THE LINEARIZED PROBLEM IN ADIABATIC MULTIDIMENSIONAL GASDYNAMICS (II)

by

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Abstract

For systems of conservation laws [(1.1), (2.1), (3.1), 4.1)] one discusses the manner in which the number of space dimensions and/or the number of equations influences the structure of the set of concepts/restrictions connected with the linearized well-posedness (see also [1],[3] - [8]). Moreover (see [4],[8]), the remarks of §§ 3,5 show that in adiabatic gasdynamics 2D in space, it is possible to formulate, for certain equations of state, an (exponential) criterion of linearized stability/well-posedness. This criterion doesn't work (for instance) for equation of state (4.3). In such a case the possibility of linearized stability/well-posedness should be studied by starting directly from the solution.

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1. LINEARIZED PROBLEM FOR A SINGLE CONSERVATION LAW, 1D IN SPACE

1.1. Wording up of linearized problem

Let us consider the Riemann problem

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \quad -\infty < x < \infty, \; t > 0 \tag{1.1}$$

$$u(x,0) = \begin{cases} u_\ell & \text{for } x < 0 \\ u_r & \text{for } x > 0 \end{cases} \tag{1.2}$$

where u_ℓ and u_r are constants, $u_r \neq u_\ell$, and $f'' \neq 0$ on a certain domain in which u takes values.

A discontinuous solution of (1.1) satisfies, at the points of a discontinuity line, the relation

$$[\![f(u)]\!] = D\,[\![u]\!] \tag{1.3}$$

where $[\![f(u)]\!] = f(u_r) - f(u_\ell)$, $[\![u]\!] = u_r - u_\ell$ and D denotes the speed with which the discontinuity propagates.

It is well-known ([7]) that (1.3) is a necessary condition and then we have to impose additionally to the piecewise constant solution of the Riemann problem, according to the method of characteristics, the conditions of determinacy (through initial data and jump relation)

$$f'(u_r) < D < f'(u_\ell). \tag{1.4}$$

We call (1.4) the <u>entropy conditions of Lax</u> (CEL).

In the perturbation theory we shall present hereinafter in this paragraph, the afore-mentioned (discontinuous) solution of (1.1),(1.2) plays the part of the "zeroth order".

Let ε be a parameter of the problem, small in comparison with the constant states adjacent to the discontinuity and also small in comparison with the magnitude of the jump through discontinuity

$$0 < \varepsilon \ll |u_d - u_s| \; . \tag{1.5}$$

In a perturbation theory, to the initial data

$$(1.6)\qquad \overset{\varepsilon}{u}_0(x) = \begin{cases} \overset{\varepsilon}{u}_{\ell 0}(x) & \text{for } x<0 \\ \overset{\varepsilon}{u}_{r0}(x) & \text{for } x>0 \end{cases}$$

the hereinbelow solution corresponds

$$(1.7)\qquad \overset{\varepsilon}{u}(x,t) = \begin{cases} \overset{\varepsilon}{u}_\ell(x,t) & \text{for } x-Dt-\overset{\varepsilon}{\psi}(t)<0 \\ \overset{\varepsilon}{u}_r(x,t) & \text{for } x-Dt-\overset{\varepsilon}{\psi}(t)>0 \end{cases}$$

The data (1.6) evolve according to the equations

$$(1.8)\qquad \begin{cases} \dfrac{\partial \overset{\varepsilon}{u}_\ell}{\partial t} + a(\overset{\varepsilon}{u}_\ell)\dfrac{\partial \overset{\varepsilon}{u}_\ell}{\partial x} = 0 & \text{for } \bar{x} = x-Dt-\overset{\varepsilon}{\psi}(t)<0 \\ \dfrac{\partial \overset{\varepsilon}{u}_r}{\partial t} + a(\overset{\varepsilon}{u}_r)\dfrac{\partial \overset{\varepsilon}{u}_r}{\partial x} = 0 & \text{for } \bar{x}>0 \end{cases}$$

where we denote $a(u)=f'(u)$, and on a line of discontinuity the jump relation

$$(1.9)\qquad f(\overset{\varepsilon}{u})\Big|_{\bar{x}=0+} - f(\overset{\varepsilon}{u})\Big|_{\bar{x}=0-} = \left[D + \overset{\varepsilon}{\psi}'(t)\right]\left(\overset{\varepsilon}{u}\Big|_{\bar{x}=0} - \overset{\varepsilon}{u}\Big|_{\bar{x}=0-}\right)$$

obtained from (1.3) is satisfied. By mapping

$$(1.10)\qquad \bar{x} = x - Dt - \overset{\varepsilon}{\psi}(t), \qquad \bar{t} = t$$

(1.8) passes into

$$(1.11)\qquad \begin{cases} \dfrac{\partial}{\partial \bar{t}}\overset{\varepsilon}{U}_\ell + \left[a(\overset{\varepsilon}{U}_\ell) - D\right]\dfrac{\partial}{\partial \bar{x}}\overset{\varepsilon}{U}_\ell = \overset{\varepsilon}{\psi}'(t)\dfrac{\partial}{\partial \bar{x}}\overset{\varepsilon}{U}_\ell, & \text{for } \bar{x}<0 \\ \dfrac{\partial}{\partial \bar{t}}\overset{\varepsilon}{U}_r + \left[a(\overset{\varepsilon}{U}_r) - D\right]\dfrac{\partial}{\partial \bar{x}}\overset{\varepsilon}{U}_r = \overset{\varepsilon}{\psi}'(t)\dfrac{\partial}{\partial \bar{x}}\overset{\varepsilon}{U}_r, & \text{for } \bar{x}>0 \end{cases}$$

where

$$\overset{\varepsilon}{U}(\bar{x},\bar{t}) = \overset{\varepsilon}{u}(x,t).$$

For separating the first order in ε we shall assume that $\overset{\varepsilon}{U}_{\ell 0}$, $\overset{\varepsilon}{U}_{r0}$, $\overset{\varepsilon}{U}_\ell$, $\overset{\varepsilon}{U}_r$, $\overset{\varepsilon}{\psi}$ depend smoothly on ε, then differentiate (1.11) and (1.9) with respect to ε and take into account

$$\left[\overset{\varepsilon}{U}_{\ell,r}(\bar{x},\bar{t})\right]_{\varepsilon=0} = u_{\ell,r}, \qquad \left[\overset{\varepsilon}{\psi}(\bar{t})\right]_{\varepsilon=0} = 0$$

$$\frac{d}{d\varepsilon}\left[\overset{\varepsilon}{U}(\bar{x},\bar{t})\right]\Big|_{\varepsilon=0} = \tilde{U}(\bar{x},\bar{t}), \quad \frac{d}{d\varepsilon}\left[\overset{\varepsilon}{\psi}(t)\right]\Big|_{\varepsilon=0} = \psi(\bar{t}), \quad \frac{d}{d\varepsilon}\left[\overset{\varepsilon}{U}_0(\bar{x})\right]\Big|_{\varepsilon=0} = \tilde{u}_0(\bar{x}).$$

ignoring, in (1.11), the dependence of $\bar{x}$ on ε. It thus results

$$(1.12)\qquad \begin{cases} \frac{\partial}{\partial \bar{t}} \tilde{U}_\ell + A_\ell \frac{\partial}{\partial \bar{x}} \tilde{U}_\ell = 0, & \bar{x} < 0 \\ \frac{\partial}{\partial \bar{t}} \tilde{U}_r + A_r \frac{\partial}{\partial \bar{x}} \tilde{U}_r = 0, & \bar{x} > 0 \end{cases}$$

$$(1.13)\qquad A_r \tilde{U}_r = A_\ell \tilde{U}_\ell + [\![u]\!] \Psi', \quad \text{for } \bar{x} = 0$$

where $A_{\ell,r} = a(u_{\ell,r}) - D$, and

$$(1.14)\qquad \tilde{U}(\bar{x},0) = \tilde{u}_0(\bar{x}), \quad \bar{x} \in \mathbb{R}; \quad \Psi(0) = 0$$

The equations of the following (allowed) orders are obtained similarly.

DEFINITION 1.1. The problem (1.12)-(1.14) is called the linearized problem associated with the Riemann problem.

1.2. Determinacy.

Since we have ignored - to separate the first order in ε - the dependence of $\bar{x}$ on ε in (1.11), in the solution of the linearized problem-depending on the nature of initial data-secular terms will appear. Therefore, as we shall show through the theorem 1.1, the method described in 1.1 "linearizes" - at the first order in ε - the problem (1.8),(1.6) only for certain classes of initial data.

Let us consider the class of initial data

(1.15) $\mathscr{E}_0 = \{ \tilde{u}_0 \mid \tilde{u}_{\ell 0}$ and $\tilde{u}_{r0}$ are smooth functions with compact support $\}$

and, correspondingly, the class of functions $\tilde{U}(\bar{x},\bar{t})$ with the properties

(a) for each $\bar{x} \in \mathbb{R}$, $\tilde{U}$ is a Laplace original (abbreviated fo) with respect to $\bar{t}$;

(b) for each $\bar{t} < \infty$, $\tilde{U}_\ell$ and $\tilde{U}_r$ are smooth functions with compact support with respect to $\bar{x}$.

Let us denote

(1.16) $\mathscr{E} = \{ \tilde{U}, \Psi \mid \tilde{U}$ with the properties (a) and (b), Ψ is fo$\}$.

For data in $\mathscr{E}_0$ we shall seek in $\mathscr{E}$ for the solution of the linearized problem. We shall thus suppose a certain type for time growing of the solution, and the study of the problem will show that this assumption is justified.

Applying the Laplace transform to (1.12), (1.13) and putting $\overset{*}{f}(\bar{x},\omega) = L[\tilde{f}] = \int_0^\infty \tilde{f} e^{-\omega\bar{t}}\, d\bar{t}$, we find

$$(1.17)\qquad \begin{cases} A_\ell \dfrac{d\overset{*}{U}_\ell}{d\bar{x}} + \omega \overset{*}{U}_\ell - \tilde{u}_0 = 0 \quad \text{for} \quad \bar{x} < 0 \\ A_r \dfrac{d\overset{*}{U}_r}{d\bar{x}} + \omega \overset{*}{U}_r - \tilde{u}_0 = 0 \quad \text{for} \quad \bar{x} > 0 \end{cases} \qquad \operatorname{Re}\omega > 0$$

$$(1.18)\qquad A_r \overset{*}{U}_r = A_\ell \overset{*}{U}_\ell + \omega [\![u]\!] \overset{*}{\psi} \quad \text{for} \quad \bar{x} = 0$$

The solution of the system (1.17) can be represented formally as

$$(1.19)\quad \overset{*}{U}(\bar{x},\omega) = \begin{cases} \left[\overset{*}{U}_\ell(0,\omega) + A_\ell^{-1} \int_0^{\bar{x}} \tilde{u}_0(\xi) e^{(\omega/A_\ell)\xi} d\xi \right] e^{-(\omega/A_\ell)\bar{x}}, \quad \text{for } \bar{x} < 0 \\ \left[\overset{*}{U}_r(0,\omega) + A_r^{-1} \int_0^{\bar{x}} \tilde{u}_0(\xi) e^{(\omega/A_r)\xi} d\xi \right] e^{-(\omega/A_r)\bar{x}}, \quad \text{for } \bar{x} > 0 \end{cases}$$

In order to divide, formally, the considerations concerning the well-posedness of linearized problem into parts reflecting the extension corresponding to the passage from § 1, through § 2, to §§ 3 and 4, we shall introduce hereinbelow the concepts of determinacy (through the initial data and jump relations), evolutionary conditions and stability.

In the context of § 1, the exposition of these concepts is trivial and will be used only to support the analogy considered at page 2.

Let us take

$$(1.20)\qquad A_\ell > 0, \quad A_r < 0$$

in (1.19) and put, correspondingly, the coefficients of $\exp[-(\omega/A_\ell)\bar{x}]$ and $\exp[-(\omega/A_r)\bar{x}]$ equal to zero

$$(1.21)\qquad \begin{cases} \overset{*}{U}_\ell(0,\omega) = -A_\ell^{-1} \int_0^{-\infty} \tilde{u}_0(\xi) e^{(\omega/A_\ell)\xi} d\xi \\ \overset{*}{U}_r(0,\omega) = -A_r^{-1} \int_0^{\infty} \tilde{u}_0(\xi) e^{(\omega/A_r)\xi} d\xi \end{cases}$$

The relations (1.21) are called relations of determinacy. The conditions (1.20) are called conditions of determinacy. If these conditions are fulfilled we say that the linearized problem is determinate.

We shall use the two relations (1.21), together with the relation (1.18) to determine the three unknowns $\overset{*}{U}_\ell(0,\omega)$, $\overset{*}{U}_r(0,\omega)$ and $\overset{*}{\Psi}(\omega)$.

The formal procedure described hereinabove becomes (as we shall see immediately) effective if the data are, for instance, in $\mathscr{C}_0$.

Carrying (1.21) into (1.19) we obtain

$$U(\bar{x},\bar{t}) = \begin{cases} \tilde{u}_0(\bar{x} - A_\ell \bar{t}), & \bar{x} < 0 \\ \tilde{u}_0(\bar{x} - A_r \bar{t}), & \bar{x} > 0 \end{cases} \tag{1.22}$$

From (1.18) we find, by (1.21),

$$-\omega[\![u]\!]\,\overset{*}{\Psi}(\omega) = \int_0^\infty [A_\ell \tilde{u}_0(-A_\ell \tau) - A_r\, \tilde{u}_0(-A_r\tau)]\, e^{-\omega\tau}\, d\tau \tag{1.23}$$

which gives

$$\Psi(\bar{t}) = -\frac{1}{[\![u]\!]} \int_{-A_\ell \bar{t}}^{-A_r \bar{t}} \tilde{u}_0(\tau)\, d\tau \tag{1.24}$$

From (1.22) and (1.24) we see that for data in $\mathscr{C}_0$ the solution of the linearized problem does not contain secularities. On the other hand, if $\tilde{u}_0(\bar{x}) \equiv \cos k\bar{x}$ in (1.24) then for each $\bar{t} < \infty$ we have $\lim_{k \to 0} \Psi(\bar{t}) = \bar{t}\{(A_\ell - A_r)/\ [\![u]\!]\}$ and so, for $|k| \approx 0$, the nonlinearity slinks even from the first order and the procedure of isolating the linearized problem is not justified any more (in the absence of its uniform validity) for $\bar{t} \sim 0\ (\varepsilon^{-1})$. Something similar happens wherein the data do not tend quickly enough to zero when $|\bar{x}| \to \infty$, because in that case the Laplace images $L[\tilde{u}_0(-A_\ell\tau)]$ and/or $L[\tilde{u}_0(-A_r\tau)]$ in (1.23) have a singularity in $\omega = 0$.

REMARK 1.1. (i) From (1.5) we can see that the picture of fig. 1 b cannot be obtained as a limit - when $|u_r - u_\ell| \to 0$ - from the picture of fig.1 a. A relation can be established only between the zeroth orders of the two pictures because the small parameter of the perturbation expansion which leads to the linearization in fig.1 b is free of restriction (1.5).

(ii) The determinacy conditions (1.20) (associated to the first order of the perturbation theory) can be transcribed

$$a(u_r) < D < a(u_\ell) \qquad (1.25)$$

and so they coincide with CEL (see (1.4)). From (1.24) we see that if these conditions are fulfilled the evolution of distortion ψ depends on the data on the whole $\bar{x}$ axis.

1.3. Linearized stability. Linearized well-posedness

DEFINITION 1.2. A solution - consisting of $\tilde{U}$ and ψ - of the linearized problem is called stable/unstable if it is kept bounded/grows boundlessly when $\bar{t} \to \infty$. We say, correspondingly, that the discontinuous solution considered for the Riemann problem is (linearized) stable/unstable. The linearized problem with data in the class K_0 is said to be well-posed in the class K if (it attaches to each element in K_0 a unique and stable solution in K, that is,) it is determined and stable in K.

THEOREM 1.1. If the conditions (1.20) are fulfilled then the linearized problem with data in $\mathscr{C}_0$ is well-posed in the class $\mathscr{C}$.

◀ According to (1.22) and (1.24). ▶

REMARK 1.2. The hypotheses of the theorem 1.1 do not impose on the value u_ℓ and u_r but the ordering restriction $u_r < u_\ell$.

2. LINEARIZED PROBLEM FOR A SYSTEM OF CONSERVATION LAWS, 1D IN SPACE

2.1. Wording up of linearized problem

Let us extend now, in case of systems of conservation laws, the results of § 1. The Riemann problem takes then the form

(2.1) $$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \quad -\infty < x < \infty \quad , t > 0$$

(2.2) $$u(x,0) = \begin{cases} u_\ell & \text{for} \quad x < 0 \\ u_r & \text{for} \quad x > 0 \end{cases}$$

where u and f are vector functions with n components and u_ℓ, u_r are constant arbitrary vectors, $u_r \neq u_\ell$.

In 2.1-2.3 we shall suppose that u_ℓ, u_r are sufficiently close in $\mathbb{R}^n$ and so related (see lemma 2.1 here in below) that the solution of Riemann problem should contain only a j-shock ([7]) together with the constant regions adjacent to it. This solution satisfies, at the points of the discontinuity line, the relations (analogous to (1.3))

(2.3) $$[\![f(u)]\!] = D\,[\![u]\!]$$

It is well-known ([7]) that, on the considered j-shock discontinuity, the determinacy conditions CEL (1.4) can be extended as (λ are the eigenvalues of matrix $a(u) = (\partial f_i/\partial u_j)$)

(2.4) $$\begin{cases} \lambda_j(u_r) < D < \lambda_j\,(u_\ell) \\ \lambda_{j-1}(u_\ell) < D < \lambda_{j+1}\,(u_r) \end{cases}$$

In the perturbation theory we shall present hereinafter in this paragraph, this (j-shock) solution plays the part of the "zeroth order".

According to [2] and [7] we can formulate

<u>LEMMA 2.1</u>. Let $f \in C^m$ in (2.1). Given u_ℓ as a state on the left, the set of vectors u_r which can be joined (as states on the right) with u_ℓ by a j-shock lay, in a conveniently close neighbourhood, on a (unique) smooth curve

(2.5) $$u = \mathcal{S}_j\,(\bar{\varepsilon}, u_s), \qquad \bar{\varepsilon} \in \mathbb{R}_-$$
$$\mathcal{S}_j\,(0, u_s) = u_s$$

which is C^{m-1} with respect to $\bar{\varepsilon}$, u_s and for which

(i) u_s is not a singular point

(ii) $$\left.\frac{du}{d\bar{\varepsilon}}\right|_{\bar{\varepsilon}=0} = \frac{\overset{j}{R}(u_s)}{\left[\overset{j}{R}(u) \cdot \operatorname{grad}_u \lambda_j(u)\right]_{u=u_s}} \qquad (2.6)$$

(iii) $D(0,u_s) = \lambda_j\,(u_s)$

(R is a right eigenvector of matrix a(u)).

In the following we shall present (2.5) (given j, u_s) as $u_r = u(\bar{\varepsilon})$ and choose the length of $\overset{j}{R}$ so that $\overset{j}{R}\cdot\operatorname{grad}_u\lambda_j = 1$.

Let $\bar{\varepsilon}$ be a small parameter of the problem, characterized the same as in 1.1.

The expressions (1.6) and (1.7) and the notations of § 1 have a vectorial analogue here. In particular motivating as in 1.1 we find for the linearized problem the following form

$$(2.7)\qquad \begin{cases} \dfrac{\partial}{\partial\bar{t}}\tilde{U}_\ell + A_\ell\dfrac{\partial}{\partial\bar{x}}\tilde{U}_\ell = 0\,, & \bar{x} < 0 \\[2ex] \dfrac{\partial}{\partial\bar{t}}\tilde{U}_r + A_r\dfrac{\partial}{\partial\bar{x}}\tilde{U}_r = 0\,, & \bar{x} > 0 \end{cases}$$

$$(2.8)\qquad A_r\tilde{U}_r = A_\ell\tilde{U}_\ell + [\![u]\!]\,\psi' \qquad \text{for} \quad \bar{x} = 0$$

$$(2.9)\qquad \tilde{U}(\bar{x},0) = \tilde{u}_0(\bar{x}), \quad \bar{x}\in\mathbb{R}\ ;\ \psi(0) = 0$$

with A(u) = a(u) - DI, I the unit matrix.

2.2. <u>Determinacy. Evolutionary conditions</u>

We suppose that the matrices A_ℓ and A_r are nonsingular and have distinct eigenvalues, $\lambda_1 < \lambda_2 < \ldots < \lambda_n$.

We shall use here the classes $\mathscr{C}_0$ and $\mathscr{C}$ introduced the same as in 1.2.

Using the Laplace transform in (2.7) and (2.8) we find

$$(2.10)\qquad \begin{cases} A_\ell\dfrac{d\overset{*}{U}_\ell}{d\bar{x}} + \omega\overset{*}{U}_\ell - \tilde{u}_0 = 0 & \text{for} \quad \bar{x} < 0 \\[2ex] A_r\dfrac{d\overset{*}{U}_r}{d\bar{x}} + \omega\overset{*}{U}_r - \tilde{u}_0 = 0 & \text{for} \quad \bar{x} > 0 \end{cases}$$

$$(2.11)\qquad A_r\overset{*}{U}_r = A_\ell\overset{*}{U}_\ell + \omega[\![U]\!]\overset{*}{\psi} \qquad \text{for} \quad \bar{x} = 0$$

The systems (2.10) can be put in the form

$$\frac{d\overset{*}{U}}{d\bar{x}} = P\overset{*}{U} + f, \quad P = -\omega A^{-1}, \quad f = A^{-1}\tilde{u}_0 \tag{2.12}$$

Since the matrices A and P have the same eigenvectors and the eigenvalues $\tilde{\lambda}$ of A, the eigenvalues $\hat{\lambda}$ of P and the eigenvalues λ of a are related by

$$\hat{\lambda}_i = -\frac{\omega}{\tilde{\lambda}_i}, \quad \tilde{\lambda}_i = \lambda_i - D \tag{2.13}$$

it follows that the solution of (2.10) can be (formally) represented by

$$\overset{*}{U}(\bar{x};\omega) = \begin{cases} \sum_{i=1}^{n} \overset{i}{R}_\ell \left\{ \overset{i}{L}_\ell \cdot \overset{*}{U}_\ell(0,\omega) + \overset{i}{L}_\ell \cdot A_\ell^{-1} \int_0^{\bar{x}} \tilde{u}_0(\xi) e^{\frac{\omega\xi}{\lambda_i(u_\ell)-D}} d\xi \right\} e^{-\frac{\omega\bar{x}}{\lambda_i(u_\ell)-D}}, & \bar{x} < 0 \\ \sum_{i=1}^{n} \overset{i}{R}_r \left\{ \overset{i}{L}_r \cdot \overset{*}{U}_r(0,\omega) + \overset{i}{L}_r \cdot A_r^{-1} \int_0^{\bar{x}} \tilde{u}_0(\xi) e^{\frac{\omega\xi}{\lambda_i(u_r)-D}} d\xi \right\} e^{-\frac{\omega\bar{x}}{\lambda_i(u_r)-D}}, & \bar{x} > 0 \end{cases} \tag{2.14}$$

$$A_r\overset{*}{U}_r = A_\ell\overset{*}{U}_\ell + \omega[\![u]\!]\overset{*}{\psi} \quad \text{for} \quad \bar{x} = 0 \tag{2.15}$$

where R,L are right/left eigenvectors of A.

Let us extend now the formal procedure introduced in 1.2 (see (1.20)-(1.24)). When $\lambda_i(u_\ell) > D/\lambda_i(u_r) < D$ we shall annul the coefficient of $\exp\{-[\omega\bar{x}/(\lambda_i(u_\ell)-D)]\}/\exp\{-[\omega\bar{x}/(\lambda_i(u_r)-D)]\}$ in (2.14) obtaining for a given i, $1 \leq i \leq n$, a relation of determinacy.

DEFINITION 2.2. We say that the linearized problem is determined if the number of linear algebraic equations - having $\overset{*}{\psi}$ and the components of $\overset{*}{U}_\ell(0,\omega)$, $\overset{*}{U}_r(0,\omega)$ as unknowns - of the system which consists of determinacy relations and jump relations (2.15) is equal to 2n+1.

Since at the zero and first orders of the perturbation theory a determinacy relation associates to an approaching (convergent) characteristic, we can easily prove

THEOREM 2.1. The linearized problem is determined iff the conditions CEL (2.4) hold.

If (2.4) are satisfied then we have to pose in (2.14):

$$\overset{j}{L}_\ell \cdot \overset{*}{U}_\ell(0,\omega) = -\overset{j}{L}_\ell A_\ell^{-1} \int_0^{-\infty} \tilde{u}_0(\xi)\, e^{\frac{\omega\xi}{\lambda_j(u_\ell)-D}}\, d\xi$$

$$\cdots\cdots\cdots\cdots$$

$$\overset{n}{L}_\ell \cdot \overset{*}{U}_\ell(0,\omega) = -\overset{n}{L}_\ell \cdot A_\ell^{-1} \int_0^{-\infty} \tilde{u}_0(\xi)\, e^{\frac{\omega\xi}{\lambda_n(u_\ell)-D}}\, d\xi$$

(2.16)

$$\overset{1}{L}_r \cdot \overset{*}{U}_r(0,\omega) = -\overset{1}{L}_r \cdot A_r^{-1} \int_0^{\infty} \tilde{u}_0(\xi)\, e^{\frac{\omega\xi}{\lambda_1(u_r)-D}}\, d\xi$$

$$\cdots\cdots\cdots\cdots$$

$$\overset{j}{L}_r \cdot \overset{*}{U}_r(0,\omega) = -\overset{j}{L}_r \cdot A_r^{-1} \int_0^{\infty} \tilde{u}_0(\xi)\, e^{\frac{\omega\xi}{\lambda_j(u_r)-D}}\, d\xi$$

The relations (2.13) and (2.16) make up a linear algebraic system of 2n+1 equations for 2n+1 unknowns, $\overset{*}{U}_\ell(0,\omega)$, $\overset{*}{U}_r(0,\omega)$ and $\overset{*}{\Psi}$. After an easy re-arrangement, taking into account that $(\lambda-D)L=LA$, we can give to (2.16), (2.13) the form

$$\overset{j}{L}_\ell \overset{*}{U}_\ell(0,\omega) = \overset{j}{L}_\ell \cdot \int_0^{\infty} \tilde{u}_0\{-[\lambda_j(u_\ell)-D]\tau\}\, e^{-\omega\tau} d\tau = \overset{j}{g}_\ell$$

$$\cdots\cdots\cdots\cdots \qquad \cdots\cdots$$

$$\overset{n}{L}_\ell \overset{*}{U}_\ell(0,\omega) = \overset{n}{L}_\ell \cdot \int_0^{\infty} \tilde{u}_0\{-[\lambda_n(u_\ell)-D]\tau\}\, e^{-\omega\tau} d\tau = \overset{n}{g}_\ell$$

(2.17)

$$\overset{1}{L}_r \overset{*}{U}_r(0,\omega) = \overset{1}{L}_r \cdot \int_0^{\infty} \tilde{u}_0\{-[\lambda_1(u_r)-D]\tau\}\, e^{-\omega\tau} d\tau = \overset{1}{g}_r$$

$$\cdots\cdots\cdots\cdots \qquad \cdots\cdots$$

$$\overset{j}{L}_r \overset{*}{U}_r(0,\omega) = \overset{j}{L}_r \cdot \int_0^{\infty} \tilde{u}_0\{-[\lambda_j(u_1)-D]\tau\}\, e^{-\omega\tau} d\tau = \overset{j}{g}_r$$

$$\overset{*}{U}_r(0,\omega) = A_r^{-1} A_\ell \overset{*}{U}_\ell(0,\omega) + \omega A_r^{-1} [\![u]\!] \overset{*}{\Psi}$$

We shall use the lemma 2.1 in order to prove

<u>LEMMA 2.2</u>. If in the problem (2.1), (2.2) u_ℓ and u_r are linked by a j-shock and are conveniently close, then for the system (2.17) there exist a unique solution.

◀ By expressing, from the last n relations (2.17), $\overset{*}{U}_r(0,\omega)$ with respect to $\overset{*}{U}_\ell(0,\omega)$ and $\overset{*}{\Psi}$ we can find $\overset{*}{U}_\ell(0,\omega)$ and $\overset{*}{\Psi}$ from the system of n+1 equations

$$(2.18)\qquad \begin{cases} \overset{k}{L}_\ell \cdot \overset{*}{U}_\ell(0,\omega) = \overset{k}{g}_\ell\,, & k=j,\dots,n \\ \overset{s}{L}_r A_r^{-1} A_\ell \overset{*}{U}_\ell(0,\omega) + \omega \overset{*}{\psi}\, \overset{s}{L}_r A_r^{-1}\, [\![U]\!] = \overset{s}{g}_r, & s=1,\dots,j \end{cases}$$

The proof comes to an end if, denoting by Δ the discriminant of the system (2.17), we show that $\Delta \neq 0$ when $[\![u]\!] \neq 0$ and u_ℓ, u_r are close enough.

According to lemma 2.1 we write

$$(2.19)\qquad \Delta = \bar{\varepsilon}\, \Delta_1$$

where

$$(2.20)\qquad \Delta_1 = \begin{vmatrix} \overset{j}{L}_{\ell 1} & \overset{j}{L}_{\ell 2} & \dots & \overset{j}{L}_{\ell n} & 0 \\ \dots & \dots & & \dots & \dots \\ \overset{n}{L}_{\ell 1} & \overset{n}{L}_{\ell 2} & & \overset{n}{L}_{\ell n} & 0 \\ (\overset{1}{L}_r A_r^{-1} A_\ell)_1 & (\overset{1}{L}_r A_r^{-1} A_\ell)_2 & \dots & (\overset{1}{L}_r A_r^{-1} A_\ell)_n & \overset{1}{L}_r A_r^{-1}\, \dfrac{u(\bar{\varepsilon})-u(0)}{\bar{\varepsilon}} \\ \dots & \dots & & \dots & \dots \\ (\overset{j}{L}_r A_r^{-1} A_\ell)_1 & (\overset{j}{L}_r A_r^{-1} A_\ell)_2 & \dots & (\overset{j}{L}_r A_r^{-1} A_\ell)_n & \overset{j}{L}_r A_r^{-1}\, \dfrac{u(\bar{\varepsilon})-u(0)}{\bar{\varepsilon}} \end{vmatrix}$$

and we have, when $\overset{i}{L}\,\overset{k}{R} = \delta_{ik}$,

$$(2.21)\qquad \lim_{\bar{\varepsilon}\to 0} [\lambda_j(u_r) - D]\, \overset{k}{L}_r A_r^{-1}\, \frac{u(\bar{\varepsilon})-u(0)}{\bar{\varepsilon}} = \delta_{kj}$$

Since $\overset{1}{L}_\ell, \dots, \overset{n}{L}_\ell$ are independent we find from (2.21) that for a j-shock

$$(2.22)\qquad \lim_{\bar{\varepsilon}\to 0} [\lambda_j(u_r) - D]\, \Delta_1 \neq 0$$

The fact that - for u_ℓ and u_r conveniently close - we have $\Delta \neq 0$, results from (2.19) and (2.22). ▶

<u>REMARK 2.2</u>. It is easy to formulate an analogue of remark 1.1(i). When $\|u_r - u_\ell\| \to 0$, the matrices A_ℓ and A_r become singular and in (2.6) we have $[\![u]\!] \to 0$ - th ough, usually, $|\psi'|$ does not tend to zero - and $\|\tilde{U}_\ell\| \to 0$, $\|\tilde{U}_r\| \to 0$.

DEFINITION 2.3. The requirements of lemma 2.2 which guarantee that u_ℓ and u_r are linked by a j-shock and that a unique solution exists for the system (2.17) are called evolutionary conditions.

In the context of § 2 the set of evolutionary conditions contains the determinacy conditions CEL together with the (possible) demand that u_ℓ and u_r should be close. From (1.22) and (1.24) it appears that in case of a single conservation law 1D in space, the evolutionary conditions come down to determinacy conditions (see remark 1.2).

2.3. Linearized stability. Linearized well-posedness

According to lemma 2.2 we can express

$$\omega\Psi^* = b_\ell^j \int_0^\infty \tilde{u}_o\{-[\lambda_j(u_\ell)-D]\tau\} e^{-\omega\tau} d\tau + \ldots + b_\ell^n \int_0^\infty \tilde{u}_o\{-[\lambda_n(u_\ell)-D]\tau\} e^{-\omega\tau} d\tau +$$

(2.23)

$$+ b_r^1 \int_0^\infty \tilde{u}_o\{-[\lambda_1(u_r)-D]\tau\} e^{-\omega\tau} d\tau + \ldots + b_r^j \int_0^\infty \tilde{u}_o\{-[\lambda_j(u_r)-D]\tau\} e^{-\omega\tau} d\tau$$

and thus find

$$(2.24)\quad \Psi(\bar{t}) = \sum_{k=j}^{n} b_\ell^k \int_0^{\bar{t}} \tilde{u}_o\{-[\lambda_k(u_\ell)-D]\tau\} d\tau + \sum_{k=1}^{j} b_r^k \int_0^{\bar{t}} \tilde{u}_o\{-[\lambda_k(u_r)-D]\tau\} d\tau$$

It is easy to formulate an analogue of definition 1.2.

We shall now extend the theorem 1.1 by

THEOREM 2.2. If the evolutionary conditions are fulfilled then the linearized problem with data in $\mathscr{C}_o$ is well-posed in the class $\mathscr{C}$.

◀ According to (2.24) and Haar estimates (see [3],[9]). ▶

3. LINEARIZED PROBLEM FOR A SYSTEM OF CONSERVATION LAWS, 2D IN SPACE

3.1. Wording up of linearized problem

Let us now extend, in case of two space dimensions, the considerations of § 2.

Instead of the problem (2.1), (2.2) we have here

(3.1) $$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} + \frac{\partial g(u)}{\partial y} = 0$$

(3.2) $$u(x,y,o) = \begin{cases} u_\ell & \text{for } x < 0 \\ u_r & \text{for } x > 0 \end{cases}$$

where u, f and g are vector functions with n components and u_ℓ, u_r are constant arbitrary vectors, $u_\ell \neq u_r$.

In 3.1 - 3.3 we shall assume that u_ℓ, u_r are conveniently close in $\mathbb{R}^n$ and so related that the solution of the problem should contain only one shock together with the constant regions adjacent to it. On the shock line the jump conditions

(3.3) $$[\![u]\!] \frac{\partial\phi}{\partial t} + [\![f(u)]\!] \frac{\partial\phi}{\partial x} + [\![g(u)]\!] \frac{\partial\phi}{\partial y} = 0$$

are fulfilled.

<u>REMARK 3.1</u>. In case of <u>steady</u> and <u>normal</u> shock discontinuity, the possibility of such a solution/the nature of demands formulated above is similar to that presented in § 2.

The small parameter ε of the problem has to be characterized the same as in § 2.

Proceeding as in 1.1 and 2.1 but using instead of (1.10) the mapping

(3.4) $$\bar{x} = x - Dt - \overset{\varepsilon}{\Psi}(y,t), \quad \bar{t} = t, \quad \bar{y} = y\ ,$$

we find for the linearized problem the following form

(3.5) $$\begin{cases} \frac{\partial}{\partial \bar{t}} \tilde{U}_\ell + A_\ell \frac{\partial}{\partial \bar{x}} \tilde{U}_\ell + b(u_\ell) \frac{\partial}{\partial \bar{y}} \tilde{U}_\ell = 0, & \bar{x} < 0 \\ \frac{\partial}{\partial \bar{t}} \tilde{U}_r + A_r \frac{\partial}{\partial \bar{x}} \tilde{U}_r + b(u_r) \frac{\partial}{\partial \bar{y}} \tilde{U}_r = 0, & \bar{x} > 0 \end{cases}$$

(3.6) $$A_r\tilde{U}_r = A_\ell\tilde{U}_\ell + [\![u]\!] \frac{\partial}{\partial \bar{t}}\Psi + [\![g(u)]\!] \frac{\partial}{\partial \bar{y}}\Psi \quad \text{for } \bar{x} = 0$$

(3.7) $$\tilde{U}(\bar{x},\bar{y},0) = \tilde{u}_0(\bar{x},\bar{y}), \quad \Psi(\bar{y},0) = 0$$

with notations similar to those of §§ 1,2.

3.2. Determinacy. Evolutionary conditions

We assume that the system (3.1) is strictly hyperbolic with $\bar{t}$ time-like which means, in 2D, imposing - in either adjacent region of discontinuity - the condition that for every λ, $\gamma \in \mathbb{R}$ we are able to find n real distinct roots $\omega(\lambda,\gamma)$ of the equation

$$\det[\omega I + \lambda A + b\gamma] = 0. \tag{3.8}$$

Since the discontinuity is normal, we shall consider solutions of the form

$$[U(\bar{x},\bar{y},\bar{t}), \psi(\bar{y},\bar{t})] = e^{-i\alpha\bar{y}}[\tilde{U}(\bar{x},\bar{t}), \psi(\bar{t})] \tag{3.9}$$

Carrying (3.9) into (3.5) and (3.6) we obtain

$$\begin{cases} \dfrac{\partial}{\partial \bar{t}}\tilde{U}_\ell + A_\ell \dfrac{\partial}{\partial \bar{x}}\tilde{U}_\ell - i\alpha b_\ell \tilde{U}_\ell = 0, & \bar{x} < 0 \\ \dfrac{\partial}{\partial \bar{t}}\tilde{U}_r + A_r \dfrac{\partial}{\partial \bar{x}}\tilde{U}_r - i\alpha b_r \tilde{U}_r = 0, & \bar{x} > 0 \end{cases} \tag{3.10}$$

$$A_r\tilde{U}_r = A_\ell\tilde{U}_\ell + [\![u]\!]\psi'(t) - i\alpha[\![g(u)]\!]\psi \quad \text{for } \bar{x} = 0 \tag{3.11}$$

Using the Laplace transform we find, as in (2.10), for either system (3.10) the form

$$\frac{d}{d\bar{x}}\overset{*}{U} = P\overset{*}{U} + f \tag{3.12}$$

where

$$P = -A^{-1}[\omega I - i\alpha b], \quad f = A^{-1}\tilde{u}_0 \tag{3.13}$$

to which we add, according to (3.11), the jump relations

$$A_r\overset{*}{U}_r = A_\ell\overset{*}{U}_\ell + \{\omega[\![u]\!] - i\alpha[\![g(u)]\!]\}\overset{*}{\psi} \quad \text{for } \bar{x} = 0 \tag{3.14}$$

As in 2.2, we denote by $\hat{\lambda}$ the eigenvalues of matrix P. Using the remark 3.1, we can prove an analogue of the theorem 2.1.

THEOREM 3.1. The linearized problem is determined iff the conditions

$$\begin{aligned} &\operatorname{Re}\hat{\lambda}_j(u_r) > 0 > \operatorname{Re}\hat{\lambda}_j(u_\ell) \\ &\operatorname{Re}\hat{\lambda}_{j-1}(u_\ell) > 0 > \operatorname{Re}\hat{\lambda}_{j+1}(u_r) \end{aligned} \tag{3.15}$$

are fulfilled for some index j, $1 \leq j \leq n$.

If (3.15) hold then j is the number of the eigenvalues $\hat{\lambda}$ for which we have $\mathrm{Re}\,\hat{\lambda} > 0$ in the right-handed region of discontinuity.

Since the eigenvalues $\hat{\lambda}$ depend on ω and α , j might depend on ω and α .

<u>THEOREM 3.2</u>. ([6]). The number j is independent on ω and α .

◀ If the number j depends on ω and α, we can find (ω_o, α_o) so that $\mathrm{Re}\,\hat{\lambda}(\omega_o,\alpha_o) = 0$. The eigenvalues $\hat{\lambda}$ can be determined, according to (3.13), by

(3.16)
$$\det[\omega I + \hat{\lambda} A - i\alpha b] = 0$$

with the restriction $\mathrm{Re}\,\omega > 0$ imposed by the Laplace transform. In (ω_o,α_o), (3.16) gives

$$\det\left[- i\omega I + (\mathrm{Im}\,\hat{\lambda})A - \alpha b\right] = 0$$

and, since the system is hyperbolic, we have (according to (3.8)) $\mathrm{Im}(i\omega) = \mathrm{Re}\,\omega = 0$ for every $\mathrm{Im}\,\hat{\lambda} \in \mathbb{R}$ and $\alpha \in \mathbb{R}$. ▶

<u>REMARK 3.2</u>.

(i) An analogue, easy to formulate, of remark 2.2 works.

(ii) In case of a steady and normal shock lemma 2.1 keeps valid (according to the remark 3.1; the formulation of the analogue of lemma 2.1 depends only on the nature of f in (3.1)) and lemma 2.2 can be easily extended (however its formulation depends on the nature of g in (3.1)).

(iii) The determinacy conditions (3.15) together with the (possible) restriction (mentioned in (ii)) that u_ℓ, u_r are close, constitute the set of <u>evolutionary conditions</u>.

3.3. <u>Linearized stability. Linearized well-posedness</u>

<u>DEFINITION 3.2</u>. We say that the discontinuous solution considered is <u>unstable</u> if we can find, at least for a value of $\alpha \in \mathbb{R}$, a solution (3.9) - consisting of $\tilde{U}$ and Ψ - of linearized problem which grows boundlessly when $\bar{t} \to \infty$. The discontinuous solution is called

stable if the solution (3.9) of the linearized problem is kept bounded, for every $\alpha \in \mathbb{R}$, when $\bar{t} \to \infty$.

To prove the well-posedness of linearized problem we have to show again that this problem is evolutionary and stable.

From theorems 3.1 and 3.2 it appears that the passing from 1D to 2D keeps unchanged the form/the nature of evolutionary conditions.

On the other hand, the stability result of theorem 2.2 cannot be obtained any more whithout a new restriction. Indeed, the Haar estimates (see [3],[9]) show that the stability of solution of linearized problem depends on the stability of Ψ. In 2D the distribution of singularities of Ψ depends on α. Let $\Psi = [d(\omega,\alpha)/L(\omega,\alpha)]$ be the expression obtained according to the analogue of lemma 2.2. The function g (see (3.1)) contributes by $L(\omega,\alpha)$ to the stability conditions. When $\alpha = 0$ this contribution vanishes [together with the dependence on $\bar{y}$; according to (3.9)]; $L(\omega,0)$ has only one zero in $\omega = 0$. However, when $\alpha \neq 0$, it is possible - depending on the form f and g in (3.1) - that some of the zeros of $L(\omega,\alpha)$ be placed in the region $\mathrm{Re}\,\omega > 0$ thus implying instability even for data in $\mathcal{E}_0$.

REMARK 3.3.

(i) In 2D we require stability for all $\alpha \in \mathbb{R}$, particularly for $\alpha = 0$. Then we shall take data in $\mathcal{E}_0$.

(ii) When $\alpha \neq 0$ we have to find the conditions for which the zeros of $L(\omega,\alpha)$ are all placed in $\mathrm{Re}\,\omega \leq 0$. This is the new restriction we mentioned hereinabove.

In the context of gasdynamics it can be shown that these conditions do not depend on α and are related, as we have already mentioned, only on the form of f and g in (3.1). This form depends in its turn on the equation of state considered. Such criteria of stability are given in [4] and [8] (see [5] for magnetodynamics). A stability criterion removes the exponentially unstable evolutions.

(iii) For certain equations of state, the condition $\mathrm{Re}\,\omega < 0$ cannot be fulfilled strictly, under stability requirements, by the set of zeros of $L(\omega,\alpha)$. In such a case, when (a part of) zeros of $L(\omega,\alpha)$ are placed on the line $\mathrm{Re}\,\omega = 0$, we have explicitly to study the possibility of (nonexponential) stability. Such a study is presented in § 4.

The schema 3.1 compares the facts of §§ 1,2,3.

4. LINEARIZED PROBLEM FOR THE SYSTEM OF CONSERVATION LAWS, 2D IN SPACE, OF ADIABATIC GASDYNAMICS

4.1. Wording up of linearized problem

Let us now rema ke the considerations of § 3 starting, in adiabatic gasdynamics (with the usual notations), from the problem

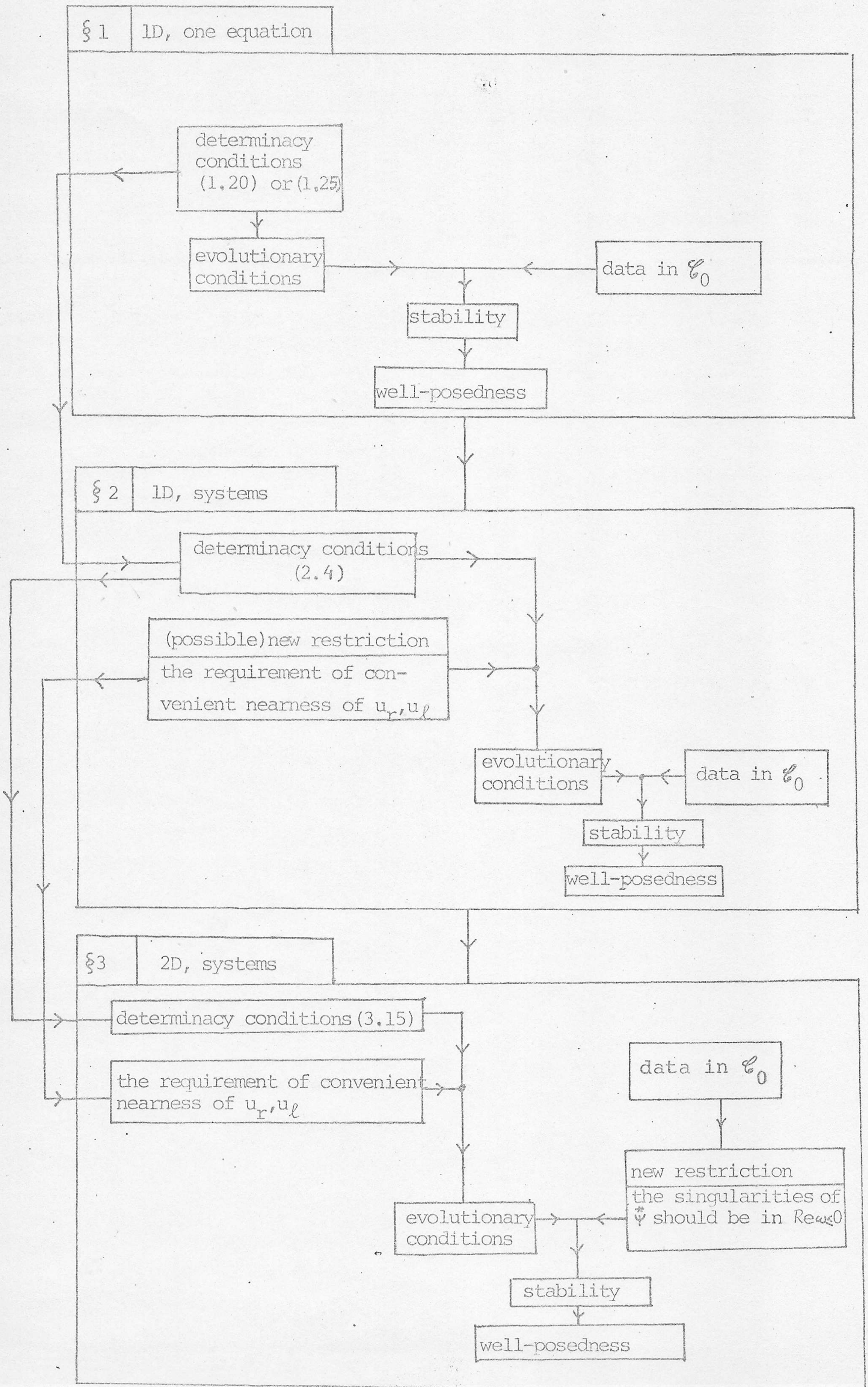
$$(4.1)\qquad \begin{aligned} &\frac{\partial \varrho}{\partial t} + \frac{\partial(\varrho u)}{\partial x} + \frac{\partial(\varrho v)}{\partial y} = 0\\ &\frac{\partial(\varrho u)}{\partial t} + \frac{\partial(\varrho u^2)}{\partial x} + \frac{\partial(\varrho uv)}{\partial y} + \frac{\partial p}{\partial x} = 0\\ &\frac{\partial(\varrho v)}{\partial t} + \frac{\partial(\varrho uv)}{\partial x} + \frac{\partial(\varrho v^2)}{\partial y} + \frac{\partial p}{\partial y} = 0\\ &\frac{\partial}{\partial t}\{\varrho[e+\tfrac{1}{2}(u^2+v^2)]\} + \frac{\partial}{\partial x}\{\varrho u[i+\tfrac{1}{2}(u^2+v^2)]\} + \frac{\partial}{\partial y}\{\varrho v[i+\tfrac{1}{2}(u^2+v^2)]\} = 0\end{aligned}$$

$$(4.2)\qquad (p,u,v,s)_{t=0} = \begin{cases} (p_1,u_1,v_1,s_1) & \text{for } x<0\\ (p_2,u_2,v_2,s_2) & \text{for } x>0\end{cases}$$

together with the equation of state $e = c_v T = \frac{p\tau}{\gamma-1}$ - which we shall write in the form

$$(4.3)\qquad p = \frac{p_0}{\varrho_0^{\gamma}}\,\varrho^{\gamma}\exp\left\{\frac{S-S_0}{c_v}\right\}$$

The jump relations on discontinuity line have the form


§ 1
1D, one equation
determinacy conditions (1.20) or (1.25)
evolutionary conditions
data in $\mathscr{C}_0$
stability
well-posedness
§ 2
1D, systems
determinacy conditions (2.4)
(possible) new restriction
the requirement of convenient nearness of u_r, u_ℓ
evolutionary conditions
data in $\mathscr{C}_0$
stability
well-posedness
§3
2D, systems
determinacy conditions (3.15)
the requirement of convenient nearness of u_r, u_ℓ
data in $\mathscr{C}_0$
new restriction
the singularities of $\hat{\psi}$ should be in $Re\omega\leq 0$
evolutionary conditions
stability
well-posedness

$$\llbracket \varrho \rrbracket \frac{\partial \phi}{\partial t} + \llbracket \varrho u \rrbracket \frac{\partial \phi}{\partial x} + \llbracket \varrho v \rrbracket \frac{\partial \phi}{\partial y} = 0$$

$$\llbracket \varrho u \rrbracket \frac{\partial \phi}{\partial t} + \llbracket p + \varrho u^2 \rrbracket \frac{\partial \phi}{\partial x} + \llbracket \varrho uv \rrbracket \frac{\partial \phi}{\partial y} = 0$$

(4.4)

$$\llbracket \varrho v \rrbracket \frac{\partial \phi}{\partial t} + \llbracket \varrho uv \rrbracket \frac{\partial \phi}{\partial x} + \llbracket p + \varrho v^2 \rrbracket \frac{\partial \phi}{\partial y} = 0$$

$$\llbracket \varrho [e + \tfrac{1}{2}(u^2+v^2)] \rrbracket \frac{\partial \phi}{\partial t} + \llbracket \varrho u [i + \tfrac{1}{2}(u^2+v^2)] \rrbracket \frac{\partial \phi}{\partial x} + \llbracket \varrho v [i + \tfrac{1}{2}(u^2+v^2)] \rrbracket \frac{\partial \phi}{\partial y} = 0$$

and, in the adjacent regions of discontinuity, we shall use - in place of $(4.1)_4$ - the concave extension of (4.1)

$(4.1)'_4$
$$\frac{d}{dt} s = 0$$

where $s = \varrho S$ is the entropy.

Given (p_1, u_1, v_1, s_1) as a state before discontinuity, we can find (p_2, u_2, v_2, s_2) on the curve of states which can be related with (p_1, u_1, v_1, s_1) by a <u>steady</u> discontinuity. In case of a <u>normal</u> discontinuity for this curve the lemma 2.1 is valid.

We shall write the problem in a dimensionless form by taking the characteristic values

$$[t] = \frac{L}{c_2},\ [x] = L,\ [\varrho] = \varrho_2,\ [u] = c_2,\ [p] = \varrho_2 c_2^2,\ [s] = c_p,\ c^2 = \gamma \frac{p}{\varrho}$$

and denoting

(4.5)
$$M = \frac{u_2}{[u]},\ \bar{M} = \frac{u_1}{[u]},\ M_y = \frac{v}{[u]},\ \bar{\varrho} = \frac{\varrho_1}{[\varrho]},\ P = \frac{p_2}{[p]},\ \bar{P} = \frac{p_1}{[p]},\ \bar{c} = \frac{c_1}{[u]}$$
$$\tilde{\varrho}_i = \frac{\varrho'_i}{[\varrho]},\ \tilde{u}_i = \frac{u'_i}{[u]},\ v_i = \frac{v'_i}{[u]},\ p_i = \frac{p'_i}{[p]},\ s_i = \frac{s'_i}{[s]}$$

(furthermore we shall ignore the labels of perturbations which correspond to the region after discontinuity).

The zeroth order of the jump relations gives

(4.6) $$M = \bar{\varrho}\bar{M},\ P - \bar{P} = M(\bar{M} - M),\ M_y = \bar{M}_y,\ \frac{\gamma}{\gamma-1}(MP - \bar{M}\bar{P}) = \frac{1}{2} M (\bar{M}^2 - M^2).$$

From (4.6) we can obtain, in particular,

(4.7) $$(\gamma-1)M^2 - (\gamma+1)M\bar{M} + 2 = 0$$

For a normal discontinuity we have $M_y = 0$ and, for a 1-shock, $0 < M < 1$.

In this context, the system (3.10) may be written

$$\frac{1}{\bar{c}^2}\left(\frac{\partial}{\partial t}+\bar{M}\frac{\partial}{\partial x}\right)\tilde{p}_1+\frac{\partial \tilde{u}_1}{\partial x}-i\alpha\tilde{v}_1=0$$

(4.8) $$\begin{aligned}&\bar{\rho}\left(\frac{\partial}{\partial t}+\bar{M}\frac{\partial}{\partial x}\right)\tilde{u}_1+\frac{\partial \tilde{p}_1}{\partial x}=0\\ &\bar{\rho}\left(\frac{\partial}{\partial t}+\bar{M}\frac{\partial}{\partial x}\right)\tilde{v}_1-i\alpha\tilde{p}_1=0\\ &\left(\frac{\partial}{\partial t}+\bar{M}\frac{\partial}{\partial x}\right)\tilde{s}_1=0\end{aligned} \qquad \text{for } x<0$$

where

(4.9) $$\tilde{p}_1=\bar{c}^2\tilde{\rho}_1+\bar{\rho}\bar{c}^2\,\tilde{s}_1$$

and

(4.10) $$\begin{aligned}&\left(\frac{\partial}{\partial t}+M\frac{\partial}{\partial x}\right)\tilde{p}+\frac{\partial \tilde{u}}{\partial x}-i\alpha\tilde{v}=0\\ &\left(\frac{\partial}{\partial t}+M\frac{\partial}{\partial x}\right)\tilde{u}+\frac{\partial \tilde{p}}{\partial x}=0\\ &\left(\frac{\partial}{\partial t}+M\frac{\partial}{\partial x}\right)\tilde{v}-i\alpha\tilde{p}=0\\ &\left(\frac{\partial}{\partial t}+M\frac{\partial}{\partial x}\right)\tilde{s}=0\end{aligned} \qquad \text{for } x>0$$

where

(4.11) $$\tilde{p}=\tilde{\rho}+\tilde{s}$$

Also, the relations (3.11) become

(4.12) $$\begin{aligned}&\tilde{s}_+=a_{11}\tilde{s}_-+a_{12}\tilde{p}_-+a_{13}\tilde{u}_-+a_{14}\tilde{v}_-+b_1\psi'-i\alpha c_1\psi\\ &\tilde{p}_+=a_{21}\tilde{s}_-+a_{22}\tilde{p}_-+a_{23}\tilde{u}_-+a_{24}\tilde{v}_-+b_2\psi'-i\alpha c_2\psi\\ &\tilde{u}_+=a_{31}\tilde{s}_-+a_{32}\tilde{p}_-+a_{33}\tilde{u}_-+a_{34}\tilde{v}_-+b_3\psi'-i\alpha c_3\psi\\ &\tilde{v}_+=a_{41}\tilde{s}_-+a_{42}\tilde{p}_-+a_{43}\tilde{u}_-+a_{44}\tilde{v}_-+b_4\psi'-i\alpha c_4\psi\end{aligned} \qquad \text{for } x=0$$

where +/- labels the after/front side of discontinuity and we have

(4.13) $$\left\{\begin{aligned}&a_{11}=1-\frac{\gamma-1}{2}(M-\bar{M})^2\\ &a_{12}=-\frac{1}{2}(\gamma^2-1)^2M\bar{M}\,\frac{(\bar{M}-M)^2}{[(\gamma-1)M^2+2][2\gamma M^2-(\gamma-1)]}\\ &a_{13}=-b_1,\ a_{14}=0,\ a_{21}=-\frac{2}{\gamma+1}M\bar{M}\\ &a_{22}=\frac{(\gamma+1)-2(\gamma-1)M\bar{M}}{2\gamma M^2-(\gamma-1)},\ a_{23}=-b_2,\ a_{24}=0=a_{42}\end{aligned}\right.$$

$$\left\{\begin{array}{l} a_{31}= M-\frac{\gamma-1}{\gamma+1}\bar{M},\ a_{32}= 2\frac{\gamma-1}{\gamma+1}\frac{1}{M},\ a_{33}= 2\frac{\gamma-1}{\gamma+1}-\frac{M}{\bar{M}} \\ a_{34}=a_{41}=a_{43}=0,\ a_{44}=1, \\ c_1=M_y b_1,\ c_2=M_y b_2,\ c_3=M_y b_3,\ c_4=\bar{M}-M \\ b_1=-\frac{2}{\bar{M}}\left(1-\frac{M}{\bar{M}}\right)(1-M\bar{M}),\ b_2=-\frac{4}{\gamma+1}M, \\ b_3=\frac{3-\gamma}{\gamma+1}+\frac{M}{\bar{M}},\ b_4=0. \end{array}\right.$$

The initial conditions are

$$(4.14)\quad \left\{\begin{array}{ll} (\tilde{s}_1,\tilde{p}_1,\tilde{u}_1,\tilde{v}_1)_{t=0}=(\tilde{s}_{10}(x),\tilde{p}_{10}(x),\tilde{u}_{10}(x),\tilde{v}_{10}(x)), & x<0 \\ (\tilde{s},\tilde{p},\tilde{u},\tilde{v})_{t=o}=(\tilde{s}_o(x),\tilde{p}_o(x),\tilde{u}_o(x),\tilde{v}_o(x)), & x>0 \\ \Psi(0)=0. & \end{array}\right.$$

4.2. <u>The expression of distortion of the discontinuity line</u>

Taking

$$(4.15)\qquad \bar{\omega}=\frac{\omega}{\bar{M}},\ \sigma(\bar{\omega})=\left[M^2\bar{\omega}^2+\alpha^2(1-M^2)\right]^{1/2}$$

we find for the four eigenvalues of matrix P (see (3.13)), which correspond to the region after discontinuity, the following expressions:

$$(4.16)\quad \hat{\lambda}_1=(1-M^2)^{-1}\left[M^2\bar{\omega}+\sigma(\bar{\omega})\right],\ \hat{\lambda}_2=(1-M^2)^{-1}\left[M^2\bar{\omega}-\sigma(\bar{\omega})\right],\ \hat{\lambda}_3=\hat{\lambda}_4=-\bar{\omega}$$

($\hat{\lambda}_1,\hat{\lambda}_2,\hat{\lambda}_3$ are <u>distinct</u> eigenvalues of reduced matrix obtained from P by deleting the row and the column corresponding to s).

Since $M<1$ we have from (4.16) [see (3.15)]

$$(4.17)\qquad Re\hat{\lambda}_1>0,\ Re\hat{\lambda}_2<0,\ Re\hat{\lambda}_3<0.$$

According to the remark 3.1 we can obtain, in the context of this paragraph, without the restriction that u_r, u_ℓ should be close:

$$(4.18)\qquad \overset{*}{\Psi}(\bar{\omega})=\frac{\gamma+1}{2}M\frac{d_1(\bar{\omega})+d_2(\bar{\omega})}{L(\bar{\omega})}$$

where

$$(4.19)\ d_1(\bar{\omega})=\frac{\sigma(\bar{\omega})}{M}\left[a_{21}\overset{*}{s}_-+a_{22}\overset{*}{p}_-+a_{23}\overset{*}{u}_-\right]-\bar{\omega}\left[a_{31}\overset{*}{s}_-+a_{32}\overset{*}{p}_-+a_{33}\overset{*}{u}_-\right]-i\alpha a_{44}\overset{*}{v}_-$$

$$(4.20)\qquad -d_2(\bar{\omega})=\int_0^\infty\left\{\frac{\sigma(\bar{\omega})}{M}\left[\tilde{p}_o\frac{M}{M^2-1}-\tilde{u}_o\frac{1}{M^2-1}\right]+\right.$$

$$+\bar{\omega}\left[\tilde{p}_0 \frac{1}{M^2-1} - \tilde{u}_0 \frac{M}{M^2-1}\right] - i\alpha\tilde{v}_0 \frac{1}{M}\Big\} e^{-\hat{\lambda}_1(\bar{\omega})\xi} \, d\xi$$

(4.21) $$L(\bar{\omega}) = 2M^2\bar{\omega}[\sigma(\bar{\omega}) + \bar{\omega}] + (1 - M^2)(\alpha^2 - \bar{\varsigma}\bar{\omega}^2)$$

For the considerations which follow it is convenient to put

(4.22) $$\frac{1}{L(\bar{\omega})} = \frac{N_1(\bar{\omega})}{N_2(\bar{\omega})}$$

where

$$N_1(\bar{\omega}) = \bar{\omega}^2\left[2M^2 - \bar{\varsigma}(1-M^2) - 2M^3\right] + \alpha^2(1-M^2) - 2M^2\bar{\omega}\left[\sigma(\bar{\omega}) - M\bar{\omega}\right]$$

$$N_2(\bar{\omega}) = \bar{\omega}^4\left\{\left[2M^2 - \bar{\varsigma}(1-M^2)\right]^2 - 4M^6\right\} + 2\bar{\omega}^2\alpha^2(1-M^2) \cdot$$
$$\cdot \left[2M^2 - \bar{\varsigma}(1-M^2) - 2M^4\right] + \left[\alpha^2(1-M^2)\right]^2$$

Since $1 < \gamma < \frac{5}{3}$, for $M < 1$ we obtain easily from (4.7)

(4.23) $$2M^2 - \bar{\varsigma}(1-M^2) > 2M^3 > 2M^4 > 0$$

By seeking for the roots of $N_2(\bar{\omega})$, we shall remark that the discriminant Δ is strictly positive :

$$\Delta = \left[\alpha^2(1-M^2)\right]^2 \left\{\left[2M^2 - \bar{\varsigma}(1-M^2) - 2M^4\right]^2 - \left[(2M^2 - \bar{\varsigma}(1-M^2))^2 - 4M^6\right]\right\} =$$
$$= (2M^2\alpha^2)^2(1-M^2)^3(\bar{\varsigma} - M^2) = \bar{\varsigma}(2M^2\alpha^2)^2(1-M^2)^3(1 - M\bar{M}) =$$
$$= [\text{by } (4.7)] = 4\,\frac{\gamma-1}{\gamma+1}\,\bar{\varsigma}\left[\alpha M(1-M^2)\right]^4 > 0.$$

Then we have

(4.24) $$\frac{1}{L(\bar{\omega})} = \varkappa\,\frac{(\bar{\omega}^2 + \bar{\omega}_3^2) - k^2\bar{\omega}\left[\sigma(\bar{\omega}) - M\bar{\omega}\right]}{(\bar{\omega}^2 + \bar{\omega}_1^2)(\bar{\omega}^2 + \bar{\omega}_2^2)}$$

where

(4.25) $$\begin{cases} \varkappa = \left[2M^2 - \bar{\varsigma}(1-M^2) + 2M^3\right]^{-1}, \quad k = \left\{2M^2/\left[2M^2 - \bar{\varsigma}(1-M^2) - 2M^3\right]\right\}^{1/2} \\ \bar{\omega}_{1,2} = \left\{\dfrac{\alpha^2(1-M^2)}{\bar{\varsigma}} \cdot \dfrac{(2M\bar{M}-1) \pm 2\left(\frac{\gamma-1}{\gamma+1}\right)^{1/2}\bar{\varsigma}^{1/2} M\bar{M}}{(2M\bar{M}-1)^2 - M^2}\right\}^{1/2} \\ \bar{\omega}_3 = \left\{\left[\alpha^2(1-M^2)\right] / \left[2M^2 - \bar{\varsigma}(1-M^2) - 2M^3\right]\right\}^{1/2} \end{cases}$$

and, according to (4.23), in (4.25) the expressions under the radical are positive.

<u>REMARK 4.1</u>. The gasdynamic context related to (4.3) has the

following peculiarities

- we do not restrict u_r and u_ℓ to be close

- for every $\alpha \in \mathbb{R}$ the singularities of $\overset{*}{\psi}$ are all placed on the line $\mathrm{Re}\,\omega = 0$.

<u>LEMMA 4.1.</u> If f is fo and $F(\omega) = L[f]$ then

$$(4.26)\quad F\left[M\omega+(\omega^2+1)^{1/2}\right] = \int_0^\infty e^{-\omega t}\left\{ \frac{1}{1+M} f\left(\frac{t}{1+M}\right) - \right.$$

$$\left. -(1-M^2)^{-3/2}\int_{Mt}^{t} \frac{J_1\left[\left(\frac{t^2-s^2}{1-M^2}\right)^{1/2}\right]}{(t^2-s^2)^{1/2}}(s-Mt)f\left(\frac{s-Mt}{1-M^2}\right)ds\right\}dt$$

◀ Since

$$J_{1/2}(z) = \frac{e^z - e^{-z}}{(2\pi z)^{1/2}} \qquad K_{1/2}(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}$$

we can calculate [1)]

$$(4.27)\quad \frac{1}{v}\left(e^{-\omega v} - e^{-v(\omega^2+1)^{1/2}}\right) = I_{1/2}\left\{\frac{1}{2}v\left[(\omega^2+1)^{1/2}-\omega\right]\right\}\cdot$$

$$\cdot K_{1/2}\left\{\frac{1}{2}v\left[(\omega^2+1)^{1/2}+\omega\right]\right\} = \int_0^\infty J_1(z)\,\frac{e^{-\omega(u^2+v^2)^{1/2}}}{(u^2+v^2)^{1/2}}\,du =$$

$$= \int_v^\infty \frac{J_1\left[(t^2-v^2)^{1/2}\right]}{(t^2-v^2)^{1/2}}\,e^{-\omega t}\,dt$$

and further

$$\int_0^\infty vf(v)\left\{\int_v^\infty \frac{J_1\left[(t^2-v^2)^{1/2}\right]}{(t^2-v^2)^{1/2}}e^{-\omega(t+Mv)}dt\right\}dv = \left\{t = \frac{t-Ms}{1-M^2},\ v = \frac{s-Mt}{1-M^2}\right\} =$$

$$= (1-M^2)^{-3/2}\int_0^\infty e^{-\omega\tau}\left\{\int_{M\tau}^{\tau} \frac{J_1\left[\left(\frac{\tau^2-s^2}{1-M^2}\right)^{1/2}\right]}{(\tau^2-s^2)^{1/2}}(s-M\tau)\,f\left(\frac{s-M\tau}{1-M^2}\right)ds\right\}d\tau$$

---

1) Y.S.Gradstein, Y.M.Rijik - Tables of integrals, summs, series and products. 5th edition. Moscow, Nauka, 1971 (in Russian); pag.733, 6.637.

However, let us remark, according to (4.27), that

$$F\left[M\omega + (\omega^2+1)^{1/2}\right] = \int_0^\infty f(v)\, e^{-v\left[M\omega+(\omega^2+1)^{1/2}\right]}\, dv =$$

$$= \int_0^\infty f(v) e^{-vM\omega}\left\{e^{-v\omega} - v\int_v^\infty \frac{J_1\left[(t^2-v^2)^{1/2}\right]}{(t^2-v^2)^{1/2}}\, e^{-\omega t}\, dt\right\} dv. \blacktriangleright$$

By lemma 4.1 it follows easily that

(4.28) $$F_f(t) = L^{-1}\left[\int_0^\infty f(t) e^{-\hat{\lambda}_1(\omega)t}\, dt\right] = (1-M) f\left[(1-M)t\right] +$$

$$+ \int_{|\alpha| M(1-M^2)^{1/2} t}^{|\alpha|(1-M)^{1/2} t} \frac{J_1\left[\left(\alpha^2 t^2 - \frac{s^2}{1-M^2}\right)^{1/2}\right]}{\left(\alpha^2 t^2 - \frac{s^2}{1-M^2}\right)^{1/2}} \left[M|\alpha| t - \frac{s}{(1-M^2)^{1/2}}\right] f\left[\frac{s}{|\alpha|(1-M^2)^{1/2}} - Mt\right] ds$$

The expression of Ψ then comes, using the tables, from (4.18), (4.24), (4.19), (4.20) and (4.28).

4.3. Linearized stability. Linearized well-posedness

THEOREM 4.1. The linearized problem (4.8) - (4.13) with data from $\mathscr{C}_0$ is well-posed in the class $\mathscr{C}$.

$\blacktriangleleft$ From (4.8) it appears, using (4.28), that for data in $\mathscr{C}_0$, Ψ and Ψ' are bounded. The theorem then follows by the Haar estimates ([9]). $\blacktriangleright$

5. THE (EXPONENTIAL) INSTABILITY CRITERION OF NYQUIST AND ERPENBECK.

To complete the picture given by the previous paragraphes we add here a review of the important paper [4].

For the case of general (dimensionless) equation of state

(5.1) $$p = p(\rho, S)$$

the expression (4.21) takes the form[1)]

$$(5.2)\quad L(\omega) = \bar{\varrho}\left\{2-M(\bar{M}-M)\,\frac{1}{\varrho T}\frac{\partial p}{\partial S}\right\}\omega\left[\sigma(\omega)+\omega\right] + (1-M^2)(\alpha^2 - \bar{\varrho}\omega^2)$$

Let us describe, first, the correspondence established by (5.2) between the complex planes ω and L .

We set $\omega = |\alpha| r \exp(i\theta)$ and consider the arithmetic determination of the function $\sigma(\omega) = |\alpha|\left[M^2 r^2 \exp(2i\theta) + (1-M^2)\right]^{1/2}$ (see (4.15)), making a cut along the imaginary axis of the plane ω, between the points $\pm\, i\left[(1-M^2)^{1/2}/M\right]$. On the axis $\mathrm{Re}\,\omega = 0$ (for the afore-mentioned cut we shall consider the points $\mathrm{Re}\,\omega = 0+$) we find the following expressions for $|\alpha|^{-2}\,L(\omega)$:

(5.3)

$$-\bar{\varrho}\left\{(1+M^2)-M(\bar{M}-M)\,\frac{1}{\varrho T}\frac{\partial p}{\partial S}\right\}r^2 - \bar{\varrho}M\left\{2-M(\bar{M}-M)\,\frac{1}{\varrho T}\frac{\partial p}{\partial S}\right\} r\left[r^2 - \frac{1-M^2}{M^2}\right]^{1/2} + (1-M^2)$$

$$\text{if}\quad \frac{(1-M^2)^{1/2}}{M} < r,\quad \theta = \frac{\pi}{2}$$

$$-\bar{\varrho}\left\{(1+M^2)-M(\bar{M}-M)\,\frac{1}{\varrho T}\frac{\partial p}{\partial S}\right\}r^2 + i\bar{\varrho}M\left\{2-M(\bar{M}-M)\,\frac{1}{\varrho T}\frac{\partial p}{\partial S}\right\} r\left[\frac{1-M^2}{M^2} - r^2\right]^{1/2} + (1-M^2)$$

$$\text{if}\quad 0 < r < \frac{(1-M^2)^{1/2}}{M},\quad \theta = \frac{\pi}{2}$$

$$-\bar{\varrho}\left\{(1+M^2)-M(\bar{M}-M)\,\frac{1}{\varrho T}\frac{\partial p}{\partial S}\right\}r^2 - i\bar{\varrho}M\left\{2-M(\bar{M}-M)\,\frac{1}{\varrho T}\frac{\partial p}{\partial S}\right\} r\left[\frac{1-M^2}{M^2} - r^2\right]^{1/2} + (1-M^2)$$

$$\text{if}\quad 0 < r < \frac{(1-M^2)^{1/2}}{M},\quad \theta = -\frac{\pi}{2}$$

$$-\bar{\varrho}\left\{(1+M^2)-M(\bar{M}-M)\,\frac{1}{\varrho T}\frac{\partial p}{\partial S}\right\}r^2 - \bar{\varrho}M\left\{2-M(\bar{M}-M)\,\frac{1}{\varrho T}\frac{\partial p}{\partial S}\right\} r\left[r^2 - \frac{1-M^2}{M^2}\right]^{1/2} + (1-M^2)$$

$$\text{if}\quad \frac{(1-M^2)^{1/2}}{M} < r,\quad \theta = -\frac{\pi}{2}$$

where (the dimensionless) $\varrho = 1$. By noting

$$(5.4)\qquad \mathcal{F} = 1 + M - M(\bar{M}-M)\,\frac{1}{\varrho T}\frac{\partial p}{\partial S}$$

---

1) For the dimensionless equation of state (4.3), $p = K\varrho^{\gamma}\exp\left[\gamma(S-S_0)\right]$, we have $\frac{\partial p}{\partial S} = 1$, $\frac{1}{T} = \gamma - 1$. From (4.7) and (5.2) it then results (4.21).

we can present the previous expressions as

$$
(5.5)\quad |\alpha|^{-2}\ \mathrm{Re}\ L(\omega)=\begin{cases} -\frac{1}{2}\bar{\rho}\mathcal{F}(1+M)\,r\,\xi_+(r)+\frac{1}{2}\bar{\rho}(1-M)(2M-\mathcal{F})\,r\,\xi_-(r)+(1-M^2) & \text{if } \frac{(1-M^2)^{1/2}}{M}<r \\ -\bar{\rho}\left[\mathcal{F}-M(1-M)\right]r^2+(1-M^2) & \text{if } 0<r<\frac{(1-M^2)^{1/2}}{M} \end{cases}
$$

$$
(5.6)\quad |\alpha|^{-2}\ \mathrm{Im}\ L(\omega)=\begin{cases} 0 & \text{if } \frac{(1-M^2)^{1/2}}{M}<r \\ (\mathrm{sign}\ \theta)\,\bar{\rho}M(\mathcal{F}+1-M)\,r\left[\frac{1-M^2}{M^2}-r^2\right]^{1/2} & \text{if } 0<r<\frac{(1-M^2)^{1/2}}{M} \end{cases}
$$

where the functions

$$
(5.7)\qquad \xi_+(r)=r+\left(r^2-\frac{1-M^2}{M^2}\right)^{1/2},\quad \xi_-(r)=r-\left(r^2-\frac{1-M^2}{M^2}\right)^{1/2}
$$

are described in fig.2.

Next, we discuss some facts of the correspondence between the planes ω and L .

LEMMA 5.1.

(i) If $\mathcal{F}>0$ then $\mathrm{Re}\ L(\omega)$ has on the axis $\mathrm{Re}\,\omega=0$ either two or six zeros symmetrically placed with respect to the origin.

(ii) If $\mathcal{F}<0$ then $\mathrm{Re}\ L(\omega)$ has no zeros on the axis $\mathrm{Re}\,\omega=0$.

◀ In the case $\mathcal{F}>0$, $r\geq\frac{(1-M^2)^{1/2}}{M}$, we shall consider first the situation

$$
(5.8)\qquad 2M<M(1-M+\bar{M})
$$

and analyse the following circumstances

(a) $0<\mathcal{F}<2M$.

By noting

$$a = -\frac{1}{2}\bar{\rho}\mathcal{F}(1+M), \quad b = \frac{1}{2}\bar{\rho}(1-M)(2M-\mathcal{F}) \tag{5.9}$$

and considering the function

$$\xi_o(r) = \frac{1-M^2}{r} \tag{5.10}$$

we observe that $a<0$, $b>0$ and we have

$$|\alpha|^{-2}\ \mathrm{Re}\ L(\omega) = r\left[a\xi_+(r)+b\xi_-(r) + \xi_o(r)\right] \tag{5.11}$$

where (according to the figure 2) $a\xi_+(r)$ is a convex function which decreases boundlessly when $r \to \infty$ and $b\xi_-(r) + \xi_o(r)$ is a convex function with a finite limit for $r \to \infty$. Therefore in the right bracket of the expression (5.11) we have a convex function which decreases boundlessly when $r \to \infty$. It results then, that, given sign θ, Re $L(\omega)$ can have a single zero if its maximum value (reached in $r = \frac{(1-M^2)^{1/2}}{M}$) is positive.

According to (5.11), (5.10) there exists a zero (and only one) if

$$-\frac{1}{2}\bar{\rho}\mathcal{F}(1+M)\frac{(1-M^2)^{1/2}}{M} + \frac{1}{2}\bar{\rho}(1-M)(2M-\mathcal{F})\frac{(1-M^2)^{1/2}}{M} + M(1-M^2)^{1/2} \geq 0$$

which (since $\bar{\rho} = (M/\bar{M})$) can be rearranged as

$$\mathcal{F} \leq M\,(1-M+\bar{M}),$$

This requirement holds according to (5.8).

(b) $2M < \mathcal{F} < M(1-M+\bar{M})$.

In this case $a<0$, $b<0$ in (5.9); the functions $a\xi_+(r)$, $-b\xi_-(r)$ and $\xi_o(r)$ have already been described for the case (a). We shall observe yet that (fig.3) in $r = \frac{(1-M^2)^{1/2}}{M}$ the function $a\xi_+(r)$ has a finite slope while the function $-b\xi_-(r)$ has an infinite slope. The figure 3 shows thus, that if $\mathcal{F} < M(1-M+\bar{M})$ then (given sign θ) Re $L(\omega)$ has a single zero.

(c) $M(1-M+\bar{M}) < \mathcal{F} < \mathcal{F}_c$

where, on the $\mathcal{F}$-axis, $\mathcal{F}_c$ is conveniently close to $M(1-M+\bar{M})$ (fig.3). Given sign θ, Re $L(\omega)$ has in this case two zeros

$$(d) \quad \mathcal{F}_c < \mathcal{F}$$

In this case (given sign θ), Re $L(\omega)$ has no zeros.

Let us now consider the situation

$$M(1-M+\bar{M}) < 2M \tag{5.12}$$

Given sign θ, we shall consider the following circumstances

$$(a) \quad 0 < \mathcal{F} < M\,(1-M+\bar{M}).$$

In this case Re $L(\omega)$ has a single zero according to analysis (a) of (5.8).

$$(b) \quad M(1-M+\bar{M}) < \mathcal{F} < 2M.$$

In this case Re $L(\omega)$ has no zeros.

$$(c) \quad 2M < \mathcal{F}$$

If $\mathcal{F}_c < 2M$ then, according to analysis (d) of (5.8), Re $L(\omega)$ has no zeros. If $\mathcal{F}_c > 2M$ then we are again in the situation (c) of (5.8).

Next, when $\mathcal{F} > 0$, $r \leq \frac{(1-M^2)^{1/2}}{M}$ we shall observe that, in the situation $\mathcal{F} < M\,(1-M)$, Re $L(\omega)$ is strictly positive. If $M(1-M) < \mathcal{F}$ then the requirement that

$$|\alpha|^{-2} \text{ Re } L(\omega) = -\bar{\xi}\,[\mathcal{F} - M(1-M)]\,r^2 + (1-M^2)$$

has a zero (given sign θ) is equivalent to the demand that in $r = \frac{(1-M^2)^{1/2}}{M}$ we should have

$$-\bar{\xi}\,[\mathcal{F} - M(1-M)]\,\frac{1-M^2}{M^2} + 1-M^2 \leq 0$$

which comes down to $\mathcal{F} \geq M\,(1-M+\bar{M})$. ▶

The results of lemma 5.1 corresponding to (5.8) and (5.12) respectively are gathered up in fig.4.

Given sign θ we thus have for Re $L(\omega)$

$$\begin{cases} \text{one zero in } r > \dfrac{(1-M^2)^{1/2}}{M} \text{ in the circumstances } S_{11}, S_{21} \\ \text{one zero in } r < \dfrac{(1-M^2)^{1/2}}{M} \text{ in the circumstances } S_{12}, S_{14}, S_{23} \\ \text{three zeros of which one in } r < \dfrac{(1-M^2)^{1/2}}{M} \text{ in the circumstances } S_{13}, S_{22}. \end{cases}$$

Another description of the results of lemma 5.1 is presented in fig.5.

According to (5.6) we can depict Im $L(\omega)$ as in fig.6.

Let us now consider in the plane ω the contour $\mathcal{C}$ depicted in fig.7 whose semicircular part has a sufficiently large radius. At the points of this arc we have

$$|\alpha|^{-2}\ L(\omega) \approx \bar{\zeta}\,\mathcal{F}(1+M)r^2 \exp\ (2i\theta).$$

The contours that correspond to $\mathcal{C}$ in the plane L , according to circumstances S_{11}, S_{21}; S_{12},S_{14}, S_{23}; S_{13}, S_{22}; S_4, are presented in fig.7.

We can now formulate the (exponential) instability criterion of Erpenbeck.

<u>THEOREM 5.1</u>. The linearized problem is exponentially unstable if $\mathcal{F} < 0$.

◀ Let N be the number of zeros of the function $L(\omega)$ in Re $\omega > 0$. According to the argument principle it appears that N = 0 if $\mathcal{F} > 0$ and N = 1 if $\mathcal{F} < 0$. ▶

<u>REMARK 5.1.</u>

(i) If $\mathcal{F} > 0$ we say that the linearized problem is <u>exponen-</u>

tially stable. The possibility of a non-exponential instability cannot be excluded generally. The theorem 4.1 shows that for the (usual) equation of state (4.3) the linearized problem is stable.

(ii) The requirement $\mathcal{F} > 0$ can be presented, taking into account (5.4), as the demand

$$\frac{\partial p}{\partial S} < \varrho T \frac{1 + M}{\bar{M}M - M^2} = \frac{\varrho T}{M^2} \frac{1 + M}{\frac{1}{\varrho} - 1}$$

which ought to be added to the Weyl (thermodynamic) assumptions in order to delimitate a class of equations of state for which the exponential instability of the linearized problem is removed.

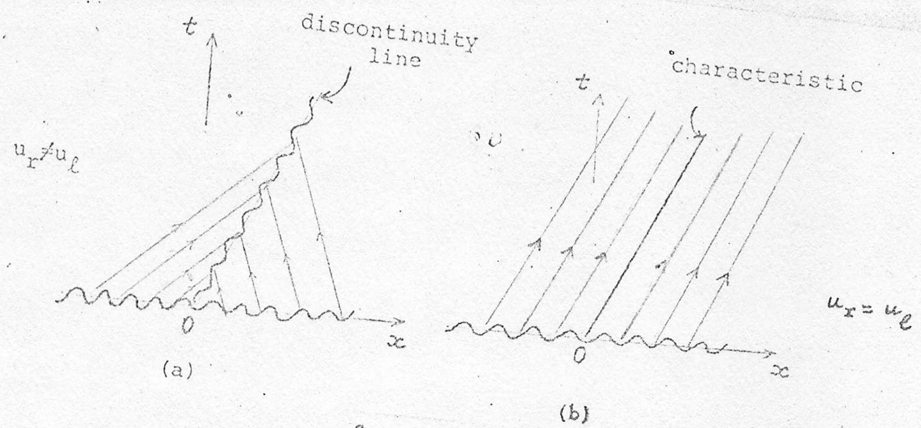


fig. 1

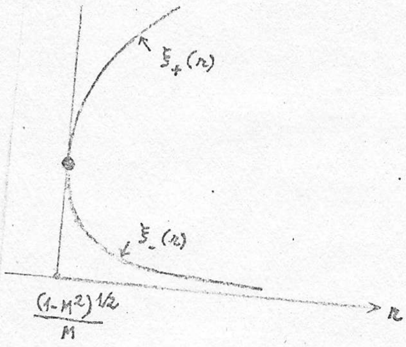


fig. 2

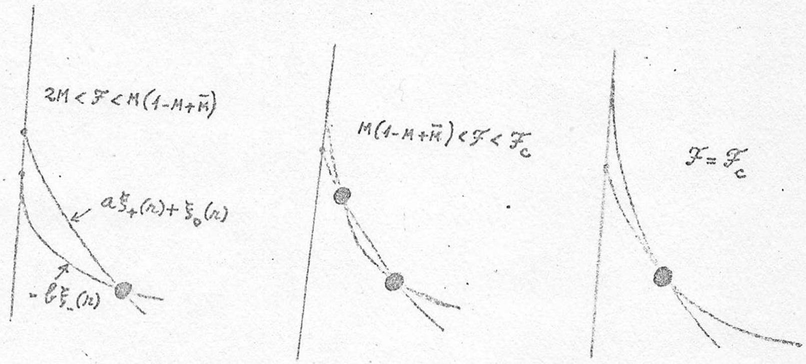


fig. 3

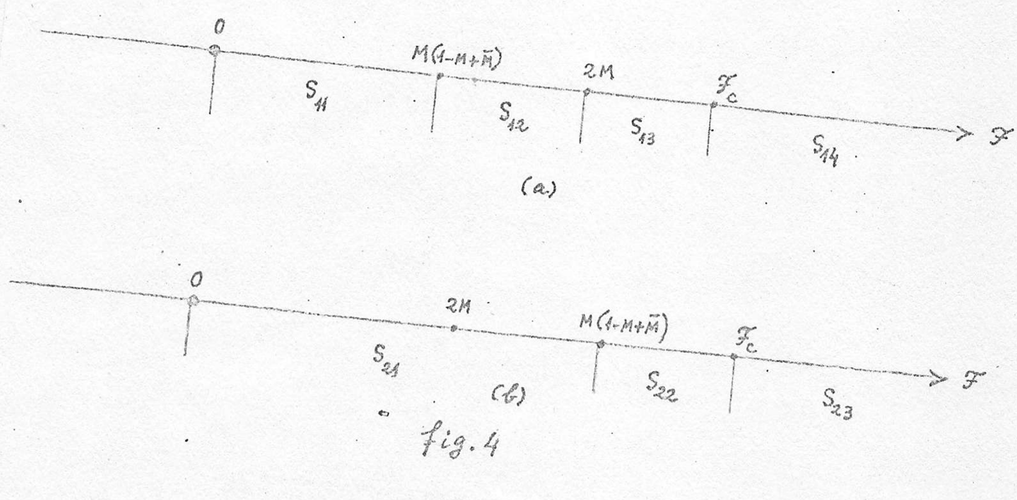
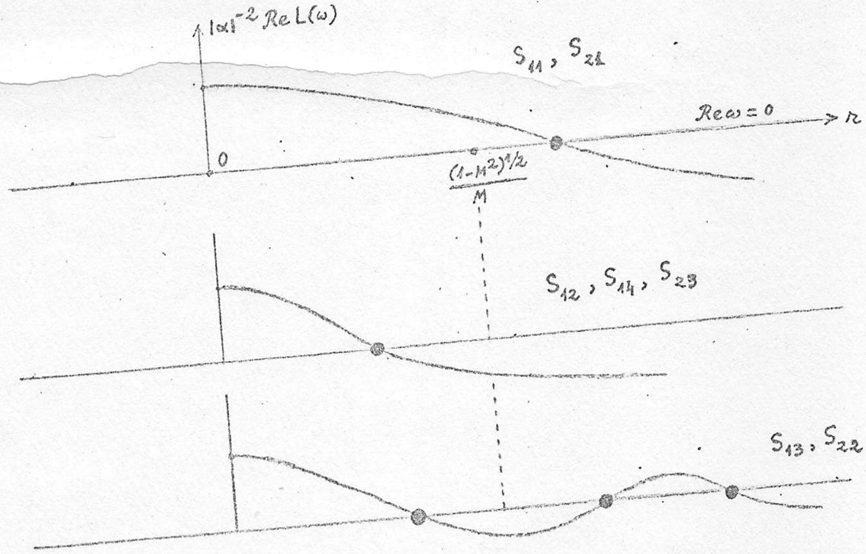
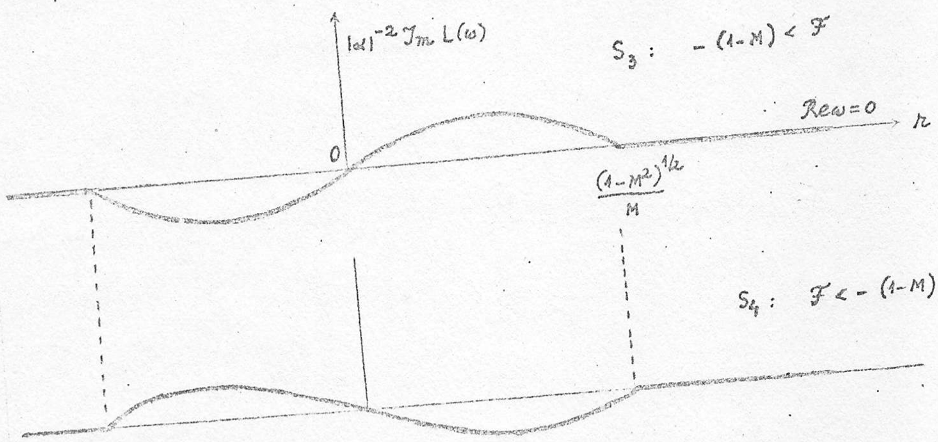


fig. 4



|α|^-2 Re L(ω)
S11 , S21
Re ω = 0
r
0
(1-M²)^1/2 / M
S12 , S14 , S23
S13 , S22

fig. 5



|α|^-2 Im L(ω)
S3 : -(1-M) < F
Re ω = 0
r
0
(1-M²)^1/2 / M
S4 : F < -(1-M)

fig. 6

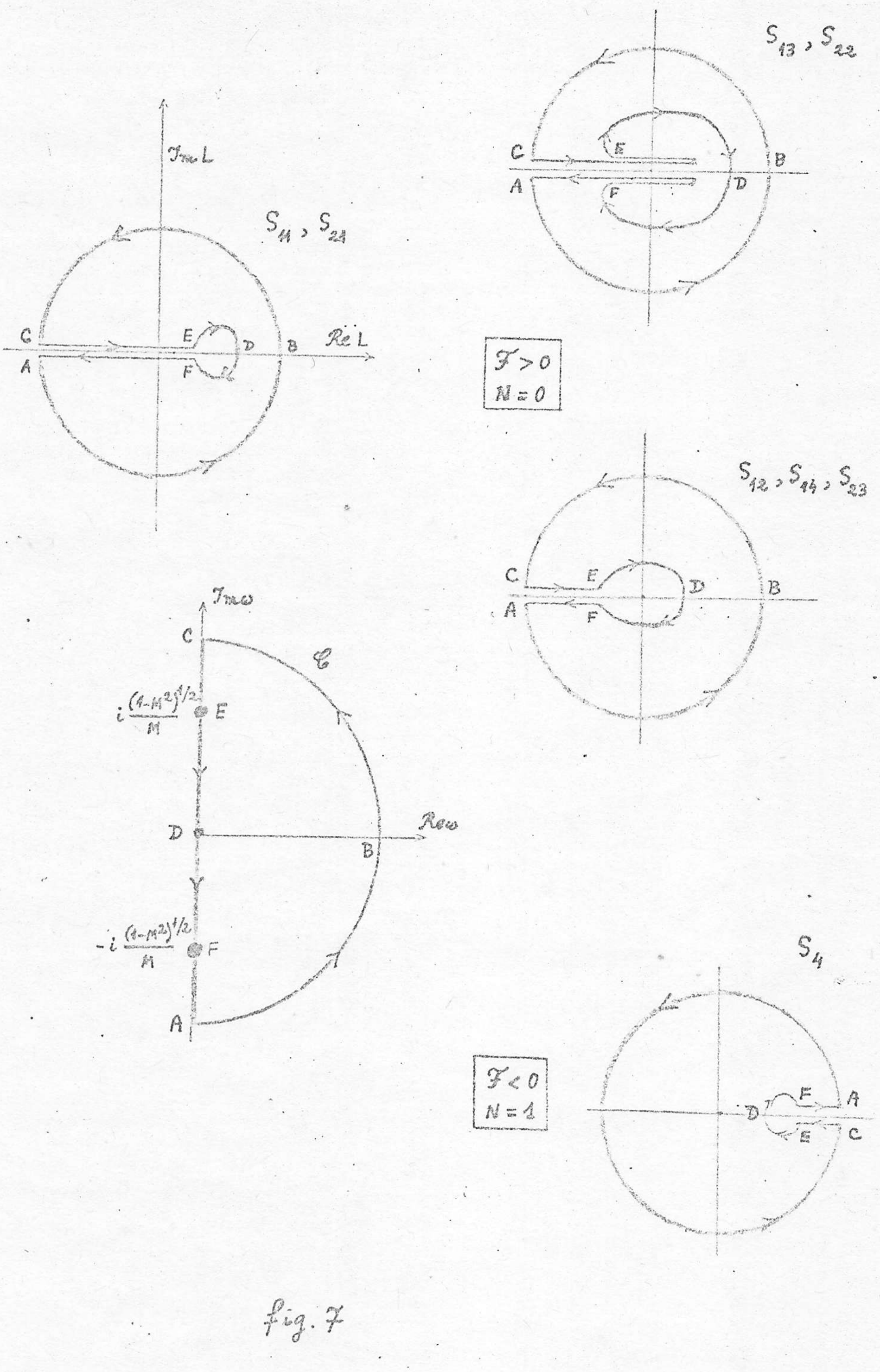

Im L
S11 , S21
C
E
D
B
Re L
A
F
S13 , S22
C
E
B
A
D
F
F > 0
N = 0
S12 , S14 , S23
C
E
D
B
A
F
Im ω
C
𝒞
E
D
Re ω
B
F
A
S4
F < 0
N = 1
F
A
D
E
C

fig. 7

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APPENDIX. The Haar estimates (see, for example, [9])

Let us consider the system

$$(1) \qquad \frac{\partial}{\partial t} q + A \frac{\partial}{\partial x} q + B_1 q = 0$$

with A and B_1 constant matrices [see (2.3)/(3.10)/(4,8), (4.10)]. The eigenvalues $\lambda_1,\ldots,\lambda_n$ of matrix A are real and distinct. Let P be a matrix which diagonalizes A :

$$(2) \qquad P^{-1}AP = D = \mathrm{diag}\{\lambda_1,\ldots,\lambda_n\}$$

and put $v=P^{-1}q$. Multiplying (1) by P^{-1} to the left we find

$$(3) \qquad \frac{\partial}{\partial t} v + D\frac{\partial}{\partial x} v + Bv = 0, \qquad B=P^{-1}B_1P$$

Let us take the point (ξ,η); $\eta>0$. By integrating (3) along the characteristics we obtain

$$(4) \quad v_i(\xi,\eta)=g_i\left[X_i(0\;;\xi,\eta)\right]-\int_0^{\eta}\sum_{j=1}^{n} B_{ij}v_j(X_i,t)dt;\ 1\leq i\leq n;\ g=P^{-1}q_0$$

where the points $\Theta(\xi,\eta)$ and $\Theta_i[X_i(0;\xi,\eta),0]$ are in corresponden-ce as belonging to the characteristic $x=X_i(t;\xi,\eta)$.

Let now $[x_1,x_2]$ be a compact interval of the real axis. We denote by $\mathcal{D}_\eta$ the closure of the intersection of the determina-cy domain of this interval with the strip $0\leq t\leq\eta$ and put

$$H = \max_{Q\in\mathcal{D}_\eta} |v(Q)| , \quad |v| \leq \max_{1\leq i\leq n} |v_i|$$

Let $R(x_R,\ t_R)\in\mathcal{D}_\eta$ be a point at which $|v(Q)|$ reaches the value H . Denoting

$$\|g\|_{[x_1,x_2]} = \max_{1\le i\le n} \sup_{x\in[x_1,x_2]} |g_i(x)| \ , \quad K = \max_{i,j} |B_{ij}|$$

we obtain from (4)

$$|v(P)| \le |v(R)| = H \le \max_{1\le i\le n} |v_i(R)| =$$
$$= \max |g_i(R_i) - \int_0^{t_R} (\sum_{j=1}^{n} B_{ij} v_j) dt| \le \|g\|_{[x_1,x_2]} + n\eta KH$$

and further

$$H \le \tilde{C}(\eta) \|g\|_{[x_1,x_2]} \ , \quad \tilde{C}(\eta) = \frac{1}{1-n\eta K} \quad \text{for} \quad \eta < \frac{1}{nK} \ .$$

When $\eta > (1/nK)$ the procedure has to be repeated. Let us advance, in this case, by strips of breadth $1/2nK$ and parallel to axis $t=0$. In such a strip $\tilde{C}(\eta) \le 2$ so that

(5) $$H \le 2 \|g\|_{[x_1,x_2]} \le 2 \|g\|$$

where the constant $\|g\|$ majorizes the initial data (on a given interval).

The mentioned procedure can be applied directly to the problem (4.8), $(4.14)_1$ because the determinacy domain of the interval $x<0$, $t = 0$ is the whole region $x<0$, $t>0$. If the initial data are bounded, then from (5) the solution (corresponding to them, in the domain of determinacy) is bounded.

To study the mixed problem (4.10), (4.12), $(4.14)_{2,3}$ we need, moreover, the expression of Ψ. In fig. 8 we depict, in such a case, the curve which carries the initial ($\tilde{u}_0$) or boundary (Ψ)

data and its domain of determinacy. The procedure expounded above keeps valid if one makes certain minor and obvious modifications related to the estimates corresponding to the points of discontinuity. The boundedness of solution depends now, moreover, on the boundedness of ψ and ψ'.

The estimate (5), and the analogue estimates which correspond to the mixed problem, have to be regarded as Haar estimates because they allow to evaluate the solution by means of initial and boundary data.

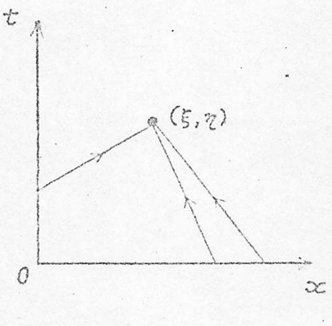

fig. 8