G$. Since T is easyly seen to be unitary, the only thing left to prove is that the commutator $[\psi(f), \tilde{T}]$ belongs to $\mathcal{K}(\mathcal{E})$, for every $f \in C(\overline{X}^1)$, i.e. that

$$T\psi_0(f) - \psi_1(f)T \in \mathcal{K}(\mathcal{E}_0, \mathcal{E}_1)$$

for every $f \in C(\overline{X}^1)$. Recall the definition of $\overline{X}^1$ (definition 3.) and lemma 2., to see that it is sufficient toprove the above statement for the constant functions, for $f \in C_0(X^1)$ and for a single function of the type χ_P , $P \in X^0$. It is easy to see that ψ_i are unital and that $\psi_i(f) \in \mathcal{K}(\mathcal{E}_i)$ for every $f \in C_0(X^1)$, $i = 0,1$. This settles the first two cases. For the remaining one, chose $P = 0$, the vertex appearing in the definition of T_0 . Then

$$T\psi_0(\chi_0)(e \oplus \xi) = T((\psi_-(\chi_0)e) \oplus \xi) =$$
$$= (T_0\xi + (1 - T_0T_0^*)\psi_-(\chi_0)e) \oplus (T_0^*\psi_-(\chi_0)e + (1 - T_0^*T_0)\xi)$$

On the other hand

$$\psi_1(\chi_0)T(e \oplus \xi) = \psi_1(\chi_0)(T_0\xi + (1 - T_0T_0^*)e) \oplus (T_0^*e +$$
$$+ (1 - T_0^*T_0)\xi) = (\psi_+(\chi_0)(T_0\xi + (1 - T_0T_0^*)e)) \oplus 0$$

Applying the preceding lemma i) and ii) we see that

$$T\psi_0(\chi_0)(e\oplus\xi) = (T_0\xi + \psi_-(\chi_0)e)\oplus \delta_0\xi$$

$$\psi_1(\chi_0)T(e\oplus\xi) = (T_0\xi + \psi_+(\chi_0)e - T_0T_0^*e)\oplus 0$$

Again by point ii) of the preceding lemma we get that

$$(T\psi_0(\chi_0) - \psi_1(\chi_0)T)(e\oplus\xi) = \delta_0\xi$$

which is clearly compact.

<u>Definition 7.</u> The class in $KK^G(C_+(X^1), C_o(X^o))$ determined by the restriction of the triplet $(\mathcal{E}, \psi, \tilde{T})$ to $C_+(X^1)$ will be denoted by β.

Our next goal is to show that $\alpha\otimes\beta\in KK^G(C_o(X^o), C_o(X^o))$ equals $1_{C_o(X^o)}$ [[14] Theorem 5. and Consequence 1] i.e. coincides with the class determined by the identic map $id : C_o(X^o)\longrightarrow C_o(X^o)$. To this end let us first explicitly describe a representative $(\mathcal{G}, \mu, \tilde{V})$ of $\alpha\otimes\beta$. As usual we shall describe the grading by putting $\mathcal{G} = \mathcal{G}_o\oplus\mathcal{G}_1$, $\mu = \mu_o\oplus\mu_1$ with $\mu_i : C_o(X^o)\longrightarrow\mathcal{L}(\mathcal{G}_i)$, and

$$\tilde{V} = \begin{pmatrix} 0 & V^* \\ V & 0 \end{pmatrix} \quad \text{with} \quad V\in\mathcal{L}(\mathcal{G}_o, \mathcal{G}_1).$$

Due to the particular form of α, it is easy to see that one may take as Hilbert $C_o(X^o)$-modules :

$$(l^2(\bar{X}^o)\otimes C(\bar{X}^1))\otimes_{C(\bar{X}^1)}\mathcal{E}_i \qquad i = 0,1$$

the $*$-homomorphisms from $C(\bar{X}^1)$ to $\mathcal{L}(\mathcal{E}_i)$ being ψ_i.

Since ψ_i are unital one sees that the above module is isomorphic to $l^2(X^o) \otimes \mathcal{E}_i$, via the map that sends $(x \otimes f) \otimes e$ to $x \otimes \psi_i(f)e$. We thus take

$$\mathcal{G}_i = l^2(X^o) \otimes \mathcal{E}_i \qquad i = o,1$$

Note that $\mathcal{G}_o = \mathcal{G}_1$; we shall denote it sometimes shortly by $\mathcal{G}'$. The maps μ_i are obtained by composing the map $d \otimes 1 : C_o(X^o) \longrightarrow \mathcal{L}((l^2(X^o) \otimes C(\bar{X}^1)) \otimes_{C(\bar{X}^1)} \mathcal{E}_i)$ with the map induced by the above described isomorphisms. (The explicit formula will be given below). Finally the operator $V \in \mathcal{L}(l^2(X^o) \otimes \mathcal{E}_o, l^2(X^o) \otimes \mathcal{E}_1)$ is defined by $V = 1 \otimes T$.

It will be more convenient to regard the Hilbert module $\mathcal{G}'$ as sections in the corresponding field of Hilbert spaces over X^o . Recall the definition of E to see that the fibre over the point $P \in X^o$ is

$$\mathcal{G}'_P = l^2(X^o) \otimes l^2(E_P) \oplus l^2(X^o)$$

and that the sections $e_{Q,y,P}$, $Q,P \in X^o$ $y \in E_P$, and $e_{Q,P}$, $Q,P \in X^o$, defined by

$$e_{Q,y,P}(R) = \begin{cases} e_Q \otimes e_y & \text{if } R = P \\ o & \text{if } R \neq P \end{cases}$$

$$e_{Q,P}(R) = \begin{cases} e_Q & \text{if } R = P \\ o & \text{if } R \neq P \end{cases}$$

generate the Hilbert module $\mathcal{G}'$. Moreover, the action of G

may be described on these sections by :

$$g\, e_{Q,y,P} = e_{gQ,gy,gP} \quad , \quad g\, e_{Q,P} = e_{gQ,gP} \quad .$$

Recall now the definitions of ψ_i , χ_p , and of the sets X_y^o (definition 4.) to get the following formulae for μ_i , $i = o,1$, :

$$\mu_o(f)\, e_{Q,y,P} = \begin{cases} f(Q)\, e_{Q,y,P} & \text{if } t(y) = P \text{ and } Q \in X_y^o \\ o & \text{otherwise} \end{cases}$$

$$\mu_o(f)\, e_{Q,P} = f(Q)\, e_{Q,P} \quad .$$

$$\mu_1(f)\, e_{Q,y,P} = \begin{cases} f(Q)\, e_{Q,y,P} & \text{if } t(y) = P \\ f(Q)\, e_{Q,y,P} & \text{if } o(y) = P \text{ and } Q \in X_y^o \\ o & \text{otherwise} \end{cases}$$

$$\mu_1(f)\, e_{Q,P} = o$$

for every $f \in C_o(X^o)$.

Let us finally record the formula of V : If $V_0 \in \mathcal{L}(\mathcal{G}_o, \mathcal{G}_1) = \mathcal{L}(\mathcal{G}')$ denotes the operator

$$V_0\, e_{Q,y,P} = o$$

$$V_0\, e_{Q,P} = \begin{cases} e_{Q,|O(P)|,P} & \text{if } P \neq 0 \\ o & \text{if } P = 0 \end{cases}$$

then $$V = V_0 + V_0^* + 1 - V_0 V_0^* - V_0^* V_0 \quad .$$

We proceed now to the proof that $\alpha\otimes\beta$ equals $1_{C_o(X^o)}$. To this end we shall first construct a $W \in \mathcal{L}(\mathcal{G}')$ such that the class of $(\mathcal{G}, \mu, \tilde{W})$ is equal to $1_{C_o(X^o)}$, and afterwards exhibit a homotopy connecting the above triplet to $(\mathcal{G}, \mu, \tilde{V})$. The definition of W is as follows : Let $W' \in \mathcal{L}(\mathcal{G}')$ be defined by

$$W' e_{Q,y,P} = o$$

$$W' e_{Q,P} = \begin{cases} e_{Q,|Q(P)|,P} & \text{if } Q \neq P \\ o & \text{if } Q = P \end{cases}$$

then W is given by

$$W = W' + W'^{*} + 1 - W'W'^{*} - W'^{*}W' .$$

<u>Lemma 9.</u> i) The above defined $W \in \mathcal{L}(\mathcal{G}')$ is a G equivariant unitary.

ii) Let $v \in \mathcal{L}(C_o(X^o), \mathcal{G}')$ be the G equivariant isometry defined by $v(\delta_P) = e_{P,P}$, $P \in X^o$. Then

$$W\mu_o(f) - \mu_1(f)W = v f v^{*}$$

for every $f \in C_o(X^o)$.(Here f is also considered as an operator in $\mathcal{K}(C_o(X^o)) \subset \mathcal{L}(C_o(X^o))$.)

iii) The class of the element $(\mathcal{G}, \mu, \tilde{W})$ in $KK^G(C_o(X^o), C_o(X^o))$ is equal to $1_{C_o(X^o)}$.

<u>Proof</u> : i) W' being a partial isometry whose range is orthogonal to its domain, it is obvious that W is unitary. The G equivariance follows from the fact that

$$g\, e_{Q,|Q(P)|,P} = e_{gQ,g(|Q(P)|),gP} = e_{gQ,|gQ(gP)|,gP}$$

which in turn follows from lemma 1.

ii) Let us first note that $W'\mu_0(f) = \mu_1(f)W'$ and that $W'\mu_1(f) = \mu_0(f)W' = 0$ for every $f \in C_0(X^0)$. Indeed :

$$W'\mu_0(f)\, e_{Q,y,P} = \mu_1(f)W'\, e_{Q,y,P} = 0 \quad \text{, while}$$

$$W'\mu_0(f)\, e_{Q,P} = f(Q)\, W' e_{Q,P} = \begin{cases} f(Q)\, e_{Q,|Q(P)|,P} & \text{if } Q \neq P \\ 0 & \text{if } Q = P \end{cases}$$

and

$$\mu_1(f)\, W'\, e_{Q,P} = \begin{cases} \mu_1(f)\, e_{Q,|Q(P)|,P} & \text{if } Q \neq P \\ 0 & \text{if } Q = P\ . \end{cases}$$

But if the origin of $|Q(P)|$ is P, then Q lies in $X^0_{|Q(P)|}$ so that we get the first equality. The equality $W'\mu_1(f) = 0$ being obvious, let us show that $\mu_0(f)\, W' = 0$. We have to consider only sections $e_{Q,P}$ with $Q \neq P$, and we get :

$$\mu_0(f)\, W' e_{Q,P} = \mu_0(f)\, e_{Q,|Q(P)|,P}\ .$$

Since it is impossible to have both $t(|Q(P)|) = P$ and $Q \in X^0_{|Q(P)|}$, it follows that $\mu_0(f)\, W' e_{Q,P} = 0$.

The above intertwining relations show that

$$W\mu_o(f) - \mu_1(f)\,W = (\mu_o(f) - \mu_1(f))(1 - W'W'^* - W'^*W')$$

But $(1 - W'W'^* - W'^*W')$ being the projection onto the sugspace generated by the sections $\{e_{P,P} ; P \in X^o\}$, $\{e_{Q,y,P} ; o(y) = P , Q \notin X^o_y\}$ and $\{e_{Q,y,P} ; t(y) = P, Q \in X^o_y\}$ we finally get

$$(W\mu_o(f) - \mu_1(f)W)\,e_{Q,P} = \begin{cases} f(P)\,e_{P,P} & \text{if } Q = P \\ o & \text{if } Q \neq P \end{cases}$$

$$(W\mu_o(f) - \mu_1(f)W)\,e_{Q,y,P} = o\,.$$

This is clearly the same as $v\cdot f\cdot v^*$ so that ii) is completely proved.

iii) This is a straightforward consequence of ii) once we rewrite it as

$$\mu_o(f) = W^*\mu_1(f)W \oplus v\cdot f\cdot v^*$$

<u>Proposition lo.</u> If $\alpha \in KK^G(C_o(X^o),C_+(X^1))$ and $\beta \in KK^G(C_+(X^1),C_o(X^o))$ are the elements introduced in definitions 6. and 7. , then

$$\alpha \otimes \beta = 1_{C_o(X^o)}\ .$$

<u>Proof</u> : Let V_0 and W' be the partial isometries appearing in the definition of V and respectively W, and denote by U' the partial isometry $V_0W'^*$. Since both V and W, together with $\mathcal{G}$ and μ define elements in $KK^G(C_0(X^0),C_0(X^0))$, it is easy to get the following properties of U' :

1. U' is G continuous,
2. $[\mu_i(f),U'] \in \mathcal{K}(\mathcal{G}')$, $i = 0,1$, $f \in C_0(X^0)$,
3. $(g(U') - U')\mu_i(f) \in \mathcal{K}(\mathcal{G}')$ and

$\mu_i(f)(g(U') - U') \in \mathcal{K}(\mathcal{G}')$, $i = 0,1$, $f \in C_0(X^0)$.

Note next that the elements satisfying the above three properties form a C^*-algebra, and that the projections $U'U'^*$ and U'^*U' mutually commute. In particular, there is a continuous path of unitaries in this C^*-algebra, connecting the selfadjoint unitary

$$U_1 = U' + U'^* + 1 - U'U'^* - U'^*U'$$

to the identity. It follows that if we define the continuous path of unitaries W_t by the formula

$$W_t = U_tW' + (U_tW')^* + 1 - (U_tW')(U_tW')^* - (U_tW')^*(U_tW')$$

then $W_0 = W$ and $W_1 = V$. This yields a homotopy connecting $(\mathcal{G},\mu,\tilde{V})$ to $(\mathcal{G},\mu,\tilde{W})$. The proposition now follows from the preceding lemma .

We turn now to the properties of the element $\beta \otimes \alpha$. Again we shall first explicitly describe a triplet $(\mathcal{H}, \eta, \tilde{U})$ whose class in $KK^G(C_+(X^1), C_+(X^1))$ is $\beta \otimes \alpha$. Due to the particular form of α, it is easy to describe $(\mathcal{H}, \eta, \tilde{U})$ formally :

$\mathcal{H} = \mathcal{H}_o \oplus \mathcal{H}_1$, where $\mathcal{H}_i = \mathcal{E}_i \otimes_{C_o(X^o)} (l^2(X^o) \otimes C_+(X^1))$, $i = o,1$, (with $*$-homomorphism $d : C_o(X^o) \longrightarrow \mathcal{L}(l^2(X^o) \otimes C_+(X^1))$

$\eta = \eta_o \oplus \eta_1$, where $\eta_i(f) = \Psi_i(f) \otimes 1$, $i = o,1$

$\tilde{U} = \begin{pmatrix} o & U^* \\ U & o \end{pmatrix}$, where $U = T \otimes 1$.

Recall that $\mathcal{E}_o = \mathcal{E}_1 = E \oplus C_o(X^o)$ and that E is generated by the sections $e_{y,P}$, with $P \in X^o$ and $y \in E_P$. It is easy to see that the map that sends

$$e_{y,P} \otimes_{C_o(X^o)} \xi \quad \text{to} \quad e_y \otimes d(\delta_P)\xi$$

defines a G equivariant isometry from $E \otimes_{C_o(X^o)} (l^2(X^o) \otimes C_+(X^1))$ to $l^2(X^1) \otimes l^2(X^o) \otimes C_+(X^1)$, whose immage is the Hilbert module generated by the vectors $e_y \otimes e_P \otimes \chi_P f$, with $P \in X^o$, $y \in E_p$ and $f \in C_+(X^1)$. Let us denote this Hilbert $C_+(X^1)$-module by $\mathcal{H}'$. By $\mathcal{H}''$, we shall denote the Hilbert $C_+(X^1)$-module generated in $l^2(X^o) \otimes C_+(X^1)$ by the elements $e_P \otimes \chi_P f$. This is the immage of $C_o(X^o) \otimes_{C_o(X^o)} (l^2(X^o) \otimes C_+(X^1))$ under the G equivariant isometry that sends $\delta_P \otimes_{C_o(X^o)} \xi$ to $d(\delta_P)\xi$. ($\xi \in l^2(X^o) \otimes C_+(X^1)$).

We shall identify $\mathcal{H}_o = \mathcal{H}_1$ with $\mathcal{H}' \oplus \mathcal{H}''$. Note next that $l^2(X^o) \otimes l^2(X^1) \otimes C_+(X^1)$ (resp. $l^2(X^o) \otimes C_+(X^1)$)

may be regarded as the Hilbert module of sections of the constant continuous field of Hilbert spaces over $\bar{X}^1$, with fibre $l^2(X^1)\otimes l^2(X^o)$ (resp. $l^2(X^o)$) that vanish at $-\infty$. It follows that $\mathcal{H}'$ (resp $\mathcal{H}''$) may be identified with those sections s, satisfying $s(y)\in \mathcal{H}'_y$ (resp. $s(y)\in \mathcal{H}''_y$), where $\mathcal{H}'_y$ (resp. $\mathcal{H}''_y$) is the Hilbert space generated by the vectors $e_z\otimes e_P$ (resp. e_P), satisfying $z\in E_P$ and $P\in X^o_y$ (resp. $P\in X^o_y$), with the convention $X^o_\infty = X^o$ and $X^o_{-\infty} = \emptyset$.

Note that for every $y\in \bar{X}^1$, the Hilbert space $\mathcal{H}'_y$ is the direct sum of $\mathcal{H}'^{o}_y$ and $\mathcal{H}'^{t}_y$, where $\mathcal{H}'^{o}_y$ (resp. $\mathcal{H}'^{t}_y$) is generated by the vectors $e_z\otimes e_P$ with the property that $o(z) = P$ (resp. $t(z) = P$). Let us denote by $e^o_{z,y}$ (resp. $e^t_{z,y}$) the vectors $e_z\otimes e_{o(z)}$ (resp. $e_z\otimes e_{t(z)}$) in the space $\mathcal{H}'^{o}_y$ (resp. in $\mathcal{H}'^{t}_y$). ($e^o_{z,y}$ and respectively $e^t_{z,y}$ are thus defined for every $y\in \bar{X}^1$ and every $z\in X^1$ such that $o(z)\in X^o_y$, respectively $t(z)\in X^o_y$).

With these notations let us describe $(\mathcal{H}, \eta, \tilde{U})$ explicitly :

- the action of the group is

$$g\, e^o_{z,y} = e^o_{gz,gy} \quad , \quad g\, e^t_{z,y} = e_{gz,gy}$$

- the representations η_i, $i = o,1$, are :

$$\eta_o(f)\, e^o_{z,y} = f(-\infty)\, e^o_{z,y} \, , \quad \eta_o(f)\, e^t_{z,y} = f(z)\, e^t_{z,y}$$

$$\eta_o(f)\, e_{P,y} = f(\infty)\, e_{P,y}$$

$$\eta_1(f)\, e^{o}_{z,y} = f(z)\, e^{o}_{z,y} \quad , \quad \eta_1(f)\, e^{t}_{z,y} = f(\infty)\, e^{t}_{z,y}$$

$$\eta_1(f)\, e_{P,y} = f(-\infty)\, e_{P,y}$$

for every $f \in C(\bar{X}^1)$, and where $e_{P,y}$, $P \in X^o_y$, is the canonical basis in $\mathcal{H}''_y$.

- U is defined by

$$U = U_0 + U_0^* + 1 - U_0 U_0^* - U_0^* U_0$$

where U_0 acts in the following way :

$$U_0\, e^{o}_{z,y} = o \ , \quad U_0\, e^{t}_{z,y} = o \ , \quad U_0\, e_{P,y} = \begin{cases} e^{o}_{|O(P)|,y} & \text{if } o(|O(P)|) = P ,\ P \neq O \\ e^{t}_{|O(P)|,y} & \text{if } t(|O(P)|) = P ,\ P \neq O \\ o & \text{if } P = O . \end{cases}$$

We shall construct now a unitary $W_0 \in \mathcal{L}(\mathcal{H}' \oplus \mathcal{H}'')$ such that the triplet $(\mathcal{H}, \eta, \tilde{W}_0)$, $\tilde{W}_0 = \begin{pmatrix} o & W^* \\ W & o \end{pmatrix}$, still represents $\beta \otimes \alpha$, but is easyier to handle than $(\mathcal{H}, \eta, \tilde{U})$. Note first that the only edge $z \in X^1$ that has one extremity in X^o_y and the other in $X^o \setminus X^o_y = X^o_{\bar{y}}$ is y . In particular, if $P \in X^o_y$ and $O \in X^o$ is some vertex, then $t(|O(P)|) \in X^o_y$, while $o(|O(P)|) \in X^o_y$ as long as $|O(P)| \neq y$. Moreover, the map $|O|$ maps $X^o_y \setminus \{O\}$ onto the set $\{z \ ; \ o(z) \in X^o_y\} = \{z \ ; \ t(z) \in X^o_y\} \setminus \{y\}$ if $O \in X^o_y$, respectively onto the set $\{z \ ; \ o(z) \in X^o_y\} \cup \{y\} = \{z \ ; \ t(z) \in X^o_y\}$ if $O \notin X^o_y$ (this forces $y \neq \infty$) .

It follows that if we fix some origin $0 \in X^0$, then the operator $W_{0,y} \in \mathcal{L}(\mathcal{H}'_y \oplus \mathcal{H}''_y)$ defined for every $y \in \overline{X}^1$ by

$$W_{0,y}\, e^0_{z,y} = e_{|0|^{-1}(z),y}$$

$$W_{0,y}\, e^t_{z,y} = \begin{cases} e^0_{z,y} & \text{if } z \neq y \\ e^t_{y,y} & \text{if } z = y \text{ and } 0 \in X^0_y \\ e_{t(y),y} & \text{if } z = y \text{ and } 0 \notin X^0_y \end{cases}$$

$$W_{0,y}\, e_{P,y} = \begin{cases} e^t_{0(P),y} & \text{if } P \neq 0 \\ e_{0,y} & \text{if } P = 0 \end{cases}$$

is a well defined unitary. Moreover, $y \longmapsto W_{0,y}$ being $*$-strongly continuous on $\overline{X}^1$, we get a unitary

$$W_0 \in \mathcal{L}(\mathcal{H}' \oplus \mathcal{H}'') .$$

Lemma 11. The above defined $W_0 \in \mathcal{L}(\mathcal{H}' \oplus \mathcal{H}'')$ has the following properties :

i) $g(W_0) = W_{g0}$ for every $g \in G$. In particular W_0 is G continuous.

ii) $W_0 - W_{0'} \in \mathcal{K}(\mathcal{H}' \oplus \mathcal{H}'')$ for every $0, 0' \in X^0$.

iii) $W_0\, \eta_0(f) - \eta_1(f) W_0 \in \mathcal{K}(\mathcal{H}' \oplus \mathcal{H}'')$, for every $f \in C_+(X^1)$.

iv) The class of $(\mathcal{H}, \eta, \widetilde{W}_0)$ in $KK^G(C_+(X^1), C_+(X^1))$ coincides with $\beta \otimes \alpha$.

Proof : i) Is easy to check from the definition using the fact that $g(|0(g^{-1}P)|) = |g0|(P)$.

ii) Note first that $\mathcal{K}(\mathcal{H}' \oplus \mathcal{H}'')$ may be identified with sections $y \longmapsto K_y \in \mathcal{K}(\mathcal{H}'_y \oplus \mathcal{H}''_y)$ that are norm continuous when regarded as functions from $\bar{X}^1$ to $\mathcal{K}(l^2(X^1)\otimes l^2(X^o) \oplus l^2(X^o))$. Next, lemma 1. implies that $W_{0,y} - W_{0',y}$ is a finite rank operator whose kernel contains the orthogonal of the space generated by the vector $e^t_{y,y}$, if both $o(y)$ and $t(y)$ belong to the geodesic $[0,0']$, together with the vectors $\{e^o_{z,y}\}$ and $\{e_{P,y}\}$, with $o(z)$, $t(z)$ and P all lying on $[0,0']$. But if y is close to $+\infty$, then the whole geodesic belongs to X^o_y, while if y is close to $-\infty$, then $[0,0']$ does not intersect X^o_y. This concludes the proof of ii).

iii) The definitions of W_0 and η_i, $i = o,1$, imply that

$$(W_0\,\eta_o(f) - \eta_1(f)W_0)\, e^o_{z,y} = o$$

$$(W_0\,\eta_o(f) - \eta_1(f)W_0)\, e^t_{z,y} = \begin{cases} (f(z) - f(\infty))\, e^t_{y,y} & \text{if } z = y \text{ and } 0 \in X^o_y \\ (f(z) - f(-\infty))\, e_{t(y),y} & \text{if } z = y \text{ and } 0 \notin X^o_y \\ o & \text{otherwise} \end{cases}$$

$$(W_0\,\eta_o(f) - \eta_1(f)W_0)\, e_{P,y} = \begin{cases} (f(\infty) - f(-\infty))\, e_{0,y} & \text{if } P = 0 \\ o & \text{otherwise .} \end{cases}$$

for every $f \in C(\bar{X}^1)$.

This shows that the above operator is determined by the section $y \longmapsto K'_y + K''_y$, where K'_y and K''_y are rank one operators such that $K'_{-\infty} = K'_{+\infty} = 0$, $K''_y = o$ if $0 \notin X^o_y$ and K''_y is constant if $0 \in X^o_y$. This shows that both K' and K'' belong to $\mathcal{K}(\mathcal{H}' \oplus \mathcal{H}'')$.

iv) An easy computation shows that W_0U^* ($= W_0U$) is selfadjoit. Moreover W_0U^* satisfies the following conditions

1) W_0U^* is G continuous

2) $W_0U^* - g(W_0U^*) \in \mathcal{K}(\mathcal{H}' \oplus \mathcal{H}'')$ for every $g \in G$

3) $[\eta_1(f), W_0U^*] \in \mathcal{K}(\mathcal{H}' \oplus \mathcal{H}'')$ for every $f \in C_+(X^1)$,

since W_0 and U both satisfy 1) and 2) and both intertwine the representations η_0 and η_1. Since the elements satisfying 1), 2) and 3) form a C^*-algebra, we can find a norm continuous path $[0,1] \ni t \longmapsto W_t \in \mathcal{L}(\mathcal{H}' \oplus \mathcal{H}'')$ with end points W_0U^* and 1, and such that W_t is a unitary still satisfying 1), 2) and 3), for every $t \in [0,1]$. But then the map $t \longmapsto W_tU$ yields the desired homotopy.

<u>Proposition 12.</u> Consider the exact sequence

$$0 \longrightarrow C_0(X^1) \xrightarrow{\ i\ } C_+(X^1) \xrightarrow{\ p\ } \mathbb{C} \longrightarrow 0$$

and let $\beta \otimes \alpha \in KK^G(C_+(X^1), C_+(X^1))$ be the cup product of the elements introduced in definitions 6. and 7. . Then :

i) $i^*(\beta \otimes \alpha) = [i]$ in $KK^G(C_0(X^1), C_+(X^1))$.

ii) $p_*(\beta \otimes \alpha) = [p]$ in $KK^G(C_+(X^1), \mathbb{C})$.

<u>Proof</u> : For the proof of both i) and ii) we shall use the representative $(\mathcal{H}, \eta, \tilde{W}_0)$ of $\beta \otimes \alpha$ (see the preceding lemma).

i) Note first that the class of the inclusion i coincides with the class of the identic map $id : C_0(X^0) \longrightarrow C_0(X^0)$ where $C_0(X^0)$ is regarded as a $C_+(X^1)$ module. (This is a general fact [14], for if $i : I \hookrightarrow A$ is an ideal, then the mapping cylinder $\{f \in C([0,1], A) \; ; \; f(0) \in I\}$, regarded as a $C([0,1], A)$ module, provides a homotopy connecting $[i]$ to $[id]$.)

Note next that $i^*(\beta \otimes \alpha)$ may be represented by the triplet $(\mathcal{H}^t \oplus \mathcal{H}^o, \eta_o^t \oplus \eta_1^o, \tilde{W})$ where :

$\mathcal{H}^t \subset \mathcal{H}_o$ is generated by those sections s_y such that $s_y \in \mathcal{H}_y'^{o}$,

$\mathcal{H}^o \subset \mathcal{H}_1$ is generated by those sections s_y such that $s_y \in \mathcal{H}_y'^{t}$,

$\eta_o^t(f)\, e_{z,y}^t = f(z)\, e_{z,y}^t$

$\eta_1^o(f)\, e_{z,y}^o = f(z)\, e_{z,y}^o$ for every $f \in C_o(X^1)$

and $\tilde{W} = \begin{pmatrix} o & W^* \\ W & o \end{pmatrix}$, where $W e_{z,y}^t = \begin{cases} e_{z,y}^o & \text{if } z \neq y \\ o & \text{if } z = y \end{cases}$

This follows since the orthogonal of $\mathcal{H}^t$ (resp. of $\mathcal{H}^o$) is contained in the kernel of $\eta_o|_{C_o(X^1)}$ (resp. of $\eta_1|_{C_o(X^1)}$) and W is the compression of W_0 to these subspaces.

Since W^* is an isometry whose kernel is generated by $e_{y,y}^t$, it is obvious that if we denote by $v \in \mathcal{L}(C_o(X^1), \mathcal{H}^t)$ the isometry $v(\delta_y) = e_{y,y}^t$, then

$$\eta_o(f) = W^* \eta_1(f) W \oplus v \circ id(f) \circ v^*$$

which is equivalent to $i^*(\beta \otimes \alpha) = [i]$.

ii) The definition of p_* shows that $p_*(\beta \otimes \alpha)$ is the class of the restriction of the triplet $(\mathcal{H}, \eta, \tilde{W}_0)$ to $y = \infty$, i.e. by the triplet $(\mathcal{H}_{o,\infty} \oplus \mathcal{H}_{1,\infty}, \eta_{o,\infty} \oplus \eta_{1,\infty}, \tilde{W}_{0,\infty})$, where

$$\mathcal{H}_{o,\infty} = \mathcal{H}_{1,\infty} = \mathcal{H}_\infty'^{o} \oplus \mathcal{H}_\infty'^{t} \oplus \mathcal{H}_\infty'' .$$

Again $\mathcal{H}'^{o}_{\infty}$ is in the kernel of η_o while $\mathcal{H}''_{\infty}$ is in the kernel of η_1 so that $p_*(\beta\otimes\alpha)$ is finally determined by $(\tilde{\mathcal{H}}_o \oplus \tilde{\mathcal{H}}_1, \tilde{\eta}_o \oplus \tilde{\eta}_1, \tilde{W})$, where :

$$\tilde{\mathcal{H}}_o = \mathcal{H}'^{t}_{\infty} \oplus \mathcal{H}''_{\infty} \simeq l^2(X^1) \oplus l^2(X^o)$$

$$\tilde{\mathcal{H}}_1 = \mathcal{H}'^{o}_{\infty} \oplus \mathcal{H}'^{t}_{\infty} \simeq l^2(X^1) \oplus l^2(X^1) ,$$

$$\tilde{\eta}_o(f)\, e^t_{z,\infty} = f(z)\, e^t_{z,\infty} \quad , \quad \tilde{\eta}_o(f)\, e_{P,\infty} = f(\infty)\, e_{P,\infty}$$

$$\tilde{\eta}_1(f)\, e^o_{z,\infty} = f(z)\, e^o_{z,\infty} \quad , \quad \tilde{\eta}_1(f)\, e_{z,\infty} = f(\infty)\, e^t_{z,\infty}$$

while $\tilde{W} = \begin{pmatrix} o & W^* \\ W & o \end{pmatrix}$, where

$$W\, e^t_{z,\infty} = e^o_{z,\infty} \qquad W\, e_{P,\infty} = \begin{cases} e^t_{|O(P)|,\infty} & \text{if } P \neq 0 \\ o & \text{if } P = 0 \end{cases}$$

The above formulae show that $p_*(\beta\otimes\alpha)$ is the sum of a degenerate triplet and of the triplet

$$(l^2(X^o) \oplus l^2(X^1),\ \eta^o_\infty \oplus \eta^1_\infty,\ \tilde{F})$$

where $\eta^o_\infty(f)\, e_P = f(\infty)\, e_P$, $\eta^1_\infty(f)\, e_y = f(\infty)\, e_y$ and $\tilde{F} = \begin{pmatrix} o & F^* \\ F & o \end{pmatrix}$ with $F : l^2(X^o) \longrightarrow l^2(X^1)$ defined by

$$F\, e_P = \begin{cases} e_{|O(P)|} & \text{if } P \neq 0 \\ o & \text{if } P = 0 . \end{cases}$$

Recall now the definition of the element $\gamma \in KK^G(\mathbb{C},\mathbb{C})$ from [11] to see that the above formulae may be written

$$p_*(\beta \otimes \alpha) = p^*(\gamma) = [p] \otimes \gamma \quad .$$

Since $\gamma = 1_{\mathbb{C}}$ ([11] Proposition 1.) , we get the proof of ii) .

The next lemma identifies the elements $i^*(\beta)$ and $p_*(\alpha)$.

Lemma 13. i) Let $\varphi_t, \varphi_o : C_o(X^1) \longrightarrow \mathcal{K}(l^2(X^1) \otimes C_o(X^o))$ be the G equivariant $*$-homomorphisms defined by :

$$\varphi_t(f)\, e_y \otimes \delta_P = \begin{cases} f(y)\, e_y \otimes \delta_P & \text{if } t(y) = P \\ o & \text{otherwise} \end{cases}$$

$$\varphi_o(f)\, e_y \otimes \delta_P = \begin{cases} f(y)\, e_y \otimes \delta_P & \text{if } o(y) = P \\ o & \text{otherwise} \end{cases}$$

for every $f \in C_o(X^1)$, $y \in X^1$ and $P \in X^o$.

Then

$$i^*(\beta) = [\varphi_t] - [\varphi_o] \quad .$$

ii) Let $\psi : C_o(X^o) \longrightarrow \mathcal{K}(l^2(X^o))$ be the G equivariant $*$-homomorphism defined by :

$$\psi(f)\, e_Q = f(Q)\, e_Q$$

for every $f \in C_o(X^o)$ and $Q \in X^o$. Then

$$p_*(\alpha) = [\psi] .$$

Proof : Point ii) is straightforward, while point i) follows from remark 1. ii) at the beginning of this section.

We conclude this section by showing the relevance of the preceding results to the Toeplitz extension introduced in the preceding section. To this end recall from [14] theorem 1. that there is a functorial homomorphism

$$j_G : KK^G(A,B) \longrightarrow KK(G \ltimes A, G \ltimes B)$$

that commutes with the intersection product and such that (in the case $A = B$) $j_G(1_A) = 1_{G \ltimes A}$. In order to fix the notation, let us briefly describe the construction.

If $\mathcal{E}$ is a right Hilbert B-module, one denotes by $C_c(G, \mathcal{E})$ the set of continuous functions with compact support, defined on G with values in $\mathcal{E}$. The right $C_c(G,B)$ action and the $C_c(G,B)$ valued inner product are defined by

$$(e \cdot b)(t) = \int_G e(s)\, s(b(s^{-1}t))\, ds$$

$$\langle e \mid f \rangle (t) = \int_G s^{-1} \langle e(s) \mid f(st) \rangle \, ds$$

for every $e,f \in C_c(G, \mathcal{E})$ and $b \in C_c(G,B)$. We shall denote the completion of $C_c(G, \mathcal{E})$ with respect to the norm $\| e \| = \| \langle e,e \rangle \|^{1/2}_{G \ltimes B}$ with $G \ltimes \mathcal{E}$. Note that the formula

$$U_g\, e\,(t) = g(e(g^{-1}t))$$

for every $e \in C_c(G, \mathcal{E})$ and $g,t \in G$, defines a unitary representation of G on $G \ltimes \mathcal{E}$. Moreover, if $T \in \mathcal{L}(\mathcal{E})$, then the formula

$$T_G e\,(t) = Te(t) \qquad e \in C_c(G, \mathcal{E}) \ , \ t \in G$$

defines a $*$-homomorphism $\mathcal{L}(\mathcal{E}) \ni T \longmapsto T_G \in \mathcal{L}(G \ltimes \mathcal{E})$, such that

$$U_g T_G U_{g^{-1}} = (gT)_G$$

for every $T \in \mathcal{L}(\mathcal{E})$ and $g \in G$. In particular if $\varphi : A \longrightarrow \mathcal{L}(\mathcal{E})$ is a $*$-representation, one gets the corresponding representation of the full crossed product, denoted by

$$\varphi_G : G \ltimes A \longrightarrow \mathcal{L}(G \ltimes \mathcal{E}).$$

The map j_G sends the triplet $(\mathcal{E}, \varphi, T) \in \mathcal{E}^G(A,B)$ to $(G \ltimes \mathcal{E}, \varphi_G, T_G)$.

Remark 2 It is also known from Kasparov, (see [16] 6.4 Page 16) that there exists an analogue of the above j_G for reduced crossed products. Let us briefly hint the construction.

It is easy to see that if we denote by $G \ltimes_r \mathcal{E}$ the Hilbert $G \ltimes_r B$ module obtained by completing $C_c(G, \mathcal{E})$ with respect to the norm $\| e \| = \| \langle e | e \rangle \|^{1/2}_{G \ltimes_r B}$, then we get $U_g \in \mathcal{L}(G \ltimes_r \mathcal{E})$ and $T_{G,r} \in \mathcal{L}(G \ltimes_r \mathcal{E})$ by the same formulae. Moreover if $\varphi : A \longrightarrow \mathcal{L}(\mathcal{E})$ is a $*$-representation, we get a $*$-homomorphism

$$\varphi_{G,r} : G \ltimes_r A \longrightarrow \mathcal{L}(G \ltimes_r \mathcal{E}).$$

One can see this by choosing for example a faithful representation of $G \ltimes_r B$ on H, and the corresponding regular representation on $L^2(G) \otimes H$. Then the representation of $\mathcal{L}(G \ltimes_r \mathcal{E})$ on the Hilbert space

$$K = (G \ltimes_r \mathcal{E}) \otimes_{G \ltimes_r B} (L^2(G) \otimes H)$$

is faithful ,[[13] §2, 8.], and $G \ni g \longmapsto \dot{U}_g \in \mathcal{L}(K)$ is a multiple of the regular representation of G. (Note that $K \simeq L^2(G) \otimes ((G \ltimes_r \mathcal{E}) \otimes_{G \ltimes_r B} H)$.) Moreover the same argument shows that $G \ltimes_r \mathcal{K}(\mathcal{E}) \simeq \mathcal{K}(G \ltimes_r \mathcal{E})$.

This shows the existence of a functorial homomorphism

$$j_{G,r} : KK^G(A,B) \longrightarrow KK(G \ltimes_r A, G \ltimes_r B)$$

defined by $j_{G,r}[(\mathcal{E}, \varphi, T)] = [(G \ltimes_r \mathcal{E}, \varphi_{G,r}, T_{G,r})]$, with the same properties as j_G .

Moreover, it is straightforward that if $f : A \longrightarrow \mathcal{K}(\mathcal{E})$ is a G equivariant $*$-homomorphism, then $j_G[f] = [f_G]$ (respectively $j_{G,r}[f] = [f_{G,r}]$) where $f_G : G \ltimes A \longrightarrow G \ltimes \mathcal{K}(\mathcal{E}) \simeq \mathcal{K}(G \ltimes \mathcal{E})$ (respectively $f_{G,r} : G \ltimes_r A \longrightarrow G \ltimes_r \mathcal{K}(\mathcal{E}) \simeq \mathcal{K}(G \ltimes_r \mathcal{E})$) is the $*$-homomorphism induced on the crossed products.

Definition 8. Let A be a fixed separable C^*-algebra, and suppose that G acts continuously by automorphisms on A . Let $\sigma_A : KK^G(B,C) \longrightarrow KK^G(B \otimes A, C \otimes A)$ be the map from [14] definition 4. .

If $\gamma \in KK^G(B,C)$ we shall denote simply by γ_G (resp. $\gamma_{G,r}$) the element $j_G \circ \sigma_A(\gamma)$ (resp. $j_{G,r} \circ \sigma_A(\gamma)$).

With these notations in mind, we can state the following proposition, which proves the claim made at the beginning of this section.

<u>Proposition 14.</u>

i) $\alpha_G \otimes \beta_G = 1_{G \ltimes C_o(X^o,A)}$, $\alpha_{G,r} \otimes \beta_{G,r} = 1_{G \ltimes_r C_o(X^o,A)}$,

ii) $\beta_G \otimes \alpha_G = 1_{G \ltimes C_+(X^1,A)}$, $\beta_{G,r} \otimes \alpha_{G,r} = 1_{G \ltimes_r C_+(X^1,A)}$.

<u>Proof</u> : i) Follows directly from proposition lo.

To prove ii) we shall show that $\beta_G \otimes \alpha_G$ (resp. $\beta_{G,r} \otimes \alpha_{G,r}$) is the identity in the ring $KK(G \ltimes C_+(X^1,A), G \ltimes C_+(X^1,A))$ (resp. $KK(G \ltimes_r C_+(X^1,A), G \ltimes_r C_+(X^1,A))$). Applying theorem 1.1 of [29] together with lemma 4. we get the following diagram

$$\begin{array}{ccccccc}
\xrightarrow{\partial} KK_n(B, G \ltimes C_o(X^1,A)) & \xrightarrow{[i_G]} & KK_n(B, G \ltimes C_+(X^1,A)) & \xrightarrow{[p_G]} & KK_n(B, G \ltimes A) \xrightarrow{\partial} \\
\downarrow id & & \downarrow \tau & & \downarrow id \\
\xrightarrow{\partial} KK_n(B, G \ltimes C_o(X^1,A)) & \xrightarrow{[i_G]} & KK_n(B, G \ltimes C_+(X^1,A)) & \xrightarrow{[p_G]} & KK_n(B, G \ltimes A) \xrightarrow{\partial}
\end{array}$$

with exact rows, and where the map τ is given by the cup product with $\beta_G \otimes \alpha_G$. Proposition 12. implies the commutativity of the above diagram, point i) shows that $(\beta_G \otimes \alpha_G)^2 = \beta_G \otimes \alpha_G$, so that we finally get that τ is the identic map. The proof for the reduced crossed product being the same, we get the proof of the proposition.

§ 4.

In this section we put together the results of the preceding sections to get the main results of this paper.

Recall from section 1., that the lifting of $G\backslash X$ provides (for every $y \in \Sigma^1$) the homomorphisms

$$\sigma_y : G_y \longrightarrow G_{\hat{t}(y)}$$

$$\sigma_{\bar{y}} : G_y \longrightarrow G_{\hat{o}(y)}$$

and the elements γ_{yt} , $\gamma_{yo} \in G$. We shall denote by α_y , $\alpha_{\bar{y}} \in \mathrm{Aut}(A)$, the automorphisms

$$\alpha_y(a) = \gamma^{-1}_{yt}(a)$$

$$\alpha_{\bar{y}}(a) = \gamma^{-1}_{yo}(a)$$

for every $a \in A$. Since the pairs (σ_y, α_y) and $(\sigma_{\bar{y}}, \alpha_{\bar{y}})$ are covariant and since the maps σ_y and $\sigma_{\bar{y}}$ are homeomorphisms onto their immage, we get the following maps on the corresponding crossed products:

$$\sigma_y \times \alpha_y : G_y \ltimes A \longrightarrow G_{\hat{t}(y)} \ltimes A \quad ; \quad \sigma_y \times_r \alpha_y : G_y \ltimes_r A \longrightarrow G_{\hat{t}(y)} \ltimes_r A$$

$$\sigma_{\bar{y}} \times \alpha_{\bar{y}} : G_{\bar{y}} \ltimes A \longrightarrow G_{\hat{o}(y)} \ltimes A \quad ; \quad \sigma_{\bar{y}} \times_r \alpha_{\bar{y}} : G_{\bar{y}} \ltimes_r A \longrightarrow G_{\hat{o}(y)} \ltimes_r A$$

<u>Remark</u> : Note that since the Haar measures on G_y and G_P , are chosen as the restriction of the Haar measure of G , the maps σ_y and $\sigma_{\bar{y}}$ do not (in general) preserve these

measures. This has to be taken into account when writing the explicit formulae of the above maps. For example, if $k \in C_c(G_y, A)$, then $\sigma_y \times \alpha_y(k)$ is the continuous function

$$\sigma_y \times \alpha_y(k)\,(t) = \begin{cases} \Delta(\gamma_{yt}^{-1})\,\alpha_y(k(\sigma_y^{-1}(t))) & \text{if } t \in \sigma_y(G_y) \\ 0 & \text{otherwise} \end{cases}$$

Moreover

$$\tau_P : G_P \rtimes A \longrightarrow G \rtimes A \quad ; \quad \tau_{P,r} : G_P \rtimes_r A \longrightarrow G \rtimes_r A$$

($P \in \Sigma^o$) will stand for the maps induced by the inclusions $G_P \hookrightarrow G$.

Corresponding to the above defined $*$-homomorphisms, we get

$$\sigma^t , \sigma^o : \bigoplus_{y \in \Sigma^1} G_y \rtimes A \longrightarrow \mathcal{K}(l^2(X^1)) \otimes (\bigoplus_{P \in \Sigma^o} G_P \rtimes A)$$

and

$$\tau : \bigoplus_{P \in \Sigma^o} G_P \rtimes A \longrightarrow \mathcal{K}(l^2(X^o)) \otimes (G \rtimes A)$$

respectively

$$\sigma_r^t , \sigma_r^o : \bigoplus_{y \in \Sigma^1} G_y \rtimes_r A \longrightarrow \mathcal{K}(l^2(X^1)) \otimes (\bigoplus_{P \in \Sigma^o} G_P \rtimes_r A)$$

and

$$\tau_r : \bigoplus_{P \in \Sigma^o} G_P \rtimes_r A \longrightarrow \mathcal{K}(l^2(X^o)) \otimes (G \rtimes_r A) .$$

These $*$-homomorphisms are defined by

$$\sigma^t\left(\bigoplus_{y\in\Sigma^1} x_y\right) = \sum e_{yy}\otimes \sigma_y\times\alpha_y(x_y)$$

$$\sigma^o\left(\bigoplus_{y\in\Sigma^1} x_y\right) = \sum e_{yy}\otimes \sigma_{\bar{y}}\times\alpha_{\bar{y}}(x_y)$$

$$\tau\left(\bigoplus_{P\in\Sigma^o} x_P\right) = \sum e_{PP}\otimes \tau_P(x_P) \quad ,$$

and respectively

$$\sigma^t_r\left(\bigoplus_{y\in\Sigma^1} x_y\right) = \sum e_{yy}\otimes \sigma_y\times_r\alpha_y(x_y)$$

$$\sigma^o_r\left(\bigoplus_{y\in\Sigma^1} x_y\right) = \sum e_{yy}\otimes \sigma_{\bar{y}}\times_r\alpha_{\bar{y}}(x_y)$$

$$\tau_r\left(\bigoplus_{P\in\Sigma^o} x_P\right) = \sum e_{PP}\otimes \tau_{P,r}(x_P) \quad ,$$

where $\{e_{y,z}\}$ and respectively $\{e_{P,Q}\}$ are the canonical matrix units in $\mathcal{K}(l^2(X^1))$ and respectively in $\mathcal{K}(l^2(X^o))$.

Our next goal is to show the way the above $*$-homomorphisms enter into the final results. Let us first make the following remark :

Remark 3. If G acts on the Hilbert B-modules $\mathcal{E}$ and $\mathcal{F}$, and $W\in\mathcal{L}(\mathcal{E},\mathcal{F})$ is a (not necessary G equivariant) unitary, then W_G defined for $e\in C_c(G,\mathcal{E})$ by

$$W_G e\,(t) = W(e(t))$$

extends to a unitary $W_G\in\mathcal{L}(G\ltimes\mathcal{E}, G\ltimes\mathcal{F})$. If we denote by

U_g and respectively by V_g the unitary representations of G on $G \ltimes \mathcal{E}$ and respectively on $G \ltimes \mathcal{F}$, then

$$V_g W_G U_{g^{-1}} = (g(W))_G$$

for every $g \in G$. It follows that the isomorphism $\alpha_W : G \ltimes \mathcal{K}(\mathcal{E}) \longrightarrow G \ltimes \mathcal{K}(\mathcal{F})$ induced by conjugation with W_G (see remark 2. in the preceding section) acts in the following way : if $e', e'' \in C_c(G, \mathcal{E})$ and $G \ni g \longmapsto \theta_{e'(g), e''(g)}$ is the corresponding function in $C_c(G, \mathcal{K}(\mathcal{E}))$ (recall that $\theta_{x,y}$ is the rank one operator $\theta_{x,y}(z) = x \langle y \mid z \rangle$), then the automorphism α_W sends this function to the continuous function $G \ni g \longmapsto \theta_{W(e'(g)), W\, g(W^*)\, W(e''(g))} \in \mathcal{K}(\mathcal{F})$.

Moreover, the same is true for the reduced crossed products. This means that the above W_G defined on $C_c(G, \mathcal{E})$ extends to a unitary $W_{G,r} \in \mathcal{L}(G \ltimes_r \mathcal{E}, G \ltimes_r \mathcal{F})$ and that conjugation with $W_{G,r}$ yields an automorphism $\alpha_{W,r} : G \ltimes_r \mathcal{K}(\mathcal{E}) \longrightarrow G \ltimes_r \mathcal{K}(\mathcal{F})$, whose action on $C_c(G, \mathcal{K}(\mathcal{E}))$ coincides with that of α_W.

In order to state the next lemma, let us fix the following natural inclusions :

$$j_o : \bigoplus_{P \in \Sigma^o} G_P \ltimes A \longrightarrow \bigoplus_{P \in \Sigma^o} (G_P \ltimes A) \otimes \mathcal{K}(l^2(X_P^o)))$$

$$j_{o,r} : \bigoplus_{P \in \Sigma^o} G_P \ltimes_r A \longrightarrow \bigoplus_{P \in \Sigma^o} (G_P \ltimes_r A) \otimes \mathcal{K}(l^2(X_P^o)))$$

defined by

$$j \left(\bigoplus x_P \right) = \bigoplus (x_P \otimes e_{PP})$$

$$j_{0,r}(\oplus x_p) = \oplus(x_p \otimes e_{pp}) .$$

Similarly

$$j_1 : \bigoplus_{y \in \Sigma^1} G_y \ltimes A \longrightarrow \bigoplus_{y \in \Sigma^1} (G_y \ltimes A) \otimes \mathcal{K}(l^2(X_y^1))$$

$$j_{1,r} : \bigoplus_{y \in \Sigma^1} G_y \ltimes_r A \longrightarrow \bigoplus_{y \in \Sigma^1} (G_y \ltimes_r A) \otimes \mathcal{K}(l^2(X_y^1))$$

are defined by

$$j_1(\oplus x_y) = \oplus(x_y \otimes e_{yy})$$

$$j_{1,r}(\oplus x_y) = \oplus(x_y \otimes e_{yy}) .$$

(see definition 5.). Let us moreover fix the isomorphisms

$$\Phi : G \ltimes C_0(X^0, A) \longrightarrow \bigoplus_{p \in \Sigma^0} (G_p \ltimes A) \otimes \mathcal{K}(l^2(X_p^0))$$

$$\Phi_r : G \ltimes_r C_0(X^0, A) \longrightarrow \bigoplus_{p \in \Sigma^0} (G_p \ltimes_r A) \otimes \mathcal{K}(l^2(X_p^0))$$

provided by proposition 5. applied to the set X^0 with transversal Σ^0. Note that Φ and Φ_r determine elements of KK , i.e. we may consider

$$[\Phi] \in KK(G \ltimes C_0(X^0, A), \bigoplus_{p \in \Sigma^0} G_p \ltimes A)$$

$$[\Phi_r] \in KK(G \ltimes_r C_0(X^0, A), \bigoplus_{p \in \Sigma^0} G_p \ltimes_r A) .$$

With these notations in mind we can state the following lemma, where for the sake of simplicity the subscript in the notation of the intersection product will be dropped (i.e.

we denote $x \otimes y$ instead of $x \otimes_B y$). (See also remark 1. in section 3.).

Lemma 15. Let $\hat{i}_G$, $\hat{i}_{G,r}$, q_G and $q_{G,r}$ be the maps arising in the total (i.e. $S = \Sigma^1$) Toeplitz extension, and α, β the elements constructed in the preceding section (definitions 6. and 7. ; see also definition 8.). Then :

i)

$$[j^1] \otimes [\hat{i}_G] \otimes \beta_G \otimes [\phi] = [\sigma^t] - [\sigma^o] \quad \text{in}$$

$KK(\bigoplus_{y \in \Sigma^1} G_y \ltimes A, \bigoplus_{P \in \Sigma^o} G_P \ltimes A)$,

$$[j^1_r] \otimes [\hat{i}_{G,r}] \otimes \beta_{G,r} \otimes [\phi_r] = [\sigma^t_r] - [\sigma^o_r] \quad \text{in}$$

$KK(\bigoplus_{y \in \Sigma^1} G_y \ltimes_r A, \bigoplus_{P \in \Sigma^o} G_P \ltimes_r A)$.

ii)

$$[j^o] \otimes [\phi^{-1}] \otimes \alpha_G \otimes [q_G] = [\tau] \quad \text{in}$$

$KK(\bigoplus_{P \in \Sigma^o} G_P \ltimes A, G \ltimes A)$,

$$[j^o_r] \otimes [\phi^{-1}_r] \otimes \alpha_{G,r} \otimes [q_{G,r}] = [\tau_r] \quad \text{in}$$

$KK(\bigoplus_{P \in \Sigma^o} G_P \ltimes_r A, G \ltimes_r A)$.

Proof: The proof for the full and reduced crossed products being the same, we shall prove only the former case.

i) Let us denote by $\tilde{\varphi}_t, \tilde{\varphi}_o : \bigoplus_{y \in \Sigma^1} G_y \ltimes A \longrightarrow$
$\longrightarrow \mathcal{K}(G \ltimes (l^2(X^1) \otimes C_o(X^o, A))) \simeq G \ltimes \mathcal{K}(l^2(X^1) \otimes C_o(X^o) \otimes A)$
the maps

$$\tilde{\varphi}_t(\oplus x_y) = \oplus \tilde{\varphi}_{t,y}(x_y)$$

$$\tilde{\varphi}_o(\oplus x_y) = \oplus \tilde{\varphi}_{o,y}(x_y)$$

where $\tilde{\varphi}_{t,y}$, $\tilde{\varphi}_{o,y} : G_y \ltimes A \longrightarrow G \ltimes \mathcal{K}(l^2(X^1) \otimes C_o(X^o) \otimes A)$ are the maps determined by the inclusions and by the $*$-homomorphisms

$$A \ni a \longmapsto e_{yy} \otimes \delta_{t(y)} \otimes a \in \mathcal{K}(l^2(X^1) \otimes C_o(X^o) \otimes A)$$

respectively

$$A \ni a \longmapsto e_{yy} \otimes \delta_{o(y)} \otimes a \in \mathcal{K}(l^2(X^1) \otimes C_o(X^o) \otimes A).$$

By lemma 13. and remark 2. at the end of the preceding section it is easy to see that

$$[j^1] \otimes [\hat{i}_G] \otimes \beta_G' = [\tilde{\varphi}_t] - [\tilde{\varphi}_o] .$$

Consider now the Hilbert $C_o(X^o,A)$ module $l^2(X^1) \otimes C_o(X^o,A)$ with trivial action on $l^2(X^1)$. In this case $G \ltimes (l^2(X^1) \otimes C_o(X^o,A)) \simeq l^2(X^1) \otimes G \ltimes C_o(X^o,A)$ .We shall use the second notation in this case, keeping the notation $G \ltimes (l^2(X^1) \otimes C_o(X^o,A))$ for the usual action of G on $l^2(X^1)$. The remark preceding the lemma shows that the identity map gives rise to a unitary

$$id_G : G \ltimes (l^2(X^1) \otimes C_o(X^o,A)) \longrightarrow l^2(X^1) \otimes G \ltimes C_o(X^o,A)$$

Conjugation with id_G , taking into account that y is fixed by G_y , shows that $[\tilde{\varphi}_t]$ and respectively $[\tilde{\varphi}_o]$ are equal to $[\varphi'_t]$ and respectively $[\varphi'_o]$, where

$$\varphi'_t , \varphi'_o : \bigoplus_{y \in Z^1} G_y \ltimes A \longrightarrow \mathcal{K}(l^2(X^1) \otimes G \ltimes C_o(X^o,A)) \simeq$$

$$\simeq \mathcal{K}(l^2(X^1)) \otimes (G \ltimes C_o(X^o,A))$$

are defined in the following way :

if $\varphi'_{t,y}$, $\varphi'_{o,y}$: $G_y \ltimes A \longrightarrow G \ltimes C_o(X^o, A)$ are the maps determined by the inclusions $G_y \hookrightarrow G$ and by $A \ni a \longmapsto \delta_{t(y)} \otimes a \in C_o(X^o) \otimes A$ respectively $A \ni a \longmapsto \delta_{o(y)} \otimes a \in C_o(X^o) \otimes A$, then

$$\varphi'_t(\oplus x_y) = \oplus\big(e_{yy} \otimes \varphi'_{t,y}(x_y)\big)$$

$$\varphi'_o(\oplus x_y) = \oplus\big(e_{yy} \otimes \varphi'_{o,y}(x_y)\big).$$

Since it is easily seen that φ'_t and $\phi^{-1} \circ \sigma^t$ (resp. φ'_o and $\phi^{-1} \circ \sigma^o$) are unitarily equivalent, we finally get

$$[j^1] \otimes [\hat{i}_G] \otimes \beta_G = ([\sigma^t] - [\sigma^o]) \otimes [\phi^{-1}]$$

ii) Again , lemma 13. and remark 2. show that

$$[j^o] \otimes [\phi^{-1}] \otimes \alpha_G \otimes [q_G] = [\tilde{\Psi}]$$

where $\tilde{\Psi} : \bigoplus_{P \in \Sigma^o} G_P \ltimes A \longrightarrow \mathcal{K}(G \ltimes (l^2(X^o) \otimes A)) \simeq G \ltimes \mathcal{K}(l^2(X^o) \otimes A)$ is the map

$$\tilde{\Psi}(\oplus x_P) = \oplus \tilde{\Psi}_P(x_P)$$

and where $\tilde{\Psi}_P : G_P \ltimes A \longrightarrow G \ltimes \mathcal{K}(l^2(X^o) \otimes A)$ is the map determined by the inclusion $G_P \hookrightarrow G$ and by $A \ni a \longmapsto e_{PP} \otimes a \in \mathcal{K}(l^2(X^o) \otimes A)$.

If we consider the Hilbert A-module $l^2(X^0)\otimes A$, first with the usual action of G (given by the action of G on X^0) on $l^2(X^0)$ and then with the trivial action on $l^2(X^0)$, then the identic map gives rise to a unitary (remark 3.)

$$\mathrm{id}_G : G\rtimes(l^2(X^0)\otimes A) \longrightarrow l^2(X^0)\otimes G\rtimes A .$$

Since conjugation with id_G sends $\tilde{\Psi}$ to τ we get the proof of ii).

We come now to the main results.

<u>Theorem 16.</u> Let G be a second countable group that acts continuously on the countable oriented tree X . Let A be a separable C^*-algebra endowed with a continuous G action. Then

i) If B is a separable C^*-algebra, then the diagram

$$\begin{array}{ccccccccc}
KK_{n-1}(B,G\rtimes A) & \xrightarrow{\partial} & KK_n(B,\bigoplus_{y\in\Sigma^1} G_y\rtimes A) & \xrightarrow{\sigma^t_*-\sigma^0_*} & KK_n(B,\bigoplus_{P\in\Sigma^0} G_P\rtimes A) & \xrightarrow{\tau_*} & KK_n(B,G\rtimes A) & \xrightarrow{\partial} & KK_{n+1}(B,\bigoplus_{y\in\Sigma^1} G_y\rtimes A) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
KK_{n-1}(B,G\rtimes_r A) & \xrightarrow{\partial} & KK_n(B,\bigoplus_{y\in\Sigma^1} G_y\rtimes_r A) & \xrightarrow{\sigma^t_{r*}-\sigma^0_{r*}} & KK_n(B,\bigoplus_{P\in\Sigma^0} G_P\rtimes_r A) & \xrightarrow{\tau_{r*}} & KK_n(B,G\rtimes_r A) & \xrightarrow{\partial} & KK_{n+1}(B,\bigoplus_{y\in\Sigma^1} G_y\rtimes_r A)
\end{array}$$

is commutative and has exact rows (for every $n\in\mathbb{Z}/2$)

ii) If B is an arbitrary C^*-algebra, then the diagram

$$\begin{array}{ccccccccc}
KK_{n-1}(G\rtimes A,B) & \xleftarrow{\partial} & KK_n(\bigoplus_{y\in\Sigma^1} G_y\rtimes A,B) & \xleftarrow{\sigma^{t*}-\sigma^{0*}} & KK_n(\bigoplus_{P\in\Sigma^0} G_P\rtimes A,B) & \xleftarrow{\tau^*} & KK_n(G\rtimes A,B) & \xleftarrow{\partial} & KK_{n+1}(\bigoplus_{y\in\Sigma^1} G_y\rtimes A,B) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
KK_{n-1}(G\rtimes_r A,B) & \xleftarrow{\partial} & KK_n(\bigoplus_{y\in\Sigma^1} G_y\rtimes_r A,B) & \xleftarrow{\sigma^{t*}_r-\sigma^{0*}_r} & KK_n(\bigoplus_{P\in\Sigma^0} G_P\rtimes_r A,B) & \xleftarrow{\tau_r^*} & KK_n(G\rtimes_r A,B) & \xleftarrow{\partial} & KK_{n+1}(\bigoplus_{y\in\Sigma^1} G_y\rtimes_r A,B)
\end{array}$$

is commutative and has exact rows (for every $n \in \mathbb{Z}/_2$).

The vertical arrows are given by the natural projection from the full crossed products onto the reduced ones, while ∂ are the boundary maps associated to the total (i.e. $S = \Sigma^1$) Toeplitz extension (modulo the isomorphisms induced by

$$j_1 : \bigoplus_{y \in \Sigma^1} G_y \ltimes A \longrightarrow \bigoplus_{y \in \Sigma^1} (G_y \ltimes A \otimes \mathcal{K}(l^2(X^1_y))) \qquad \text{respectively}$$

$$j_{1,r} : \bigoplus_{y \in \Sigma^1} G_y \ltimes_r A \longrightarrow \bigoplus_{y \in \Sigma^1} (G_y \ltimes_r A \otimes \mathcal{K}(l^2(X^1_y)))\ ).$$

<u>Proof:</u> Applying theorem 1.1 of [29] to the total Toeplitz extension and then proposition 14. we get the above diagrams. Finally the preceding lemma shows that we get precisely the maps described in the theorem.

<u>Theorem 17.</u> In the conditions of the preceding theorem, we get the following commutative diagrams with exact rows.

i) If B is separable and the fundamental domain $G\backslash X$ is finite :

$$\begin{array}{ccccccccc}
KK_{n-1}(B, G\ltimes A) & \xrightarrow{\partial} & \bigoplus_{y\in\Sigma^1} KK_n(B, G_y\ltimes A) & \xrightarrow{\sigma} & \bigoplus_{p\in\Sigma^0} KK_n(B, G_p\ltimes A) & \xrightarrow{\tau} & KK_n(B, G\ltimes A) & \xrightarrow{\partial} & \bigoplus_{y\in\Sigma^1} KK_{n+1}(B, G_y\ltimes A) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
KK_{n-1}(B, G\ltimes_r A) & \xrightarrow{\partial} & \bigoplus_{y\in\Sigma^1} KK_n(B, G_y\ltimes_r A) & \xrightarrow{\sigma'} & \bigoplus_{p\in\Sigma^0} KK_n(B, G_p\ltimes_r A) & \xrightarrow{\tau'} & KK_n(B, G\ltimes_r A) & \xrightarrow{\partial} & \bigoplus_{y\in\Sigma^1} KK_{n+1}(B, G_y\ltimes A)
\end{array}$$

where $\sigma = \sum_{y\in\Sigma^1}((\sigma_y \times \alpha_y)_* - (\sigma_{\bar{y}} \times \alpha_{\bar{y}})_*)$, $\sigma' = \sum_{y\in\Sigma^1}((\sigma_y \times_r \alpha_y)_* - (\sigma_{\bar{y}} \times_r \alpha_{\bar{y}}))$, $\tau = \sum_{p\in\Sigma^0} \tau_{p*}$, $\tau' = \sum_{p\in\Sigma^0} \tau_{p}$

ii) If B and $G\backslash X$ are arbitrary :

$$\begin{array}{ccccccccc}
KK_{n-1}(G\ltimes A, B) & \xleftarrow{\partial} & \bigoplus_{y\in\Sigma^1} KK_n(G_y\ltimes A, B) & \xleftarrow{\sigma} & \bigoplus_{p\in\Sigma^0} KK_n(G_p\ltimes A, B) & \xleftarrow{\tau} & KK_n(G\ltimes A, B) & \xleftarrow{\partial} & \bigoplus_{y\in\Sigma^1} KK_{n+1}(G_y\ltimes A, B) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
KK_{n-1}(G\ltimes_r A, B) & \xleftarrow{\partial} & \bigoplus_{y\in\Sigma^1} KK_n(G_y\ltimes_r A, B) & \xleftarrow{\sigma'} & \bigoplus_{p\in\Sigma^0} KK_n(G_p\ltimes_r A, B) & \xleftarrow{\tau'} & KK_n(G\ltimes A, B) & \xleftarrow{\partial} & \bigoplus_{y\in\Sigma^1} KK_{n+1}(G_y\ltimes_r A, B)
\end{array}$$

where $\sigma = \sum_{y\in\Sigma^1}((\sigma_y \times \alpha_y)^* - (\sigma_{\bar{y}} \times \alpha_{\bar{y}})^*)$, $\sigma' = \sum_{y\in\Sigma^1}((\sigma_y \ltimes_r \alpha_y)^* - (\sigma_{\bar{y}} \times_r \alpha_{\bar{y}})^*)$, $\tau = \sum_{p\in\Sigma^0} \tau_p^*$

Proof: This follows from the preceding theorem and from the aditivity of KK [[14] theorem 3.], respectively from the countable additivity of KK^G in the first variable [31].

When specialising to K-theory, one can get rid of the separability condition on A and of the finitness condition on $G\backslash X$ to get :

Theorem 18. Suppose that the second countable group G acts on the oriented tree X and on the C^*-algebra A. Then the following diagram is commutative and has exact rows

$$\begin{array}{ccccccccc}
K_{n-1}(G\ltimes A) & \xrightarrow{\partial} & \bigoplus_{y\in\Sigma^1} K_n(G_y\ltimes A) & \xrightarrow{\sigma} & \bigoplus_{P\in\Sigma^0} K_n(G_P\ltimes A) & \xrightarrow{\tau} & K_n(G\ltimes A) & \xrightarrow{\partial} & \bigoplus_{y\in\Sigma^1} K_{n+1}(G_y\ltimes A) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
K_{n-1}(G\ltimes_r A) & \xrightarrow{\partial} & \bigoplus_{y\in\Sigma^1} K_n(G_y\ltimes_r A) & \xrightarrow{\sigma'} & \bigoplus_{P\in\Sigma^0} K_n(G_P\ltimes_r A) & \xrightarrow{\tau'} & K_n(G\ltimes_r A) & \xrightarrow{\partial} & \bigoplus_{y\in\Sigma^1} K_{n+1}(G_y\ltimes_r A)
\end{array}$$

Proof: For A separable one gets the above diagram directly from theorem 16., and then one uses direct limits to cover the general case.

The next corollary is a sharpening of the result of P.Julg and A.Valette of [11]. The case of free products of groups is due to J.Cuntz [7].

Corollary 19. If G acts on some tree, then G is KK-amenable if and only if every stabilizer is KK-amenable.

Proof: One applies the five lemma to the diagram of theorem 17. to get that the map from the full C^*-algebra onto the reduced one induces an isomorphism on K-homology. By theorem 2.1 of [7] this is equivalent to the KK-amenability of G.

The above results are especially easy to apply in the case G (discrete) is the fundamental group of a graph of groups. This is due to the fact [27] that G acts on the universal covering relative to the graph of groups, which is a tree whose fundamental domain is the initial graph of groups. In particular, (if the graph contains only one edge) we get exact sequences for the KK groups of crossed products by groups that are either amalgamated products or HNN extensions. (see [6] [7] [18] [20] and [1])

We conclude this paper by briefly mentioning the results that one gets for the other Toeplitz extensions. Recall that we have worked only with the total (i.e. $S = \Sigma^1$) Toeplitz extension. This is due to the following fact :

If $S \subset \Sigma^1$ is an arbitrary nonempty subset, then one can construct the oriented tree X_S in the following way. Call two vertices $P, Q \in X^0$ S-equivalent ($P \sim_S Q$) if the geodesic $[P,Q]$ contains only edges y such that $\hat{y} \in \Sigma^1 \setminus S$. Let $(X_S)^0 = X^0/\sim_S$ and $(X_S)^1 = X^1_S$ (definition 5.). If we define the origin (resp. the terminus) of the edge $y \in X^1_S$ to be the class of $o(y)$ (resp. of $t(y)$) in $(X_S)^0$, it is easy to see that we get an oriented tree and that G acts on it. Its fundamental domain is the graph with vertices $\Sigma^0/\sim_S$ and edges S. Moreover the total Toeplitz extension for X_S, is the Toeplitz extension of proposition 6. for $S \subset \Sigma^1$. We thus can apply the preceding results to get exact sequences for all Toeplitz extensions. (Note that if $\tilde{P} \in \Sigma^0/\sim_S$ denotes the class of the vertex $P \in \Sigma^0$, then $G_{\tilde{P}} = \{ g \in G \ ; \ gP \sim_S P \}$). In particular we get the exact sequence of theorem 3.1 of [23].

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