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by

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GENERALIZATIONS OF CONVEX SUPREMIZATION DUALITY

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Abstract. We give some extensions of our duality theorems of [13]-[15], to the optimization problems (P) $\alpha=\sup h(G)$ and (P) $\alpha=\sup(h-f)(F)$, where G is a subset of a set F, $h:F\to\overline{R}=[-\infty,+\infty]$ is a W-quasi-convex ([5], [21]) or a W-convex [4] functional, W being a subset of $\overline{R}^F$, and $f:F\to\overline{R}$ is arbitrary.

§1. Introduction

Given a set F, a subset G of F (assumed to be non-empty, throughout the sequel) and a functional $h:F\to\overline{R}=[-\infty,+\infty]$, we shall consider the following (global, scalar) primal supremization problem:

$$(P)=(P_{G,h}) \qquad \alpha=\alpha_{G,h}=\sup h(G). \tag{1.1}$$

In the paper [13] (see also [6]) we have proved some theorems of "unperturbational surrogate duality" type (in a sense similar to [20], [22]) for the particular case of problem (1.1), in which F is a locally convex space, G is a bounded subset of F and h is convex and lower semi-continuous, with values in $R\cup\{-\infty\}$ (where $R=(-\infty, +\infty)$). Furthermore, in [15] we have proved some theorems of "unperturbational Lagrangian duality" type (in a sense similar to [22]), for (1.1) with F a locally convex space, G a bounded subset of F and $h:F\to\overline{R}$ a proper lower semi-continuous convex functional. In [14] we have extended the main duality theorem of [15] to a duality theorem for the problem of supremization of the difference h_1-h_2 on a locally convex space F, where $h_1:F\to\overline{R}$ is a proper lower semi-continuous convex functional on F and $h_2:F\to\overline{R}$ is arbitrary, with the convention $+\infty-(+\infty)=-\infty$ (thus, taking h_2=the indicator functional of a bounded subset G of F, i.e., $h_2(y)=0$ for $y\in G$ and $h_2(y)=+\infty$ for $y\in F\setminus G$, we obtain the case of [15]). The result of [14] has been also obtained, independently, in an equivalent form (namely, as a duality theorem for the infimization problem $\inf(h_1-h_2)(F)$, where $h_1:F\to\overline{R}$ is arbitrary and $h_2:F\to\overline{R}$ is proper lower semi-continuous, with $+\infty-(+\infty)=+\infty$), and from a different starting point (namely, some non-linear problems in the calculus of variations, which arise in mechanics, such as the analysis of a steadily rotating heavy chain) by Toland [25]; moreover, Toland has developed, in [24], a theory of "perturbational Lagrangian duality" type (in a sense similar to [22]), which contains, as a special case, the duality theory of [25]. The importance of problem (1.1) with $F=R^n$, G a closed convex (possibly unbounded) subset of R^n, and $h:F \longrightarrow R$ a finite convex functional, has been stressed by Tuy [26], who has shown that it includes a wide class of mathematical programming problems (such as linear and convex programming, 0-1 integer programming, bilinear programming, linear and convex complementarity problems, and "convex-difference" programming).

Motivated by the above mentioned results of [13], [15] (see e.g. corollary 3.2 and remark 5.3 below), we introduce here the following concept of "dual problem" to (P) of (1.1) (without any assumptions on F, G, h):

Definition 1.1. By a dual problem to (P) we shall mean any supremization problem of the form

$$(Q)=(Q^{G,h}) \qquad \beta=\beta^{G,h}=\sup \lambda(W), \tag{1.2}$$

where $W=W^{G,h}$ is a set (assumed non-empty, without loss of generality) and $\lambda=\lambda^{G,h}:W \longrightarrow \overline{R}$ is a functional.

Remark 1.1. a) We assume no relation between α and β.

b) There is a marked difference between the above dual problems (1.2) and the "usual" dual problems [22] to (P) (extending the usual dual problems for concave supremization, i.e., for (P) of (1.1) with F a linear space, h concave and G convex), in which $\beta=\inf \lambda(W)$, or, equivalently (see e.g. [8]), $\beta=-\sup \lambda(W)$. Therefore, as in [23], we shall call the dual problems (1.2) "unusual" dual problems to (P).

We shall first consider "unperturbational surrogate dual problems" to (P), in a sense similar to [22] (see also [20]), namely, the case when λ of (1.2) is of the form

$$\lambda(w)=\lambda_{W\Delta}^{G,h}(w)=\inf h(\Delta_{G,w}) \qquad (w\in W), \tag{1.3}$$

where $\Delta_{G,w}\subseteq F$ $(w\in W)$ is a given family of ("surrogate constraint") sets; thus, by (1.2) and (1.3), we have

$$\beta=\sup_{w\in W} \inf h(\Delta_{G,w}). \tag{1.4}$$

Remark 1.2. If we interchange everywhere sup and inf, then (P) and (Q) become infimization problems and β of (1.4) will be replaced by

$$\beta'=\inf_{w\in W}\sup h(\Delta_{G,w}). \tag{1.5}$$

There is a marked difference between (1.5), the values $\beta'=\sup_{w\in W}\inf h(\Delta_{G,w})$ of the "usual" surrogate dual problems to

$$(P') \quad \alpha'=\inf h(G), \tag{1.6}$$

and the values $\beta'=\inf_{w\in W}\inf h(\Delta_{G,w})$ of the "unusual" surrogate dual problems to (P′) of (1.6), studied in [23] (see also [16], [7]); in [23], the latter ones and the dual problems with β' of (1.5) have been called "unusual surrogate dual problems of the first type" and "unusual surrogate dual problems of the second type", respectively.

In §2 we shall give some necessary and sufficient conditions for $\alpha\geq\beta$, for $\alpha\leq\beta$ and for $\alpha=\beta$, with $\alpha\in\overline{R}$ arbitrary and β of (1.4), and some simultaneous characterizations of "solutions" of (P) (of (1.1)) and of "weak duality" for (P), (Q) (i.e., conditions in order to have $\alpha=\beta$, with α,β of (1.1), (1.4)), involving the level sets

$$A_c(h)=\{y\in F \mid h(y)<c\} \qquad (c\in R), \tag{1.7}$$

$$S_c(h)=\{y\in F \mid h(y)\leq c\} \qquad (c\in R) \tag{1.8}$$

of h; we recall that, by definition, the "solutions" of (P) (of (1.1)) are the elements of the (possibly empty) set

$$\mathcal{M}_G(h)=\{g_o\in G \mid h(g_o)=\sup h(G)\}. \tag{1.9}$$

In §3 we shall apply the results of §2 to $\alpha=\sup h(G)$ and to certain families of surrogate constraint sets $\Delta^i_{G,w}\subseteq F$ ($w\in W$, $1=1,\ldots,6$), where $W\subseteq\overline{R}^F$ (we recall that $\overline{R}^F$ denotes the family of all functionals $w:F\longrightarrow\overline{R}$); in the particular case when F is a locally convex space and $W\subseteq F^*$ (where F^* denotes the conjugate space of F), these sets $\Delta^i_{G,w}$ admit convenient geometric interpretations.

In §4 we shall show how the results of surrogate duality of §3 can be applied to problem (P) of (1.1) with $G=u^{-1}(\Omega)$, where u is a mapping of F into a "parameter set" X and Ω is a subset of X, with $u(F)\cap\Omega\neq\emptyset$ (where $\emptyset$ denotes the empty set); in this case we shall take $W\subseteq R^X$ (rather than $W\subseteq\overline{R}^F$) and we shall define surrogate constraint sets $\Delta^i_{u^{-1}(\Omega),w}\subseteq F$ ($w\in W$) corresponding to those of §3.

Finally, in §5, considering the "Lagrangian dual problem" to (P) of (1.1), i.e., problem (1.2), with $W\subseteq\overline{R}^F$ and λ of the form

$$\lambda(w)=\lambda_W^{G,h}(w)=\sup w(G)\dot{+}\inf_{y\in F}\{h(y)\dot{+}-w(y)\} \qquad (w\in W) \tag{1.10}$$

(for $\underset{\cdot}{+}$, $\dot{+}$, see (5.2) and (5.3)), we shall show that the main result of [15] can be extended to W-convex [4] functionals h on a set F, where $W\subseteq\overline{R}^F$, and to arbitrary subsets G of F. The usefulness of such an extension consists in the possibility of applying it to various choices of W's, which permits a unified treatment of "augmented Lagrangians" (for the corresponding theory for problem (P') of (1.6), see e.g. [4]). Also, we shall extend the main result of [14] to the W-convex case, where $W\subseteq\overline{R}^F$.

Throughout the paper, we adopt the usual conventions

$$\inf \emptyset=+\infty , \quad \sup \emptyset=-\infty . \tag{1.11}$$

Also, as in [17]-[19], we make the convention that if $A_c(h)=\emptyset$ or $S_c(h)=\emptyset$ for some $c\in R$, then the conditions involving these $A_c(h)$, $S_c(h)$ (see e.g. (2.1), (2.2), etc.) will be considered satisfied (vacuously). By "linear space" (with or without a topology) we shall mean: real linear space.

§2. Surrogate duality results in the general case

Let us first recall

Lemma 2.1 ([19], proposition 1.1 and corollary 1.1). *Let* F *be a set*, $\Delta\subseteq F$, $h:F\longrightarrow\overline{R}$ *and* $c\in R$.

a) *We have* $\inf h(\Delta)\geqslant c$ *if and only if* $\Delta\cap A_c(h)=\emptyset$.

b) *If* $\inf h(\Delta)>c$, *then* $\Delta\cap S_c(h)=\emptyset$.

Proof [19]. If $y_o\in\Delta\cap A_c(h)$, then $\inf h(\Delta)\leqslant h(y_o)<c$. The proof of b) is similar. Finally, if $\inf h(\Delta)<c$, then there exists $y_o\in\Delta$ such that $h(y_o)<c$, so $y_o\in\Delta\cap A_c(h)$.

Proposition 2.1. Let F, G, h, W *and* $\Delta_{g,w}\subseteq F$ $(w\in W)$ *be as in* §1, *and let* $\alpha\in\overline{R}$ *be arbitrary. The following statements are equivalent:*

1^o. *We have*

$$\Delta_{G,w}\cap A_c(h)\neq\emptyset \qquad (w\in W,\ c\in R,\ c>\alpha). \tag{2.1}$$

2^o. *We have*

$$\Delta_{G,w}\cap S_c(h)\neq\emptyset \qquad (w\in W,\ c\in R,\ c>\alpha). \tag{2.2}$$

3^o. *We have*

$$\alpha\geqslant\beta=\sup_{w\in W}\inf h(\Delta_{G,w}). \tag{2.3}$$

Proof. The implication $1^\circ \Rightarrow 2^\circ$ is obvious.

$2^\circ \Rightarrow 3^\circ$. If 2° holds, say $y_{w,c} \in \Delta_{G,w} \cap S_c(h)$, then

$$\lambda(w) = \inf h(\Delta_{G,w}) \leq h(y_{w,c}) \leq c \qquad (w \in W,\ c \in R,\ c > \alpha),$$

whence $\beta = \sup \lambda(W) \leq \inf_{c>\alpha} c = \alpha$.

$3^\circ \Rightarrow 1^\circ$. If 3° holds, then for each $c \in R$, $c > \alpha$, we have $c > \beta = \sup_{w \in W} \inf h(\Delta_{G,w})$, whence, by lemma 2.1 a), we obtain (2.1).

Remark 2.1. In particular, if $\alpha = \inf h(G)$ and $G \subseteq \Delta_{G,w}$ $(w \in W)$, then $\emptyset \neq G \cap A_c(h) \subseteq \Delta_{G,w} \cap A_c(h)$ $(w \in W,\ c \in R,\ c > \alpha)$, so we have 1°-3°. Hence, proposition 2.1 permits an improvement of the results of [19].

Proposition 2.2. *Let $\alpha \in \bar{R}$ be arbitrary. The following statements are equivalent:*

1°. *For each $c \in R$, $c < \alpha$, there exists $w_c \in W$ such that*

$$\Delta_{G,w_c} \cap A_c(h) = \emptyset. \tag{2.4}$$

2°. *For each $c \in R$, $c < \alpha$, there exists $w_c \in W$ such that*

$$\Delta_{G,w_c} \cap S_c(h) = \emptyset. \tag{2.5}$$

3°. *We have*

$$\alpha \leq \beta = \sup_{w \in W} \inf h(\Delta_{G,w}). \tag{2.6}$$

Proof. $1^\circ \Rightarrow 3^\circ$. If c and w_c are as in 1°, then, by lemma 2.1 a), we have

$$\lambda(w_c) = \inf h(\Delta_{G,w_c}) \geq c,$$

whence $\beta = \sup \lambda(W) \geq \sup_{c<\alpha} \lambda(w_c) \geq \sup_{c<\alpha} c = \alpha$.

$3^\circ \Rightarrow 2^\circ$. If 3° holds and $c \in R$, $c < \alpha$, then $c < \beta$, and hence, by (1.4), there exists $w_c \in W$ such that $c < \inf h(\Delta_{G,w_c})$. Then, by lemma 2.1 b), we have (2.5).

Finally, the implication $2^\circ \Rightarrow 1^\circ$ is obvious.

Remark 2.2. For the particular case when $\alpha = \inf h(G)$ and $G \subseteq \Delta_{G,w}$ $(w \in W)$, whence $\alpha \geq \beta$, the above argument has been given, essentially, in [19], proof of theorem 1.1.

Combining propositions 2.1 and 2.2, we obtain

Theorem 2.1. *Let $\alpha \in \bar{R}$ be arbitrary. The following statements are equivalent:*

1^o. *We have* (2.1), *and for each* $c \in R$, $c < \alpha$, *there exists* $w_c \in W$ *satisfying* (2.4).

2^o. *We have* (2.2), *and for each* $c \in R$, $c < \alpha$, *there exists* $w_c \in W$ *satisfying* (2.5).

3^o. *We have*

$$\alpha = \sup_{w \in W} \inf h(\Delta_{G,w}) . \tag{2.7}$$

Concerning simultaneous characterizations of solutions of (P) and of weak duality for $\{(P),(Q)\}$ of (1.1), (1.4), let us prove

Theorem 2.2. *For an element* $g_o \in G$, *and for* $\alpha = \sup h(G)$, *the following statements are equivalent:*

1^o. *We have*

$$\Delta_{G,w} \cap A_c(h) \neq \emptyset \qquad (w \in W,\ c \in R,\ c > h(g_o)), \tag{2.8}$$

and for each $c \in R$, $c < \alpha$, *there exists* $w_c \in W$ *satisfying* (2.4).

2^o. *We have*

$$\Delta_{G,w} \cap S_c(h) \neq \emptyset \qquad (w \in W,\ c \in R,\ c > h(g_o)), \tag{2.9}$$

and for each $c \in R$, $c < \alpha$, *there exists* $w_c \in W$ *satisfying* (2.5).

3^o. *We have* $g_o \in \mathcal{M}_G(h)$ *and* (2.7).

Proof. $1^o \Longrightarrow 3^o$. Assume 1^o. Then, by (2.8) and proposition 2.1 (with $\alpha = h(g_o)$), we have $h(g_o) \geq \beta$. Furthermore, by the second condition of 1^o and by proposition 2.2, we have (2.6). Hence, by $g_o \in G$, we obtain

$$\beta \geq \alpha = \sup h(G) \geq h(g_o) \geq \beta . \tag{2.10}$$

$3^o \Longrightarrow 1^o$. If 3^o holds, then $h(g_o) = \sup h(G) = \alpha$, and hence, by theorem 2.1, we have 1^o.

Finally, the proof of the equivalence $2^o \Longleftrightarrow 3^o$ is similar.

Remark 2.3. Similarly, one can prove the following result for infimization, which extends [19], theorem 1.4 (and hence also the particular cases of [19], theorem 1.4, given in [18]): *For an element* $g_o \in G$ *and for* $\alpha = \inf h(G)$, *the following statements are equivalent:*

1^o. *We have* (2.1), *and for each* $c \in R$, $c < h(g_o)$, *there exists* $w_c \in W$ *satisfying* (2.4).

2^o. *We have* (2.2), *and for each* $c \in R$, $c < h(g_o)$, *there exists* $w_c \in W$ *satisfying* (2.5).

3^o. *We have*

$$h(g_o)=\inf h(G)=\sup_{w\in W} \inf h(\Delta_{G,w}). \qquad (2.11)$$

Indeed, in the proof, the inequalities (2.10) are now replaced by

$$\beta \leq \alpha = \inf h(G) \leq h(g_o) \leq \beta. \qquad (2.12)$$

§3. Applications to surrogate duality for supremization

In this section we shall assume that F is a set and $W \subseteq \overline{R}^F$. Also, as before, let $G \subset F$ and $h: F \longrightarrow \overline{R}$.

1) Let us define a family of sets $\Delta^1_{G,w} \subseteq F$ $(w\in W)$ by

$$\Delta^1_{G,w} = \{y\in F \mid w(y) \geq \sup w(G)\} \qquad (w\in W). \qquad (3.1)$$

Remark 3.1. a) If $0\in W$ (where 0 denotes the zero functional on F), then $\Delta^1_{G,0}=F$, whence, by (1.3), $\lambda(0)=\inf h(\Delta^1_{G,0})=\inf h(F)$. Hence,

$$\beta = \sup_{0\neq w\in W} \inf h(\Delta^1_{G,w}). \qquad (3.2)$$

b) If F is a locally convex space, then for $0\neq w\in F^*$ such that $\sup w(G)=+\infty$, we have $\Delta^1_{G,w}=\emptyset$, while for $0\neq w\in F^*$ such that $\sup w(G)<+\infty$, $\Delta^1_{G,w}$ is a closed half-space in F, supporting the set G (i.e., $G\cap \mathrm{Int}\, \Delta^1_{G,w}=\emptyset$ and the boundary of $\Delta^1_{G,w}$ is a support hyperplane of G; for the definition of support hyperplanes, see e.g. [17], I.0), and we have

$$\beta = \sup_{w\in G^s} \inf h(\Delta^1_{G,w}), \qquad (3.3)$$

where

$$G^s = \{w\in W \mid w\neq 0,\ \sup w(G)<+\infty\}. \qquad (3.4)$$

Thus, if $W=F^*$ or $W=F^*\setminus\{0\}$, formula (3.3) means that

$$\beta = \sup_{D\in \mathfrak{D}_G} \inf h(D), \qquad (3.5)$$

where $\mathfrak{D}_G$ denotes the collection of all closed half-spaces in F, which support the set G. We shall omit the corresponding geometric interpretations of the β's occurring in the sequel, and, for simplicity, we shall work only with β's written similarly to (3.2).

For a set F and functionals $h,w: F \longrightarrow \overline{R}$, $w\neq 0$, let

$$\varphi(c)=\varphi_w(c)=\inf_{\substack{y\in F\\ w(y)\geq c}} h(y) \qquad (c\in R); \qquad (3.6)$$

in the particular case $F=R^n$, $0\neq w\in(R^n)^*$, the non-decreasing functions $\varphi_w: R\to\overline{R}$ have been studied in [3], [2], [9]. Extending [3], p.214, we shall say that h is regular (or, extending [9], p.66, one might use the term "semi-regular") with respect to w, if

$$\varphi_w(c)=\sup_{\substack{c'\in R\\ c'<c}} \varphi_w(c') \qquad (c\in R). \qquad (3.7)$$

Remark 3.2. If $h:R^n\to\overline{R}$ is convex, then, by [2], theorem 11 i), φ_w is convex, for all $w\in(R^n)^*\setminus\{0\}$, and hence, if h is convex and $h(R^n)\subseteq R$, then, by [1], p.48 and [9], p.66, h is regular with respect to all $w\in(R^n)^*\setminus\{0\}$ (alternatively, one can prove these statements similarly to [13], lemma 2.1). Let us also mention that, conversely, if $h:R^n\to\overline{R}$ is quasi-convex and lower semi-continuous and if all φ_w ($w\in(R^n)^*\setminus\{0\}$) are convex, then, by [2], theorem 11 ii), h is convex.

Proposition 3.1. Let F be a set, $W\subseteq\overline{R}^F$, G a subset of F, and $h: F\to\overline{R}$ a functional, which is regular with respect to all $w\in W\setminus\{0\}$. Then, for $\alpha=\sup h(G)$ and β of (3.2), we have (2.3).

Proof. For any $w\in W\setminus\{0\}$ we have

$$\varphi_w(w(g))=\inf_{\substack{y\in F\\ w(y)\geq w(g)}} h(y)\leq h(g)\leq\alpha \qquad (g\in G), \qquad (3.8)$$

whence, by (3.7) (with $c=\sup w(G)$) and since φ_w is non-decreasing,

$$\lambda(w)=\inf_{\substack{y\in F\\ w(y)\geq\sup w(G)}} h(y)=\varphi_w(\sup w(G))=\sup_{\substack{c'\in R\\ c'<\sup w(G)}} w(c')\leq$$

$$\leq\sup_{g\in G}\varphi_w(w(g))\leq\alpha \qquad (w\in W\setminus\{0\});$$

hence, by (3.2), we obtain $\beta\leq\alpha$.

We recall that, following Ky Fan [5], a subset M of a set F is said to be W-convex, where $W\subseteq\overline{R}^F$, if for each $y\notin M$ there exists $w\in W$, $w\neq 0$, such that $\sup w(M)<w(y)$. In particular, for a locally convex space F and $W=F^*$ or $W=F^*\setminus\{0\}$, from the strict separation theorem it follows that a set $M\subseteq F$ is W-convex if and only if it is closed and convex (see e.g. [4]).

Proposition 3.2. Let F be a set, $W\subseteq\overline{R}^F$, G a subset of F, and $h:F\to\overline{R}$ a functional, such that for each $c<\alpha=\sup h(G)$, the level set

$S_c(h)$ is W-convex. Then, for β of (3.2), we have (2.6).

Proof. For each $c<\alpha=\sup h(G)$ there exists $g_c \in G$ such that $h(g_c)>c$, that is, $g_c \notin S_c(h)$. Hence, since $S_c(h)$ is W-convex, there exists $w_c \in W$, $w_c \neq 0$, such that

$$w_c(g_c) > \sup w_c(S_c(h)). \tag{3.9}$$

Then, by $g_c \in G$ and (3.9), we obtain

$$\sup w_c(G) \geqslant w_c(g_c) > w_c(y) \qquad (y \in S_c(h)), \tag{3.10}$$

and thus w_c satisfies (2.5) (with $\Delta=\Delta^1$). Hence, by proposition 2.2, we have (2.6).

Remark 3.3. The assumption of proposition 3.2 is satisfied for each $h: F \longrightarrow \overline{R}$ which is "W-quasi-convex" in the sense of [21], i.e., for which all level sets $S_c(h)$ $(c \in R)$ are W-convex. In particular, if F is a locally convex space and $W=F^*$ or $F^* \setminus \{0\}$, then h is W-quasi-convex if and only if it is quasi-convex (in the usual sense) and lower semi-continuous (see [21]).

Combining propositions 3.1 and 3.2, we obtain

Theorem 3.1. Let F be a set, $W \subseteq R^F$, G a subset of F, and $h: F \longrightarrow \overline{R}$ a functional, which is regular with respect to all $w \in W \setminus \{0\}$, and such that, for each $c < \sup h(G)$, the level set $S_c(h)$ is W-convex. Then

$$\sup h(G) = \sup_{0 \neq w \in W} \inf_{\substack{y \in F \\ w(y) \geqslant \sup w(G)}} h(y). \tag{3.11}$$

From theorem 3.1 and remarks 3.2, 3.3, there follows

Corollary 3.1. Let F be a locally convex space, G a subset of F and $h: F \longrightarrow R$ a finite lower semi-continuous convex functional. Then we have (3.11) with $W=F^*$.

2) If $W=-W$, then the family of sets

$$\Delta^2_{G,w} = \{y \in F \mid w(y) \leqslant \inf w(G)\} \qquad (w \in W) \tag{3.12}$$

coincides with (3.1), since

$$\Delta^1_{G,w} = \Delta^2_{G,-w} \qquad (w \in W). \tag{3.13}$$

Hence, if $W=-W$, then formula (3.11) is equivalent to

$$\sup h(G) = \sup_{0 \neq w \in W} \inf_{\substack{y \in F \\ w(y) \leqslant \inf w(G)}} h(y). \tag{3.14}$$

3) Let us define a family of sets $\Delta^3_{G,w} \subseteq F$ $(w \in W)$ by

$$\Delta^3_{G,w} = \{y \in F \mid w(y) = \sup w(G)\} \qquad (w \in W). \tag{3.15}$$

Remark 3.4. a) If $0 \in W$, then $\Delta^3_{G,0} = F$. Hence,

$$\beta = \sup_{0 \neq w \in W} \inf h(\Delta^3_{G,w}). \tag{3.16}$$

b) If F is a locally convex space, then for $0 \neq w \in F^*$ such that $\sup w(G) = +\infty$ we have $\Delta^3_{G,w} = \emptyset$, while for $0 \neq w \in F^*$ such that $\sup w(G) < +\infty$, $\Delta^3_{G,w}$ is a support hyperplane of G.

For a set F and functionals $h, w: F \longrightarrow \overline{R}$, $w \neq 0$, let

$$\gamma(c) = \gamma_w(c) = \inf_{\substack{y \in F \\ w(y) = c}} h(y) \qquad (c \in R); \tag{3.17}$$

in the particular case when F is a linear space and $0 \neq w \in F^\#$ (the algebraic conjugate space of F), the functions $\gamma_w: R \longrightarrow \overline{R}$ have been studied in [13]. In contrast with φ_w (of (3.6)), the functions $\gamma_w: R \longrightarrow \overline{R}$ need not be non-decreasing. Nevertheless, we have

Proposition 3.3 ([13], lemma 2.1 and remark 2.2 a)). *Let F be a linear space, $h: F \longrightarrow R$ a finite convex functional and $w \neq 0$ a linear functional on F. Then the function γ_w of (3.17) is finite, convex and continuous on R. Hence, for $\alpha = \sup h(G)$ and β of (3.16), we have (2.3).*

From the above, we obtain

Corollary 3.2 ([13], theorem 2.1). *Under the assumptions of corollary 3.1, we have*

$$\sup h(G) = \sup_{0 \neq w \in F^*} \inf_{\substack{y \in F \\ w(y) = \sup w(G)}} h(y). \tag{3.18}$$

Proof. By proposition 3.3 and corollary 3.1, we have

$$\sup h(G) \geqslant \sup_{0 \neq w \in F^*} \inf_{\substack{y \in F \\ w(y) = \sup w(G)}} h(y) \geqslant$$

$$\geqslant \sup_{0 \neq w \in F^*} \inf_{\substack{y \in F \\ w(y) \geqslant \sup w(G)}} h(y) = \sup h(G).$$

4) Considering the family of sets

$$\Delta^4_{G,w} = \{y \in F \mid w(y) = \inf w(G)\} \qquad (w \in W), \tag{3.19}$$

we see that (since $F^* = -F^*$) formula (3.18) *is equivalent to*

$$\sup h(G)= \sup_{0\neq w\in F^{*}} \inf_{\substack{y\in F \\ w(y)=\inf w(G)}} h(y). \tag{3.20}$$

5) Let us define a family of sets $\Delta^{5}_{G,w}\subseteq F$ $(w\in W)$ by

$$\Delta^{5}_{G,w}=\{y\in F \mid w(y)>\sup w(G)\} \qquad (w\in W). \tag{3.21}$$

Remark 3.5. a) If $0\in W$, then $\Delta^{5}_{G,0}=\emptyset$, whence, by (1.3), $\lambda(w)=\inf \emptyset=$ $=+\infty$, and hence $\beta=\sup \lambda(W)=+\infty$; thus, in general, for β of (1.4), with $\Delta=\Delta^{5}$, we need not have the equality corresponding to (3.2) and (3.16). One can avoid this problem by working directly with $W\setminus\{0\}$, instead of W.

b) If F is a locally convex space, then for $0\neq w\in F^{*}$ such that $\sup w(G)=+\infty$ we have $\Delta^{5}_{G,w}=\emptyset$, while for $0\neq w\in F^{*}$ such that $\sup w(G)<+\infty$, $\Delta^{5}_{G,w}$ is an open half-space in F, supporting the set G (i.e., $G\cap\Delta^{5}_{G,w}=\emptyset$ and the boundary of $\Delta^{5}_{G,w}$ is a support hyperplane of G).

From theorem 3.1, there follows

Corollary 3.3. *Under the assumptions of theorem* 3.1, *if* F *is a locally convex space*, $W\subseteq F^{*}$ *and* $h:F\to\overline{R}$ *is upper semi-continuous, then*

$$\sup h(G)= \sup_{0\neq w\in W} \inf_{\substack{y\in F \\ w(y)>\sup w(G)}} h(y). \tag{3.22}$$

Proof. Since h is upper semi-continuous on F, for every subset M of F we have $\inf h(\overline{M})=\inf h(M)$ (where $\overline{M}$ is the closure of M). Thus, observing that for each $0\neq w\in F^{*}$ we have

$$\overline{\{y\in F \mid w(y)>\sup w(G)\}}=\{y\in F \mid w(y)\geq\sup w(G)\}, \tag{3.23}$$

and applying theorem 3.1, we obtain (3.22).

Similarly, from corollary 3.1 there follows

Corollary 3.4. *Under the assumptions of corollary* 3.1, *if* h *is also continuous on* F, *then we have* (3.22) *with* $W=F^{*}$.

6) Considering the family of sets

$$\Delta^{6}_{G,w}=\{y\in F \mid w(y)<\inf w(G)\} \qquad (w\in W), \tag{3.24}$$

we see that *if* $W=-W$, *then formula* (3.22) *is equivalent to*

$$\sup h(G)= \sup_{0\neq w\in W} \inf_{\substack{y\in F \\ w(y)<\inf w(G)}} h(y). \tag{3.25}$$

§4. Applications to surrogate dual problems for systems

By a "system" we shall mean a triple $(F \xrightarrow{u} X)$, consisting of two sets F, X and a mapping u of F into X. For a system $(F \xrightarrow{u} X)$, we shall consider now the primal supremization problem

$$(P)=(P_{u^{-1}(\Omega),h}) \qquad \alpha=\alpha_{u^{-1}(\Omega),h} = \sup_{\substack{y\in F\\ u(y)\in\Omega}} h(y), \tag{4.1}$$

i.e., (1.1) with $G=u^{-1}(\Omega)$, where $h:F\longrightarrow\overline{R}$ and $\Omega\subset X$, $u(F)\cap\Omega\neq\emptyset$ (Ω is called a "target set"). Furthermore, we shall assume that $W\subseteq\overline{R}^X$.

1) Let us define a family of sets $\Delta^1_{u^{-1}(\Omega),w}\subseteq F$ $(w\in W)$ by

$$\Delta^1_{u^{-1}(\Omega),w}=\{y\in F\,|\,w(u(y))\geqslant\sup w(u(F)\cap\Omega)\} \qquad (w\in W). \tag{4.2}$$

The main tool in studying surrogate duality for (4.2), is the following observation:

<u>Remark 4.1. Surrogate duality for</u> (4.2) <u>is equivalent to surrogate duality for a family of type</u> (3.1). Indeed, clearly, (3.1) is the particular case $X=F$, $u=I_F$ (the identity operator) and $\Omega=G$, of (4.2). Conversely, given (4.2) as above, let

$$V=V_W=\{v_w\,|\,w\in W\}\subseteq\overline{R}^F, \tag{4.3}$$

where

$$v_w=wu \qquad (w\in W). \tag{4.4}$$

Then, for $G=u^{-1}(\Omega)\subseteq F$, we have

$$\sup w(u(F)\cap\Omega)=\sup_{\substack{y\in F\\ u(y)\in\Omega}} wu(y)=\sup v_w(G) \qquad (w\in W), \tag{4.5}$$

whence, by (4.2) and (3.1),

$$\Delta^1_{u^{-1}(\Omega),w}=\Delta^1_{G,v_w} \qquad (w\in W). \tag{4.6}$$

Thus, <u>from each result of</u> §3, <u>on</u> (3.1), <u>one can obtain a corresponding result for</u> (4.2), <u>replacing</u> G, W <u>and</u> w <u>by</u> $u^{-1}(\Omega)$, V <u>and</u> $v_w=wu$, <u>respectively</u>. Note that the condition $w\neq0$ of §3 will now be replaced by $wu\neq0$; also, the assumption occurring in some results of §3, that F is a locally convex space, will now be replaced by the assumption that F and X are locally convex spaces and $u:F\longrightarrow X$ is a continuous linear mapping (which will ensure that $v_w=wu\in F^*$ for all $w\in X^*$). As an example, let us mention that formula (3.11) will be replaced by

$$\sup_{\substack{y\in F\\ u(y)\in\Omega}} h(G)=\sup_{\substack{w\in W\\ wu\neq 0}}\ \inf_{\substack{y\in F\\ w(u(y))\geq \sup w(u(F)\cap\Omega)}} h(y). \qquad (4.7)$$

2)-6) One can define, similarly to (4.2), families of surrogate constraint sets $\Delta^2_{u^{-1}(\Omega),w},\dots,\Delta^6_{u^{-1}(\Omega),w}\subseteq F$ $(w\in W)$, where $W\subseteq\overline{R}^X$, corresponding to 2)-6) of §3. For these sets, again, there hold similar remarks to remark 4.1 (mutatis mutandis). We omit the details.

§5. Lagrangian duality for supremization

Motivated by the results of [15] (see e.g. remark 5.3 below), we define here the "Lagrangian dual problem" to (P) of (1.1) (without any assumptions on F, G, h), as the dual problem (1.2), with λ of the form

$$\lambda(w)=\sup w(G)\underset{\cdot}{+}\inf_{y\in F}\{h(y)\dot{+}-w(y)\} \qquad (w\in W); \qquad (5.1)$$

we recall that $\dot{+}$ and $\underset{\cdot}{+}$ denote the "upper addition" and the "lower addition" on $\overline{R}$, defined (see [10], [11]) by

$$a\dot{+}b=a\underset{\cdot}{+}b=a+b \quad \text{if} \quad R\cap\{a,b\}\neq\emptyset \quad \text{or} \quad a=b=\pm\infty, \qquad (5.2)$$

$$a\dot{+}b=+\infty, \quad a\underset{\cdot}{+}b=-\infty \quad \text{if } a=-b=\pm\infty. \qquad (5.3)$$

Remark 5.1. In [15] we have used, for problem (1.2), (5.1) above, the term "quasi-Lagrangian dual problem", since it corresponds to the Lagrangian dual problem to (P′) of (1.6), defined (see [12], [21], [22]) as problem (1.2), with λ of the form

$$\lambda'(w)=\inf w(G)\underset{\cdot}{+}\inf_{y\in F}\{h(y)\dot{+}-w(y)\} \qquad (w\in W); \qquad (5.4)$$

however, in subsequent papers we have used the term "quasi-Lagrangian" in a different sense, and therefore, we call here problem (1.2), (5.1) above, simply, the "Lagrangian dual problem" to (P) of (1.1). Note that, by remark 1.1 b), this is an "unusual" dual problem to (P).

Now we shall show that the main results of [15] and [14] on Lagrangian type duality for supremization, involving proper lower semi--continuous convex functionals and bounded subsets in locally convex spaces F, can be extended to W-convex functionals on a set F, where $W\subseteq\overline{R}^F$, and to arbitrary subsets of F. We recall that the "W-convex hull" of $h:F\to\overline{R}$ is the functional $h_{\mathcal{H}(W)}:F\to\overline{R}$ defined [4] by

$$h_{\mathcal{H}(W)}=\sup_{\substack{w\in W\\ w\leq h}} w, \tag{5.5}$$

and that h is said to be "W-convex" [4], if $h_{\mathcal{H}(W)}=h$. The "W-conjugate" of $h:F\to\overline{R}$ and the "second W-conjugate" of h are (see e.g. [11], [4]) the functionals $h^W:W\to\overline{R}$ and $h^{WW}:F\to\overline{R}$ defined by

$$h^W(w)=\sup_{y\in F}\{w(y)\underset{\cdot}{+}-h(y)\} \qquad (w\in W), \tag{5.6}$$

$$h^{WW}(y)=\sup_{w\in W}\{w(y)\underset{\cdot}{+}-h^W(w)\}, \qquad (y\in F). \tag{5.7}$$

Lemma 5.1. Let F be a set, $W\subseteq\overline{R}^F$, $y_o\in F$, and $h:F\to\overline{R}$ a W-convex functional. Then

$$h(y_o)=h^{WW}(y_o)=\sup_{w\in W}\{w(y_o)\underset{\cdot}{+}\inf_{y\in F}\{h(y)\dot{+}-w(y)\}\}. \tag{5.8}$$

Proof. By [21], theorem 4.1, for any $h:F\to\overline{R}$ we have $h^{WW}=h_{\mathcal{H}(W+R)}\leq h$, where $R=(-\infty,+\infty)$ is identified with the family of all real-valued constant functionals on F; furthermore, by (5.5), $h_{\mathcal{H}(W)}\leq h_{\mathcal{H}(W+R)}$. Hence, if h is W-convex, then

$$h=h_{\mathcal{H}(W)}\leq h_{\mathcal{H}(W+R)}=h^{WW}\leq h, \tag{5.9}$$

whence $h=h^{WW}$. Finally, by [21], formula (4.26), for any $h:F\to\overline{R}$ and $y_o\in F$ we have the second equality in (5.8).

Remark 5.2. a) In general, $h^{WW}\neq h_{\mathcal{H}(W)}$ (see [21]); the problem of the existence of a concept of "conjugation" for which the "second conjugate" of h coincides with $h_{\mathcal{H}(W)}$, raised in [22], has been solved, in the affirmative, in [27].

b) When F is a locally convex space, a functional $h:F\to\overline{R}$ is (F^*+R)-convex if and only if either $h\equiv-\infty$, or $h\equiv+\infty$, or $h(F)\subseteq R\cup\{+\infty\}$ and h is lower semi-continuous convex (see [4], p.279). Hence, observing that, by (5.8) we have

$$h^{F^*+R,F^*+R}=h^{F^*,F^*}, \tag{5.10}$$

it follows that for a locally convex space F and for $W=F^*+R$ (the family of all continuous affine functionals on F), lemma 5.1 yields again [15], lemma 2.1.

We recall that, by [11], formula (3.2), we have

$$a\dot{+}(b\underset{\cdot}{+}c)\geq(a\dot{+}b)\underset{\cdot}{+}c \qquad (a,b,c\in\overline{R}). \tag{5.11}$$

Theorem 5.1. Let F be a set, $W\subseteq\overline{R}^F$, G a subset of F, and $h:F\longrightarrow\overline{R}$ a W-convex functional. Then

$$\sup h(G)=\sup_{w\in W}\left\{\sup w(G)\underset{\cdot}{+}\inf_{y\in F}\{h(y)\dot{+}-w(y)\}\right\}. \tag{5.12}$$

Proof. Let $w\in W$ and $c<\sup w(G)$. Then there exists $g'=g'_{w,c}\in G$ such that $w(g')\geq c$, whence, by [11], formula (2.1) and p.120, corollary, we have $0\geq -w(g')\underset{\cdot}{+}c$. Consequently, by (5.11),

$$\sup h(G)\geq h(g')\geq h(g')\dot{+}(-w(g')\underset{\cdot}{+}c)\geq$$

$$\geq(h(g')\dot{+}-w(g'))\underset{\cdot}{+}c\geq c\underset{\cdot}{+}\inf_{y\in F}\{h(y)\dot{+}-w(y)\}, \tag{5.13}$$

whence, since $c<\sup w(G)$ and $w\in W$ were arbitrary, we obtain

$$\sup h(G)\geq\sup_{w\in W}\left\{\sup w(G)\underset{\cdot}{+}\inf_{y\in F}\{h(y)\dot{+}-w(y)\}\right\}; \tag{5.14}$$

note that this is valid for any functional $h:F\longrightarrow\overline{R}$.

On the other hand, if h is W-convex, then, by lemma 5.1, we have

$$h(g)=\sup_{w\in W}\left\{w(g)\underset{\cdot}{+}\inf_{y\in F}\{h(y)\dot{+}-w(y)\}\right\}\leq$$

$$\leq\sup_{w\in W}\left\{\sup w(G)\underset{\cdot}{+}\inf_{y\in F}\{h(y)\dot{+}-w(y)\}\right\}\qquad(g\in G). \tag{5.15}$$

Hence, by (5.14) and (5.15), we obtain (5.12).

Remark 5.3. In the particular case when F is a locally convex space and $W=F^*+R$, by remark 5.1 b) we see that theorem 5.1 yields an improvement of [15], theorem 2.1 (namely, the assumption of boundedness of G, made in [15], is omitted).

We recall that, by [11], formulae (4.8) and (2.1), for any set E and any $k:E\longrightarrow\overline{R}$ and $a,b,c\in\overline{R}$ we have

$$\sup_{x\in E}k(x)\underset{\cdot}{+}c=\sup_{x\in E}\{k(x)\underset{\cdot}{+}c\}, \tag{5.16}$$

$$-(a\underset{\cdot}{+}b)=-a\dot{+}-b. \tag{5.17}$$

Theorem 5.2. Let F be a set, $W\subseteq\overline{R}^F$, $h:F\longrightarrow\overline{R}$ a W-convex functional, and $f:F\longrightarrow\overline{R}$ an arbitrary functional. Then

$$\sup_{y\in F}\{h(y)\underset{\cdot}{+}-f(y)\}=\sup_{w\in W}\{f^W(w)\underset{\cdot}{+}-h^W(w)\}. \tag{5.18}$$

Proof. Since h is W-convex, by lemma 5.1, (5.16), (5.6) and (5.17), we obtain

$$\sup_{y\in F}\{h(y)\dot{+}-f(y)\}=$$

$$=\sup_{y\in F}\Big\{\sup_{w\in W}\big[w(y)\dot{+}\inf_{y'\in F}\{h(y')\dot{+}-w(y')\}\big]\dot{+}-f(y)\Big\}=$$

$$=\sup_{y\in F}\Big\{\sup_{w\in W}\big[w(y)\dot{+}-f(y)\dot{+}\inf_{y'\in F}\{h(y')\dot{+}-w(y')\}\big]\Big\}=$$

$$=\sup_{w\in W}\Big\{\sup_{y\in F}\big[w(y)\dot{+}-f(y)\dot{+}\inf_{y'\in F}\{h(y')\dot{+}-w(y')\}\big]\Big\}=$$

$$=\sup_{w\in W}\Big\{\sup_{y\in F}\{w(y)\dot{+}-f(y)\}\dot{+}\inf_{y'\in F}\{h(y')\dot{+}-w(y')\}\Big\}=$$

$$=\sup_{w\in W}\{f^W(w)\dot{+}-h^W(w)\}.$$

Remark 5.4. a) In the particular case when $f=\chi_G$, the indicator functional of a subset G of F (i.e., $\chi_G(y)=0$ for $y\in G$ and $\chi_G(y)=+\infty$ for $y\in F\setminus G$), we have

$$f^W(w)=\sup_{y\in F}\{w(y)\dot{+}-\chi_G(y)\}=\sup w(G) \qquad (w\in W), \qquad (5.19)$$

and hence theorem 5.2 yields again theorem 5.1.

b) In the particular case when F is a locally convex space and $W=F^*+R$, by remark 5.2 b) we see that theorem 5.2 yields the main result of [14].

REFERENCES

[1] J.-P-Crouzeix, Contributions à l'étude des fonctions quasiconvexes. Thèse. Université de Clermont, 1977.

[2] J.-P.Crouzeix, Continuity and differentiability properties of quasiconvex functions on R^n. In: Generalized concavity in optimization and economics (S.Schaible and W.T.Ziemba, eds.), Acad. Press, New York, 1981, pp.109-130.

[3] J.-P.Crouzeix, A duality framework in quasiconvex programming. In: Generalized concavity in optimization and economics (S.Schaible and W.T.Ziemba, eds.), Acad. Press, New York, 1981, pp.207-225.

[4] S.Dolecki and S.Kurcyusz, On Φ-convexity in extremal problems. SIAM J.Control Optim. 16(1978), 277-300.

[5] K.Fan, On the Krein-Milman theorem. In: Convexity (V.Klee, ed.). Proc. Symposia Pure Math. 7. Amer.Math.Soc., Providence, 1963, pp.211-220.

[6] C.Franchetti and I.Singer, Deviation and farthest points in normed linear spaces. Rev.Roum.Math.Pures Appl. 24(1979), 373-381.

[7] C.Franchetti and I.Singer, Best approximation by elements of caverns in normed linear spaces. Boll.Un.Mat.Ital. (5) 17-B(1980), 33-43.

[8] P.-J.Laurent, Approximation et optimisation. Hermann, Paris, 1972.

[9] J.-E.Martínez-Legaz, Un concepto generalizado de conjugación. Applicación a las funciones quasiconvexas. Thesis, Barcelona, 1981.

[10] J.-J.Moreau, Fonctionnelles convexes. Sémin.Eq.Dériv.Part.Collège de France, Paris, 1966-1967, no.2.

[11] J.-J.Moreau, Inf-convolution, sous-additivité, convexité des fonctions numériques. J.math.pures et appl. 49 (1950), 109-154.

[12] I.Singer, Some new applications of the Fenchel-Rockafellar duality theorem: Lagrange multipier theorems and hyperplane theorems for convex optimization and best approximation. Nonlinear Anal.Theory, Methods, Appl. 3(1979), 239-248.

[13] I.Singer, Maximization of lower semi-continuous convex functionals on bounded subsets of locally convex spaces. I: Hyperplane theorems. Appl.Math.Optim. 5(1979), 349-362.

[14] I.Singer, A Fenchel-Rockafellar type duality theorem for maximization. Bull.Austral.Math.Soc. 20(1979), 193-198.

[15] I.Singer, Maximization of lower semi-continuous convex functionals on bounded subsets of locally convex spaces. II: Quasi-Lagrangian duality theorems. Result. Math. 3(1980), 235-248.

[16] I.Singer, Minimization of continuous convex functionals on complements of convex subsets of locally convex spaces. Math.Operationsforsch. Stat. Ser. Optim. 11 (1980), 221-234.

[17] I.Singer, Optimization by level set methods. I :Duality formulae. In: Optimization: Theory and algorithms (Proceedings.Internat. Confer. in Confolant, March 1981; J.-B.Hiriart-Urruty, W.Oettli and J.Stoer, eds.), Lecture Notes in Pure and Appl. Math. 86, Marcel Dekker, New York, 1983, pp.13-43.

[18] I.Singer, Optimization by level set methods. III: Characterizations of solutions in the presence of duality. Numer.Funct.Anal. Optim. 4 (1981-1982), 151-170.

[19] I.Singer, Optimization by level set methods. IV: Generalizations and complements. Numer. Funct. Anal. Optim. 4 (1981-1982), 279--310.

[20] I.Singer, A general theory of surrogate dual and perturbational extended surrogate dual optimization problems. J.Math.Anal. Appl. 104(1984), 351-389.

[21] I.Singer, Generalized convexity, functional hulls and applications to conjugate duality in optimization. In: Selected topics in operations research and mathematical economics (G.Hammer and D.Pallaschke, eds.). Lecture Notes in Econ. and Math. Systems 226, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1984, pp.49-79.

[22] I.Singer, A general theory of dual optimization problems. J.Math. Anal.Appl. (to appear). Preprint INCREST (Bucharest) 67/1984.

[23] I.Singer, Optimization by level set methods. VI: Generalizations of surrogate type reverse convex duality (to appear). Preprint INCREST (Bucharest) 74/1985.

[24] J.F.Toland, Duality in nonconvex optimization. J.Math.Anal.Appl. 66(1978), 399-415.

[25] J.F.Toland, A duality principle for non-convex optimisation and the calculus of variations. Arch.Rat.Mech.Anal. 71(1979), 41-61.

[26] H.Tuy, Global maximization of a convex function over a closed, convex, not necessarily bounded set. Preprint CEREMADE (Paris) 8223/1982.

[27] J.-E.Martínez-Legaz and I.Singer, Dualities between complete lattices (in preparation).

OPTIMIZATION BY LEVEL SET METHODS. VI: GENERALIZATIONS OF SURROGATE TYPE REVERSE CONVEX DUALITY

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We give some extensions of our duality theorems of [9] on the optimization problem (P) $\alpha=\inf h(G)$, where G is a subset of a locally convex space F such that $F\setminus G$ and $h:F\to\overline{R}=[-\infty,+\infty]$ are convex, to the case when G is a subset of an arbitrary set F and $h:F\to\overline{R}$ is an arbitrary functional. We give some applications to the case when G is embedded into a family of sets $\Gamma(x)\subseteq F$ $(x\in X)$, where X is a parameter set.

§1. Introduction

Given a set F, a subset G of F (assumed to be non-empty, throughout the sequel) and a functional $h:F\to\overline{R}=[-\infty,+\infty]$, let us consider the following (global, scalar) primal infimization problem:

$$(P)=(P_{G,h}) \qquad \alpha=\alpha_{G,h}=\inf h(G). \tag{1.1}$$

In the paper [9] (see also [4]) we have proved some theorems of "unperturbational surrogate duality" type (in a sense similar to [15],[16]), for a certain "reverse convex infimization" problem, namely, for the particular case of problem (1.1), in which F is a linear space and $F\setminus G$ and h are convex (the general "reverse convex infimization", as studied e.g. in [18] and the references therein, is the case when F is a linear space, h is convex and $G=G_1\cap G_2$, where G_1 and $F\setminus G_2$ are convex; taking $G_1=F$, we obtain the case of [9]).

Motivated by the above mentioned results of [9] (see e.g. remark 5.4 below), we introduce here the following concept of "dual problem" to (P) of (1.1) (without any assumptions on F, G, h):

Definition 1.1. By a dual problem to (P) we shall mean any infimization problem of the form

$$(Q)=(Q^{G,h}) \qquad \beta=\beta^{G,h} = \inf \lambda(W), \tag{1.2}$$

where $W=W^{G,h}$ is a set (assumed non-empty, without loss of generality) and $\lambda=\lambda^{G,h}:W\to\overline{R}$ is a functional.

Remark 1.1. a) We assume no relation between α and β.

b) There is a marked difference between the above dual problems (1.2) and the "usual" dual problems [16] to (P) (extending the usual dual problems for convex infimization, i.e., for (P) of (1.1) with F a linear space and h, G convex), in which $\beta=\sup\lambda(W)$, or, equivalently (see e.g. [5]), $\beta=-\inf\lambda(W)$. Therefore, we shall call the above problems (1.2) "unusual" dual problems to (P).

We shall first consider "unperturbational surrogate dual problems" to (P), in a sense similar to [16] (see also [15]), namely, the case when λ of (1.2) is of the form

$$\lambda(w)=\lambda_{W\Delta}^{G,h}(w)=\inf h(\Delta_{G,w}) \qquad (w\in W), \tag{1.3}$$

where $\Delta_{G,w}\subseteq F$ ($w\in W$) is a given family of ("surrogate constraint") sets; thus, by (1.2) and (1.3), we have

$$\beta=\inf_{w\in W}\inf h(\Delta_{G,w}). \tag{1.4}$$

Remark 1.2. There is a marked difference between problems (1.3), (1.4), which may be called "unusual surrogate dual problems of the first type", and the problems where λ of (1.2) is of the form $\lambda(w)=\sup h(\Delta_{G,w})$ (and hence $\beta=\inf_{w\in W}\sup h(\Delta_{G,w})$), studied in [17], which may be called "unusual surrogate dual problems of the second type".

In §2 we shall give some necessary and sufficient conditions for $\alpha\leq\beta$, for $\alpha\geq\beta$ and for $\alpha=\beta$, with β of (1.4) and $\alpha\in\overline{R}$ arbitrary or, in particular, $\alpha=\inf h(G)$, and some simultaneous characterizations of "solutions" of (P) (of (1.1)) and of "weak duality" for $\{(P), (Q)\}$ (i.e., conditions in order to have $\alpha=\beta$, with α,β of (1.1), (1.4)), involving the level sets

$$A_c(h)=\{y\in F \mid h(y)<c\} \qquad (c\in R), \tag{1.5}$$

$$S_c(h)=\{y\in F \mid h(y)\leq c\} \qquad (c\in R), \tag{1.6}$$

of h, which correspond to our results of [11]-[13] on the "usual" dual problems of remark 1.1 b) above; we recall that, by definition, the "solutions" of (P) are the elements of the (possibly empty) set

$$\mathcal{S}_G(h)=\{g_o\in G \mid h(g_o)=\inf h(G)\}. \tag{1.7}$$

In §3 we shall apply the results of §2 to surrogate dual problems defined by "perturbation multifunctions" $\Gamma: X \rightarrow 2^F$, where X is a parameter set and 2^F denotes the collection of all subsets of F, and to certain families of "surrogate constraint sets" $\Delta^i_{\Gamma(x_o),w} \subseteq F$ ($w \in W$, $i=1,\ldots,7$), where $x_o = x_o^G \in X$ is such that $\Gamma(x_o)=G$ and where $W \subseteq \overline{R}^X$ (we recall that $\overline{R}^X$ denotes the family of all functionals $w: X \rightarrow \overline{R}$). In §4 we shall consider the particular case of the "natural perturbation multifunction $\Gamma = \Gamma^n$ associated to (u, Ω), where $u: F \rightarrow X$ is a mapping and $\Omega \subset X$ is a "target set", and where $G = \Gamma(x_o) = u^{-1}(\Omega)$. The particular case when $X=F$, $u=I_F$ (the identity operator) and $\Omega = G \subset F$, will be considered in §5. For locally convex spaces, the surrogate constraint sets of §4 and §5 admit convenient geometric interpretations.

Throughout the paper, we adopt the usual conventions

$$\inf \emptyset = +\infty, \quad \sup \emptyset = -\infty, \tag{1.8}$$

where $\emptyset$ denotes the empty set. Also, as in [11]-[13], we make the convention that if $A_c(h)=\emptyset$ or $S_c(h)=\emptyset$ for some $c \in R$, then the conditions involving these $A_c(h)$, $S_c(h)$ (see e.g. (2.10), (2.11), etc.) will be considered satisfied (vacuously). By "linear space" (with or without a topology) we shall mean: real linear space.

§2. Surrogate duality results in the general case

Let us first recall

Lemma 2.1 ([13], proposition 1.1). *Let F be a set, $\Delta \subseteq F$, $h: F \rightarrow \overline{R}$ and $c \in R$. We have $\inf h(\Delta) \geq c$ if and only if $\Delta \cap A_c(h) = \emptyset$.*

Proof [13]. If $y_o \in \Delta \cap A_c(h)$, then $\inf h(\Delta) \leq h(y_o) < c$. Conversely, if $\inf h(\Delta) < c$, then there exists $y_o \in \Delta$ such that $h(y_o) < c$, so $y_o \in \Delta \cap A_c(h)$.

Proposition 2.1. *For $F, G, h, W, \Delta_{G,w} \subseteq F$ ($w \in W$) as in §1, and any $\alpha \in \overline{R}$, the following statements are equivalent:*

1^o. *We have*

$$\Delta_{G,w} \cap A_c(h) = \emptyset \qquad (w \in W,\ c \in R,\ c < \alpha). \tag{2.1}$$

2^o. *We have*

$$\Delta_{G,w} \cap S_c(h) = \emptyset \qquad (w \in W,\ c \in R,\ c < \alpha). \tag{2.2}$$

3^o. *We have*

$$\beta = \inf_{w \in W} \inf h(\Delta_{G,w}). \tag{2.3}$$

Proof. $1^o \Leftrightarrow 3^o$. By lemma 2.1, condition 1^o is equivalent to

$$\inf h(\Delta_{G,w}) \geq c \qquad (w \in W,\ c \in R,\ c < \alpha), \tag{2.4}$$

i.e., to $\inf h(\Delta_{G,w}) \geq \alpha$ $(w \in W)$, which is equivalent to 3^o.

Finally, the equivalence $1^o \Leftrightarrow 2^o$ follows from the inclusions

$$A_c(h) \subseteq S_c(h) \subseteq A_d(h) \qquad (c,d \in R,\ c<d<\alpha). \quad (2.5)$$

Corollary 2.1. a) For F, G, h, W and $\Delta_{G,w} \subseteq F$ $(w \in W)$ as in §1, if we have

$$\Delta_{G,w} \subseteq G \qquad (w \in W), \qquad (2.6)$$

then $\alpha = \inf h(G)$ satisfies (2.3).

b) If F is a topological space, $h: F \to \overline{R}$ is upper semi-continuous and

$$\Delta_{G,w} \subseteq \overline{G} \qquad (w \in W) \qquad (2.7)$$

(where $\overline{G}$ denotes the closure of G), then $\alpha = \inf h(G)$ satisfies (2.3).

Proof. a) Clearly,

$$G \cap A_c(h) = \emptyset \qquad (c \in R, c < \alpha = \inf h(G)).$$

Hence, if (2.6) holds, then

$$\Delta_{G,w} \cap A_c(h) \subseteq G \cap A_c(h) = \emptyset \qquad (w \in W, c \in R, c < \alpha),$$

so the result follows from proposition 2.1.

b) If h is upper semi-continuous on F, then $\inf h(\overline{G}) = \inf h(G)$, and hence the conclusion follows from part a) (applied with G replaced by $\overline{G}$, and with $\Delta_{\overline{G},w} = \Delta_{G,w}$).

Remark 2.1. a) If $w \in W$, $\Delta_{G,w} = \emptyset$, then, by (1.3), we have $\lambda(w) = \inf \emptyset = +\infty$. Hence, by (1.4),

$$\beta = \inf_{w \in G^r} \inf h(\Delta_{G,w}), \qquad (2.8)$$

where

$$G^r = \{w \in W \mid \Delta_{G,w} \neq \emptyset\}. \qquad (2.9)$$

b) If $\Delta_{G,w} = \emptyset$ $(w \in W)$, then (2.6) is satisfied and, by (2.8), (2.9), we have $\beta = \inf \emptyset = +\infty \geq \alpha$.

Proposition 2.2. For $F, G, h, W, \Delta_{G,w} \subseteq F$ $(w \in W)$ as in §1, and any $\alpha \in \overline{R}$, the following statements are equivalent:

1^o. For each $c \in R$, $c > \alpha$, there exists $w_c \in W$ such that

$$\Delta_{G,w_c} \cap A_c(h) \neq \emptyset. \qquad (2.10)$$

2^o. For each $c \in R$, $c > \alpha$, there exists $w_c \in W$ such that

$$\Delta_{G,w_c} \cap S_c(h) \neq \emptyset. \qquad (2.11)$$

3^o. We have

$$\alpha \geq \beta = \inf_{w\in W} \inf h(\Delta_{G,w}). \quad (2.12)$$

Proof. The implication $1^o \Rightarrow 2^o$ is obvious.

$2^o \Rightarrow 3^o$. If $c\in R$, $c>\alpha$ and $w_c\in W$ satisfy (2.10), say $y_c\in\Delta_{G,w_c}\cap S_c(h)$, then, by (1.4),

$$\beta = \inf_{w\in W} \inf h(\Delta_{G,w}) \leq \inf h(\Delta_{G,w_c}) \leq h(y_c) \leq c; \quad (2.13)$$

hence, $\beta \leq \inf_{c>\alpha} c = \alpha$. On the other hand, if there exists no $c\in R$ such that $c>\alpha$, then $\beta \leq +\infty = \alpha$.

$3^o \Rightarrow 1^o$. If 3^o holds and $c\in R$, $c>\alpha\geq\beta=\inf_{w\in W} \inf h(\Delta_{G,w})$, then there exists $w_c\in W$ such that $c>\inf h(\Delta_{G,w_c})$, whence, by lemma 2.1, we obtain (2.10).

Corollary 2.2. For $F,G,h,W,\Delta_{G,w}\subseteq F$ $(w\in W)$ as in §1, if there holds

$$G \subseteq \bigcup_{w\in W} \Delta_{G,w}, \quad (2.14)$$

then $\alpha=\inf h(G)$ satisfies (2.12).

Proof. If $c\in R$, $c>\alpha=\inf h(G)$, then there exists $g_c\in G$ such that $c>h(g_c)$. Hence, if (2.14) holds, then there exists $w_c\in W$ such that $g_c\in G\subseteq\Delta_{G,w_c}$, so $g_c\in\Delta_{G,w_c}\cap A_c(h)\neq\emptyset$. Thus, by proposition 2.2, we obtain (2.12).

Combining propositions 2.1 and 2.2, we obtain

Theorem 2.1. For F,G,h,W and $\Delta_{G,w}\subseteq F$ $(w\in W)$ as in §1, and any $\bar{\alpha}\in\bar{R}$, the following statements are equivalent:

1^o. We have (2.1), and for each $c\in R$, $c>\alpha$, there exists $w_c\in W$ satisfying (2.10).

2^o. We have (2.2), and for each $c\in R$, $c>\alpha$, there exists $w_c\in W$ satisfying (2.11).

3^o. We have

$$\alpha = \inf_{w\in W} \inf h(\Delta_{G,w}). \quad (2.15)$$

Combining corollary 2.1 and proposition 2.2, we obtain

Corollary 2.3. For F,G,h,W and $\Delta_{G,w}\subseteq F$ $(w\in W)$ satisfying the assumptions of corollary 2.1 a) or b) and for $\alpha=\inf h(G)$, the following statements are equivalent:

1^o. We have 1^o of proposition 2.2.

2^o. We have 2^o of proposition 2.2.

3^o. We have (2.15).

Remark 2.2. The assumptions of corollary 2.1 a) or b) are needed only in the proofs of the implications $1^o \Rightarrow 3^o$ and $2^o \Rightarrow 3^o$.

Combining corollaries 2.1 and 2.2, we obtain

Corollary 2.4. Under the assumptions of corollary 2.1 a) or b), if there holds (2.14), then we have (2.15).

Remark 2.3. If (2.6) holds, then (2.14) is equivalent to

$$G = \bigcup_{w \in W} \Delta_{G,w} . \qquad (2.16)$$

Concerning simultaneous characterizations of solutions of (P) and of weak duality for $\{(P), (Q)\}$ of (1.1), (1.4), let us prove

Theorem 2.2. For an element $g_o \in G$ and for $\alpha = \inf h(G)$, the following statements are equivalent:

1^o. We have

$$\Delta_{G,w} \cap A_c(h) = \emptyset \qquad (w \in W, c \in R, c < h(g_o)). \quad (2.17)$$

and for each $c \in R$, $c > \alpha$, there exists $w_c \in W$ satisfying (2.10).

2^o. We have

$$\Delta_{G,w} \cap S_c(h) = \emptyset \qquad (w \in W, c \in R, c < h(g_o)). \quad (2.18)$$

and for each $c \in R$, $c > \alpha$, there exists $w_c \in W$ satisfying (2.11).

3^o. We have $g_o \in \mathcal{S}_G(h)$ and (2.15).

Proof. $1^o \Rightarrow 3^o$. Assume 1^o. Then, by (2.17) and proposition 2.1 (with $\alpha = h(g_o)$), we have $h(g_o) \leq \beta$. Furthermore, by the second condition of 1^o and by proposition 2.2, we have (2.12). Hence, by $g_o \in G$, we obtain

$$\alpha = \inf h(G) \leq h(g_o) \leq \beta \leq \alpha . \qquad (2.19)$$

$3^o \Rightarrow 1^o$. If 3^o holds, then $h(g_o) = \inf h(G) = \alpha$, and hence, by theorem 2.1, we have 1^o.

Finally, the proof of the equivalence $2^o \Leftrightarrow 3^o$ is similar.

Remark 2.4. Similarly, one can prove the following result for supremization (instead of the infimization problem (1.1)): Let F, G, h, W and $\Delta_{G,w} \subseteq F$ $(w \in W)$ be as above. Then, for $g_o \in G$ and $\alpha = \sup h(G)$, the following statements are equivalent:

1^o. We have (2.1) (with $\alpha = \sup h(G)$), and for each $c \in R$, $c > h(g_o)$, there exists $w_c \in W$ satisfying (2.10).

2°. <u>We have</u> (2.2), <u>and for each</u> $c\in R$, $c>h(g_o)$, <u>there exists</u> $w_c \in W$ <u>satisfying</u> (2.11).

3°. <u>We have</u>

$$h(g_o)=\sup h(G)=\inf_{w\in W} \inf h(\Delta_{G,w}). \tag{2.20}$$

Indeed, in the proof, the inequalities (2.19) are now replaced by

$$\beta \leq h(g_o) \leq \sup h(G) \leq \beta .$$

§3. Applications to surrogate dual problems defined by perturbation multifunctions

Assume that problem (P) of (1.1) is "embedded" into a family of "perturbed" constrained optimization problems

$$(P^x)=(P^x_{G,h}) \qquad \alpha^x=\alpha^x_{G,h}=\inf h(\Gamma(x)) \qquad (x\in X), \tag{3.1}$$

where X is a parameter set and $\Gamma=\Gamma_G: X \to 2^F$ is a "perturbation multifunction", such that for some $x_o=x_o^G \in X$ there holds

$$G=\Gamma(x_o); \tag{3.2}$$

then, by (1.1) and (3.2), we can write (1.1) in the form

$$(P)=(P_{\Gamma(x_o),h}) \qquad \alpha=\alpha_{\Gamma(x_o),h}=\inf h(\Gamma(x_o)). \tag{3.3}$$

Furthermore, <u>in this section we shall assume that</u> $W\subseteq \overline{R}^X$.

1) Let us define a family of sets $\Delta^1_{\Gamma(x_o),w} \subseteq F$ $(w\in W)$ by

$$\Delta^1_{\Gamma(x_o),w}=\{y\in F \mid w(x_o)<\inf w(X\setminus \Gamma^{-1}(y))\} \qquad (w\in W), \tag{3.4}$$

where $\Gamma^{-1}: F\to 2^X$ is the "inverse multifunction", defined by

$$\Gamma^{-1}(y)=\{x\in X \mid y\in \Gamma(x)\} \qquad (y\in F). \tag{3.5}$$

The family (3.4) satisfies (2.6); indeed, for G=F, (2.6) is obvious, while if $G\neq F$ and $y\notin G=\Gamma(x_o)$, then $x_o \notin \Gamma^{-1}(y)$, whence $w(x_o)\geq \inf w(X\setminus \Gamma^{-1}(y))$, so $y\notin \Delta^1_{\Gamma(x_o),w}$. Note also that if $0\in W\subseteq R^X$ (where 0 denotes the zero functional on X), then $\Delta^1_{\Gamma(x_o),0}=\emptyset$, whence, by (1.3), $\lambda(0)=\inf h(\Delta^1_{\Gamma(x_o),0})=+\infty$. Hence,

$$\beta=\inf_{0\neq w\in W} \inf h(\Delta^1_{\Gamma(x_o),w}). \tag{3.6}$$

We recall that, following Ky Fan [3], a subset M of a set X is said to be "W-<u>convex</u>", where $W\subseteq \overline{R}^X$, if for each $x\notin M$ there exists $w\in W$, $w\neq 0$, such that $\sup w(M)<w(x)$; hence, M is (-W)-convex if for each $x\notin M$ there exists $w\in W$, $w\neq 0$, such that $\sup(-w)(M)<(-w)(x)$, i.e., $\inf w(M)>w(x)$. In particular, for a locally convex space X and $W=X^*$ or $W=X^*\setminus\{0\}$ (where X^* is the conjugate space of X), from the strict separation theorem it follows that a set $M\subset X$ is W-convex if and only if it is closed and convex (see e.g. [2]).

Theorem 3.1. Let F,G and h be as in §1, let X be a locally convex space, $W\subseteq\overline{R}^X$, and $\Gamma:X\to 2^F$ a multifunction satisfying (3.2) for some $x_o=x_o^G\in X$, such that $X\setminus\Gamma^{-1}(g)$ is $(-W)$-convex, for each $g\in G$. Then

$$\inf h(\Gamma(x_o))=\inf_{0\neq w\in W}\ \inf_{\substack{y\in F\\ w(x_o)<\inf w(X\setminus\Gamma^{-1}(y))}} h(y). \qquad (3.7)$$

Proof. For each $g\in G=\Gamma(x_o)$ we have $x_o\in\Gamma^{-1}(g)$. Hence, since $X\setminus\Gamma^{-1}(g)$ is $(-W)$-convex, there exists a functional $w_g\in W$, $w_g\neq 0$, such that

$$w_g(x_o)<\inf w_g(X\setminus\Gamma^{-1}(g)), \qquad (3.8)$$

so $g\in\Delta^1_{\Gamma(x_o),w_g}$ of (3.4). Hence, by corollary 2.4 and formula (3.6), we obtain (3.7).

Remark 3.1. We recall that the "complementary multifunction" $\Gamma^c:X\to 2^F$ is defined (see e.g. [1]) by

$$\Gamma^c(x)=F\setminus\Gamma(x) \qquad (x\in X). \qquad (3.9)$$

By (3.5), we have

$$(\Gamma^c)^{-1}(y)=\{x\in X\mid y\in F\setminus\Gamma(x)\}=X\setminus\Gamma^{-1}(y) \qquad (y\in F), \qquad (3.10)$$

and thus the assumption of theorem 3.1 means that $(\Gamma^c)^{-1}(g)$ is $(-W)$-convex, for each $g\in G$. This assumption is satisfied e.g. when $W=X^*$ (or $X^*\setminus\{0\}$), F is a topological linear space, Γ^c is upper semi-continuous (i.e., $(\Gamma^c)^{-1}(A)=\bigcup_{y\in A}(\Gamma^c)^{-1}(y)=\{x\in X\mid(\Gamma^c)(x)\cap A\neq\emptyset\}$ is closed for each closed subset of F) and $(\Gamma^c)^{-1}$ is quasi-convex in the sense of Oettli [6] (i.e., $(\Gamma^c)^{-1}(A)$ is convex for each convex subset A of F).

2) The family of sets $\Delta^2_{\Gamma(x_o),w}\subseteq F$ $(w\in W)$, defined by

$$\Delta^2_{\Gamma(x_o),w}=\{y\in F\mid w(x_o)>\sup w(X\setminus\Gamma^{-1}(y))\} \qquad (w\in W), \qquad (3.11)$$

satisfies (2.6). Moreover, if $W\subseteq\overline{R}^X$ satisfies $W=-W$, then the family (3.11) coincides with (3.4), since

$$\Delta^1_{\Gamma(x_o),w}=\Delta^2_{\Gamma(x_o),-w} \qquad (w\in W); \qquad (3.12)$$

hence, if $W=-W$, then formula (3.7) is equivalent to

$$\inf h(\Gamma(x_o))=\inf_{0\neq w\in W}\ \inf_{\substack{y\in F\\ w(x_o)>\sup w(X\setminus\Gamma^{-1}(y))}} h(y). \qquad (3.13)$$

3) The family of sets $\Delta^3_{\Gamma(x_o),w}\subseteq F$ $(w\in W)$, defined by

$$\Delta^3_{\Gamma(x_o),w}=\{y\in F\,|\,w(x_o)\notin w(X\setminus\Gamma^{-1}(y))\} \qquad (w\in W), \qquad (3.14)$$

satisfies (2.6); indeed, for G=F this is obvious, while if $G\neq F$ and $y\notin G=\Gamma(x_o)$, then $x_o\notin\Gamma^{-1}(y)$, whence $w(x_o)\in w(X\setminus\Gamma^{-1}(y))$, so $y\notin\Delta^3_{\Gamma(x_o),w}$. Moreover, we have

$$\Delta^1_{\Gamma(x_o),w}\subseteq\Delta^3_{\Gamma(x_o),w} \qquad (w\in W), \qquad (3.15)$$

and hence, by corollary 2.1 a),

$$\inf_{0\neq w\in W}\ \inf h(\Delta^1_{\Gamma(x_o),w})\geqslant\inf_{0\neq w\in W}\ \inf h(\Delta^3_{\Gamma(x_o),w})\geqslant\inf h(G); \qquad (3.16)$$

consequently, from theorem 3.1 we obtain

<u>Corollary 3.1. Under the assumptions of theorem 3.1, we have</u>

$$\inf h(\Gamma(x_o))=\inf_{0\neq w\in W}\ \inf_{\substack{y\in F\\ w(x_o)\notin w(X\setminus\Gamma^{-1}(y))}} h(y). \qquad (3.17)$$

4) Let us consider now family of sets

$$\Delta^4_{\Gamma(x_o),w}=\{y\in F\,|\,w(x_o)=\inf w(X\setminus\Gamma^{-1}(y))\} \qquad (w\in W), \qquad (3.18)$$

which, in general, need not satisfy (2.6). In particular, if $0\in W\subseteq\overline{R}^X$, then $\Delta^4_{\Gamma(x_o),0}=F\nsubseteq G$, and therefore we shall assume now that

$$0\notin W. \qquad (3.19)$$

<u>Proposition 3.1. Let F be a topological space, G a subset of F, $h:F\to\overline{R}$ a functional, X a locally convex space, and $\Gamma:X\to 2^F$ a multifunction satisfying</u> (3.2) <u>for some $x_o=x_o^G\in X$, such that</u>

$$\{y\in F\,|\,x_o\in\overline{\Gamma^{-1}(y)}\}\subseteq\overline{\Gamma(x_o)}. \qquad (3.20)$$

<u>Then, for $W=X^*\setminus\{0\}$ (where X^* is the conjugate space of</u> X), <u>we have</u>

$$\Delta^4_{\Gamma(x_o),w}\subseteq\overline{\Gamma(x_o)}=\overline{G} \qquad (w\in W). \qquad (3.21)$$

<u>Proof</u>. If $0\neq w\in X^*$ and $y\in\Delta^4_{\Gamma(x_o),w}$, then $x_o\notin\mathrm{Int}(X\setminus\Gamma^{-1}(y))$ (see e.g. [11], proof of lemma 2.1), whence $x_o\in\overline{\Gamma^{-1}(y)}$. Hence, by (3.20) and (3.2), we obtain (3.21).

<u>Remark 3.2. If we have</u>

$$\overline{\Gamma^{-1}(y)}\subseteq\overline{\Gamma}^{-1}(y) \qquad (y\in F), \qquad (3.22)$$

<u>where</u> $\overline{\Gamma}:X\to 2^F$ <u>is the multifunction defined</u> (see e.g.[1]) <u>by</u>

$$\overline{\Gamma}(x)=\overline{\Gamma(x)} \qquad (x\in X), \qquad (3.23)$$

<u>then</u> Γ <u>satisfies</u> (3.20). Indeed, if $y\in F$ and $x_o\in\overline{\Gamma^{-1}(y)}$, then, by (3.22), $x_o\in\overline{\Gamma}^{-1}(y)$, and hence, by (3.5), we obtain $y\in\overline{\Gamma}(x_o)=\overline{\Gamma(x_o)}$.

<u>Theorem 3.2. Let F be a topological space, G a subset of F,</u> $h:F\to\overline{R}$ <u>an upper semi-continuous functional, X a normed linear space, and</u> $\Gamma:X\to 2^F$ <u>a multifunction, satisfying</u> (3.2) <u>and</u> (3.20) <u>for some</u> $x_o=x_o^G\in X$, <u>and such that each</u> $X\setminus\Gamma^{-1}(g)$ $(g\in G)$ <u>is either a bounded convex set with non-empty interior, or a bounded closed convex set. Then</u>

$$\inf h(\Gamma(x_o))=\inf_{0\neq w\in X^*}\ \inf_{\substack{y\in F\\ w(x_o)=\inf w(X\setminus\Gamma^{-1}(y))}} h(y). \qquad (3.24)$$

<u>Proof</u>. For each $g\in G=\Gamma(x_o)$ we have $x_o\in\Gamma^{-1}(g)$. Hence, by our assumptions on $X\setminus\Gamma^{-1}(g)$ and by [8], corollary 2 and [9], Addendum, for each $g\in G$ there exists $w_g\in X^*\setminus\{0\}=W$ such that

$$w_g(x_o)=\inf w_g(X\setminus\Gamma^{-1}(g)), \qquad (3.25)$$

so $g\in\Delta^4_{\Gamma(x_o),w_g}$ of (3.18). Hence, by (3.21) and corollary 2.4 (with $W=X^*\setminus\{0\}$), we obtain (3.24).

5) If $W\subseteq\overline{R}^X$ satisfies $W=-W$, then the family of sets

$$\Delta^5_{\Gamma(x_o),w}=\{y\in F\mid w(x_o)=\sup w(X\setminus\Gamma^{-1}(y))\} \qquad (w\in W) \qquad (3.26)$$

coincides with (3.18), since

$$\Delta^5_{\Gamma(x_o),w}=\Delta^4_{\Gamma(x_o),-w} \qquad (w\in W); \qquad (3.27)$$

hence, <u>formula</u> (3.24) <u>is equivalent to</u>

$$\inf h(\Gamma(x_o))=\inf_{0\neq w\in X^*}\ \inf_{\substack{y\in F\\ w(x_o)=\sup w(X\setminus\Gamma^{-1}(y))}} h(y). \qquad (3.28)$$

6) Let us consider now the family of sets

$$\Delta^6_{\Gamma(x_o),w}=\{y\in F\mid w(x_o)\leq\inf w(X\setminus\Gamma^{-1}(y))\} \qquad (w\in W). \qquad (3.29)$$

The remarks made on $\Delta^4_{\Gamma(x_o),w}$, as well as proposition 3.1, remain valid, with similar proofs, for $\Delta^4_{\Gamma(x_o),w}$ replaced by $\Delta^6_{\Gamma(x_o),w}$. Moreover, we have

$$\Delta^4_{\Gamma(x_o),w} \subseteq \Delta^6_{\Gamma(x_o),w} \qquad (w\in W), \qquad (3.30)$$

and hence, by corollary 2.1 b),

$$\inf_{w\in W} \inf h(\Delta^4_{\Gamma(x_o),w}) \geq \inf_{w\in W} \inf h(\Delta^6_{\Gamma(x_o),w}) \geq \inf h(G), \qquad (3.31)$$

whenever h is a topological space and $h:F\to\overline{R}$ is upper semi-continuous; consequently, from theorem 3.2 we obtain

Corollary 3.2. Under the assumptions of theorem 3.2, we have

$$\inf h(\Gamma(x_o)) = \inf_{0\neq w\in X^*} \inf_{\substack{y\in F \\ w(x_o)\leq \inf w(X\setminus\Gamma^{-1}(y))}} h(y). \qquad (3.32)$$

7) If $W\subseteq \overline{R}^X$ satisfies $W=-W$, then the family of sets

$$\Delta^7_{\Gamma(x_o),w} = \{y\in F \mid w(x_o) \geq \sup w(X\setminus\Gamma^{-1}(y))\} \qquad (w\in W) \qquad (3.33)$$

coincides with (3.29), since

$$\Delta^7_{\Gamma(x_o),w} = \Delta^6_{\Gamma(x_o),-w} \qquad (w\in W); \qquad (3.34)$$

hence, formula (3.32) is equivalent to

$$\inf h(\Gamma(x_o) = \inf_{0\neq w\in X^*} \inf_{\substack{y\in F \\ w(x_o)\geq \sup w(X\setminus\Gamma^{-1}(y))}} h(y). \qquad (3.35)$$

§4. Applications to surrogate duality for systems

Generalizing [14], definition 3.1, by a "system" we shall mean a triple $(F \xrightarrow{u} X)$, consisting of two sets F,X and a mapping u of F into X. Given a system $(F\xrightarrow{u} X)$, where X is a linear space, a "target set" $\Omega\subset X$, with $u(F)\cap\Omega\neq\emptyset$, and a functional $h:F\to\overline{R}$, we shall consider now the primal infimization problem

$$(P)=(P_{u^{-1}(\Omega),h}) \quad \alpha=\alpha_{u^{-1}(\Omega),h} = \inf_{\substack{y\in F \\ u(y)\in\Omega}} h(y), \qquad (4.1)$$

i.e., (1.1) with $G=u^{-1}(\Omega)$. Then, the "natural" multifunction $\Gamma^n:X\to 2^F$, defined (see [10],[15]) by

$$\Gamma^n(x)=u^{-1}(\Omega+x) \qquad (x\in X), \qquad (4.2)$$

satisfies (3.2) with $x_o=0$ and, clearly,

$$(\Gamma^n)^{-1}(y)=\{x\in X \mid u(y)\in\Omega+x\}=u(y)-\Omega \qquad (y\in F). \qquad (4.3)$$

Remark 4.1. a) For $G=u^{-1}(\Omega)$, $\Gamma=\Gamma^n$ of (4.2), $x_o=0$ and

$W\subseteq X^{\#}$ (where $X^{\#}$ is the algebraic conjugate space of X), the family (3.4) becomes

$$\Delta^{1}_{u^{-1}(\Omega),w}=\{y\in F \mid w(u(y))>\sup w(X\setminus\Omega)\} \qquad (w\in W). \qquad (4.4)$$

Indeed, by (3.4) and (4.3),

$$\Delta^{1}_{\Gamma^{n}(0),w}=\{y\in F \mid 0<\inf_{x'\notin u(y)-\Omega} w(x')\} \qquad (w\in W). \qquad (4.5)$$

Observe now that $x'\notin u(y)-\Omega$ if and only if $u(y)-x'\in X\setminus\Omega$, and thus, writing $x'=u(y)-(u(y)-x')$, we have

$$X\setminus(u(y)-\Omega)=u(y)-(X\setminus\Omega) \qquad (y\in F). \qquad (4.6)$$

Hence, by $\Gamma^{n}(0)=G=u^{-1}(\Omega)$ and $W\subseteq X^{\#}$, we obtain

$$\Delta^{1}_{u^{-1}(\Omega),w} = \{y\in F \mid 0<w(u(y))+\inf_{x\in X\setminus\Omega} w(-x)\} = $$
$$=\{y\in F \mid 0<w(u(y))-\sup w(X\setminus\Omega)\} \qquad (w\in W),$$

i.e., (4.4). Note also that, by (4.3) and (4.6), $X\setminus(\Gamma^{n})^{-1}(y)$ (*where* $y\in F$) *is* $(-W)$-*convex if and only if* $X\setminus\Omega$ *is* W-*convex.*

b) If $(F\xrightarrow{u}X)$ is a "*linear system*" in the sense of [14] (i.e., F and X are locally convex spaces and $u:F\to X$ is a continuous linear mapping), then for $0\neq w\in X^{*}$ such that $\sup w(X\setminus\Omega)=+\infty$ we have $\Delta^{1}_{u^{-1}(\Omega),w}=\emptyset$, while for $0\neq w\in X^{*}$ such that $\sup w(X\setminus\Omega)<+\infty$, $\Delta^{1}_{u^{-1}(\Omega),w}$ is an open half-space in F.

From theorem 3.1 and remark 4.1 a), we obtain

Theorem 4.1. Let $(F\xrightarrow{u}X)$ *be a system, in which X is a linear space, let* $W\subseteq X^{\#}$, *let* Ω *be a subset of X, with* $u(F)\cap\Omega\neq\emptyset$, *such that* $X\setminus\Omega$ *is* W-*convex, and let* $h:F\to\overline{R}$. *Then*

$$\inf_{\substack{y\in F\\ u(y)\in\Omega}} h(y)=\inf_{0\neq w\in W}\ \inf_{\substack{y\in F\\ w(u(y))>\sup w(X\setminus\Omega)}} h(x). \qquad (4.7)$$

Replacing w by -w (or, alternatively, using (3.13)), we see that *if* $W=-W$, *then formula* (4.7) *is equivalent to*

$$\inf_{\substack{y\in F\\ u(y)\in\Omega}} h(y)= \inf_{0\neq w\in W}\ \inf_{\substack{y\in F\\ w(u(y))<\inf w(X\setminus\Omega)}} h(y). \qquad (4.8)$$

Remark 4.2. For $G=u^{-1}(\Omega)$, $\Gamma=\Gamma^{n}$ *of* (4.2), $x_{o}=0$ *and* $W\subseteq X^{\#}$, *the family* (3.14) *becomes*

$$\Delta^{3}_{u^{-1}(\Omega),w}=\{y\in F \mid w(u(y))\notin w(X\setminus\Omega)\} \qquad (w\in W). \qquad (4.9)$$

Indeed, by $\Gamma^n(0)=G=u^{-1}(\Omega)$, (4.3) and (4.6), we have

$$\Delta^3_{u^{-1}(\Omega),w}=\Delta^3_{\Gamma^n(0),w}=\{y\in F|\ 0\notin w(X\setminus(u(y)-\Omega))\}=$$

$$=\{y\in F|\ 0\notin w(u(y))-w(X\setminus\Omega)\} \qquad (w\in W).$$

From corollary 3.1 and remarks 4.2, 4.1 a) (or, from theorem 4.1), we obtain

<u>Corollary 4.1.</u> <u>Under the assumptions of theorem</u> 4.1, <u>we have</u>

$$\inf_{\substack{y\in F\\ u(y)\in\Omega}} h(y)=\inf_{0\neq w\in W}\ \inf_{\substack{y\in F\\ w(u(y))\notin w(X\setminus\Omega)}} h(y). \qquad (4.10)$$

<u>Remark 4.3.</u> a) Similarly to remark 4.1 a), <u>for</u> $G=u^{-1}(\Omega)$, $\Gamma=\Gamma^n$ <u>of</u> (4.2), $x_o=0$ <u>and</u> $W\subseteq X^{\#}$, <u>the family</u> (3.18) <u>becomes</u>

$$\Delta^4_{u^{-1}(\Omega),w}=\{y\in F|\ w(u(y))=\sup w(X\setminus\Omega)\} \qquad (w\in W). \qquad (4.11)$$

If $(F\xrightarrow{u}X)$ is a linear system, then for $0\neq w\in X^*$ such that $\sup w(X\setminus\Omega)=+\infty$ we have $\Delta^4_{u^{-1}(\Omega),w}=\emptyset$, while for $0\neq w\in X^*$ such that $\sup w(X\setminus\Omega)<+\infty$, $\Delta^4_{u^{-1}(\Omega),w}$ is a (closed) hyperplane in F.

b) When F is a topological space and X is a locally convex space, for $\Gamma=\Gamma^n$ of (4.2) and $x_o=0$ we have, by (4.3),

$$\overline{(\Gamma^n)^{-1}(y)}=\overline{u(y)-\Omega}=u(y)-\overline{\Omega} \qquad (y\in F), \qquad (4.12)$$

and hence Γ^n <u>satisfies</u> (3.20) <u>(with</u> $x_o=0$<u>)</u> <u>if and only if</u>

$$u^{-1}(\overline{\Omega})=\{y\in F|\ u(y)\in\overline{\Omega}\}\subseteq\overline{\{y\in F|\ u(y)\in\Omega\}}=\overline{u^{-1}(\Omega)}; \qquad (4.13)$$

in particular, when $u:F\to X$ is one-to-one, (4.13) is equivalent to the continuity of u^{-1}. Let us also note that if u is continuous, then $u^{-1}(\overline{\Omega})$ is closed, and hence we have the opposite inclusion to (4.13); thus, in this case, (4.13) is equivalent to

$$u^{-1}(\overline{\Omega})=\overline{u^{-1}(\Omega)}. \qquad (4.14)$$

From theorem 3.2 and remark 4.3, we obtain

<u>Theorem 4.2.</u> <u>Let</u> $(F\xrightarrow{u}X)$ <u>be a system, in which</u> F <u>is a topological space and</u> X <u>is a normed linear space, let</u> Ω <u>be a subset of</u> X, <u>with</u> $u(F)\cap\Omega\neq\emptyset$, <u>satisfying</u> (4.13) <u>and such that</u> $X\setminus\Omega$ <u>is either a bounded convex set with non-empty interior, or a bounded closed convex set, and let</u> $h:F\to\overline{R}$ <u>be an upper semi-continuous functional. Then</u>

$$\inf_{\substack{y\in F\\ u(y)\in\Omega}} h(y)=\inf_{0\neq w\in X^*}\ \inf_{\substack{y\in F\\ w(u(y))=\sup w(X\setminus\Omega)}} h(y). \qquad (4.15)$$

Replacing w by $-w$ (or, alternatively, using (3.26)), we see that formula (4.15) is equivalent to

$$\inf_{\substack{y\in F\\ u(y)\in\Omega}} h(y)=\inf_{0\neq w\in X^*}\ \inf_{\substack{y\in F\\ w(u(y))=\inf w(X\setminus\Omega)}} h(y). \qquad (4.16)$$

Remark 4.4. Similarly to remark 4.1 a) above, for $G=u^{-1}(\Omega)$, $\Gamma=\Gamma^n$ of (4.2), $x_o=0$ and $W\subseteq X^\#$, the family (3.29) becomes

$$\Delta^6_{u^{-1}(\Omega),w}=\{y\in F\,|\,w(u(y))\geqslant\sup w(X\setminus\Omega)\} \qquad (w\in W). \qquad (4.17)$$

If $(F\xrightarrow{u}X)$ is a linear system, then for $0\neq w\in X^*$ such that $\sup w(X\setminus\Omega)=+\infty$ we have $\Delta^6_{u^{-1}(\Omega),w}=\emptyset$, while for $0\neq w\in X^*$ such that $\sup w(X\setminus\Omega)<+\infty$, $\Delta^6_{u^{-1}(\Omega),w}$ is a closed half-space in F.

From corollary 3.2 and remark 4.4 (or, from theorem 4.2 and (3.31) for $\Gamma=\Gamma^n$, $x_o=0$), we obtain

Corollary 4.2. Under the assumptions of theorem 4.2, we have

$$\inf_{\substack{y\in F\\ u(y)\in\Omega}} h(y)=\inf_{0\neq w\in X^*}\ \inf_{\substack{y\in F\\ w(u(y))\geqslant\sup w(X\setminus\Omega)}} h(y). \qquad (4.18)$$

Replacing w by $-w$ (or, alternatively, using (3.33)), we see that formula (4.18) is equivalent to

$$\inf_{\substack{y\in F\\ u(y)\in\Omega}} h(y)=\inf_{0\neq w\in X^*}\ \inf_{\substack{y\in F\\ w(u(y))\leqslant\inf w(X\setminus\Omega)}} h(y). \qquad (4.19)$$

§5. The particular case of systems $(F\xrightarrow{I_F}F)$

Let us consider separately the particular case when $X=F$, $u=I_F$ (the identity operator) and $\Omega=G\subset F$. In this case, $(F\xrightarrow{u}X)$ is a system, problem (4.1) reduces to problem (1.1), and the natural multifunction Γ^n of (4.2) reduces to the "standard" multifunction $\Gamma^s:F\to 2^F$, defined (see [10], [15]) by

$$\Gamma^s(x)=G+x \qquad (x\in F), \qquad (5.1)$$

which satisfies (3.2) with $x_o=0$ and

$$(\Gamma^s)^{-1}(y)=y-G \qquad (y\in F). \qquad (5.2)$$

In this section we shall assume that $W\subseteq\bar{R}^F$.

Remark 5.1. a) The family (4.4) becomes now

$$\Delta^1_{G,w} = \{y \in F \mid w(y) > \sup w(F \setminus G)\} \qquad (w \in W). \qquad (5.3)$$

b) If F is a locally convex space, then for $0 \neq w \in F^*$ such that $\sup w(F\setminus G) = +\infty$ we have $\Delta^1_{G,w} = \emptyset$, while for $0 \neq w \in F^*$ such that $\sup w(F\setminus G) < +\infty$, $\Delta^1_{G,w}$ is an open half-space, supporting the set $F\setminus G$ (i.e., $\Delta^1_{G,w} \cap (F\setminus G) = \emptyset$ and the boundary of $\Delta^1_{G,w}$ is a support hyperplane of $F\setminus G$; for the definition of support hyperplanes, see e.g. [11], § I.0), and we have

$$\beta = \inf_{w \in (F\setminus G)^s} \inf h(\Delta^1_{G,w}), \qquad (5.4)$$

where

$$(F\setminus G)^s = \{w \in W \mid w \neq 0,\ \sup w(F\setminus G) < +\infty\}. \qquad (5.5)$$

Thus, if $W = F^*$ or $W = F^* \setminus \{0\}$, formula (5.4) means that

$$\beta = \inf_{D \in \mathcal{D}_{F\setminus G}} \inf h(D), \qquad (5.6)$$

where $\mathcal{D}_{F\setminus G}$ denotes the collection of all open half-spaces which support the set $F\setminus G$. We shall omit the corresponding geometric interpretations of the β's occurring in the sequel, and, for simplicity, we shall work only with β's written similarly to (3.6).

From theorem 4.1 and remark 5.1 a), we obtain

Theorem 5.1. *Let F be a linear space, let $W \subseteq F^\#$, let G be a subset of F such that $F\setminus G$ is W-convex, and let $h: F \to \overline{R}$. Then*

$$\inf h(G) = \inf_{0 \neq w \in W} \inf_{\substack{y \in F \\ w(y) > \sup w(F\setminus G)}} h(y). \qquad (5.7)$$

Again, if $W = -W$, then formula (5.7) is equivalent to

$$\inf h(G) = \inf_{0 \neq w \in F^*} \inf_{\substack{y \in F \\ w(y) < \inf w(F\setminus G)}} h(y). \qquad (5.8)$$

Remark 5.2. The family (4.9) becomes now

$$\Delta^3_{G,w} = \{y \in F \mid w(y) \notin w(F\setminus G)\} \qquad (w \in W). \qquad (5.9)$$

From corollary 4.1 (or, from theorem 5.1), we obtain

Corollary 5.1. *Under the assumptions of theorem 5.1, we have*

$$\inf h(G) = \inf_{0 \neq w \in F^*} \inf_{\substack{y \in F \\ w(y) \notin w(F\setminus G)}} h(y). \qquad (5.10)$$

Remark 5.3. a) The family (4.11) becomes now

$$\Delta^4_{G,w}=\{y\in F \mid w(y)=\sup w(F\setminus G)\} \qquad (w\in W). \qquad (5.11)$$

If F is a locally convex space, then for $0\neq w\in F^*$ such that $\sup w(F\setminus G)=+\infty$ we have $\Delta^4_{G,w}=\emptyset$, while for $0\neq w\in F^*$ such that $\sup w(F\setminus G)<+\infty$, $\Delta^4_{G,w}$ is a support hyperplane of $F\setminus G$.

b) When $X=F$ is a locally convex space, $u=I_F$ and $\Omega=G\subset F$, conditions (4.13), (4.14) are obviously satisfied (and so is even (3.22) for $\Gamma=\Gamma^S$).

From theorem 4.2 and remark 5.3, we obtain

Theorem 5.2. Let F be a normed linear space, G a subset of F such that $F\setminus G$ is either a bounded convex set with non-empty interior, or a bounded closed convex set, and let $h:F\to\overline{R}$ be an upper semi-continuous functional. Then

$$\inf h(G)=\inf_{0\neq w\in F^*}\ \inf_{\substack{y\in F\\ w(y)=\sup w(F\setminus G)}} h(y). \qquad (5.12)$$

Remark 5.4. In the case when h is a finite continuous convex functional on F, theorem 5.2 has been given in [9], theorem 2.2 and Addendum, but the proof given there remains also valid for any upper semi-continuous functional h.

Note that, again, formula (5.12) is equivalent to

$$\inf h(G)=\inf_{0\neq w\in F^*}\ \inf_{\substack{y\in F\\ w(y)=\inf w(F\setminus G)}} h(y). \qquad (5.13)$$

Remark 5.5. The family (4.17) becomes now

$$\Delta^6_{G,w}=\{y\in F \mid w(y)\geq \sup w(F\setminus G)\} \qquad (w\in W). \qquad (5.14)$$

If F is a locally convex space, then for $0\neq w\in F^*$ such that $\sup w(F\setminus G)=+\infty$ we have $\Delta^6_{G,w}=\emptyset$, while for $0\neq w\in F^*$ such that $\sup w(F\setminus G)<+\infty$, $\Delta^6_{G,w}$ is a closed half-space supporting the set $F\setminus G$ (i.e., $(\operatorname{Int}\Delta^5_{G,w})\cap(F\setminus G)=\emptyset$ and the boundary of $\Delta^6_{G,w}$ is a support hyperplane of $F\setminus G$).

From corollary 4.2 and remarks 5.5 and 5.3 b) (or, from theorem 5.2 and (3.31) for $\Gamma=\Gamma^S$, $x_0=0$), we obtain

Corollary 5.2. Under the assumptions of theorem 5.2, we have

$$\inf h(G)=\inf_{0\neq w\in F^*}\ \inf_{\substack{y\in F\\ w(y)\geq \sup w(F\setminus G)}} h(y). \qquad (5.15)$$

Note that, again, formula (5.15) is equivalent to

$$\inf h(G)=\inf_{0\neq w\in F^*}\ \inf_{y\in F} h(y). \qquad (5.16)$$

Finally, let us mention that, in [17], §§2-4, using level set methods, we give similar generalizations of the results of [7] on maximization of convex functionals on convex sets in linear spaces.

REFERENCES

[1] S.Dolecki, Semicontinuity in constrained optimization. Part I.1. Metric spaces. Control Cybern. 7(1978), 5-26.

[2] S.Dolecki and S.Kurcyusz, On Φ-convexity in extremal problems. SIAM J.Control Optim.16(1978), 277-300.

[3] K.Fan, On the Krein-Milman theorem. In: Convexity (V.Klee, ed.), Proc.Symposia Pure Math.7.Amer.Math.Soc., Providence, 1963, pp.211-220.

[4] C.Franchetti and I.Singer , Best approximation by elements of caverns in normed linear spaces Boll.Un.Mat.Ital.(5) 17-B (1980), 33-43.

[5] P.-J.Laurent, Approximation et optimisation. Hermann, Paris, 1972.

[6] W.Oettli, Optimality conditions involving generalized convex mappings. In: Generalized concavity in optimization and economics (S.Schaible and W.T.Ziemba, eds.), Acad. Press, New York, 1981, pp.227-238.

[7] I.Singer, Maximization of lower semi-continuous convex functionals on bounded subsets of locally convex spaces. I: Hyperplane theorems. Appl.Math.Optimiz.5(1979), 349-362.

[8] I.Singer, Extension with larger norm and separation with double support in normed linear spaces. Bull.Austral.Math. Soc.21(1980), 93-105.

[9] I.Singer, Minimization of continuous convex functionals on complements of convex subsets of locally convex spaces. Math.Operationsforsch.Stat.Ser.Optim.11(1980), 221-234.

[10] I.Singer, Duality theorems for constrained convex optimization. Control Cybern.9(1980), 37-45.

[11] I.Singer, Optimization by level set methods. I: Duality formulae. In: Optimization: Theory and algorithms (Proceedings, Internat.Confer.in Confolant, March 1981; J.-B.Hiriart-Urruty, W.Oettli and J.Stoer, eds.), Lecture Notes in Pure and Appl.Math.86, Marcel Dekker, New York, 1983, pp.13-43.

[12] I.Singer, Optimization by level set methods. III: Characterizations of solutions in the presence of duality. Numer. Funct.Anal.Optim.4(1981-1982), 151-170.

[13] I.Singer, Optimization by level set methods. IV: Generalizations and complements. Numer.Funct.Anal.Optim.4(1981-1982), 279-310.

[14] I.Singer, Surrogate dual problems and surrogate Lagrangians, J.Math.Anal.Appl.98(1984), 31-71.

[15] I.Singer, A general theory of surrogate dual and perturbational extended surrogate dual optimization problems.J.Math. Anal.Appl.104(1984), 351-389.

[16] I.Singer, A general theory of dual optimization problems. J.Math.Anal.Appl.(to appear). Preprint INCREST (Bucharest) 67/1984.

[17] I.Singer, Generalizations of convex supremization duality. (to appear). Preprint INCREST (Bucharest) 74/1985.

[18] H.Tuy, Global minimization of a concave function subject to mixed linear and reverse convex constraints. Preprint Inst. Math.Hanoi 26/1982.