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RIGIDITY PROPERTIES OF COMPACT LIE GROUPS MODULO MAXIMAL TORI

by

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Stefan PAPADIMA

1. Introduction and Statement of Main Results

When trying to understand the topological symmetry of homogenous spaces of the form $M = G/K$, where G and K are compact connected Lie groups of the same rank, one is quickly led to concentrate on the cohomological aspect, for several reasons. First of all, the cohomology algebras (for characteristic zero coefficients $\mathbb{F}$) have nice convenient descriptions in terms of invariants of Weyl groups [4]. Next, it is known that, for two such manifolds M and M', the set of homotopy classes of maps between their rationalizations is in natural bijection with the set of graded algebra morphisms between their rational cohomology algebras (see the proof of Theorem 1.1 [10]). Moreover since they are 1-connected finite formal complexes [24], it follows, again by [24], that the knowledge of $[M_0, M'_0]$, i.e. of rational cohomology morphisms, offers the homotopy classification of maps between M and M', up to finite ambiguity (see also [10], [22], [16]). Various kinds of applications are possible, see e.g. Corollary 1.3 and Theorems 1.4 - 1.6 below.

The main aim of this paper is to begin a systematic study of cohomology automorphisms of such M, both over $\mathbb{Q}$ and over $\mathbb{Z}$; here we shall be concerned only with the case $K = T$, a maximal torus, which is the most natural to start with. In the last decade much work has been done in the direction of determining the rational cohomology endomorphisms and/or automorphisms of complex flag manifolds $M = G/K = U(n_1 + \dots + n_k) \bmod U(n_1) \times \dots \times U(n_k)$, see e.g. [10], [19], [17] and their references. The methods were based more or less on direct computations using the special features of the cohomological structure of complex flags and complete results are available only in a few particular cases (up to our present knowledge).

For $M = G/T$, G compact connected arbitrary, we obtain a complete and simple description of the cohomological symmetry (both rational and integral). Our method relies on the relationship between the invariants of the Weyl group and the geometry of the Stiefel diagram of G and it was inspired by the results for classifying spaces of [2]; it has the advantage of working uniformly and of minimizing the computational effort.

The proofs of the results on cohomology autormorphisms occupy the next section. In more detail, recall that the Weyl group W acts on the Lie algebra V of T and preserves the integral lattice Γ (more on notations may be found at the beginning of § 2). The classical description by Borel [4] of $H^*(G/T;\mathbb{F})$ in terms of invariants of W in the polynomial graded algebra on $\Gamma \otimes \mathbb{F}$ implies that the graded algebra automorphisms of G/T over $\mathbb{F}$ may be identified with those $\mathbb{F}$ - linear automorphisms of $\Gamma \otimes \mathbb{F}$ whose polynomial extension preserves the ideal generated by the positive degree invariants of W (Proposition 2.1). Obvious examples are the elements of the normalizer of W in $GL(\Gamma \otimes \mathbb{F})$; when $\mathbb{F} = \mathbb{Q}$, this normalizer coincides with the admissible automorphisms of [2]. Our first result establishes that there are no other cohomology automorphisms of G/T.

1.1. Theorem. The group of graded algebra automorphisms of $H^*(G/T; \mathbb{F})$ is antiisomorphic to the normalizer of W in $GL(\Gamma \otimes \mathbb{F})$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{Q}$.

For convenience we first give the proof for $\mathbb{F} = \mathbb{R}$, and then deduce the result for $\mathbb{F} = \mathbb{Q}$ in a straightforward manner.

1.2. Theorem. The group of graded algebra automorphisms of $H^*(G/T;\mathbb{Z})$ is antiisomorphic to the group of automorphisms of the root system of G.

The result over $\mathbb{Z}$ is also derived from our knowledge of the picture over $\mathbb{R}$.

1.3. Corollary. The group of homotopy classes of self-homotopy equivalences of G/T is finite.

Proof. By the above theorem the group of integral cohomology automorphisms is finite. On the other hand we have seen that there are only finitely many homotopy classes of self-maps inducing the identity in rational cohomology.

We chose to say that $M = G/K$ (or $M = BG$) has the rigidity property with respect to some question related to its topological symmetry if the answer may be formulated in terms of the corresponding Lie theory. Examples: what is the structure of the group of self-homotopy equivalences of M_0? For $M = G/K$ a complex flag manifold, in all known cases this group turns out to be generated by grading automorphisms (which act on each $H^{2i}(M; \mathbb{Q})$ as $\lambda^i.\mathrm{id}$, for some nonzero $\lambda \in \mathbb{Q}$, and which come from Frobenius self-maps in positive characteristic Lie theory, by [9]) together with the rational automorphisms coming from the action of the normalizer $N_G(K)$ on M. A subtler question was formulated, for $M = BG$, in [2]: which self-equivalences of M_0 are defined after finite localization (for a finite 1-connected complex M the answer is: all of them, see e.g. [16])? By [2] this subgroup of the rational automorphisms of $M = BG$ may be identified with $N_{GL(\Gamma \otimes \mathbb{Q})}(W)/W$.

The second aim of this paper is to improve the rather vaguely formulated definition of the rigidity properties. Theorems 1.1 and 1.2 above may be considered as typical examples in this direction. We shall next state in precise form three more examples of rigidity (theorems 1.4, 1.5 and 1.6 below) and later give the proofs as applications of our results on cohomology automorphisms (in sections 3,4 and 5).

The main application is devoted to geometry. Let M be a closed 1-connected Riemannian manifold and let f be an isometry of M. A geodesic curve c is called <u>f-invariant</u> if it is nonconstant and there exists a period t such that $f(c(x)) = c(x + t)$, any x. When $f = \mathrm{id}$ one recovers the classical notion of closed geodesic. The question of the existence and of the abundance of various kinds of geodesics is a central problem in Riemannian geometry. A major development in this area is contained in the paper [25]; they pointed out the relationship between the rational homotopy properties of M and the existence of closed geodesics on M. Further refinements of both Morse theory and rational homotopy theory involved here led to the conclusion that the nonexistence of (many) f-invariant geodesics imposes severe restrictions on the rational homotopy properties of f. The following result in this direction will be strong enough for our present purposes (subtler statements may be found in [12], [14], see also 3.4). Denoting by $[\pi_*(M) \otimes \mathbb{Q}]^f$ the fixed points of the obvious action of f, one has:

Theorem (see [12], [14], [13]).

(i) If there are no f-invariant geodesics then $\dim[\pi_*^{odd}(M) \otimes \mathbb{Q}]^f = 0$.

(ii) If there are only finitely many geometrically distinct f-invariant geodesics then $\dim[\pi_*^{odd}(M) \otimes \mathbb{Q}]^f \leq 1$.

Using this approach, strong existence theorems were obtained in [13]: if M is odd-dimensional every isometry has an invariant geodesic; if $\dim \pi_*(M) \otimes \mathbb{Q} = \infty$, every isometry has infinitely many invariant geodesics. These leave still open the case M = G/K, K a closed connected subgroup of maximal rank. For K = T we are able, by computing the rational homotopy fixed points of the self homotopy equivalences of M, to obtain a complete solution of the existence problem for invariant geodesics on M.

1.4. Theorem. Let f be an isometry of M = G/T (for an arbitrary metric).

(i) $\dim[\pi_*^{odd}(M) \otimes \mathbb{Q}]^f > 0$.

(ii) $\dim[\pi_*^{odd}(M) \otimes \mathbb{Q}]^f > 1$, unless $M = (SU(n)/T)^\ell$, n = 2 or 3, and f^* equals the cohomology automorphism corresponding by the isomorphism of Theorem 1.2 to the root system automorphism given by

$$a(v_1,\ldots,v_{\ell-1},v_\ell) = (a_\ell(v_\ell),a_1(v_1),\ldots,a_{\ell-1}(v_{\ell-1}))$$

where a_i are automorphisms of the corresponding type A_1 or type A_2 root systems, and (in the A_2 case) they are subject to the condition that $a_1 . a_2 \ldots a_\ell$ induces the nontrivial automorphism of the Dynkin diagram; in all these cases

$\dim[\mathcal{J}^{odd}_*(M) \otimes \mathbb{Q}]^f = 4.$

This result is the best one can hope, see 3.4.

Section 4 deals with fixed points of transformation groups. On $M = G/K$ (where rank K = rank G) there is a natural free action of the finite group $N_G(K)/K$. For $K = T$, by using our knowledge of cohomology automorphisms of M together with a computation relating Lefschetz numbers to rational homotopy fixed points, we deduce that this natural action represents an upper bound for the free symmetry of M. This result helps to make more precise in this case the general result, conjectured for $M = G/K$ by W.Y.Hsiang [20] and proved in [3], which states that any circle action on M must have a fixed point; this becomes an immediate consequence of the theorem below (by looking at the action of a generator of the circle).

1.5. Theorem. The cardinality of a group acting freely on G/T does not exceed the cardinality of the Weyl group of G.

This result will be proved in a slightly strengthened form (Theorem 4.1).

By Sullivan [23] a homotopy type M may be described as the collection of its localizations $\{M_p \mid p \text{ a prime}\}$ together with the coherence information provided by the rationalization maps $\{M_p \longrightarrow M_o\}$. The simplest situations arise when the collection of the localizations $\{M_p\}$ already determines M. Thus, a homotopy type M is called <u>generically rigid</u> ([11]) if the genus of M, defined as the set of homotopy types M' with the property that $M'_p \simeq M_p$, for all primes p, consists of M alone.

1.6. Theorem. G/T is generically rigid, for any G.

This result is entitled to be called a rigidity property (in our sense) by more than philological reasons. The method developed in [11] indicates that, for a 1-con nected finite formal complex M, the generic rigidity is a consequence of the fact that the group of self-homotopy equivalences of M_o, to be denoted in the sequel by $E_o(M)$, is generated by rationalizations of self-homotopy equivalences which are defined after inverting at most one prime. When $M = G/K$, this in turn is a direct consequence, via etale homotopy theory ([8], [9]), of the rigidity of $E_o(M)$. We mean by this that $E_o(M)$ is generated by self-maps which come from the purely inseparable isogenies of the corresponding Lie theory. For $K = T$ this rigidity property follows from the detailed description of rational cohomology automorphisms of M given in 2.8 and 2.9 (see the proof of Proposition 5.1 and the remarks preceding its statement); the basic arguments for establishing the rigidity of $E_o(M)$ are extracted from [2], propositions 2.13 and 2.15. As far as generic rigidity is concerned, we follow [11], obtaining a little more (see Theorem 5.3).

2. Cohomology Automorphisms

Let G be a compact connected Lie group and let T be a maximal torus. By classical Lie theory (see [4]), $G/T = G_1/T_1$, where in addition G_1 is 1-connected (and its Lie algebra equals the semisimple part of the Lie algebra of G). Therefore we may and we shall indeed from now on suppose that G is 1-connected. We can further write $G = \prod G_i$ as the product of its simple components. Denoting by V the Lie algebra of T and by Γ the kernel of the exponential map of T, there is a corresponding product splitting for T, V and Γ.

The real vector space V is endowed with the euclidean metric coming from the Killing form of G. This choice of metric provides a canonical isomorphism $V \xrightarrow{\simeq} V^*$ and an euclidean structure on the dual space V^* (everything being compatible with the splittings). Denote by $\Phi^* \subset V^*$ the root system (in the axiomatic sense of [21]) consisting of the roots of the adjoint representation of T in the Lie algebra of G (see [1]). The group $N_G(T)/T$ is canonically isomorphic to the Weyl group of this root system, and both will be denoted in the sequel by W (see again [1]). The decomposition of G into simple components corresponds to the decomposition of Φ^*, $\Phi^* = \coprod \Phi_i^*$, into irreducible components; similarly, the Weyl group decomposes as a direct product. It will be convenient to normalize the metric on each component of V ($V = \prod V_i$) in order to make all short roots of the corresponding Φ_i^* have length equal to $\sqrt{2}$ (for types A, D and E we consider that all roots are short). Denoting by (,) the resulting metric on V and by $\tau : V \to V^*$ the corresponding isometry we obtain a root system $\Phi \subset V$ which is isometrically isomorphic to Φ^* and whose Weyl group action on V corresponds to the adjoint action of the Weyl group of G([1]).

Recall next from [4] that, with characteristic zero coefficients $\mathbb{F}$, the spectral sequence of the fibration $G/T \hookrightarrow BT \to BG$ gives the isomorphism $H^*(G/T;\mathbb{F}) = H^*(BT;\mathbb{F})/\text{ideal}\,(H^+(BT;\mathbb{F})^W)$. Denoting by $\mathbb{F}[\Gamma \otimes \mathbb{F}]$ the graded $\mathbb{F}$-algebra of polynomial functions on $\Gamma \otimes \mathbb{F}$ (with degree of the generators = 1), on which W naturally acts by $p^w = p \circ w$, for $p \in \mathbb{F}[\Gamma \otimes \mathbb{F}]$ and $w \in W$, the natural isomorphism $\Gamma \otimes \mathbb{F} \to H^2(BT;\mathbb{F})$ gives rise to an algebra isomorphism

$$\mathbb{F}[\Gamma \otimes \mathbb{F}]/\text{ideal}\,(\mathbb{F}[\Gamma \otimes \mathbb{F}]^{+W}) \xrightarrow{\sim} H^*(G/T;\mathbb{F}) \qquad (1)$$

which doubles the degrees.

Denote by $\mathcal{E} \subset gl(\Gamma \otimes \mathbb{F})$ the submonoid consisting of those $\mathbb{F}$-linear maps $b: \Gamma \otimes \mathbb{F} \to \Gamma \otimes \mathbb{F}$ with the property that, for any $p \in \mathbb{F}[\Gamma \otimes \mathbb{F}]^{+W}$, $p^b = p \circ b \in \text{ideal}\,(\mathbb{F}[\Gamma \otimes \mathbb{F}]^{+W})$, and by $\mathcal{A}$ the group of invertible elements of $\mathcal{E}$.

2.1. Proposition. The correspondence which associates to each $b \in \mathcal{E}$ the induced graded algebra endomorphism of $H^*(G/T;\mathbb{F})$, via (1), establishes an antimultiplicative isomorphism onto the cohomology endomorphisms of G/T over $\mathbb{F}$, under which the cohomology automorphisms correspond to the elements of $\mathcal{A}$.

Proof. Since G is in particular semisimple each Weyl group W_i acts irreducibly on V_i [21]. It follows that there are no nonzero degree one invariants of the Weyl group W in $\mathbb{F}[\Gamma \otimes \mathbb{F}]$ for $\mathbb{F} = \mathbb{R}$ and consequently (see [6], p.126) for any $\mathbb{F}$. Hence the projection $\mathbb{F}[\Gamma \otimes \mathbb{F}] \to H^*(G/T;\mathbb{F})$ induces an isomorphism $\mathbb{F}[\Gamma \otimes \mathbb{F}]^1 \xrightarrow{\sim} H^2(G/T;\mathbb{F})$ at the level of indecomposable algebra generators. Using this fact, all the assertions of the proposition follow easily.

From now on we shall use for the inverse of the above isomorphism, End $H^*(G/T;\mathbb{F}) \xrightarrow{\sim} \mathcal{E}$, the notation $f \mapsto H_2(f;\mathbb{F})$.

We are now moving towards the proof of Theorem 1.1. Set $\mathbb{F} = \mathbb{R}$. Consider the subgroup $D \subset \prod GL(V_i)$ consisting of automorphisms which act as multiplication by some positive real number on each V_i. Since D centralizes W, we plainly have $D \subset \mathcal{A}$.

2.2. Lemma. For each $a \in \mathcal{A}$ there exists $d \in D$ such that ad is an isometry.

Proof. Assume $a \in \mathcal{E}$. Since we just saw that there are no nonzero elements in $\mathbb{R}[V]^{1W}$ it follows from the definition of $\mathcal{E}$ that the action of a on $\mathbb{R}[V]$ must preserve the linear subspace $\mathbb{R}[V]^{2W} = \bigoplus \mathbb{R}[V_i]^{2W_i}$. By irreducibility (see [6], p.66) each $\mathbb{R}[V_i]^{2W_i}$ is one dimensional, generated by the invariant quadratic form q_i defined by $q_i(x) = (x_i,x_i)$, for $x = \sum x_i \in V$. Hence $q_i^a = \sum_j A_{ij} q_j$, any i. Summation gives $(ax,ax) = \sum A_j q_j(x)$, for any $x \in V$. If $a \in \mathcal{A}$ pick a nonzero $x_i \in V_i$ and deduce that $A_i > 0$, for any i. Define then d by $d = \prod \lambda_i$, with $\lambda_i = 1/\sqrt{A_i}$.

The same method gives the following result, which will be useful in §4:

2.3. Lemma. For any $a \in \mathcal{E}$, ker a is W-invariant.

Proof. If $ax = 0$ then we may write, for any $w \in W$, $0 = (ax,ax) = \sum q_i^a(x) = \sum q_i^{aw}(x) = (awx,awx)$.

2.4. Proof of Theorem 1.1 for $\mathbb{F} = \mathbb{R}$

We have to show that $\mathcal{A} \subset N$, N being the normalizer of the Weyl group in GL(V). By Lemma 2.2 it is enough to show that any isometric $a \in \mathcal{A}$ normalizes W. For each $\alpha \in \Phi$ denote by H_α the hyperplane orthogonal to α, and set $H = \bigcup H_\alpha$. We claim that it suffices to prove that any such a leaves H invariant. Indeed, assuming that for any $\alpha \in \Phi$ there exists $\beta \in \Phi$ such that $a(H_\alpha) = H_\beta$, it is immediate to see, using the fact that a is isometric, that $a s_\alpha a^{-1} = s_\beta$, where s_α and s_β are the reflections in the corresponding hyperplanes; since the reflections

s_α generate W it follows that $a \in N$.

In order to prove that the action of a preserves H we proceed to the determination of the cohomology classes in $H^2(G/T;\mathbb{R})$ which have maximal height (a computation for complex flag manifolds may be found in [19]).

Denoting by n the number of positive roots (and recalling that $2n =$ dimension of G/T) we shall consider the following polynomial function $p \in \mathbb{R}[V]^n$, constructed by evaluating n-th powers of 2-dimensional cohomology classes of G/T on the fundamental class of G/T: $p(x) = \langle \tau(x)^n,[G/T]\rangle$, any $x \in V$. For any $w \in W$: $p^{w^{-1}}(x) = \langle(\tau(x)^{w}\ )^n,[G/T]\rangle = \langle(\tau(x)^n)^{w}\ ,[G/T]\rangle =$ $= \langle \deg(w\ )\cdot\tau(x)^n,[G/T]\rangle = \det(w)\cdot p(x)$. It is now easy to infer that $p(x) = \lambda\cdot\prod_{\alpha\in\Phi^+}(\alpha,x)$, any $x \in V$, for some $\lambda \in \mathbb{R}$ ([6], p.113). Moreover, λ must be nonzero, for otherwise (use [21], p.134) we would have $H^{2n}(G/T;\mathbb{R}) = 0$.

Coming back to our given orthogonal $a \in \mathcal{A}$, it is clear that a^* must preserve the zeroes of $p\circ\tau^{-1}$, which implies that a^{-1} preserves the zero set of p, which is just H.

Our next task is to describe the group structure of $\tilde{N}$. Let us choose a system of simple roots $S \subset \Phi$; if $\Phi = \coprod \Phi_i$ is the decomposition into irreducible components, there is a corresponding splitting $S = \coprod S_i$, with $S_i \subset \Phi_i$ a system of simple roots. We shall denote by Graphaut (S) the group of permutations of S which are automorphisms of the Coxeter graph structure. It contains the subgroup Dgraut (S), consisting of graph automorphisms which preserve short and long roots.

2.5. Proposition. There is a split exact sequence

$$1 \longrightarrow D \times W \longrightarrow N \underset{\sigma}{\overset{\gamma}{\rightleftarrows}} \mathrm{Graphaut}(S) \longrightarrow 1$$

Proof. The proof is inspired by [2], Proposition 2.13. The novelty consists in the construction of the splitting (in [2] the surjectivity of γ is verified by case-by-case checking); though perhaps known to Adams and Mahmud, this explicit construction will play a key role in what follows, therefore we shall treat this point carefully. As a word of caution, note that our graph-automorphisms are called in [2] diagram isomorphisms.

Denote by C the Weyl chamber corresponding to S. The arguments of 2.4 show that the elements of N act on Weyl chambers. Call the stability group of C, N_C. An element $a \in N_C$ permutes the walls of C and thus induces a permutation of S, denoted by $\gamma(a)$, characterized by: $a(H_\alpha) = H_{\gamma(a)\alpha}$, any $\alpha \in S$. It can be checked ([2]) that $\gamma(a)$ is a graph-automorphism and that we have an exact sequence $1 \to D \to N_C \xrightarrow{\gamma}$ Graphaut(S). Granting for the moment the existence of a splitting σ: Graphaut(S) $\to N_C$, we can easily finish the proof. The transitivity of the Weyl group action on chambers provides a natural group surjection $N_C/D \to N/D\cdot W$, which is in fact an isomorphism (using the simple transitivity). The existence of γ and σ for N_C gives thus rise to the corresponding constructions for N and the asserted split exact sequence is established. It remains to construct the splitting for N_C.

Pick $g \in$ Graphaut(S). We claim that there exists uniquely $\mu: S \to \mathbb{R}_+$ such that defining $b \in GL(V)$ by $b(\alpha) = \mu(\alpha)\cdot g(\alpha)$, for any $\alpha \in S$, we have

$$b s_\alpha b^{-1} = s_{g(\alpha)}, \text{ for any } \alpha \in S \tag{1}$$

$$\text{and } \prod_{\beta \in S_i} \mu(\beta) = 1, \text{ for any } i \tag{2}$$

where $S = \coprod S_i$ is the splitting given by the irreducible components of Φ. Defering the proof, notice that, by (1), $b \in N$, and that in order to show $b \in N_C$ and $\gamma(b) = g$ it is harmless to assume (by Lemma 2.2) that b is an isometry, eventually changing $\mu(\alpha)$ to some other positive $\nu(\alpha)$. If $x \in C$, then, for any $\alpha \in S$, we have $0 < (x,\alpha) =$ $= (b(x),b(\alpha)) = \nu(\alpha)\cdot(b(x),g(\alpha))$, which shows that $b \in N_C$, and again by orthogonality $b(H_\alpha) = H_{g(\alpha)}$, any $\alpha \in S$, hence $\gamma(b) = g$. Put then $\sigma(g) = b$ and emphasize the dependence on g writing μ_g instead of μ. The fact that σ is a group morphism is equivalent to $\mu_{gg'} = \mu_{g'}\cdot(\mu_g \circ g')$, for any g,g' (3). Since an easy computation shows that conditions (1) are equivalent to

$$\mu(\beta)/\mu(\alpha) = \langle\beta,\alpha\rangle/\langle g(\beta),g(\alpha)\rangle, \text{ for any } \alpha,\beta \in S \text{ such that } \langle\beta,\alpha\rangle \neq 0 \quad (4)$$

(where $\langle,\rangle$ denote Cartan integers, as in [21]) one may use (4) and (2) for a rapid proof of (3). Observing further that conditions (4) are involving independently the various irreducible components and that it is enough to check them only when $|\alpha| \leq |\beta|$, it is clear how to prove the existence and the uniqueness of a solution μ of (1) with arbitrarily prescribed values at the "central nodes" of each component (specifically, we may choose the node 1 for each A_ℓ, C_ℓ and for G_2, the node ℓ for each B_ℓ, the node $\ell-2$ for each D_ℓ, the node 4 for each E_ℓ and for F_4, in the notations of [21], p.58). For each component Φ_i choose such a "central node" β_o and, for any other $\beta \in S_i$, by choosing a string of nodes joining β_o to β and iterating (4), deduce that $\mu(\beta) = \mu(\beta_o)\cdot c_\beta$, where the positive constant c_β is independent of μ; therefore $\prod_{\beta \in S_i} \mu(\beta) = \mu(\beta_o)^{|S_i|}\cdot c_i$, where c_i is positive and independent of μ. This helps to complete the proof.

2.6. Remark. Let us say that two Dynkin diagrams are <u>$\mathbb{Q}$-isomorphic</u> if the underlying graphs are isomorphic; among connected ones the only $\mathbb{Q}$-isomorphic but not isomorphic ones are B_ℓ and C_ℓ. Let us say that a diagram is <u>$\mathbb{Q}$-isotypic</u> <u>(isotypic)</u> if all its irreducible components are $\mathbb{Q}$-isomorphic (isomorphic); there are the obvious notions of decomposition into <u>$\mathbb{Q}$-isotypic components</u> and into <u>isotypic components</u>. A similar terminology applies to root systems, Lie algebras and Lie groups. These notions naturally arise in connection with graph (diagram) automorphism groups, which obviously split as direct products, according to the decomposition into $\mathbb{Q}$-isotypic (isotypic) components. As a byproduct of the above proof, if $G = \prod G_j$ is the $\mathbb{Q}$-isotypic decomposition, it follows that all the groups in the statement of 2.5 split as the direct product of the groups corresponding to each G_j, in a manner compatible with γ and σ.

2.7. Proof of Theorem 1.1 for $\mathbb{F} = \mathbb{Q}$. By extension of scalars the group of graded algebra automorphisms of $H^*(G/T;\mathbb{Q})$ is identified with the subgroup of graded algebra automorphisms of $H^*(G/T;\mathbb{R})$ which preserve the rational structure $H^2(G/T;\mathbb{Q}) \subset H^2(G/T;\mathbb{R})$. Using the inverse of the isomorphism established in Proposition 2.1 we may further identify the rational cohomology automorphisms with $\mathcal{A} \cap GL(\Gamma \otimes \mathbb{Q})$ which, by the result for $\mathbb{F} = \mathbb{R}$, is nothing else but the normalizer of W in $GL(\Gamma \otimes \mathbb{Q})$.

We shall denote in the sequel by $N_{\mathbb{Q}}$ this normalizer group; set also $D_{\mathbb{Q}}$ = group of automorphisms of V which act as rational positive scalars on each irreducible component V_i. As far as the group structure of $N_{\mathbb{Q}}$ is concerned, we have the following replica of Proposition 2.5:

2.8. Proposition. The split exact sequence of Proposition 2.5 restricts to an exact sequence

$$1 \to D_{\mathbb{Q}} \times W \to N_{\mathbb{Q}} \xrightarrow{\gamma} \mathrm{Graphaut}(S) \to 1$$

which splits as the direct product of the analogous exact sequences corresponding to the $\mathbb{Q}$-isotypic components of V.

Proof. We only have to check that the restriction of γ to $N_{\mathbb{Q}}$ is still onto. Actually it can be shown that for any $g \in \mathrm{Graphaut}(S)$ there exists $d \in D$ such that $\sigma(g)d \in GL(\Gamma \otimes \mathbb{Q})$, using the construction of σ given in the proof of Proposition 2.5 and keeping in mind that $S \subset \Gamma$ (see 2.10). Here is an alternative proof, based on [2], Proposition 2.13, which disposes of the case when G is simple. For a $\mathbb{Q}$-isotypic G it is immediate to see that we have a split exact sequence (which will be useful again later on)

$$1 \to \prod_{i=1}^{m} \mathrm{Graphaut}(S_i) \to \mathrm{Graphaut}(S) \overset{\pi_0}{\rightleftarrows} \Sigma_m \to 1. \qquad (1)$$

where $\Phi = \coprod_{i=1}^{m} \Phi_i$ is the decomposition into irreducible components and $\pi_0(g)$ represents the permutation of the connected components of the Coxeter graph induced by $g \in \mathrm{Graphaut}(S)$. It is equally immediate to see that $\pi_0 \circ \gamma : N_{\mathbb{Q}} \to \Sigma_m$ is onto, which implies the result for the $\mathbb{Q}$-isotypic case. The general case follows by Remark 2.6.

2.9. Remarks. Assume G is simple. In most cases, namely excepting $\Phi = B_2, F_4$ or G_2, we have equality between graph-automorphisms and diagram-automorphisms; in the exceptional cases there is only one automorphism of the diagram but there is one more exotic graph-automorphism, which turns the graph end for end. If $g \in$ Dgraut then g preserves Cartan integers hence, by (4) and (2) in the proof of Proposition 2.5, $\mu_g = 1$; since $S \subset \Gamma$ (2.10), it follows that the splitting σ of Proposition 2.5 also splits the exact sequence of Proposition 2.8, when Graphaut (S) contains only diagram automorphisms. However it is not difficult to see that the exact sequence does not split, for $G = B_2, F_4$ or G_2 (due to the presence of a square root factor in the expression of μ_g, g being the nontrivial graph-automorphism). In the general case, one can see that $N_{\mathbb{Q}}$ is generated by direct products of grading automorphisms and the automorphisms of the root system, eventually together with the exotic admissible isomorphisms of [2], corresponding to the graph isomorphisms between irreducible Dynkin diagrams which do not respect the length of the roots (for more details see the proof of Proposition 5.1).

The following simple lemma is very useful.

2.10. Lemma. Γ coincides with the free abelian group generated by $2\alpha/(\alpha,\alpha)$, $\alpha \in S$. In particular $S \subset \Gamma$.

Proof. The simple connectivity of G implies that Γ is generated by $2\alpha/(\alpha,\alpha)$, $\alpha \in \Phi$, see [1], p.129. On the other hand, it is standard that, for any $\alpha \in \Phi$, $2\alpha/(\alpha,\alpha)$ is a $\mathbb{Z}$-linear combination of $2\beta/(\beta,\beta)$, $\beta \in S$, see e.g. [20], p.27. With our choice of metric, it follows immediately that $S \subset \Gamma$.

2.11. Proof of Theorem 1.2. Since $H^*(G/T;\mathbb{Z})$ is torsion free [5], extension of scalars identifies Aut $H^*(G/T;\mathbb{Z})$ with a certain subgroup of Aut $H^*(G/T;\mathbb{R})$, which is plainly contained in the subgroup of those elements of Aut $H^*(G/T;\mathbb{R})$ which induce unimodular self-maps of $H^2(G/T;\mathbb{Z})$. Due to the 1-connectedness assumption on G this latter group is identified, via the inverse of the isomorphism described in Proposition 2.1, with $N \cap GL(\Gamma)$. Our first aim is to show that $N \cap GL(\Gamma) = Aut(\Phi)$.

The exact sequence of Proposition 2.5 easily describes Aut (Φ) as $W \cdot \sigma(Dgraut(S))$. Since all short roots have the same length, we infer from the previous lemma that Aut $(\Phi) \subset N \cap GL(\Gamma)$.

In order to prove the other inclusion, we have to start with an element $a \in N \cap GL(\Gamma)$, of the form $a = d \cdot \sigma(g)$, with $d \in D$ and $g \in Graphaut(S)$, and show that necessarily $g \in Dgraut(S)$. (Since $D \cap GL(\Gamma) = \{1\}$ it will follow that $a \in Aut(\Phi)$).

Writing $d = diag(d_\alpha)_{\alpha \in S}$, with $d_\alpha \in \mathbb{R}_+$, we know that $a(\alpha) = d_{g\alpha} \mu_g(\alpha) \cdot g(\alpha)$ for any $\alpha \in S$. By Lemma 2.10 the condition $a(\Gamma) \subset \Gamma$ simply means that for every $\alpha \in S$, $a(2\alpha/(\alpha,\alpha)) = n_\alpha \cdot (2g(\alpha)/(g(\alpha),g(\alpha)))$, for some $n_\alpha \in \mathbb{Z}$, which may be rewritten as

$$d_{g\alpha} \mu_g(\alpha) = [(\alpha,\alpha)/(g\alpha,g\alpha)] \cdot n_\alpha \quad (n_\alpha \in \mathbb{Z}_+), \text{ any } \alpha \in S \tag{1}$$

Adding the condition that $\det(a) = \pm 1$ and recalling that $\det(\sigma(g)) = \pm 1$, for any $g \in Graphaut(S)$ (by construction), and then multiplying the conditions (1), we find out that we must have $n_\alpha = 1$, for any $\alpha \in S$. Take any $\alpha, \beta \in S$ such that $\langle \alpha, \beta \rangle \neq 0$. Since g is a graph-automorphism and $d \in D$, we know that $d_{g\alpha} = d_{g\beta}$. Dividing the equality (1) corresponding to β by that corresponding to α and using the defining properties of μ_g (namely (4) in the proof of Proposition 2.5) we deduce that $\langle g\alpha, g\beta \rangle = \langle \alpha, \beta \rangle$, therefore $g \in Dgraut(S)$ and the proof of our first claim is completed.

On the other hand, it follows by classical Lie theory that for any $g \in Dgraut(S)$ there exists a group automorphism h of G which leaves T invariant and such that $\gamma(H_2(\bar{h}^*;\mathbb{R})) = g$, where $\bar{h}$ denotes the induced map on G/T (see [7], § 33, and also [2], p.14). Therefore all automorphisms of the root system are induced, by extension of scalars, by automorphisms of $H^*(G/T;\mathbb{Z})$, which concludes the proof of our theorem (and also shows that all integral cohomology automorphisms are induced by self-maps of G/T, by contrast with the case of classifying spaces).

3. Invariant Geodesics

As explained in the introduction, the results of K.Grove, S.Halperin and M.Vigué reduce the geometric problem to the computation of rational homotopy fixed points of rational homotopy equivalences of G/T. These in turn correspond bijectively [10] to the automorphisms of the cohomology, which may be identified as in the previous section with the normalizer of the Weyl group. We are going to associate a number F(a) to each $a \in N$ in such a way that $F(a) = \dim[\pi_*^{odd}(G/T) \otimes \mathbb{Q}]^h$ whenever $a = H_2(h^*;\mathbb{R})$ and h is an isometry of G/T, and then proceed to the effective computation of these numbers.

Denote by I the subalgebra of the invariants of the Weyl group in $\mathbb{R}[V]$ (which is a commutative graded algebra freely generated by r elements, r = dim V, see [6], p.107) and by I^+ the positive degree invariants. In the notations of section 2 (which will be used throughout this paper) the natural action on $\mathbb{R}[V]$ of any $a \in N$ induces a linear map, denoted by $\pi(a) : I^+/I^+ \cdot I^+ \rightarrow I^+/I^+ \cdot I^+$. Define then: $F(a) = \dim(I^+/I^+ \cdot I^+)^{\pi(a)}$.

3.1. Proposition. For any rational homotopy equivalence $h : G/T \rightarrow G/T$ we have $\dim[\pi_*^{odd}(G/T) \otimes \mathbb{Q}]^h = F(H_2(h^*;\mathbb{R}))$.

Proof. The proof uses rational homotopy theory ([24]). We may safely replace $\mathbb{Q}$ by $\mathbb{R}$ in the left hand side and then use minimal models over $\mathbb{R}$. Since G/T is formal ([24]) its minimal model coincides with the minimal model of its cohomology algebra considered with trivial differential. The minimal model $\rho : (\mathcal{M},d) \rightarrow (H^*(G/T,0)$ is constructed as follows. Pick homogenous elements $p_1,\dots,p_r \in I^+$ which freely generate the graded algebra I. Construct $\mathcal{M}$ as a free commutative graded algebra by setting $\mathcal{M} = VZ_o \otimes \Lambda Z_1$ (where V and Λ indicate symmetric, respective exterior algebras) with $Z_o = V^*$ and deg(z) = 2, for any $z \in Z_o$, and $Z_1 = \mathbb{R}\text{-span}\{y_1, \dots, y_r\}$ with $\deg(y_i) = \deg(p_i) - 1$ (where $\deg(p_i)$ is considered by identifying $\mathbb{R}[V]$ and VZ_o as graded algebras). Set dz = 0 for $z \in Z_o$ and $dy_i = p_i$ for any i. Since there are no nonzero invariants of W in V^* (remember that G is still assumed to be 1-connected), it follows that $(\mathcal{M}, d)$ is indeed minimal. Define $\varrho z = z$ for $z \in Z_o$ and $\varrho y_i = 0$ for any i. Since $(p_1, \dots, p_r)$ is a regular sequence in $\mathbb{R}[V]$ (see [6], p.115), it follows from [24] that ϱ is a minimal model map. Given $h^*: H^*(G/T) \to H^*(G/T)$, it is easy to construct a differential graded algebra map $\hat{h}: (\mathcal{M}, d) \to (\mathcal{M}, d)$ such that $\varrho\hat{h} = h^*\varrho$. By [10] $\hat{h}$ will represent a minimal model of h, therefore, by rational homotopy theory, the dual of the action of h on $\pi_*^{odd}(G/T) \otimes \mathbb{R}$ is identified with the action of $\hat{h}$ on the indecomposables of odd degree of $\mathcal{M}$, i.e. with the linear part of the restriction $\hat{h}|Z_1$. Notice that $\hat{h}(VZ_o) \subset VZ_o$ and that the restriction $\hat{h}|VZ_o$ coincides with the action on $\mathbb{R}[V]$ of $a = H_2(h^*; \mathbb{R})$, by construction. Denoting by J the ideal of $\mathbb{R}[V]$ generated by I^+, i.e. $J = (VZ_o) \cdot dZ_1$ we infer that $\hat{h}(J) \subset J$. To be more precise, decompose $\hat{h}|Z_1$ as follows: $\hat{h}|Z_1 = h_1 + h_2$, where $h_1 : Z_1 \to (VZ_o) \cdot Z_1$ and $h_2 : Z_1 \to (VZ_o \otimes \Lambda Z_1) \cdot Z_1 \wedge Z_1$, write the commutation condition with d and find out that $\hat{h}d|Z_1 = dh_1$. Writing further $h_1 = \pi^{odd}(\hat{h}) + h_1'$, where $\pi^{odd}(\hat{h}) : Z_1 \to Z_1$ and $h_1' : Z_1 \to (V^+ Z_o) \cdot Z_1$ deduce that $\hat{h}d|Z_1 = d\pi^{odd}(\hat{h})$, in $J/(V^+Z_o) \cdot J$. All these considerations together imply that the action of h on $\pi_*^{odd}(G/T) \otimes \mathbb{R}$ may be identified with the action of a in $J/(V^+Z_o) \cdot J$. Consider now the natural surjection: $I^+/I^+ \cdot I^+ \to J/(V^+Z_o) \cdot J$, which is an isomorphism, due to the fact that $\mathbb{R}[V]$ is a free graded module over I ([6], p.105), and conclude the proof.

Since in the geometric applications h will be a self-homotopy equivalence and since we know in that case, by Theorem 1.2, that $H_2(h^*; \mathbb{R})$ must lie in $W \cdot \sigma(\text{Dgraut}(S))$ and since obviously F(wa) = F(a), for any $w \in W$ and $a \in N$, we could just compute $F(\sigma(g))$, $g \in Dgraut(S)$, checking case by case, in order to give the proof of Theorem 1.4. However, we shall not pursue this way, but we choose to use the following general result, which provides both a very convenient new description of $F(\sigma(g))$, $g \in Dgraut(S)$, and a useful information related to Lefschetz numbers (see the next section).

3.2. Lemma. If $a \in N$ has finite order and leaves some Weyl chamber invariant, then $F(a) = \dim V^a$ and this number is positive.

Proof. Assume $a(C) = C$, for some Weyl chamber C, and let $ord(a) = m < \infty$. We first show that a has a fixed point in C. Indeed, starting with any $y \in C$, set $x = \sum_{i=1}^{m} a^i(y)$; then $x \in C$ and $a(x) = x$. In particular this proves the second assertion.

For any linear map b, denote by $m_1(b)$ the multiplicity of the eigenvalue 1. In order to compute $F(a) = m_1(\pi(a))$ choose homogenous elements $p_1, \dots, p_r \in I^+ (r = \dim V)$ which freely generate the algebra I and, writing that $p_i^a \in I$, for any i, deduce the existence of a polynomial function $A : \mathbb{R}^r \to \mathbb{R}^r$ with the property that $p \circ a = A \circ p$ $(*)$, where $p : V \to \mathbb{R}^r$ has p_i as its i-th component; by the very definition of $\pi(a)$, it has the same characteristic polynomial as the linear part of A, hence $F(a) = m_1(D_o A)$. By taking derivatives in $(*)$, in a point $x \in C$ which is fixed by a, and recalling that the jacobian of p is nonsingular in all points of C (see [6], p.113), we infer that $m_1(a) = m_1(D_{p(x)}A)$, which of course also equals $\dim V^a$. We know in fact that, for any $t \in \mathbb{R}_+$, $D_{p(tx)}A$ has the same characteristic polynomial as a (using tx in place of x), hence, letting t go to zero, we conclude that a and $D_o A$ have the same characteristic polynomial, and this finishes the proof.

3.3. Proof of Theorem 1.4. If $f \in Isom(G/T)$ then, by Theorem 1.2, $H_2(f^*; \mathbb{R}) = w \cdot \sigma(g)$, with $w \in W$ and $g \in Dgraut(S)$, and, by Proposition 3.1, $\dim[\pi_*^{odd}(G/T) \otimes \mathbb{Q}]^f = F(\sigma(g))$. Lemma 3.2 applies then to $\sigma(g)$ (see the construction of σ in Proposition 2.5) and clarifies the first assertion of the

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theorem. Moreover, recalling that $\mu_g = 1$, for any $g \in \mathrm{Dgraut}(S)$, the same lemma gives that $F(\sigma(g))$ equals the number of cycles of g, considered as a permutation of S. The fact that any diagram automorphism respects the isotypic components (see Remark 2.6) implies that $\dim[\pi_*^{odd}(G/T) \otimes \mathbb{Q}]^f > 1$ unless Φ is isotypic, say $\Phi = \Phi_1 \amalg \ldots \amalg \Phi_\ell$ with $\Phi_1 = \ldots = \Phi_\ell$ = irreducible. For such an isotypic root system, the split exact sequence (1) constructed in the proof of Proposition 2.8 restricts to a similar split exact sequence, in which Dgraut replaces Graphaut. We thus see that $\dim[\pi_*^{odd}(G/T) \otimes \mathbb{Q}]^f > 1$ unless $\pi_0(g)$ is a cycle, say $(1\ 2 \ldots \ell)$, and in this case g acts on S_i as $g_i : S_i \to S_{i+1}$ where g_i is a diagram isomorphism, for any i. Since it is clear that g acts as a cycle on S if and only if $g_\ell g_{\ell-1} \cdots g_1$ acts as a cycle on S_1 and since the only cyclic diagram automorphisms of the connected Dynkin diagrams are the identity of the type A_1 and the nontrivial diagram automorphism of the type A_2, the second assertion of our theorem follows (remember that the order of taking products in Dgraut (A_2) is irrelevant!).

3.4. Remarks. It is shown in [14] that the finiteness assumption on the number of f-invariant geodesics imposes a stronger restriction on f than the one we quoted in the Introduction, namely

$$\dim[\pi_*^{even}(M) \otimes \mathbb{Q}]^f \leq \dim[\pi_*^{odd}(M) \otimes \mathbb{Q}]^f \leq 1$$

However, in our case the first inequality always holds.

This can be seen as follows: for any $f \in \mathrm{Isom}(M)$, $\pi_*(f) \otimes \mathbb{Q}$ has finite order, by Corollary 1.3.

The desired inequality may thus be rewritten as

$$m_1(\pi_{even}(f) \otimes \mathbb{Q}) \leq m_1(\pi_{odd}(f) \otimes \mathbb{Q})$$

which is a direct consequence of a result of S.Halperin, see [15], Theorem 3 (for a detailed statement of this result, see the proof of Theorem 4.1; the key fact here is that $\dim \pi_*(G/T) \otimes \mathbb{Q} < \infty$, see the construction of the minimal model of G/T given in the proof of Proposition 3.1).

We finally mention the existence of examples of isometries f having only finitely many invariant geodesics and with $H_2(f^*;\mathbb{R})$ as in the statement of our theorem (see also [13]).

4. Free Transformation Groups

The result stated in Theorem 1.5 was obtained in the special case $G = U(n)$ in [18] as a corollary of a slightly more general assertion about noncoincidence indices. The proof of Theorem 1.5 to be given below, albeit independent of this kind of generalization, also works and produces information about the noncoincidence index, with only minor changes.

We shall therefore reformulate the statement of Theorem 1.5, using the terminology of [18]. Define the noncoincidence index of M, n(M), as the maximum k for which there exist k-1 fixed point free self-maps of M, no two of which having a coincidence, see [18]; also define the free symmetry index of M, f(M), as the maximum of the cardinalities of the groups which can act freely on M. We always have $f(M) \leq n(M)$, and, for $M = G/T$, $f(m) \geq$ order of W. We shall prove Theorem 1.5 in the following strengthened form:

4.1. Theorem. Set $G/T = M$.

(i) $f(M) =$ order of W.

(ii) $n(M) = \infty$ unless G is simple, and in this case $n(M) = f(M)$.

Proof. If $G = G_1 \times G_2$ then $M = M_1 \times M_2$ and it is immediate to produce arbitrarily large families of fixed point free self-maps of M without coincidences. From now on we shall treat (i) and (ii) simultaneously, and assume that G is simple in (ii).

We have to show that, given a family $\mathcal{S}$ consisting of k fixed point free homeomorphisms (resp. self-maps) without coincidences, we must have $k <$ order of W. Notice first that the family $\mathcal{S}^* = \{H^*(\varphi;\mathbb{R}) \mid \varphi \in \mathcal{S}\}$ must consist only of automorphisms of $H^*(G/T;\mathbb{R})$. This follows from the fact that $H^*(\varphi;\mathbb{R})$ is induced by the action on $\mathbb{R}[V]$ of some $a \in \mathcal{E}$, see Proposition 2.1; if a is not a linear isomorphism then the irreducibility of the Weyl group action on V implies, via Lemma 2.3, that $H^*(\varphi;\mathbb{R})$ is trivial, which contradicts the fact that φ has no fixed points, by the Lefschetz fixed point theorem. Observe next that $\mathcal{S}^*$ has the same cardinality as $\mathcal{S}$, by the Lefschetz coincidence theorem [26] (see also [18], Theorem 4.1 and Proposition 4.2). Considering the family $\mathcal{F} = \{H_2(\varphi^*;\mathbb{R}) \mid \varphi \in \mathcal{S}\}$, consisting of k distinct elements, we know that $L(a) = 0$ for any $a \in \mathcal{F}$ and $L(a,b) = 0$ for any $a,b \in \mathcal{F}$, $a \neq b$ (by the Lefschetz theorems), and $\mathcal{F} \subset W \cdot \sigma(\text{Graphaut})$ (see the proof of Theorem 1.2), respectively $\mathcal{F} \subset N_{GL(V)}(W)$ (by the previous remarks); we have denoted here by $L(a,b)$, for $a,b \in \mathcal{E}$, the Lefschetz coincidence number (see [26], [18]) of the endomorphisms of $H^*(G/T;\mathbb{R})$ which correspond to a and b by Proposition 2.1, and $L(a) = L(a,\text{id})$, for any $a \in \mathcal{E}$, as usual. The key step of the proof is contained in the following:

Claim. Write, according to Proposition 2.5, $a = w \cdot \sigma(g)$, with $w \in W$ and $g \in \text{Graphaut}$ (respectively $a = \lambda w \cdot \sigma(g)$, with $\lambda \in \mathbb{R}_+$, $w \in W$ and $g \in \text{Graphaut}$). If $L(a) = 0$ then $w \neq \text{id}$ (resp. $\lambda = 1$ and $w \neq \text{id}$).

Granting the claim, we are going to finish quickly the proof of the theorem. Since we know that $\mathcal{F} \subset (W \setminus \{\text{id}\}) \cdot \sigma(\text{Graphaut})$, we may write $\mathcal{F} = \coprod_g W_g \cdot \sigma(g)$, where

$$W_g = \{ w \in W,\ w \neq \text{id} \mid w \cdot \sigma(g) \in \mathcal{F} \},\ \text{for any } g \in \text{Graphaut}$$

If $w \in W_g \cap W_h$ then $L(w\sigma(g), w\sigma(h)) = \pm L(\sigma(gh^{-1}))$, see e.g. [18], Proposition 4.2, and, since this number is nonzero by the previous claim, our hypotheses imply that $g = h$. This shows that $k <$ order of W.

The proof of the claim uses rational homotopy theory. We recall the following result, due to S.Halperin ([15], Theorem 3), which relates Lefschetz numbers to rational homotopy fixed points: let X be a 1-connected rational space with the property that $\dim H_*(X) < \infty$ and $\dim \pi_*(X) < \infty$, and let $\varphi : X \to X$ be any map; then $m_1(\pi_{odd}(\varphi)) \geq m_1(\pi_{even}(\varphi))$ (compare with Remark 3.4), where

m_1 denotes the multiplicity of the eigenvalue 1, and equality holds if and only if $L(\varphi^*) \neq 0$. Setting $X = (G/T)_o$, we may apply this result, taking φ to be the formal map which induces the cohomology automorphism corresponding to a (remember that G/T is a formal space by [24]). Denoting by $(\mathcal{M},d)$ the minimal model of X (see the proof of Proposition 3.1) and by $\hat{\varphi}$ the minimal model of φ, we may compute m_1 using the induced map on de Rham homotopy, denoted by $\pi^*(\hat{\varphi})$. Recall from the proof of Proposition 3.1 that $\pi^{even}(\mathcal{M}) = V^*$ and $\pi^{odd}(\mathcal{M}) = I^+/I^+ \cdot I^+$, and that $\pi^{even}(\hat{\varphi}) = a^*$ and $\pi^{odd}(\hat{\varphi}) = \pi(a)$. If $\lambda \neq 1$, it is easy to see that $m_1(a^*) = 0$ and $m_1(\pi(a)) = 0$, due to the fact that $ord(w\sigma(g)) < \infty$, which implies in turn that $L(a) \neq 0$. It remains to show that $L(\sigma(g)) \neq 0$, for any $g \in$ Graphaut. By the previous remarks, this is equivalent to $F(\sigma(g)) = \dim V^{\sigma(g)}$, in the notations of § 3, and it is a consequence of Lemma 3.2, which is available since $\sigma(g)$ leaves some Weyl chamber invariant by construction (see the proof of Proposition 2.5). The proof of Theorem 4.1 is now complete.

5. Generic Rigidity

Our first task will be to clarify the assertions made in the Introduction in connection with the rigidity of $E_o(M)$ and its relationship with the amount of localization needed to construct generators for $E_o(M)$.

Given a set of primes P, we shall denote by $E_P(M)$ the subgroup of $E_o(M)$ consisting of rationalizations of self-homotopy equivalences of M_P, the localization of M at P; the same construction, applied to the complementary set of primes, will be denoted by $E_{1/P}(M)$; for notational convenience, $E_1(M)$ will stand for the rationalizations of self-homotopy equivalences of M. The isomorphism established in Section 2, given by $f \to H_2(f^*;\mathbb{Q})$, will serve to identify $E_o(M)$ and $N_{\mathbb{Q}}$, when $M = G/T$. In the notations of Proposition 2.8, which describes the group structure of $N_{\mathbb{Q}}$, we obviously have $W \subset E_1$ and, as far as the products of grading automorphisms are concerned, we know by [9] that, for any simple G, if $d \in D_{\mathbb{Q}}$ is a prime, then $d \in E_{1/d}$ and it is induced by the corresponding Frobenius isogeny. The generators of

$N_{\mathbb{Q}}$ corresponding to graph-automorphisms are settled in the next proposition (plainly it is enough to consider the $\mathbb{Q}$-isotypic case).

5.1. Proposition. If G is $\mathbb{Q}$-isotypic then $\gamma | E_{1/q}$ is still onto, where q denotes the maximum number of bonds appearing in the Dynkin diagram.

Proof. The group structure of Graphaut (S) is described by the exact sequence (1) which was derived in the proof of Proposition 2.8. If Φ is not isotypic then all its irreducible components are of type B_r or C_r, for some $r \geq 3$, and have no nontrivial graph-automorphisms, and our claim amounts to the existence of a homotopy equivalence between $(B_r/T)_{\frac{1}{2}}$ and $(C_r/T)_{\frac{1}{2}}$. This is provided by [9] and comes from the exceptional isogeny in characteristic 2 relating the orthogonal and symplectic groups. Since for an isotypic G the composition $\pi_0 \circ \gamma | E_1$ is plainly onto, we are reduced to the case when G is simple. Since moreover Dgraut $\subset \gamma(E_1)$ by characteristic zero Lie theory (see the proof of Theorem 1.2), we are finally left with three cases: $\Phi = B_2, F_4$ or G_2, each one with a single graph-automorphism which does not respect the lengths. The first one follows by using the already mentioned homotopy equivalence $(SO(5)/T)_{\frac{1}{2}} \xrightarrow{\sim} (Sp(2)/T)_{\frac{1}{2}}$ and the other ones by recalling the self-homotopy equivalences of $(G/T)_{1/q}$ constructed in [8] for $G = F_4$ and G_2 from the exceptional isogenies in characteristic 2 (respectively 3) of these groups (see also [2], Proposition 2.15).

5.2. Corollary. For any $G, M = G/T$ has the following property:

(*) given a set of primes P, for any $f \in E_0(M)$ there exist $f_1 \in E_P(M)$ and $f_2 \in E_{1/P}(M)$ such that $f = f_1 f_2$.

Proof. Reduce to the $\mathbb{Q}$-isotypic case, recall the structure of $E_0(M) = N_{\mathbb{Q}}$ and use Proposition 5.1 and the remarks preceding it.

Given the corollary, the theorem below readily implies Theorem 1.6.

5.3. Theorem. Let M be a 1-connected finite formal complex. If M has the property (*) stated in Corollary 5.2, then M is generically rigid.

Proof. If $M = G/K$ is a complex flag manifold, the generic rigidity of M was derived in [11] from the assumption that $E_o(M)$ is generated by grading automorphisms together with $N_G(K)/K$; using [9] it is easy to see that this assumption implies the property (*). Our contribution consists in observing that the arguments of [11] still work for an arbitrary 1-connected finite formal complex, provided the property (*) holds. The generic rigidity of M follows immediately if M satisfies the hypotheses of Lemma 1.3 [11] and the conclusion of Lemma 2.2 [11]. The argument showing that Lemma 1.3 is available for our M is the same as in [11]. Writing $P = \{p_1, \ldots, p_n\}$, an easy induction which uses property (*) shows that for any $f_1, \ldots, f_n \in E_o(M)$ there exists $f \in E_{1/P}(M)$ such that $ff_i \in E_{p_i}(M)$, for any i. In particular the conclusion of Lemma 2.2 [11] holds for M.

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