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MATRICES HAVING KAPPA NEGATIVE SQUARES

by

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by T. Constantinescu

I INTRODUCTION

Writing the form associated with a positive matrix as a sum of positive squares, there appears a sequence of complex numbers determining the given matrix. In a certain sense, these calculations are equivalent to the classical algorithm of I.Schur which associates a similar sequence of parameters to an analytic contractive function on the unit disc (such kind of parameters are known as Schur-Szegö parameters). The formalism using these Schur-Szegö parameters (we call it Schur analysis) can be extended to a general framework (operators on Hilbert spaces- see [4] , where the operatorial version of the Schur-Szegö parameters is called choice sequence) and can be used to solve some extension problems (as Carathéodory-Fejér problem , Nehari problem,...), to describe the Kolmogorov decomposition of positive-definite kernels on the set of integers,....

On the other hand, a Schur analysis can be given for a larger class of functions. Thus, in [11], it is showed that with a meromorphic function f on the unit disc, having $\varkappa$ poles different from zero and $|f(z)|\leq 1$ for $|z|=1$ one associates a sequence of complex numbers with the property that from a certain rank it becomes a sequence of Schur-Szegö parameters.

This phenomen proves to be generic for the generalization of the classical extension problems as stated in [26] ,[21] ,....

The aim of this note is to adapt the computations in [14] for a matrix having $\varkappa$ negative squares (section 2). Then, we shall develop some elements of orthogonal polynomials in the Pontrjagin space associated with an ascendent sequence of matrices having $\varkappa$

negative squares (section 4). In the last section we shall discuss variants for the Szegö limit theorems.

Actually, many of our considerations constitute the subject of some classical papers. For example, in section 3 we shall treat of indefinite trigonometric moment problem which was first considered in [17] . Several kinds of orthogonal polynomials in spaces with indefinite metric appear in [20], [21], [19],...

The extension problems which are connected with the trigonometric moment problem are treated in [1] and in the vast program developed in the series of papers [21] (see also [8]).

What we want to point out are only the Schur analysis aspects.

II PRELIMINARIES

In this section we introduce some objects we will need and some of their properties. We consider $\mathcal{H}=\mathbb{C}^N$ with the usual Hilbertian structure (the inner product is noted $\langle\cdot,\cdot\rangle$) and $\mathcal{L}(\mathcal{H})$ the set of linear operators on $\mathcal{H}$. For a selfadjoint operator A in $\mathcal{L}(\mathcal{H})$, let E be its spectral measure and define:

$$J_A=\operatorname{sgn}A=\int_{-\infty}^{\infty}\operatorname{sgn}t\,dE(t),$$
$$Q_A=|A|^{\frac{1}{2}}=\int_{-\infty}^{\infty}|t|^{\frac{1}{2}}dE(t).$$

The following relations easily hold: $J_A=J_A^*$, $J_A^2=I$, $Q_A\geq 0$, $A=Q_AJ_AQ_A$. For a selfadjoint operator A in $\mathcal{L}(\mathcal{H})$, we will say that A has $\varkappa$ negative squares if the associated quadratic form has $\varkappa$ negative squares. The following elementary and well-known result will be basic for our analysis in section 2.

1.1 PROPOSITION Let there be given the integers $0\leq\varkappa_1\leq\varkappa_2$, the invertible operator A having $\varkappa_1$ negative squares and the invertible selfadjoint operator C.

(1) $H=\begin{pmatrix}A, & B\\ B^*, & C\end{pmatrix}$ has $\varkappa_2$ negative squares if and only if $B=Q_AGQ_C$ and $J_C-G^*J_AG$ has $\varkappa_2-\varkappa_1$ negative squares.

(2) In the precedent situation, let $D=J_C-G^*J_AG$; then the

following factorization holds:

$$(1.1)\qquad H=\begin{bmatrix} Q_A & 0 \\ 0 & Q_C \end{bmatrix}\begin{bmatrix} I & 0 \\ G^*J_A & Q_D \end{bmatrix}\begin{bmatrix} J_A & 0 \\ 0 & J_D \end{bmatrix}\begin{bmatrix} I & J_AG \\ 0 & Q_D \end{bmatrix}\begin{bmatrix} Q_A & 0 \\ 0 & Q_C \end{bmatrix}.$$

PROOF (1) We have the equality:

$$H=\begin{bmatrix} I & 0 \\ B^*A^{-1} & I \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & C-B^*A^{-1}B \end{bmatrix}\begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix}$$

so H has $\varkappa_2$ negative squares if and only if $\begin{bmatrix} A & 0 \\ 0 & C-B^*A^{-1}B \end{bmatrix}$ has $\varkappa_2$ negative squares. As A has $\varkappa_1$ negative squares, this happens if and only if $C-B^*A^{-1}B$ has $\varkappa_2-\varkappa_1$ negative squares or $Q_C(J_C-Q_C^{-1}B^*Q_A^{-1}J_AQ_A^{-1}BQ_C^{-1})Q_C$ has $\varkappa_2-\varkappa_1$ negative squares. Define $G=Q_A^{-1}BQ_C^{-1}$. Therefore, $B=Q_AGQ_C$ and $D=J_C-G^*J_AG$ has $\varkappa_2-\varkappa_1$ negative squares.

(2) It is only a simple computation. ∎

We next describe the form of the row-operators

$$X_n=(G_1,K_2,\dots K_n):\ \mathcal{H}_n(=\underbrace{\mathcal{H}\oplus\dots\oplus\mathcal{H}}_{n})\longrightarrow\mathcal{H}$$

having the property:

$$(1.2)\qquad \begin{bmatrix} J_{S_o} & 0 \\ 0 & I_{n-1} \end{bmatrix} - X_n^*J_{S_o}X_n\geqslant 0$$

where: $S_o\in\mathcal{L}(\mathcal{H})$ has $\varkappa$ negative squares (this is not an essential restriction), $G_1\in\mathcal{L}(\mathcal{H})$ and $J_{S_o}-G_1^*J_{S_o}G_1$ is positive and invertible.

The condition (1.2) is eqivalent with:

$$(1.3)\qquad J_{S_o}\ -\ X_n\begin{bmatrix} J_{S_o} & 0 \\ 0 & I_{n-1} \end{bmatrix}X_n^*\geqslant 0$$

and we define $D_1=(J_{S_o}-G_1^*J_{S_o}G_1)^{\frac{1}{2}}$, $D_{*1}=(J_{S_o}-G_1J_{S_o}G_1^*)^{\frac{1}{2}}$. For a contraction $T\in\mathcal{L}(\mathcal{H},\mathcal{H}')$, $\mathcal{H}'$ beeing a Hilbert space, we also have the usual notation: $D_T=(I-T^*T)^{\frac{1}{2}}$ and $\mathcal{D}_T=\overline{D_T\mathcal{H}}$.

1.2 PROPOSITION Let G_1 be in $\mathcal{L}(\mathcal{H})$ with $J_{S_o}-G_1^*J_{S_o}G_1$ a positive invertible operator.

(1) The row-operator X_n satisfies (1.2) if and only if

there exist a contraction $G_2 \in \mathcal{L}(\mathcal{H})$ and the contractions

$G_p : \mathcal{H} \to \mathcal{D}_{G^*_{p-1}}$, $p > 2$, such that

(1.5) $$K_p = D_{*1} D_{G^*_2} \dots D_{G^*_{p-1}} G_p \quad , \quad 2 \leq p \leq n .$$

(2) The following factorization holds:

(1.6) $$\begin{bmatrix} J_{S_o} , & 0 \\ 0 , & I_{n-1} \end{bmatrix} - X^*_n J_{S_o} X_n = E^*_n E_n$$

where: $E_1 = D_1$ and for $n > 1$

$$E_n = \begin{bmatrix} E_{n-1} , & -Z_{n-1} G_n \\ 0 , & D_{G_n} \end{bmatrix} \quad ; \quad Z_1 = D_1^{-1} G_1^* J_{S_o} D_{*1} \quad \text{and for}$$

$$n > 1, \quad Z_n = \begin{bmatrix} Z_{n-1} D_{G^*_n} \\ G^*_n \end{bmatrix}.$$

<u>PROOF</u> For (1) we have only to remark that if we proved (1.5) for the first p steps, then

$$J_{S_o} - X_p \begin{bmatrix} J_{S_o} , & 0 \\ 0 , & I_{p-1} \end{bmatrix} X^*_p = D_{*1} \dots D^2_{G^*_p} \dots D_{*1}$$

and then, if

$$J_{S_o} - (X_p , K_{p+1}) \begin{bmatrix} J_{S_o} , & 0 \\ 0 , & I_p \end{bmatrix} \begin{bmatrix} X^*_p \\ K^*_{p+1} \end{bmatrix} \geq 0$$

it results $K_{p+1} K^*_{p+1} \leq D_{*1} \dots D^2_{G^*_p} \dots D_{*1}$, consequently

$K_{p+1} = D_{*1} \dots D_{G^*_p} G_{p+1}$ where $G_{p+1} : \mathcal{H} \longrightarrow \mathcal{D}_{G^*_p}$ is a contraction.

(2) We have

(1.7) $$I - Z^*_n Z_n = D_{G^*_n} \dots D_{*1} J_{S_o} D_{*1} \dots D_{G^*_n}$$

and , by induction,

(1.8) $$X^*_n J_{S_o} D_{*1} D_{G^*_2} \dots D_{G^*_n} = E^*_n Z_n \ .$$

Using (1.7) and (1.8) it results again by induction, (1.6). ∎

We define $d_{*n} = D_{G^*_2} \dots D_{G^*_n}$, $d_n = D_{G_n} \dots D_{G_2}$, $n \geq 2$ and

$$X^{(2)}_n = (G_2 , d_{*2} G_3 , \dots , d_{*,n-1} G_n) \text{ so } X_n = (G_1 , D_{*1} X^{(2)}_n).$$

<u>1.3 PROPOSITION</u> Let G_1 be in $\mathcal{L}(\mathcal{H})$ with $J_{S_o} - G^*_1 J_{S_o} G_1$ a positive invertible operator. Then the sequence of operators $\left\{ X_n P^{\ell^2(\mathbb{N},\mathcal{H})}_{\mathcal{H}_n} \right\}_{n=1}^{\infty}$

converges to an operator X_∞ which satisfies the condition:

$$\begin{bmatrix} J_{S_0} , & 0 \\ 0 , & I \end{bmatrix} - X_\infty^* J_{S_0} X \geq 0$$

($P^{\mathcal{K}}_{\mathcal{K}'}$ denotes the orthogonal projection of $\mathcal{K}$ onto $\mathcal{K}' \subset \mathcal{K}$ and $\mathcal{K}_n$ is viewed as a subspace of $\ell^2(\mathbb{N},\mathcal{K})$).

PROOF Taking advantage of the results in section 1 [13], it results that $X_n^{(2)} P^{\ell^2(\mathbb{N},\mathcal{K})}_{\mathcal{K}_{n-1}}$ are contractions converging to a contraction $X_\infty^{(2)}$ hence $\{X_n P^{\ell^2(\mathbb{N},\mathcal{K})}_{\mathcal{K}_n}\}$ converges to the operator $X_\infty = (G_1, D_{*1} X_\infty^{(2)})$ which satisfies the condition

$$J_{S_0} - X_\infty \begin{bmatrix} J_{S_0} , & 0 \\ 0 , & I \end{bmatrix} X_\infty^* = D_{*1}(I - X_\infty^{(2)} X_\infty^{(2)*}) D_{*1} \geq 0 .$$

Moreover, having also notation and results from [13], E_n converges to

$$E_\infty = \begin{bmatrix} D_1 , & -Z_1 X_\infty^{(2)} \\ 0 , & D_\infty^{(2)} \end{bmatrix}$$

and

$$\begin{bmatrix} J_{S_0} , & 0 \\ 0 , & I \end{bmatrix} - X_\infty^* J_{S_0} X_\infty = E_\infty^* E_\infty$$

$$J_{S_0} - X_\infty \begin{bmatrix} J_{S_0} , & 0 \\ 0 , & I \end{bmatrix} X_\infty^* = \lim_{n\to\infty} (D_{*1} d_{*n} d_{*n}^* D_{*1}).$$

Now let us define for a further use the unitary operators:

$$\alpha_\infty : \mathcal{D}_\infty \longrightarrow \mathcal{D}_1 \oplus \mathcal{D}_{G_2} \oplus \mathcal{D}_{G_3} \oplus \dots$$

$$\alpha_\infty D_\infty = E_\infty$$

$$\tilde{\alpha}_\infty : \mathcal{D}_{*\infty} \longrightarrow \mathcal{D}_*$$

$$\tilde{\alpha}_\infty D_{*\infty} = (\lim_{n\to\infty} (D_{*1} d_{*n} d_{*n}^* D_{*1}))^{\frac{1}{2}} . \blacksquare$$

Similar considerations hold for column operators , which will denote by $\tilde{X}_n$, $\tilde{X}_\infty$, ...

1.4 REMARK Proposition 1.1 can be used in many other situations.

For example, let $T_1 \in \mathcal{L}(\mathcal{K})$ such that $I - T_1^* T_1$ has $\varkappa$ negative squares and we define $Q_{T_1} = Q_{I - T_1^* T_1}$, $J_{T_1} = J_{I - T_1^* T_1}$ (we keep the above notation only in this remark). We first establish the form of the operator $L=(T_1,T_2)$ satisfying the property that $I-L^*L$ has $\varkappa$ negative squares. Using Proposition 1.1, this is equivalent with:

$$I-T_2^*T_2-T_2^*T_1(I-T_1^*T_1)^{-1}T_1^*T_2 \geqslant 0$$

consequently, $T_2=Q_{T_1^*}G_2$ and $I-G_2^*J_{T_1^*}G_2 \geqslant 0$.

In a similar way, $R=\begin{bmatrix} T_1 \\ T_3 \end{bmatrix}$ has the property that $I-R^*R$ has negative squares if and only if $T_3=G_3Q_{T_1}$ and $I-G_3J_{T_1}G_3^* \geqslant 0$.

Now we cosider the problem of determining the form of the operator $T=\begin{bmatrix} T_1 , & T_2 \\ T_3 , & X \end{bmatrix}$ for which $I-T^*T$ has $\varkappa$ negative squares.

Using again Proposition 1.1, we find:

$$X=-G_3J_{T_1}T_1^*G_2 + Q_{G_3^*}GQ_{G_2}$$

where we denoted $Q_{G_3^*}=(I-G_3J_{T_1}G_3^*)^{\frac{1}{2}}$, $Q_{G_2}=(I-G_2^*J_{T_1^*}G_2)^{\frac{1}{2}}$ and G is an arbitrary contraction.

The case $\varkappa=0$ of the above problem is used in [2] for a solution of the Nehari problem and the general form appears in [6]. ∎

III The structure of Toeplitz forms having $\varkappa$ negative squares

This section contains the main result of the paper. We shall really treat a particular case, but illustrating the differences from $\varkappa=0$. The general case can be then easily stated.

Let $S_n \in \mathcal{L}(\mathcal{K})$, $n \geqslant 0$ and S_0 has $\varkappa \leqslant N$ negative squares. Our purpose is to determine the form of S_n in order that the matrices

$$\mathcal{T}_n = \begin{bmatrix} S_0 , & S_1 , & \dots , & S_n \\ S_1^* , & S_0 , & \dots , & S_{n-1} \\ & & & \\ S_n^* & & & S_0 \end{bmatrix}$$

have $\varkappa$ negative squares. We need some notation (for $\varkappa=0$, see [4] or [12])

For a contraction $T \in \mathcal{L}(\mathcal{K},\mathcal{K}')$ we define the unitary operator

$$R(T): \mathcal{K} \oplus \mathcal{D}_{T^*} \longrightarrow \mathcal{K}' \oplus \mathcal{D}_T$$

$$R(T)=\begin{bmatrix} T , & D_{T^*} \\ D_T , & -T^* \end{bmatrix}$$

and for $G_1 \in \mathcal{L}(\mathcal{H})$ for which $J_{S_o} - G_1^* J_{S_o} G_1$ is positive and invertibl

we define:

$$R(G_1) = \begin{bmatrix} J_{S_o} G_1 J_{S_o} , & J_{S_o} D_{*1} \\ D_1 J_{S_o} , & -Z_1 \end{bmatrix}$$

and $R(G_1)$ is a $\begin{bmatrix} J_{S_o} , & 0 \\ 0 , & I \end{bmatrix}$ -unitary operator.

We fix now a sequence $\{G_n\}_{n=0}^{\infty}$ such that $G_o = S_o$, G_1 as above and $G_n \in \mathcal{L}(\mathcal{D}_{G_{n-1}}, \mathcal{D}_{G^*_{n-1}})$ are comtractions for $n \geqslant 2$. We define $V_o = J_{S_o}$, $U_o = J_{S_o}$, $V_1 = R(G_1)$, $U_1 = V_1$,

$$V_n = \begin{bmatrix} R(G_1) & , 0 \\ 0 & , I_{n-1} \end{bmatrix} \begin{bmatrix} I , & 0 , & 0 \\ 0, & R(G_2), & 0 \\ 0, & 0 , & I_{n-2} \end{bmatrix} \cdots \begin{bmatrix} I_{n-1} , & 0 \\ 0 & , R(G_n) \end{bmatrix}$$

$$U_n = V_n \begin{bmatrix} J_{S_o} & , 0 \\ 0 & , I_n \end{bmatrix} \begin{bmatrix} U_{n-1} & , 0 \\ 0 & , I \end{bmatrix} .$$

$$F_0 = I \ , \ F_1 = \begin{bmatrix} I & , J_{S_o} G_1 \\ 0 & , D_1 \end{bmatrix} , \quad F_n = \begin{bmatrix} F_{n-1} & , U_{n-1} \widetilde{X}_n \\ 0 & , d_n D_1 \end{bmatrix} .$$

We first need some preliminaries on F_n.

<u>2.1 LEMMA</u> For $n \geqslant 1$ we have the equality:

$$F_n = V_n \begin{bmatrix} X_n^* & , \begin{bmatrix} J_{S_o} & , 0 \\ 0 & , I \end{bmatrix} F_{n-1} \\ d^*_{*n} D_{*1} , & 0 \end{bmatrix} \tag{2.1}$$

<u>PROOF</u> First, we obtain by induction that

$$V_n = \begin{bmatrix} J_{S_o} & , 0 \\ 0 & , I \end{bmatrix} \begin{bmatrix} X_n , & D_{*1} d_{*n} \\ E_n , & -Z_n \end{bmatrix} \begin{bmatrix} J_{S_o} & , 0 \\ 0 & , I \end{bmatrix} \tag{2.2}$$

Then,

$$U_{n-1} \widetilde{X}_n = \begin{bmatrix} J_{S_o} & , 0 \\ 0 & , I_{n-1} \end{bmatrix} \begin{bmatrix} X_{n-1} , & D_{*1} d_{*n-1} \\ E_{n-1} , & -Z_{n-1} \end{bmatrix} \begin{bmatrix} U_{n-2} \widetilde{X}_{n-1} \\ G_n d_{n-1} D_1 \end{bmatrix} =$$

$$= \begin{bmatrix} J_{S_o} X_{n-1} U_{n-2} \widetilde{X}_{n-1} + J_{S_o} D_{*1} d_{*n-1} G_n d_{n-1} D_1 \\ E_{n-1} U_{n-2} \widetilde{X}_{n-1} - Z_{n-1} G_n d_{n-1} D_1 \end{bmatrix}$$

Using the last relation , it results by induction

$$F_n=\begin{bmatrix} I, & J_{S_o}X_nF_{n-1} \\ 0, & E_nF_{n-1}\end{bmatrix} \tag{2.3}$$

and by a direct compitation using (2.3) , (2.1) follows. ∎

Having these preliminaries , we can prove the main result.

<u>2.2 THEOREM</u> There exists a one-to-one correspondence between the set of the sequences $\{\mathcal{T}_n\}_{n\geq 0}$ where $\mathcal{T}_n$ have $\varkappa$ negative squares and are invertible operators and the set of the sequences $\{G_n\}_{n=0}^{\infty}$, $G_o=S_o$, $J_{S_o}-G_1^*J_{S_o}G_1$ is a positive invertible operator and G_n are strict contractions ($\|G_n\|<1$) for $n\geq 2$, given by the formulae:

$$S_1=Q_{S_o}G_1Q_{S_o},\quad S_2=Q_{S_o}(G_1J_{S_o}G_1+D_{*1}G_2D_1)Q_{S_o}$$

$$S_n=Q_{S_o}(X_{n-1}U_{n-2}\widetilde{X}_{n-1}+D_{*1}d_{*n-1}G_nd_{n-1}D_1)Q_{S_o} \qquad ,n\geq 3 .$$

<u>PROOF</u> Formally, the proof is as in the definite case ($\varkappa$ =0)(see [12]).

$\begin{bmatrix} S_o & ,S_1 \\ S_1^* & ,S_o\end{bmatrix}$ has $\varkappa$ negative squares if and only if

$S_o-S_1^*S_o^{-1}S_1\geq 0$ so $S_1=Q_{S_o}G_1Q_{S_o}$ and $J_{S_o}-G_1^*J_{S_o}G_1\geq 0$.

Moreover, by a direct computation,

$$\mathcal{T}_1=\begin{bmatrix} Q_{S_o} & ,0 \\ 0 & ,Q_{S_o}\end{bmatrix}\begin{bmatrix} I & ,0 \\ G_1^*J_{S_o} & ,D_1\end{bmatrix}\begin{bmatrix} J_{S_o} & ,0 \\ 0 & ,I\end{bmatrix}\begin{bmatrix} I & ,J_{S_o}G_1 \\ 0 & ,D_1\end{bmatrix}\begin{bmatrix} Q_{S_o} & ,0 \\ 0 & ,Q_{S_o}\end{bmatrix}$$

consequently, $\mathcal{T}_1$ is invertible if and only if D_1 is invertible.

Further, using again Proposition 1.1, $\mathcal{T}_2$ has $\varkappa$ negative squares if and only if

$$\begin{bmatrix} S_o, & S_1 \\ S_1^*, & S_o\end{bmatrix}-\begin{bmatrix} S_1^* \\ S_2^*\end{bmatrix}S_o^{-1}(S_1\ ,\ S_2)\geq 0$$

Using the factorization of $\mathcal{T}_1$ it results:

$$Q_{S_o}^{-1}(S_1,S_2)\begin{bmatrix} Q_{S_o}^{-1} & ,0 \\ 0, & Q_{S_o}^{-1}\end{bmatrix}F_1^{-1}=(K_1,K_2);$$

consequently, $K_1=G_1$ and as

$\begin{bmatrix} J_{S_o} & , 0 \\ 0 & , I \end{bmatrix} - \begin{bmatrix} G_1^* \\ K_2^* \end{bmatrix} J_{S_o} (G_1, K_2) \geq 0$ it results from Proposition 1.2 that $K_2 = D_{*1} G_2$, where $G_2^* G_2 \leq I$ and it follows that $S_2 = Q_{S_o}(G_1 J_{S_o} G_1 + D_{*1} G_2 D_1) Q_{S_o}$. Further, we prove by induction the statements:

$$(2.4)_n \quad (S_1, \ldots, S_n) = Q_{S_o} X_n F_{n-1} \begin{bmatrix} Q_{S_o} & , \ldots, & 0 \\ & \ddots & \\ 0 & , \ldots, & Q_{S_o} \end{bmatrix}$$

$$(2.5)_n \quad \begin{bmatrix} S_n \\ S_{n-1} \\ \vdots \\ S_1 \end{bmatrix} = \begin{bmatrix} Q_{S_o} & , \ldots, & 0 \\ & \ddots & \\ 0 & , \ldots, & Q_{S_o} \end{bmatrix} F_{n-1}^* \begin{bmatrix} J_{S_o} & , 0 \\ 0 & , I_{n-1} \end{bmatrix} U_{n-1} \tilde{X}_n$$

$$(2.6)_n \quad \mathcal{T}_n = \begin{bmatrix} Q_{S_o} & \ldots, & 0 \\ & \ddots & \\ 0, & \ldots, & Q_{S_o} \end{bmatrix} F_n^* \begin{bmatrix} J_{S_o} & , 0 \\ 0 & , I \end{bmatrix} F_n \begin{bmatrix} Q_{S_o} & \ldots, & 0 \\ & \ddots & \\ 0 & \ldots, & Q_{S_o} \end{bmatrix}$$

$(2.7)_n$ there exists a contraction G_{n+1} such that

$$S_{n+1} = Q_{S_o}(X_n U_{n-1} \tilde{X}_n + D_{*1} d_{*n} G_{n+1} d_n D_1) Q_{S_o}.$$

Suppose the four statements verified for the first n-1 steps.

Then $(2.4)_n$ results by a direct computation taking into account $(2.4)_{n-1}$ and (2.3); $(2.5)_n$ follows from $(2.5)_{n-1}$ and Lemma 2.1 . For (2.6) we use $(2.6)_{n-1}$,$(2.5)_n$ and the definition of F_n.

Finally, using Proposition 1.1, $\mathcal{T}_{n+1}$ has $\varkappa$ negative squares if and only if

$$F_n^* \begin{bmatrix} J_{S_o} & , 0 \\ 0 & , I_n \end{bmatrix} F_n - \begin{bmatrix} S_1^* \\ \vdots \\ S_{n+1}^* \end{bmatrix} Q_{S_o}^{-1} J_{S_o} Q_{S_o}^{-1} (S_1, \ldots, S_{n+1}) \geq 0 ;$$

consequently, $(S_1, \ldots, S_{n+1}) = (K_1, \ldots, K_{n+1}) F_n =$

$$((K_1, \ldots, K_n) F_{n-1}, (K_1, \ldots, K_n) U_{n-1} \tilde{X}_n + K_{n+1} d_n D_1)$$

so $(K_1, \ldots, K_n) = X_n$ and as

$$\begin{bmatrix} J_{S_o}, & 0 \\ 0, & I_n \end{bmatrix} - \begin{bmatrix} X_n^* \\ K_{n+1}^* \end{bmatrix} J_{S_o} (X_n, K_{n+1}) \geq 0$$

we have from Proposition 1.2 that $K_{n+1} = D_{*1} d_{*n} G_{n+1}$ with $G_{n+1}^* G_{n+1} \leq I$ and $S_{n+1} = Q_{S_o}(X_n U_{n-1} \tilde{X}_n + D_{*1} d_{*n} G_{n+1} d_n D_1) Q_{S_o}$. ∎

<u>2.3 REMARK</u> Let $\{\mathcal{T}_n\}_{n=0}^{\infty}$ be a sequence of Toeplitz matrices such that $\mathcal{T}_p$ has $\varkappa_p$ negative squares for $p \geq 0$. Then the sequence of para-

meters producedby the corresponding variant of the algorithm in Theorem 2.2 is: $G_o=S_o$, G_1 with the property that $J_{S_o}-G_1^*J_{S_o}G_1$ has $\varkappa_1-k_0$ negative squares; define $J_{G_1}=J_{J_{S_o}-G_1^*J_{S_o}G_1}$, then G_2 has the property that $J_{G_1}-G_2^*J_{G_1^*}G_2$ has $\varkappa_2-k_1$ negative squares, and so on. ∎

2.4 REMARK Let

$$\mathcal{T}_n=\begin{bmatrix} -1, & s_1, \ldots, s_n \\ \overline{s}_1, & -1, \ldots, s_{n-1} \\ & \\ \overline{s}_n, & \overline{s}_{n-1}, \ldots, -1 \end{bmatrix}$$

where $s_n \in \mathbb{C}$ for $n \in \mathbb{N}$. Then:

$$\mathcal{T}_1 \quad \text{has} \begin{cases} 1 \text{ negative square iff } |g_1| \geq 1 \\ 2 \text{ negative squares iff } |g_1| < 1 \end{cases}$$

$$\mathcal{T}_2 \quad \text{has} \begin{cases} 1 \text{ negative square iff } |g_1| \geq 1, |g_2| \leq 1 \\ 2 \text{ negative squares iff } |g_1| \geq 1, |g_2| > 1 \text{ or} \\ \qquad |g_1| < 1, |g_2| \geq 1 \\ 3 \text{ negative squares iff } |g_1| < 1, |g_2| < 1 \end{cases}$$

and so on. ∎

2.5 REMARK Actaully, we can handle in a similar manner arbitrary matrices with a fixed signature. The resulting algorithm is the adaptation of the one in [14]. ∎

2.6 REMARK The invertibility condition in Theorem 2.2 is not necessary. The general object in parametrizing the Toeplitz forms having $\varkappa$ negative squares with the only requirement that S_o is invertible, is the sequence of operators $\{G_n\}_{n=o}^{\infty}$ such that $G_o=S_o$, $J_{S_o}-G_1^*J_{S_o}G_1 \geq 0$, $G_2 \in \mathcal{L}(\mathcal{D}_1, \mathcal{D}_{*1})$, $G_n \in \mathcal{L}(\mathcal{D}_{G_{n-1}}, \mathcal{D}_{G_{n-1}^*})$, where $\mathcal{D}_1=\overline{D_1\mathcal{H}}$, $\mathcal{D}_{*1}=\overline{D_{*1}\mathcal{H}}$. ∎

We end this section with a formula for computing the determinants of the matrices $\mathcal{T}_n$.

2.7 PROPOSITION In the conditions of Theorem 2.2,

$$\det \mathcal{T}_n = (-1)^{\varkappa n} \prod_{m=2} (\det D_{G_m})^{2(n-m+1)} \det D_1^{2n} \det Q_{S_0}^{2(n+1)}.$$

PROOF From $(2.6)_n$. ∎

IV THE INDEFINITE TRIGONOMETRIC MOMENT PROBLEM

In this section we shall obtain a Pontrjagin space $\Pi_\varkappa$ and a unitary operator W in this space , such that:

$$(W^n h,h)_{\Pi_\varkappa} = \langle Q_{S_0}^{-1} S^n Q_{S_0}^{-1} h,h \rangle \qquad , h \in \mathcal{H}$$

In [17] , the classical idea for constructing the Naimark dilation is adapted in oeder to obtain this fact . We want only to point out the connection between W and the parameters $\{G_n\}$ as was showed for the definite case in [13] .

We define : $\hat{W}_1 = G_1$,

$$\hat{W}_n = \begin{bmatrix} J_{S_0} & , 0 \\ 0 & , I_{n-1} \end{bmatrix} V_{n-1} \begin{bmatrix} J_{S_0} & , 0 \\ 0 & , I_{n-1} \end{bmatrix} \begin{bmatrix} I_{n-1}, & 0 \\ 0, & G_n \end{bmatrix} \quad : \mathcal{H}_n \longrightarrow \mathcal{H}_n$$

We have:

$$(3.1) \qquad \hat{W}_n = \begin{bmatrix} \begin{matrix} G_1, \\ \begin{bmatrix} D_1 \\ 0_{n-2} \end{bmatrix} \end{matrix} , & \begin{matrix} D_{*1} X_n^{(2)} \\ \begin{bmatrix} -Z_1 & , 0 \\ 0 & , I_{n-2} \end{bmatrix} W_n^{(2)} \end{matrix} \end{bmatrix}$$

where $W_n^{(2)}$ is the operator in [13] associated with the choice sequence $\{G_n\}_{n=2}^{\infty}$. According toLemma 2.2 in [13], there exists $W_+^{(2)} = \lim_{n\to\infty} W_n^{(2)} P_{\mathcal{H}_{n-1}}^{\ell^2(\mathbb{N},\mathcal{H})}$, then, there exists $\hat{W}_+ = \lim_{n\to\infty} \hat{W}_n P_{\mathcal{H}_n}^{\ell^2(\mathbb{N},\mathcal{H})}$. We define

$$J_+ = \begin{bmatrix} J_{S_0} & , 0, \ldots, \ldots \\ 0 & , I, \ldots \\ & \ddots \\ & & I \end{bmatrix}$$

3.1 PROPOSITION $P_{\mathcal{H}}^{\ell^2(\mathbb{N},\mathcal{H})} \hat{W}_+ J_+ \hat{W}_+ \ldots J_+ \hat{W}_+ P_{\mathcal{H}}^{\ell^2(\mathbb{N},\mathcal{H})} = Q_{S_0}^{-1} S_n Q_{S_0}^{-1}$.

PROOF Having the definition of $\hat{W}_+$,it results:

$$P_{\mathcal{H}}^{\ell^2(\mathbb{N},\mathcal{H})} \hat{W}_+ J_+ \hat{W}_+ \ldots J_+ \hat{W}_+ P_{\mathcal{H}}^{\ell^2(\mathbb{N},\mathcal{H})} = P_{\mathcal{H}}^{\mathcal{H}_n} \hat{W}_n \begin{bmatrix} J_{S_0}, & 0 \\ 0, & I_{n-1} \end{bmatrix} \hat{W}_n \ldots \hat{W}_n P_{\mathcal{H}}^{\mathcal{H}_n} .$$

From the formula (2.2) ,

$$(3.2)\qquad \hat{W}_n=\begin{bmatrix} X_{n-1} , & D_{*1}d_{*n-1}G_n \\ E_{n-1} , & -Z_{n-1}G_n \end{bmatrix}$$

consequently,

$$(3.3)\qquad (I,0_{n-1})\hat{W}_n=X_n .$$

Then , we prove by induction

$$(3.4)\qquad \begin{bmatrix} J_{S_o} , & 0 \\ 0 , & I_{n-1} \end{bmatrix}\hat{W}_n \cdots\cdots \begin{bmatrix} J_{S_o} , & 0 \\ 0 , & I_{n-1} \end{bmatrix}\hat{W}_n \begin{bmatrix} I \\ 0_{n-1} \end{bmatrix} = \begin{bmatrix} U_{n-2}\tilde{X}_{n-1} \\ d_{n-1}D_1 \end{bmatrix}$$

Using (3.3),(3.4) and Theorem 2.2 , the proof is finished. ■

We define $\mathcal{K}_+ =(\ell^2(\mathbb{N},\mathcal{H}) ,(\ ,\ )_{J_+})$ where

$$(x,y)_{J_+}= \langle J_+x,y\rangle$$

$\mathcal{K}_+$ is a Π_λ-space (see [18]) and we define $W_+=J_+\hat{W}_+$ then

$$(3.5)\qquad (W_+^n h ,h)_{J_+} = \langle J_+W_+J_+W_+\ldots W_+h,h\rangle = \langle Q_{S_o}^{-1}S_nQ_{S_o}^{-1}h,h\rangle$$

As $\hat{W}_n^* \begin{bmatrix} J_{S_o} , & 0 \\ 0 , & I_{n-1} \end{bmatrix} \hat{W}_n = \begin{bmatrix} J_{S_o} , & 0 \\ & I & \\ & & G_n^*G_n \end{bmatrix}$ we have

$\hat{W}_+^*J_+\hat{W}_+=J_+$, consequently, $W_+^*J_+W_+=J_+$. Then, let $k\in\mathcal{K}_+$,$(k,W_+^n h)_{J_+}=0$ for $n\geqslant 0,h\in\mathcal{H}$. First, $(k,h)_{J_+}=0$ or $\langle J_{S_o}k_o ,h\rangle =0$ so $J_{S_o}k_o=0$ and $k_o=0$; using the triangular form of W_+ we successively obtain: $k_1=k_2=\ldots=k_n=\ldots=0$ and

$$(3.6)\qquad \bigvee_{n=0}^{\infty} W_+^n \mathcal{H} = \mathcal{K}_+ .$$

As in [13] we can obtain the unitary extension of W_+. We define

$$\hat{W}_{red}= \begin{bmatrix} I , & 0 \\ 0, & \alpha_\infty \end{bmatrix} R(X_\infty) \begin{bmatrix} 0 , & I \\ \tilde{\alpha}_\infty^* , & 0 \end{bmatrix}$$

and

$$\hat{W}:\ell^2(\mathbb{Z},\mathcal{K}) \longrightarrow \ell^2(\mathbb{Z},\mathcal{H})$$

$$\hat{W} = I \oplus \hat{W}_{red}$$

then

$$\hat{W} = \begin{bmatrix} * , & 0 \\ * , & \hat{W}_+ \end{bmatrix}$$

and $\hat{W}J\hat{W}^{*}=J$ where $J= I \oplus J_{S_o} \oplus I$; if we define $W=J\hat{W}$, we have

$$W = \begin{bmatrix} * & , & 0 \\ * & , & W_+ \end{bmatrix}$$

Now , we define $\mathcal{K}=(\ell^2(\mathbb{Z},\mathcal{H}), (\ ,\ )_J)$ where $(x,y)_J=\langle Jx,y\rangle$ and $\mathcal{K}$ is a Π_h - space cotaining $\mathcal{K}_+$, W is a unitary operator extending W_+ and

$$(W^n h,h)_J=\langle Q_{S_o}^{-1}S_n Q_{S_o}^{-1}h,h\rangle \quad ,h\in\mathcal{H}, \bigvee_{n=-\infty}^{\infty} W^n \mathcal{K}=\mathcal{K}.$$

More will be transparent since in the next section we will develop orthogonal polynomials in $\mathcal{K}$.

V ORTHOGONAL POLYNOMIALS IN $\mathcal{K}$

In this section we consider the polynomials:

$$\varphi_n(z)=L_{nn}z^n+L_{n,n-1}z^{n-1}+\ldots+L_{no} \qquad , L_{nk}\in\mathcal{L}(\mathcal{K}) \ , \ L_{nn}\geqslant 0$$

with the following properties:

(4.1) $\qquad \varphi_o(z)S_o\varphi_o^{*}(z)=J_{S_o}$

(4.2) $\qquad \varphi_n(z)\tilde{\mathcal{T}}_n\varphi_k^{*}(z)=0 \qquad , 0<k<n$

(4.3) $\qquad \varphi_n(z)\tilde{\mathcal{T}}_n\varphi_n^{*}(z)=I \qquad , n>0$

where

$$\tilde{\mathcal{T}}_n=\begin{bmatrix} S_o, & S_1^{*}, & \ldots, & S_n^{*} \\ S_1, & S_o, & & \\ \vdots & & \ddots & \\ S_n & & & S_o \end{bmatrix}$$

From (4.1) we have $\varphi_o(z)=Q_{S_o}^{-1}$. Next, we observe that (4.2) and (4.3) are equivalent with the system:

(4.4) $\qquad (L_{no},L_{n1},\ldots L_{nn})\,\mathcal{T}_n\begin{bmatrix} I_{n-1} & , & 0 \\ 0 & , & L_{nn} \end{bmatrix}=(0_n,I).$

4.1 PROPOSITION The following relations hold:

$$L_{nn}=(Q_{S_o}^{-1}D_1^{-1}d_n^{-1}d_n{}^{-1}D_1^{-1}Q_{S_o}^{-1})^{\frac{1}{2}}$$

and

$$(L_{no},\dots,L_{n,n-1})=-L_{nn}Q_{S_o}\tilde{X}_n^{*}U_{n-1}^{*}F_{n-1}^{-1*}\begin{bmatrix}Q_{S_o}^{-1},\dots,0\\ 0,\dots,Q_{S_o}^{-1}\end{bmatrix}.$$

PROOF Let us consider the system:

$$(\overset{\circ}{L}_{no},\dots,\overset{\circ}{L}_{nn})\mathcal{T}_n=(0_n,I)$$

and using (2.6),

$$(\overset{\circ}{L}_{no},\dots,\overset{\circ}{L}_{nn})=(0_n,I)\begin{bmatrix}Q_{S_o}^{-1},\dots,0\\ 0,\dots,Q_{S_o}\end{bmatrix}F_n^{-1}\begin{matrix}J_{S_o}, & 0\\ 0, & I_n\end{matrix}F_n^{*-1}\begin{bmatrix}Q_{S_o}^{-1},\dots,0\\ 0,\dots,Q_{S_o}\end{bmatrix}$$

But,

$$F_n^{-1}=\begin{bmatrix}F_{n-1}^{-1}, & -F_{n-1}^{-1}U_{n-1}\tilde{X}_nD_1^{-1}d_n^{-1}\\ 0, & D_1^{-1}d_n^{-1}\end{bmatrix}$$

then

$$\overset{\circ}{L}_{nn}=Q_{S_o}^{-1}D_1^{-1}d_n^{-1}d_n^{-1}D_1^{-1}Q_{S_o}^{-1}$$

and

$$(\overset{\circ}{L}_{no},\dots,\overset{\circ}{L}_{n,n-1})=-Q_{S_o}^{-1}D_1^{-1}d_n^{-1}d_n^{*-1}D_1^{-1}\tilde{X}_n^{*}U_{n-1}^{*}F_{n-1}^{*-1}\begin{bmatrix}Q_{S_o}^{-1},\dots,0\\ 0,\dots,Q_{S_o}\end{bmatrix}$$

and the desired relations follow. ■

Now, we define the polynomials:

$$\Phi_o(z)=I\ ,\quad \Phi_1(z)=z-G_1^{*}J_{S_o}\quad ,\quad \Phi_n(z)=Q_{S_o}^{-1}L_{nn}^{-1}\varphi_n(z)Q_{S_o}.$$

We also define the polynomials:

$$\tilde{\varphi}_n(z)=R_{nn}z^n+R_{n,n-1}z^{n-1}+\dots+R_{no}\quad ,\ R_{nk}\in\mathcal{L}(\mathcal{K})\ ,\ R_{nn}\geqslant 0$$

with the following properties:

(4.5) $\tilde{\varphi}_o^{*}(z)S_o\tilde{\varphi}_o(z)=J_{S_o}$

(4.6) $\tilde{\varphi}_n^{*}(z)\mathcal{T}_n\tilde{\varphi}_k(z)=0 \quad ,0<k<n$

(4.7) $\tilde{\varphi}_n^{*}(z)\mathcal{T}_n\tilde{\varphi}_n(z)=I \quad , n>0$

These polynomials will satisfy the relation:

(4.8) $$(R_{no}^{*},\dots,R_{nn}^{*})\tilde{\mathcal{T}}_n\begin{bmatrix}I_n, & 0\\ 0, & R_{nn}\end{bmatrix}=(0_n,I).$$

We define $\tilde{\Phi}_o(z)=I$, $\tilde{\Phi}_n(z)=Q_{S_o}\tilde{\varphi}_n(z)R_{nn}^{-1}Q_{S_o}^{-1}$

and is easy to see that $R_{nk}=\tilde{L}_{nk}$ where $\sim$ has the same meaning as in the definite case ($\widetilde{\text{formula}}\,(G_1,\dots,G_n)=\text{formula}\,(G_1^*,\dots,G_n^*)^*$) .

4.2 THEOREM The following formulas hold:

$$\Phi_0(z)=I,\quad \Phi_1(z)=z-G_1^*J_{S_0},\qquad \widetilde{\Phi}_0(z)=I,\quad \widetilde{\Phi}_1(z)=z-J_{S_0}G_1^*,$$

$$(4.9)\qquad \Phi_{n+1}(z)=z\,\Phi_n(z)-D_1 d_n^* G_{n+1}^* d_{*n}^{-1} D_{*n}^{-1}\widetilde{\Phi}_n^{\circledast}(z),\qquad n\geq 1$$

$$(4.10)\qquad \widetilde{\Phi}_{n+1}(z)=z\,\widetilde{\Phi}_n(z)-\Phi_n^{\circledast}(z)D_1^{-1}d_n^{-1}G_{n+1}^* d_{*n}D_{*1},\qquad n\geq 1.$$

*(for a polynomial p of degree n we use the notation : $p^{\circledast}(z)=z^n\bar{p}(\frac{1}{z})$, $\bar{p}(z)=\overline{p(\bar{z})}$.)

PROOF We denote $\Phi_n(z)=C_{n0}z^n+\dots+C_{nn}$, $\widetilde{\Phi}_n(z)=\widetilde{C}_{n0}z^n+\dots+\widetilde{C}_{nn}$.

Using the results established in section II,

$$(4.11)\qquad F_{n-1}^{-1}U_{n-1}\widetilde{X}_n=\begin{bmatrix} J_{S_0}(D_{*1}d_{*n-1}-X_{n-1}E_{n-1}^{-1}Z_{n-1})G_n d_{n-1}D_1 \\ F_{n-2}^{-1}U_{n-2}\widetilde{X}_{n-1}+F_{n-2}^{-1}E_{n-1}^{-1}Z_{n-1}G_n d_{n-1}D_1 \end{bmatrix}.$$

By induction , we prove that for $n\geq 1$,

$$(4.12)\qquad D_{*1}d_{*n-1}-X_{n-1}E_{n-1}^{-1}Z_{n-1}=J_{S_0}D_{*1}^{-1}d_{*n-1}^{-1}$$

then, having also Proposition 4.1 and the formulas for $\Phi_n(z)$ and $\widetilde{\Phi}_n(z)$, it results:

$$C_{nn}=-D_1 d_{n-1}^* G_n^* d_{*n-1}^{-1}D_{*1}^{-1}.$$

Next, again by induction,

$$(4.13)\qquad U_{n-1}\widetilde{F}_{n-1}^*\begin{bmatrix} O_{n-k} \\ I \\ O_{k-1} \end{bmatrix}=F_{n-1}\begin{bmatrix} O_{k-1} \\ I \\ O_{n-k} \end{bmatrix},$$

consequently, by a direct computation,

$$(4.14)\qquad C_{n+1,k}=C_{n,k}-D_1 d_n^* G_{n+1}^* d_{*n}^{-1}D_{*1}^{-1}\widetilde{C}_{n,n-k+1}^*$$

and this is exactly (4.9). ∎

Having the recurrence formulas in Theorem 4.2 , we can state now, the analogous of the Christoffel-Darboux formulas.

4.3 THEOREM The following formulas hold:

$$(4.15)\qquad (1-\bar{z}\zeta)\Big(\varphi_0^*(z)J_{S_0}\varphi_0(\zeta)+\sum_{k=1}^{n}\varphi_k^*(z)\varphi_k(\zeta)\Big)=$$
$$=\widetilde{\varphi}_{n+1}^{\circledast *}(z)\,\widetilde{\varphi}_{n+1}^{\circledast}(\zeta)-\varphi_{n+1}^*(z)\,\varphi_{n+1}(\zeta)$$

and

$$(4.16)\qquad (1-z\bar{\zeta})\Big(\widetilde{\varphi}_0(z)J_S\widetilde{\varphi}_0^*(\zeta)+\sum_{k=1}^{n}\widetilde{\varphi}_k(z)\widetilde{\varphi}_k^*(\zeta)\Big)=$$

$$= \varphi^{\circledast}_{n+1}(z)\varphi^{\circledast *}_{n+1}(\zeta) - \tilde{\varphi}_{n+1}(z)\tilde{\varphi}^{*}_{n+1}(\zeta) .$$

PROOF By a direct computation:

$$(4.17) \qquad (1-\bar{z}\zeta)\varphi^{*}_{o}(z)J_{S_o}\varphi_o(\zeta) = \tilde{\varphi}^{\circledast *}_{1}(z)\tilde{\varphi}^{\circledast}_{1}(\zeta) - \varphi^{*}_{1}(z)\varphi_1(\zeta).$$

Then, using the reccurence formulas (4.9) and (4.10), we prove the identity (for $n \geqslant 1$):

$$(4.18) \qquad \tilde{\varphi}^{\circledast *}_{n+1}(z)\tilde{\varphi}^{\circledast}_{n+1}(\zeta) - \varphi^{*}_{n+1}(z)\varphi_{n+1}(\zeta) = \tilde{\varphi}^{\circledast *}_{n}(z)\tilde{\varphi}^{\circledast}_{n}(\zeta) - z\bar{\zeta}\varphi^{*}_{n}(z)\varphi_n(\zeta)$$

and adding up (4.17) and (4.18) for k=1,n, one obtains (4.15). ∎

From now on, we restrict ourselves to the scalar case. So , we consider

$$\mathcal{S}_n = \begin{bmatrix} -1, & s_1, & \dots, & s_n \\ \bar{s}_1, & -1, & \dots, & s_{n-1} \\ \\ \bar{s}_n, & & \dots & ,-1 \end{bmatrix} \qquad , n \geqslant 1$$

such that $\mathcal{S}_k$ has $\varkappa_k$ negative squares , $k < r$ and $\mathcal{S}_k$ has $\varkappa$ negative squares for $k \geqslant r$. According to Theorem 2.2, to the family $\{\mathcal{S}_n\}_{n=o}$ we associate a sequence of complex numbers $\{g_n\}_{n=1}^{\infty}$ such that $|g_k| < 1$ for $k \geqslant r+2$. From Theorem 4.2 it results: $\Phi_1(z) = z + \bar{g}_1$, $\Phi_n(z) = z\,\Phi_{n-1}(z) - \bar{g}_n\Phi^{\circledast}_{n-1}(z)$.

The next result establishes the explicit connection between the orthogonal polynomials and dilation(for the definite case see [12]).

4.4 PROPOSITION For $n \geqslant 1$, $\Phi_n(z) = \det(z - W^{*}_n)$, where $W_n = P^{\ell^2(\mathbb{N})}_{\mathbb{C}^n} W P^{\ell^2(\mathbb{N})}_{\mathbb{C}^n}$.

PROOF As in the definite case , by a direct computation. ∎

Let J_n be the signature of $\mathcal{S}_n$, then , for a sufficiently large n,

$$W^{*}_n \begin{bmatrix} J_{\varkappa} & , 0 \\ 0 & , I \end{bmatrix} W_n = \begin{bmatrix} J_{\varkappa} & , 0 \\ 0 & , |g_n|^2 \end{bmatrix} \leq \begin{bmatrix} J_{\varkappa} & , 0 \\ 0 & , I \end{bmatrix}$$

and using Proposition 4.4 we can obtain some information on the zeros of the polynomials (see [20]).

Let us consider the function

$$\mathcal{F}(z)=1-2\sum_{n=1}^{\infty} s_n z^n$$

then (Satz 6.2 in [21]) , $\mathcal{F}$ is in the class C_λ and is holomorphic at 0 (the class C_λ is defined in [21] as the set of the meromorphic functions in the unit disc for which the kernel $C_{\mathcal{F}}(z,\zeta)=\dfrac{\mathcal{F}(z)-\overline{\mathcal{F}(\zeta)}}{1-z\bar{\zeta}}$ has λ negative squares).

Now , let us consider the positive measure on the unit circle with $\mu(1)=1$ associated with the choice sequence $\{g_k\}_{k=r+2}$ and the function

$$\mathcal{F}_\mu(z)=\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\,d\mu(t) \qquad , \quad \mathcal{F}_\mu\in C_0 .$$

Let $\{a_k\}$,$\{b_k\}$ be the orthogonal polynomials of first and second kind associated with μ and $\psi_n(z)=\varphi_n(z;-g_1,-g_2,\dots,-g_n)$

It is easy to determine the connection between $\{a_k\}$,$\{b_k\}$ and $\{\varphi_n\}$, $\{\psi_n\}$.

<u>4.5 PROPOSITION</u> $\psi_{r+k+1}=\frac{1}{2}(P_1^{\circledast} b_k+P_2^{\circledast} a_k)$, $\varphi_{r+k+1}=\frac{1}{2}(Q_1^{\circledast} b_k+Q_2^{\circledast} a_k)$,

where

$$P_1=\psi_{r+1}+\psi^{\circledast}_{r+1} \quad , \quad P_2=\varphi^{\circledast}_{r+1}-\varphi_{r+1}$$
$$Q_1=\psi^{\circledast}_{r+1}-\psi_{r+1} \quad , \quad Q_2=\varphi_{r+1}+\varphi^{\circledast}_{r+1} .$$

<u>PROOF</u> By induction. ∎

Now, the connection between $\mathcal{F}$ and $\mathcal{F}_\mu$ is almost clear.

<u>4.6 PROPOSITION</u>

$$\mathcal{F}(z)=\frac{P_1(z)\cdot\mathcal{F}_\mu(z)+P_2(z)}{Q_1(z)\cdot\mathcal{F}_\mu(z)+Q_2(z)}$$

<u>PROOF</u> The Schur analysis shows that the computations are as in the definite case. Comparing the developments around 0 we obtain the desired formula. ∎

Let us suppose that $\log\frac{d\mu}{dt}\in L^1(\mathbb{T})$ (that is $\prod_{k=r+2}^{\infty}(1-|g_k|^2)\neq 0$) and let g be the outer function factorizing $\frac{d\mu}{dt}$, $\tilde{g}$ the outer function factorizing $\frac{d\tilde{\mu}}{dt}$ ($\tilde{\mu}$ is the positive measure on the unit circle associated with the choice sequence $\{-g_k\}_{k=r+2}$). Then , having in mind several classical results (see for instance [15]), $\{a_k^{\circledast}\}$ converges to $\frac{1}{g}$ uniformly on the compact set in the unit disc and $\{b_k^{\circledast}\}$

converges to $\frac{1}{g^-}$, consequently, $\{\varphi_n^{*}\}$ converges to $\frac{1}{2}(\frac{Q_1}{g^-}+\frac{Q_2}{g})$

and ψ_n^{*} to $\frac{1}{2}(\frac{P_1}{g^-}+\frac{P_2}{g})$ and let us define the meromorphic functions

$G=2(\frac{1}{\frac{Q_1}{g^-}+\frac{Q_2}{g}})$ and $G^-=2(\frac{1}{\frac{P_1}{g^-}+\frac{P_2}{g}})$. Then , from Proposition 4.6,

(4.19) $$\sigma_f = \frac{G}{G^-}$$

and we also note that

(4.20) $$G(0)=\prod_{n=1}^{\infty}(1-|g_n|^2)\operatorname{sgn}(1-|g_n|^2).$$

One more remark on the Christoffel-Darboux formulas. From Theorem 4.3 ,

$$(1-\bar{z}\zeta)\sum_{k=0}^{n}\overline{\varphi_k(z)}\varepsilon_k\varphi_k(\zeta)=\overline{\varphi_{n+1}^{*}(z)}\varphi_{n+1}^{*}(\zeta)-\overline{\varphi_{n+1}(z)}\varphi_{n+1}(\zeta)$$

then , for $|z|<1, |\zeta|<1$, different from the poles and zeros of G,

(4.21) $$\sum_{n=0}^{\infty}\overline{\varphi_n(z)}\,\varepsilon_n\,\varphi_n(\zeta)=\frac{1}{1-\bar{z}\zeta}\,\frac{1}{\overline{G(z)}\,G(\zeta)}$$

(here we denoted $J_n=(\varepsilon_k)$).

VI ASYMPTOTIC PROPERTIES FOR THE TOEPLITZ DETERMINANTS

In this section we discuss some asyptotic properties for the Toeplitz determinants $D_n=\det\mathcal{T}_n$. For the definite case we have the classical theorems of Szegö. As a consequence of Proposition 2.7,

$$\frac{D_{n+1}}{D_n}=\det{}^2 D_{S_0}\det{}^2 D_1\ldots\det{}^2 D_{r+1}\det{}^2 D_{G_{r+2}}\ldots\det{}^2 D_{G_{n+1}}$$

and there exists the limit

$$\lim_{n\to\infty}\frac{D_{n+1}}{D_n}=\det{}^2 D_{S_0}\prod_{k=1}^{r+1}\det{}^2 D_k\prod_{n=r+2}^{\infty}\det{}^2 D_{G_n}\qquad(=G(0))$$

We denote this quantity by $G(\mathcal{T})$, and then there exists the limit

$$\lim\frac{D_n}{G(\mathcal{T})^{n+1}}=(-1)^n\left(\prod_{n=1}^{\infty}\det{}^{2n}D_n\right)^{-1}\qquad(\text{here , } D_n=D_{G_n} \text{ for } n\geq r+2).$$

Thus, the Schur formalism permits an easy derivation for the asymptotics for the Toeplitz determinants (actually even for the general case as in [14][5]).

Without difficulties we can obtain the geometrical variants of these results (for the definite case see [22] and [5]). For instance, let us define $\mathcal{K}_n^- = \bigvee_{k=0}^{n} W^{*k}\mathcal{L}$ and $P_{J,n}$ the J-projection of $\mathcal{K}$ onto $\mathcal{K}_n$. Then

$$\det((I-P_{\mathcal{L}}^{\mathcal{K}} P_{J,n} P_{\mathcal{L}}^{\mathcal{K}})J)=\frac{D_n}{D_{n-1}}$$ and so on.

What we want to do is a remark connected with the paper [3].

For same simplicity we shall treat the scalar case only. We begin by adapting a particular case in [3] to our context. For a positive measure μ on the unit circle, with $\mu(1)=1$, we define

$$h(\mu,z)=-\frac{1}{4\pi}\int_0^{2\pi}\mathrm{Re}\,\frac{e^{it}+z}{e^{it}-z}\,\ln\frac{d\mu}{dt}(t)dt$$

and $\{s_n\}$ are the Fourier coefficients of μ. Having the complex numbers $s_1, \ldots, s_n$ with the property that

$$\mathcal{T}_n=\begin{bmatrix} 1, s_1, \ldots, s_n \\ \overline{s}_1, 1, \ldots, s_{n-1} \\ \\ \overline{s}_n, \ldots, 1 \end{bmatrix}$$

is positive, we define

$$\mathcal{G}_n=\{\mu \text{ a positive measure on } \mathbb{T} \text{ with } s_k(\mu)=s_k, k=1,n\}$$

and we consider the problem "$\min_{\mu\in\mathcal{G}_n} h(\mu,z)$". A simple solution can be done in the following way:

if g is the outer function factorizing $\frac{d\mu}{dt}$ then

$h(\mu,z)=-\ln g(z)$; using Christoffel-Darboux formula and the fact that $\mu\in\mathcal{G}_n$ if and only the first n terms in the choice sequence associated to μ are those associated to $\mathcal{T}_n$, we have

$$h(\mu,z)=\ln(1-|z|^2)^{\frac{1}{2}}+\tfrac{1}{2}\ln\sum_{k=0}^{\infty}|\varphi_k(z)|^2 \geq$$
$$\geq \ln(1-|z|^2)^{\frac{1}{2}}+\tfrac{1}{2}\sum_{k=0}^{n}|\varphi_k(z)|^2$$

where $\{\varphi_k\}$ are the orthogonal polynomials associated with μ.

Now we have only to find a choice sequence $\{g_1, g_2, \ldots, g_n, g_{n+1}(z), g_{n+2}(z), \ldots\}$ with the property that in the fixed point z,

$$\varphi_{n+1}(z)=\varphi_{n+2}(z)=\ldots=0$$

and is easy to see that the only choice is $g_{n+1}(z)=\overline{\frac{z\varphi_n(z)}{\varphi_n^{*}(z)}}$, $g_{n+2}=\ldots$ $\ldots=0$.

The paper [3] is more elaborate and is connected with the paper [10] where the case z=0 is considered. For this case,

$$h(\mu,0)=-\tfrac{1}{2}\ln\det(1-B_{1,1}) \text{ where } B_{1,1}=P^{\mathcal{K}}_{\mathcal{K}_1^{\perp}\mathcal{K}}\, P^{\mathcal{K}}_{\mathcal{K}_+}\, P^{\mathcal{K}}_{\mathcal{K}_1^{\perp}\mathcal{K}}$$

(see [5]). This geometrical variant leads to an operatorial variant of the problem (see also [5]) and to same considerations in the indefinite case. Let $s_1,\ldots,s_n$ be complex numbers such that

$$\mathcal{T}_n=\begin{bmatrix} -1, s_1,\ldots,s_n \\ \overline{s_1},-1,\ldots,s_{n-1} \\ \overline{s_n},\ldots\quad,-1\end{bmatrix}$$

has $\varkappa$ negative squares and define the set

$$\mathcal{B}_n=\{\text{Toeplitz form extending } \mathcal{T}_n \text{ and having } \varkappa \text{ negative squares}\}$$

and we consider the problem: "extremum $|G(z)|^2$," where G is the function defined and the section V. From (4.21),

$$|G(z)|^2=\frac{1}{(1-|z|^2)\sum_{k=0}^{\infty}\varepsilon_k|\varphi_k(z)|^2}$$

and if n is sufficiently large (the choice sequence is stabilized: $|g_n|<1$) then

$$|G(z)|^2=\frac{1}{(1-|z|^2)\sum_{k=0}^{\infty}{}'\varepsilon_k|\varphi_k(z)|^2}\le\frac{1}{(1-|z|^2)\sum_{k=0}^{n}\varepsilon_k|\varphi_k(z)|^2}$$

with the same choice for the solution attaining the maximum. For illustrating the rest, let us consider the following example: fix $|g_1|<1, |g_2|<1$ ($\mathcal{T}_1$ has 2 negative squares, $\mathcal{T}_2$ has 3 negative squares) and let $\mathcal{T}\in\mathcal{B}_3$, consequently, $|g_3|>1$, $|g_4|<1,\ldots$ and $G(0)=(1-|g_1|^2)(1-|g_2|^2)(|g_3|^2-1)(1-|g|^2)\ldots$; if $g_3\to\infty$, then $G(0)\to\infty$.

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