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DERIVATIONS OF VON NEUMANN ALGEBRAS INTO THE COMPACT IDEAL SPACE OF A SEMIFINITE ALGEBRA ARE INNER

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Sorin POPA* and Florin RADULESCU*

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*) Department of Mathematics, The National Institute for Scientific and Technical Creation, Bd.Pacii 220, 79622 Bucharest, Romania.

1. INTRODUCTION

Let M be a semifinite von Neumann algebra and denote by J(M) the norm closed two sided ideal generated by the finite projections of M. Let $N \subseteq M$ be a subalgebra of M. A derivation of N into J(M) is a linear application $\delta : N \longmapsto J(M)$ satisfying $\delta(xy) = \delta(x)y + x\delta(y)$, for $x, y \in N$. For instance if $K \in J(M)$ then the derivation $\delta(x) = (\mathrm{ad}K)(x) = Kx - xK$ is of this type. Such derivations implemented by elements in $\mathcal{J}(M)$ are called inner. A typical example of a derivation which is not inner is as follows: take $M = \mathcal{B}(L^2(\mathbb{T}, \mu))$, where μ is the Lebesgue measure on the thorus $\mathbb{T}$, let $N = C(\mathbb{T})$ act on $L^2(\mathbb{T}, \mu)$ by left multiplication and define $\delta(x) = (\mathrm{ad}\, P_{H^2})(x)$ where P_{H^2} is the projection onto the Hardy subspace $H^2(\mathbb{T}, \mu)$. Then it is easy to see that $\delta(x) \in \mathcal{K}(\mathcal{H}) = J(\mathcal{B}(\mathcal{H}))$ for $x \in C(\mathbb{T})$ and that δ is not implemented by a compact operator.

We will however show that if N is selfadjoint and weakly closed in M then all its derivations into J(M) are inner and thus obtain the following general theorem:

1.1. THEOREM. Let N be a w^*-subalgebra of M and $\delta : N \longmapsto J(M)$ a derivation. Then there exists an element $K \in J(M)$ such that $\delta = \mathrm{ad}\, K$.

Results of this type first appear in a paper by Johnson and Parrott in the early 70's ([3]). In that paper Johnson and Parrott were interested wanted to characterise the commutant modulo the ideal $\mathcal{K}(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H})$ of a von Neumann algebra $N \subseteq \mathcal{B}(\mathcal{H})$. They noted that in order to identify it with the compact perturbations of the commutant of N in $\mathcal{B}(\mathcal{H})$ it suffices to show that any derivation

$\delta: N \mapsto \mathcal{K}(\mathcal{H})$ is inner. They proved that this is indeed the case if N has no certain type II_1 factors as direct summands. To do this they first solved the case when N is abelian the other cases being rather easy consequences of it. The general type II_1 case was proved recently in [7] by different techniques and using more of the ergodic theory of the type II_1 factors.

In [4] it is studied this derivation problem in the more general setting when $\mathcal{B}(\mathcal{H})$ is replaced by a semifinite von Neumann algebra, $\mathcal{K}(\mathcal{H})$ by the ideal J(M) and the center of N is assumed to contain the center of M. Under this hypothesis it is proved that if N is either an abelian or a properly infinite von Neumann algebra then any derivation of N into J(M) is inner.

Although the proof at the general theorem 1.1 that we present in this paper is inspired in certain places from [3] and [7] our approach is rather new even for $M=\mathcal{B}(\mathcal{H})$. We will now present some of the ideas behind our proof.

We begin by considering a new norm on the algebra M by $|||T||| = \sup\{\|Tx\|_\varphi \mid x \in M, \|x\| \leq 1, \|x\|_\varphi \leq 1\}$, where φ is a semifinite trace on M. It turns out that in many situations the right correspondent, in an arbitrary semifinite algebra M, of the uniform norm on $\mathcal{B}(\mathcal{H})$ is the norm $|||\ |||$.

We then prove theorem 1.1 in the case N is atomic and abelian. In the proof we define the operator implementing δ as $\sum_i \delta(e_i)e_i$, where e_i are the atoms of N and the series is strongly convergent, and we use an adaption of a trick in [3] to show that $\sum_i \delta(e_i)e_i \in J(M)$.

By the atomic abelian case and by the same argument as in 4.1 [7] (for $M=\mathcal{B}(\mathcal{H})$) we prove a continuity result namely that if N is finite and countably decomposable then δ is continuous from the unit ball of N with the strong operator topology into J(M) with the norm $|||\ |||$. Using this result we prove that if an element T is in $K_\delta = \overline{co}^w\{\delta(u)u^* \mid u$ unitary element in $N\} \subseteq M$ and implements δ on N then it is in J(M). From this we easily get the proof of the theorem for finite type I and properly infinite algebras and also reduce the remaining type II_1 case to the situation when N is separable and M is countably decomposable. Moreover, by using the Ryll-Nardzewski fixed point theorem in the same way it is used to prove the Kadison--Sakai theorem on derivations of von Neumann algebras and other derivation problems (see e.g. [9]), we make the reduction to the case when $N' \cap M$ contains no finite projections of M.

Finally we prove the type II_1 case under the above assumptions: To construct a candidate for the operator $K \in J(M)$ implementing δ on N we show that N has a maximal abelian *-subalgebra $A \subset N$ such that $A' \cap M$ contains no finite projections of M. The proof of this fact is inspired from [6]. Since A is abelian by the type I case there exists $K \in J(M)$ implementing δ on A and the rest of the proof shows that in fact this K implements δ on all N. To this end we proceed by contradiction following the lines of the proof in [7]. The assumption $\delta_o = \delta - \mathrm{ad}\, K \neq 0$ shows that $\delta_o(v) \neq 0$ for some unitary element $v \in N$. Then with the help of A and v and using some technical devices symilar to 2.1 in [7] we construct a sequence of abelian subalgebras A_n in N on which δ_o behaves as bad as possible. More precisely we construct the algebras A_n together with some finite projections $e_n \in M$ so that if we consider M as acting on $L^2(M,\varphi)$ then the compressions of $\delta_o | A_n$ to the spaces $\overline{A_n e_n} \subseteq L^2(M,\varphi)$ are spatially isomorphic to a sequence of derivations $\delta_n : L^\infty(T,\mu) \longmapsto \mathcal{B}(L^2(T,\mu))$. We do this in such a way that the derivations δ_n behave more and more like ad P_{H^2} and moreover so that by the continuity result the limit ad P_{H^2} follows so--normic continuous. This is easily seen to be a contradiction. We mention that the construction of the finite projections e_n, which doesn't appear in [7], is essential here and carry most of the technical difficulties of passing from the case $M=\mathcal{B}(\mathcal{H})$ to the general case. Moreover the consideration of e_n can be used to slightly simplify the proof of the case $M=\mathcal{B}(\mathcal{H})$ in [7].

2. SOME PRELIMINARIES

2.1. Let M be a semifinite von Neumann algebra with a fixed normal semifinite faithful trace φ and assume M and φ are so that a projection $e \in M$ is finite if and only if $\varphi(e) < \infty$. Moreover assume that for any minimal projection $e \in M$, $\varphi(e) \leq 1$. Denote $M_\varphi = \{x \in M \mid \varphi(x^*x) < \infty\}$ and, for $x \in M$, $\|x\|_\varphi = \varphi(x^*x)^{1/2}$. Let H_φ be the Hilbert space completion of M_φ in the norm $\| \ \|_\varphi$.

If T is a linear bounded operator acting on H_φ then we denote by

$$|||T||| = \sup\{ \|Tx\|_\varphi \mid x \in M_\varphi , \ \|x\| \leq 1, \ \|x\|_\varphi \leq 1\} .$$

This norm will play an important role in the sequel. Note that $|||T||| \leq \|T\|$ and that the equality holds if M is the algebra of all linear bounded operators on a Hilbert space but fails if M is nonatomic.

2.2. Let J(M) be the norm closed two sided ideal of M generated by the finite projections of M. Thus an element $x \in M$ is in J(M) if and only if all the spectral projections $E_{[t,\infty)}(|x|)$ of $|x|$ corresponding to intervals $[t,\infty)$ with $t>0$ are finite projections, or equivalently $\varphi(E_{[t,\infty)}(|x|)) < \infty$.

Note that if $M = \mathcal{B}(\mathcal{H})$ then J(M) is just the ideal of compact operators on $\mathcal{H}$.

2.3. Let $K \in J(M)$ and $\{e_n\}_{n\in N}$ a sequence of mutually orthogonal projections in M. If $M = \mathcal{B}(\mathcal{H})$ then it follows that $\|Ke_n\| \longrightarrow 0$ and $\|e_n K\| \longmapsto 0$. In general this is no longer true but still we have $|||Ke_n||| \longmapsto 0$, $|||e_n K||| \longmapsto 0$. Indeed, to prove this, since

K is a linear combination of four positive elements in $J(M)$, we may assume K is positive. Let $\varepsilon > 0$ and $e = E_{[\varepsilon/2,\infty)}(K)$ then $\varphi(e) < \infty$ and since $\{e_n\}$ tends weakly to zero, $\|ee_n\|_\varphi^2 =$ $= \|e_n e\|_\varphi^2 = \varphi(e\, e_n e) \longmapsto 0$. But if $x \in M_\varphi$, $\|x\|_\varphi \leq 1$, $\|x\| \leq 1$ then

$$\|Ke_n x\|_\varphi \leq \|Kee_n x\|_\varphi + \|K(1-e)e_n x\|_\varphi \leq$$

$$\|K\| \ \|x\| \ \|ee_n\|_\varphi + \|K(1-e)\| \ \|e_n x\|_\varphi \leq \|K\| \ \|ee_n\|_\varphi + \varepsilon/2 \ ,$$

so that if n is big enough $\|Ke_n x\|_\varphi \leq \varepsilon$ independently of x and thus $|||Ke_n||| \longmapsto 0$. Similary $|||e_n K||| \longmapsto 0$.

2.4. Another feature of the norm $|||\ |||$ is that in the Calkin algebra $M/J(M)$ it gives the same norm as does the usual uniform norm. More precisely we have for any $x \in M$,

$$\inf\{\|x+K\| \mid K \in J(M)\} = \inf\{|||x+K||| \mid K \in J(M)\}.$$

To prove this we only need to show that if $y \in M$ and $\varepsilon > 0$ then there exists $K \in J(M)$ such that $\|y+K\| \leq |||y||| + \varepsilon$.

So let $e_t = E_{(t,\infty)}(|y|)$ and $t_o = \inf\{t \geq 0 \mid \varphi(e_t) < \infty\}$. Note that for any $t \geq 0$, $\|y(1-e_t)\| \leq t$. Let $K = -ye_{t_o+\varepsilon/2}$. Then $K \in J(M)$ and $\|y+K\| = \|y(1-e_{t_o+\varepsilon/2})\| \leq t_o + \varepsilon/2$. Since $\varphi(e_{t_o-\varepsilon/2}) = \infty$, there exists a projection $e_o \leq e_{t_o-\varepsilon/2}$ such that $\varphi(e_o) \leq 1$. Then

$$|||y||| \geq \|ye_o\|_\varphi \geq t_o - \varepsilon/2 \text{ so that } |||y||| \geq (t_o + \varepsilon/2) - \varepsilon \geq \|y+k\| - \varepsilon .$$

2.5. Since the norm $|||\ |||$ is a supremum of vector norms it is inferior semicontinuous with respect to the weak operator topology. Indeed if T_i tends in the weak operator topology to T then $\|T\xi\| \leq \limsup_i \|T_i \xi\|$ so that

$$|||T||| = \sup\{\|T\xi\| \mid \xi \in M, \varphi(\xi^*\xi) \leq 1, \|\xi\| \leq 1\} \leq$$

$$\lim_i \sup(\sup\{\|T_i\xi\| \mid \xi \in M, \varphi(\xi^*\xi) \leq 1, \|\xi\| \leq 1\}) =$$

$$= \lim_i \sup |||T_i||| .$$

2.6. We now prove a version of Johnson and Parrott trick in [3].

LEMMA. Let $N \subseteq M$ be a von Neumann subalgebra and $T \in M$ such that $[T,N] \subseteq J(M)$ and $T \in J(M)$. Suppose the set $\mathcal{P} = \{f \in \mathcal{P}(N) \mid \|fTf\|_{ess} = \|T\|_{ess}\}$ contains no minimal projections of A.

Then there exists a sequence of mutually orthogonal projections $\{e_n\}$ in N such that

$$|||e_n T e_n||| > \|T\|_{ess}/2.$$

Proof. Let $\mathcal{F}$ be a maximal chain in $\mathcal{P}$ and let $f_o = \inf \mathcal{F}$. Suppose $f_o \in \mathcal{P}$. Since $\mathcal{P}$ has no minimal projections of N, there exist nonzero mutually orthogonal projections f_1, f_2 in A with $f_o = f_1 + f_2$. Since $[T, f_i] \in J(M)$ for i=1,2 we have

$$\|T\|_{ess} = \|f_o T f_o\|_{ess} = \max\{\|f_1 T f_1\|_{ess}, \|f_2 T f_2\|_{ess}\}$$

which contradicts the maximality of $\mathcal{F}$. Thus $f_o \notin \mathcal{P}$ so that $\|f_o T f_o\|_{ess} < \|T\|_{ess}$. Then the chain $\mathcal{F}' = \{f - f_o \mid f \in \mathcal{F}\}$ decreases to 0 and since

$$\max\{\|(f-f_o)T(f-f_o)\|_{ess}, \|f_o T f_o\|_{ess}\} = \|T\|_{ess} ,$$

we have that $\|f'Tf'\|_{ess} = \|T\|_{ess}$ for any f' in $\mathfrak{F}'$.

We can now construct recursively the required sequence $\{f_n\}_{n\in N}$. Assume $f'_1, \dots, f'_n$ are n projections in $\mathfrak{F}'$ with $|||(f'_k - f'_{k-1})T(f'_k - f'_{k-1})||| > \|T\|_{ess}/2$, $n \geq k > 1$. Since $\mathfrak{F}'$ is a chain decreasing to zero, by the inferior semicontinuity of the norm $|||\ |||$ it follows that there exists a projection $f'_{n+1} \in \mathfrak{F}'$ with $f'_{n+1} \leq f'_n$ such that

$$|||(f'_n - f'_{n+1})T(f'_n - f'_{n+1})||| > |||f'_n T f'_n||| \ /2 \ .$$

But by 2.4 $|||f'_n T f'_n||| \geq \|f'_n T f'_n\|_{ess} = \|T\|_{ess}$. Thus $|||(f'_n - f'_{n+1})T(f'_n - f'_{n+1})||| > \|T\|_{ess}/2$ so that $f_n = f'_{n+1} - f'_n$ will do.

Q.E.D.

2.7. Let now M be an arbitrary semifinite von Neumann algebra and $N \subseteq M$ a weakly closed *-subalgebra of it. Let $\delta : N \mapsto J(M)$ be a derivation. By [3] δ is norm continuous and by [2] it is weakly continuos. Let p be the unit of N and $K = \delta(p)p - p\delta(p) \in J(M)$. Then $Kp - pK = \delta(p)p - 2p\delta(p)p + p\delta(p) =$ $= (\delta(p) - p\delta(p)) - (2\delta(p^2)p - 2\delta(p)p^2) + p\delta(p) = \delta(p)$ so that $(\delta - \text{ad}\ k)(p) = 0$ and $(\delta - \text{ad}\ K)(x) = (\delta - \text{ad}\ K)(pxp) = p(\delta - \text{ad}\ K)(x)p$ which shows that $\delta - \text{ad}\ K$ takes values in pMp.

This shows that in order to prove the theorem 1.1 we may assume the weakly closed *-subalgebra $N \subseteq M$ has the same unit as M, i.e. N is a von Neumann subalgebra of M. Therefore in all the rest of the paper the subalgebra N will be considered to have the same unit as M.

2.8. Let $\{p_i\}_{i\in I}$ be a family of mutually orthogonal projections in the center of M with $\sum_i p_i = 1$. Assume that for each i there exists $K_i \in J(M)_{p_i} = J(M_{p_i})$ such that $\delta(x)p_i = \text{ad}\ K_i(x)$

for all $x \in N$. Then $K = \sum_{i\in I} K_i$ is in $J(M)$ and $\delta = \mathrm{ad}\, K$ on N.

Since in a semifinite von Neumann algebra M there exist mutually orthogonal central projections p_i with $\sum p_i = 1$ such that each M_{p_i} has a normal semifinite faithful trace φ_i with "a projection $f \in M_{p_i}$ is finite if and only if $\varphi_i(f) < \infty$ and if f is minimal then $\varphi_i(f) \leq 1$", it follows by the preceding observation that it is sufficient to prove theorem 1.1 for each M_{p_i} , i.e. under the assumptions of 2.1.

2.9. Let $N_o \subseteq N$ be a finite dimensional von Neumann subalgebra of N, $\mathcal{U}_o$ the unitary compact group of N_o and λ the normalized Haar measure on $\mathcal{U}_o$.

Then $K = \int \delta(u) u^* d\lambda(u) \in J(M)$ satisfies for any $u_o \in \mathcal{U}_o$:

$$Ku_o - u_oK = \int \delta(u) u^* u_o d\lambda(u) - \int u_o \delta(u) u^* d\lambda(u) =$$

$$= \int \delta(u)(u_o^* u)^* d\lambda(u) - \int u_o \delta(u) u^* d\lambda(u) =$$

$$= \int \delta(u_o u) u^* d\lambda(u) - \int u_o \delta(u) u^* d\lambda(u) =$$

$$= \delta(u_o) \int d\lambda(u) + \int u_o \delta(u) u^* d\lambda(u) - \int u_o \delta(u) u^* d\lambda(u) = \delta(u_o) \ .$$

Thus $(\delta - \mathrm{ad}K)(x_o) = 0$ for any $x_o \in N_o$. In particular this shows that if N is a finite direct sum, then to prove 1.1 for $N \subseteq M$ it is sufficient to prove it for each summand.

3. THE ATOMIC ABELIAN CASE

In this section we prove the theorem in the case N is isomorphic to the algebra $l^\infty(I)$ for a set I of arbitrary cardinality.

To do this let $\{e_i\}_{i\in I}$ be the minimal projections of

$N = l^{\infty}(I)$ and note first that the series $\sum_{i\in I}\delta(e_i)e_i$ is convergent in the strong operator topology. Indeed, the sequence is bounded because if $e_1, e_2, \ldots, e_n \in \{e_i\}_{i\in I}$ then

$$(*)\qquad \sum_{k=1}^{n}\delta(e_k)e_k = \sum_{k,l=1}^{n}\int z_k\bar{z}_l\ \delta(e_k)e_l\ d\lambda(z) =$$

$$= \int\delta\Big(\sum_{k=1}^{n} z_k e_k\Big)\Big(\sum_{l=1}^{n}\bar{z}_l e_l\Big)d\lambda(z)\ ,$$

where λ is the normalized Harr measure on the thorus $\mathbb{T}^n$ and $z=(z_1, z_2, \ldots, z_n)\in\mathbb{T}^n$, so that

$$\Big\|\sum_{k=1}^{n}\delta(e_k)e_k\Big\| \le \int\Big\|\delta\Big(\sum_{k=1}^{n} z_k e_k\Big)\Big(\sum_{i=1}^{n}\bar{z}_i e_i\Big)\Big\|\ d\lambda(z) \le \|\delta\|.$$

Now if M is normally represented on some Hilbert space $\mathcal{H}$, $\xi\in\mathcal{H}$ and $\varepsilon>0$ then there exists a finite set $I_o\subseteq I$ such that $\|\xi - (\sum_{i\in I_o} e_i)\xi\| < \varepsilon$ and thus for any finite set $J_o\subseteq I$ with $J_o\cap I_o=\emptyset$ we have

$$\Big\|\sum_{i\in J_o}\delta(e_j)e_j\xi\Big\| \le \varepsilon\|\delta\| + \Big\|\Big(\sum_{j\in J_o}\delta(e_j)e_j\Big)\Big(\sum_{i\in I_o}e_i\Big)\xi\Big\| = \varepsilon\|\delta\| .$$

which shows that $\sum_{i\in I}\delta(e_i)e_i\xi$ is convergent for any $\xi\in\mathcal{H}$.

Let $T=\sum_{i\in I}\delta(e_i)e_i$. Since δ is a derivation and

$(\sum_{i\in I}\delta(e_i)e_i)e_{i_o} = \delta(e_{i_o})e_{i_o}$ we have

$$Te_{i_o} - e_{i_o}T = \delta(e_{i_o})e_{i_o} - \sum_{i\in I} e_{i_o}\delta(e_i)e_i =$$

$$= \delta(e_{i_o})e_{i_o} - \sum_{i\in I}\delta(e_{i_o}e_i)e_i + \delta(e_{i_o})\sum_{i\in I}e_i =$$

$$= \delta(e_{i_o})e_{i_o} - \delta(e_{i_o})e_{i_o} + \delta(e_{i_o}) = \delta(e_{i_o}).$$

Since both δ and ad T are weakly continuous on M and the linear span of $\{e_i\}_{i\in I}$ is weakly dense in N= $l^\infty(I)$ it follows that δ=ad T on N.

We show that T is in J(M). Suppose $T\notin J(M)$. Denote by

$$\mathcal{P} = \{ f \in \mathcal{P}(N) \mid \| fTf \|_{ess} = \| T \|_{ess} \}.$$

Then $\mathcal{P}$ contains no minimal projections of N. Indeed, because by the definition of T, for any i, $e_i T e_i = 0$. Thus by Lemma 2.6, there exists a sequence of mutually orthogonal projections $\{f_n\}_{n\in\mathbb{N}}$ in N such that

$$|\!|\!| f_n T f_n |\!|\!| > \| T \|_{ess}/2 > 0.$$

Moreover, by the inferior semicontinuity of the norm $|\!|\!|\ \ |\!|\!|$ we may assume each projection f_n is the sum of a finite set $J_n \subseteq J$ of minimal projections in N. But by (*) we have

$$Tf_n = \sum_{j\in J_n} \delta(e_j)e_j = \int \delta(\sum_{i\in J_n} z_i e_i)(\sum_{j\in J_n} \bar{z}_j e_j)\, d\lambda(z) ,$$

so that

$$\int |\!|\!| f_n \delta(\sum_{i\in J_n} z_i e_i)(\sum_{j\in J_n} \bar{z}_j e_j) f_n |\!|\!| \ d\lambda(z) \geq |\!|\!| f_n T f_n |\!|\!| \geq \| T \|_{ess}/2 ,$$

which implies that for some $u_n = \sum_{i\in J_n} z_i e_i$,

$$|\!|\!| f_n \delta(u_n) u_n^* f_n |\!|\!| \geq \| T \|_{ess}/2 .$$

Let now $u = \sum_{n\in\mathbb{N}} u_n$. Then, for each n,

$$f_n \delta(u) u^* f_n = f_n \delta(f_n u) u_n^* f_n - f_n \delta(f_n) f_n = f_n \delta(u_n^*) u_n f_n ,$$

so that

$$|||f_n\delta(u)u^*f_n||| = |||f_n\delta(u_n^*)u_nf_n||| \geq \|T\|_{ess}/2 .$$

Since $\delta(u)u^*$ is in $J(M)$, by Lemma 2.3 this is a contradiction.

Thus $\sum_{i\in I}\delta(e_i)e_i$ is in $J(M)$, and the case $N=l^\infty(I)$ is solved.

4. THE CONTINUITY RESULT

For the next result we assume $N\subseteq M$ is a finite von Neumann algebra with a normal faithful finite trace τ, $\tau(1)=1$. We denote by $\|x\|_2=\tau(x^*x)^{1/2}$, $x\in N$.

4.1. PROPOSITION. Let $\delta:N\mapsto J(M)$ be a derivation. Then δ is continuous from the unit ball of N with the strong operator topology into $J(M)$ with the norm $|||\ \ |||$.

PROOF. We first prove that if $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of projections in M with $\tau(f_n)\longmapsto 0$ then $|||\delta(f_n)|||\longleftrightarrow 0$. Suppose $|||\delta(f_n)|||$ does not converge to 0. By taking a subsequence if necessary, we may assume $|||\delta(f_n)|||\geq c>0$ for all n and $\sum\tau(f_n)<\infty$. Let g_n be the supremum of $\{f_k\}_{k\geq n}$. Then $\tau(g_n)\leq\sum_{k\geq n}\tau(f_k)$ tends to zero with n. Denote by $s_{n,m}$ the support of $f_mg_nf_m$. Then $s_{nm}\leq f_m$ and $s_{n,m}$ is majorized by g_n and thus τ being a trace, $\tau(s_{nm})\leq\tau(g_n)\xrightarrow[n]{} 0$, for each m. Since $\{g_n\}_{n\in\mathbb{N}}$ is decreasing, $\{f_ng_nf_m\}_{n\in\mathbb{N}}$ is decreasing so that $\{s_{nm}\}_{n\in\mathbb{N}}$ is decreasing for each m. Thus $\{f_m-s_{nm}\}_{n\in\mathbb{N}}$ increases to f_m so that $\{\delta(f_m-s_{nm})\}_{n\in\mathbb{N}}$ is weakly convergent to $\delta(f_m)$. By the inferior semicontinuity of the norm $|||\ \ |||$ (cf. 2.5) it follows that for a fixed m if n is big enough $|||\delta(f_m-s_{nm})|||\geq c/2$.

We may thus get by induction an increasing sequence of integers $n_1, n_2, \ldots$ such that the projections $h_k = f_{n_k} - s_{n_{k+1}, n_k}$ satisfy $|||\delta(h_k)||| \geq c/2$. These projections also satisfy $\tau(h_k) \leq \tau(f_{n_k}) \xrightarrow[k]{} 0$.

Moreover since $h_k \leq f_{n_k}$ and s_{n_{k+1}, n_k} is the support of $f_{n_k} g_{n_{k+1}} f_{n_k}$, by the definition of h_k we get

$$h_k g_{n_{k+1}} h_k = h_k f_{n_k} g_{n_{k+1}} f_{n_k} h_k \leq h_k s_{n_{k+1}, n_k} h_k = 0 \quad .$$

Thus $h_k g_{n_{k+1}} = 0$, in particular $h_k f_l$ for $l \geq k+1$ and so $h_k h_l = 0$ which means that h_k are all mutually orthogonal projections. Since we also have $|||\delta(h_k)||| \geq c/2$ we obtain a contradiction, by the atomic abelian case (§3) and 2.3.

Now we turn to the general case. Since $\| \ \|_2$ induces the strong operator topology on the unit ball of M, we have to show that if $(x_n)_n$ is a bounded sequence in M with $\|x_n\|_2 \longmapsto 0$ then $|||\delta(x_n)||| \longmapsto 0$. It is clear that we only need to prove this implication in the case x_n are selfadjoint elements and $\|x_n\| \leq 1$. Moreover, since $\| |x_n| \|_2 = \|x_n\|_2$, it follows, that if $\|x_n\|_2 \longmapsto 0$ then $\|(x_n)_+\|_2 \longmapsto 0$ and $\|(x_n)_-\|_2 \longmapsto 0$, so that it is sufficient to prove that if x_n are positive elements and $\|x_n\|_2 \longmapsto 0$ (equivalently $\tau(x_n) \longmapsto 0$) then $|||\delta(x_n)||| \longmapsto 0$.

Let $x_n = \sum_{m \geq 1} 2^{-m} e_m^n$ be the diadic decomposition of x_n. It follows that $\tau(e_m^n) \xrightarrow[n]{} 0$ for each $m \geq 1$. Let $\varepsilon > 0$ and $m_o \geq 1$ so that $2^{-m_o} \leq \varepsilon/2$. Then by the first part of the proof there exists n_o such that for $n \geq n_o$, $|||\delta(e_m^n)||| < \varepsilon/2$ for any

$m \leq m_o$. Thus, for $n \geq n_o$ we get

$$|||\delta(x_n)||| \leq \sum_{m=1}^{m_o} 2^{-m} \; |||\delta(e_m^n)||| \; + \; \|\delta\| \sum_{m \geq m_o} 2^{-m} \leq \varepsilon \, .$$

Q.E.D.

The above continuity result will enable us to reduce the theorem to more tractable situations and to prove it in several cases. We will actually use the following consequence of 4.1.

4.2. COROLLARY. Let $K_\delta = \overline{\mathrm{co}}^w \{\delta(u)u^* \mid u \text{ unitary element in } N\}$. Assume N is finite and countably decomposable and denote by τ a normal finite faithful trace on it, $\tau(1)=1$. Given $\beta > 0$ there exists $\alpha > 0$ such that if $x \in N$, $\|x\| \leq 1$, $\|x\|_2 \leq \alpha$ then
$|||Tx||| \leq \beta$ and $|||xT||| \leq \beta$ for all $T \in K_\delta$.

PROOF. By the preceding proposition there exists $\alpha > 0$ such that $\|y\| \leq 1$, $\|y\|_2 < \alpha$ implies $|||\delta(y)||| < \beta/3$. Since $\delta(u)u^*y = \delta(y) - u\delta(u^*y)$ and $\|u^*y\|_2 = \|y\|_2$ it follows that

$$|||\delta(u)u^*y||| \leq |||\delta(y)||| + |||\delta(u^*y)||| < 2\beta/3$$

for any unitary element u in M. Taking convex combinations and since the norm $|||\ \ |||$ is weak inferior semicontinuous we get $|||Ty||| < \beta$ for all $T \in K_\delta$. Similary $|||yT||| \leq \beta$. Q.E.D.

Actually 4.1 and 4.2 will be used through the following technical result which roughly shows that whenever there exists $T \in K_\delta$ (defined as in 4.2) with ad $T = \delta$ then $T \in J(M)$.

4.3. PROPOSITION. Let $N \subseteq M$ be a finite von Neumann algebra and $\{p_i\}_{i \in I} \subseteq Z(N)$ a partition of the unity with central projections of N such that N_{p_i} is of countable type for all i. Assume the derivation $\delta : N \mapsto J(M)$ satisfies $\delta(p_i) = 0$. If $T \in K_\delta$ is such that ad $T = \delta$ then either

$$\mathcal{P} = \{e \in \mathcal{P}(N) \mid \|eTe\|_{ess} = \|T\|_{ess}\}$$

contains no minimal projections of N or then $T \in J(M)$.

PROOF. Suppose $\|T\|_{ess}>0$ and $\wp$ has no minimal projections of N. By 2.6 there exists a sequence of mutually orthogonal projections $\{e_n\}_{n\in\mathbb{N}}$ in N such that $|||Te_n||| > \|T\|_{ess}/2$, $n\geq 1$. By the inferior semicontinuity of the norm $|||\ \ |||$ for each n we can find projection p_n in the von Neumann algebra generated by $\{p_i\}_{i\in I}$ such that N_{p_n} is countably decomposable and

$|||Te_np_n||| \geq \|T\|_{ess}/2$.

Let p be the supremum of the sequence $\{p_n\}_{n\in\mathbb{N}}$. Then p belongs to $\{p_i\}_i''$ (so that $\delta(p)=0$) and N_p is countably decomposable. Hence we obtain that

$$(*) \qquad |||Te_np||| \geq \|T\|_{ess}/2.$$

Let τ_p be a normal faithful finite trace on N_p, so that $\|pe_n\|_2$ converges to zero. If we consider the derivation δ' induced by δ on N_p, then by the preceding Corrolary, and since obviously $Tp\in K_{\delta'}$, it follows that $|||Te_np||| = |||Tpe_n|||$ also converges to 0, which contradicts (*).

Q.E.D.

We end this section by proving a useful converse to the preceding proposition. Note that the proof doesn't use the continuity result 4.1.

4.4. LEMMA. Let N be an arbitrary von Neumann subalgebra of M and $\delta : N \longmapsto J(M)$ a derivation. If there exists $K\in J(M)$ such that $\delta = \mathrm{ad}\ K$ there exists $T\in K_\delta$ such that $\delta = \mathrm{ad}\ T$.

PROOF. Assume first that $\varphi(K^*K)<\infty$. Let

$$C=\overline{\mathrm{co}}^w\{uKu^* \mid u \text{ unitary element in } N\}.$$

Then $\|y\|_\varphi \leq \|K\|_\varphi$ for all y in C and C is a weakly compact convex subset of M. By the inferior semicontinuity of the norm $\| \ \|_\varphi$ it follows that there exists a unique element $y_o \in C$ with $\|y_o\|_\varphi \leq \|y\|_\varphi$ for all $y \in C$. Since $uy_o u \in C$ and $\|uy_o u^*\|_\varphi = \|y_o\|_\varphi$ it follows that $uy_o u^* = y_o$ for all unitary elements $u \in N$. Thus $y_o \in C \cap N'$.

Let's show now that also for arbitrary K, there exists some $y_o \in C \cap N'$. Let K_n be a sequence in $J(M)$ with $\varphi(K_n^* K_n) < \infty$, $\|K_n\| \leq \|K\|$ and $\|K-K_n\| \longrightarrow 0$. Let

$$C_n = \overline{co}^w \{ uK_n u^* \mid u \text{ unitary element in } N \}$$

and $y_n \in C_n \cap N'$. Let y be a weak limit point of $\{y_n\}_n$ (which is bounded in the uniform norm by $\|K\|$). Then clearly $y \in N'$ and since $\|K-K_n\| \longmapsto 0$, by the weak inferior semicontinuity of the uniform norm, $y \in C$.

Now denote by $T = K-y$. Then $K-y \in K-C = \overline{co}^w \{K-u^*Ku \mid u \text{ unitary element in } N\} = K_\delta$ and moreover, since $y \in N'$, ad T = ad $K = \delta$.

Q.E.D.

5. THE TYPE I AND PROPERLY INFINITE CASES

We first prove the theorem when N is a finite type I von Neumann algebra. Since N is finite, there exists a partition of the unity $\{p_i\}_{i \in I}$ in the center of N such that N_{p_i} is countably decomposable for each i. By §3 there exists an element $K_o \in J(M)$ such that $(\delta - \text{Ad } K_o)(p_i) = 0$ for all i. Thus we may assume that δ vanishes on $\{p_i\}_{i \in I}$.

The unitary group of N has an amenable subgroup $\mathcal{U}$ such that $\mathcal{U}'' = N$. . Let $T = \int \delta(u) u^* d\mu(u)$ where μ is an invariant mean of $\mathcal{U}$ and the integral has the usual signifiance (see e.g. [2]) . Then obviously $T \in K_\delta$ and , by the same computations as in 2.9 we have $Tu_o - u_o T = \delta(u_o)$, for any $u_o \in \mathcal{U}$.

Since both δ and ad T are weakly continuous and N is the closed linear span of $\mathcal{U}$, it follows that $\delta = \operatorname{ad} T$ on N. By 2.9 we can now consider separately the case N is completely nonatomic and the case N is atomic.

In the first case, Proposition 4.3 trivially shows that $T \in J(M)$.

In the second case (when N is atomic) we may assume the projections $\{p_i\}_{i \in I}$ are the atoms of Z(N). But then the integral $p_i \int \delta(u)u^* d\mu(u) p_i = \int \delta(up_i)(up_i)^* d\mu(u)$ is norm convergent (since N_{p_i} is finite dimensional) and thus $p_i T p_i \in J(M)$. Hence if $\|eTe\|_{ess} = \|T\|_{ess} > 0$ for some minimal projection e of N, and p is the central support of e, then $0 = \|pTp\|_{ess} \geq \|eTe\|_{ess} > 0$ a contradiction. Thus, by 4.3, $T \in J(M)$.

Assume now that N is properly infinite. Then N and M are isomorphic to $N_1 \bar{\otimes} \mathcal{B}(l^2(\mathbb{Z}))$ and $M_1 \bar{\otimes} \mathcal{B}(l^2(\mathbb{Z}))$ respectively, (where $N_1 \subset M_1$ are von Neumann algebras), so that the inclusion $N \subset M$ becomes $N_1 \bar{\otimes} \mathcal{B}(l^2(\mathbb{Z})) \subset M_1 \bar{\otimes} \mathcal{B}(l^2(\mathbb{Z}))$. Note first that if the derivation $\delta : N \mapsto J(M)$ vanishes on $\mathbb{C}I \bar{\otimes} \mathcal{B}(l^2(\mathbb{Z})) \subset N = N_1 \bar{\otimes} \mathcal{B}(l^2(\mathbb{Z}))$ then given a unitary $u \in N_1 \bar{\otimes} \mathbb{C}1$ we have for any $x \in \mathbb{C}I_{M_1} \bar{\otimes} \mathcal{B}(l^2(\mathbb{Z}))$,

$$\delta(u)x = \delta(ux) = \delta(xu) = x\,\delta(u) ,$$

so that $\delta(u) \in J(M) \cap (\mathbb{C}I \bar{\otimes} \mathcal{B}(l^2(\mathbb{Z})))' \cap M_1 \bar{\otimes} \mathcal{B}(l^2(\mathbb{Z})) = J(M) \cap (M_1 \bar{\otimes} \mathbb{C}I_{\mathcal{B}(l^2(\mathbb{Z}))}) = 0$. Thus $\delta = 0$ on N.

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From this it follows that to prove the properly infinite case it is sufficient to prove the case when $\delta : N = \mathcal{B}(l^2(\mathbb{Z})) \longmapsto J(M)$.

Let D be the diagonal von Neumann subalgebra of $\mathcal{B}(l^2(\mathbb{Z}))$ and L the von Neumann algebra generated by the bilateral shift u. Let $\sigma(x) = uxu^*$ for $x \in D$ be the automorphism of D implemented by the shift u. By §3 we may assume δ vanishes on D. Then for any $x \in D$ we have

$$x\delta(u^n)u^{-n} = \delta(xu^n)u^{-n} = \delta(u^n\sigma^{-n}(x))u^{-n} =$$

$$= \delta(u^n)\sigma^{-n}(x)u^{-n} = \delta(u^n)u^{-n}\sigma^n(\sigma^{-n}(x)) = \delta(u^n)u^{-n}x \quad ,$$

which shows that $\delta(u^n)u^{-n} \in D' \cap M$ for all $n \in \mathbb{Z}$.

But if we take T to be a (weak) mean (after n) of $\delta(u^n)u^{-n}$ then $T \in D' \cap M$ and by the preceding proof of the type I case

$$\delta|_L = \text{ad } T \quad \text{and} \quad T \in J(M) \ .$$

Thus ad T equals δ on both D and L. Since δ and ad T are weakly continuous derivations if follows that $\delta = \text{ad } T$ on the von Neumann algebra generated by D and L, which is easily seen to be $\mathcal{B}(l^2(Z))$.

6. SOME TECHNICAL RESULTS

To prove the remaining type II_1 case of the theorem we need some technical devices that we prove bellow.

6.1. LEMMA. Let N be a von Neumann algebra without atoms, Ψ a normal faithful state on N and $\{w_n\}_n$ a sequence of unitary elements in N such that $\psi(w_n^k) \xrightarrow[n]{} 0$ for all $k \neq 0$.

Then there exist unitary elements $\{v_n\}_n$ in N such that $\Psi(v_n^k)=0$, $k\neq 0$, and $\|w_n-v_n\|\longrightarrow 0$.

PROOF. The proof is the same as the proof of 1.3 in [7] but we give it here anyway for the sake of completeness.

Since N has no atoms each w_n is contained in some diffuse abelian von Neumann subalgebra $A_n\subset N$ with separable predual and $(A_n, \Psi|_{A_n})$ can be identified by some measure preserving isomorphism φ_n with $L^\infty(\mathbb{T},\mu)$ where μ is the normalized Lebesgue measure on the thorus $\mathbb{T}$. Morover φ_n can be chosen so that $\varphi_n(w_n)=f_n$, where $f_n(e^{2\pi it})=e^{2\pi ih_n(t)}$ for some nondécreasing function $h_n:[0,1]\longmapsto[0,1]$. By Helly's selection principle there exists a subsequence $\{h_{k_n}\}_n$ tending everywhere to some nondecreasing function $h:[0,1]\longmapsto[0,1]$. Thus, if $f(e^{2\pi it})=$ $=e^{2\pi ih(t)}$ then $\{f_{k_n}\}_n$ tends everywhere to f so that by Lebesgue's theorem $\int f_{k_n}^p\,d\mu\longmapsto\int f^p d\mu$ for all p, which by the hypothesis implies $\int f^p d\mu=0$ for $p\neq 0$. Thus $\int q(f)\,d\mu=\int q\,d\mu$ for Laureant polynomials q so that $\int gf d\mu=\int g d\mu$ for any $g\in L^\infty(\mathbb{T},\mu)$. In particular if we define $g_z(e^{2\pi is})=\begin{cases}1 & \text{if } 0\leq s<t\\ 0 & \text{if } t\leq s<1\end{cases}$, where $z=e^{2\pi it}$, then we get $\int_{h(s)\leq t} d\lambda(s)=\int g_z\circ f d\mu=\int g_z d\mu=t$, λ being the Lebesgue measure on $[0,1]$. This implies $h(t)=t$ and hence $f(z)=z$ is the identity function on $\mathbb{T}$. Now, since h_{k_n} are monotone and converge everywhere to a continuous function it follows that h_{k_n} converge uniformly to h, so that $\|f_{k_n}-f\|\longrightarrow 0$. Since any limit point of f_{k_n} was shown to be equal to the identity f, it follows that $\|f_n-f\|\longrightarrow 0$.

We can now take $v_n = \varphi_n^{-1}(f)$. Since $\int f^p d\mu = 0$, $\psi(v_n^p) = 0$ for all $p \neq 0$. Moreover $\|w_n - v_n\| = \|\varphi_n(w_n) - \varphi_n(v_n)\| = \|f - f_n\| \longrightarrow 0$.

Q.E.D.

6.2. LEMMA. 1^o. Let $N \subset M$ be a von Neumann subalgebra such that $N' \cap M$ contains no finite projections of M. Let $\varepsilon > 0$ and e, f two finite projections of M. There exists a unitary element $u \in N$ such that $\|fue\|_\varphi < \varepsilon$. Moreover if N is abelian then given any $n \geq 1$ there exists a unitary element $u \in N$ such that $\|fu^k e\|_\varphi < \varepsilon$ for $k \neq 0$, $|k| \leq n$.

2^o. If N is finite, M is countable decomposable and $N' \cap M$ contains no finite projections of M then there exists a maximal abelian *-subalgebra $A \subset N$ such that $A' \cap M$ contains no finite projections of M.

PROOF. 1^o. Let φ_n be the semifinite faithful trace on M^{2n} given by $\varphi_n((x_k)_{|k| \leq n, k \neq 0}) = \sum \varphi(x_k)$. Denote by $K_e^n = \overline{co}^w \{ (u^k e u^{-k})_{|k| \leq n, k \neq 0} \mid u \text{ unitary element of } N \} \subset M^{2n}$. Then $\varphi_n(\overline{x}) \leq 2n\, \varphi(e)$ and $\|\overline{x}\|_{\varphi_n} \leq 2n \|e\|_\varphi$ for any $\overline{x} \in K_e^n$. By the inferior semicontinuity of the norm $\|\ \|_{\varphi_n}$, there exists a unique element $\overline{x}_o \in K_e^n$ with $\|\overline{x}_o\|_{\varphi_n} \leq \|\overline{x}\|_{\varphi_n}$ for all $\overline{x} \in K_e^n$. But if N is abelian then for any unitary element $u \in N$, if $\widetilde{u} = (u^k)_{|k| \leq n, k \neq 0}$ then $\widetilde{u} K_e^n \widetilde{u}^* \subset K_e^n$ and $\|\widetilde{u}\overline{x}_o \widetilde{u}^*\|_{\varphi_n} = \|\overline{x}_o\|_{\varphi_n}$ so that, by the uniqueness of $\overline{x}_o$, $\widetilde{u}\overline{x}_o\widetilde{u}^* = \overline{x}_o$. Thus if $\overline{x}_o = (x_k)_{|k| \leq n, k \neq 0} \neq 0$ then $x_k \neq 0$ for some k and $u^k x_k = x_k u^k$ for any unitary element $u \in N$. Since in a von Neumann algebra N any unitary element $v \in N$ can be written as u^k for some $u \in N$, it follows that $vx_k = x_k v$ for unitary elements $v\in N$ and by taking linear combinations, $yx_k=x_ky$ for all $y\in N$. But $0<\|x_k\|_\varphi \leq \|e\|_\varphi$ and $x_k\in N'\cap M$, a contradiction. If N is arbitrary we take M instead of M^{2n} and the proof is the same.

2^o. The argument we use is symilar to the one in [6], 2.4. Let $\{f_n\}_n$ be an increasing sequence of finite projections in M with $f_n \uparrow 1$. We construct recursively an increasing sequence of finite dimensional abelian von Neumann subalgebras $\{A_n\}_n$ of N such that if $\{e_i^n\}_{1\leq i\leq k_n}$ are the minimal projections of A_n then $\| E_{A_n'\cap M}(f_n)\|_\varphi^2=\|\sum_i e_i^n f_n e_i^n\|_\varphi^2<(3/4)^n$. Suppose we constructed these algebras up to n. We fist prove that if $p\in N$ then $N_p'\cap M_p$ contains no finite projections of M_p. To show this let $f\neq 0$ be a projection in $N_p'\cap M_p$ and z a projection in the center of N. Then $zf\in N_p'\cap M_p$ and if f is finite in M_p then zf is finite in M_{zp}. Take z to be so that $fz\neq 0$ and pz divides z, say n times. It follows that the inclusion $N_z\subset M_z$ is the same as $N_{zp}\otimes M_{n\times n}\subset M_{zp}\otimes M_{n\times n}$ and that $f'=zf\otimes I_n\in(N_{zp}\otimes M_{n\times n})'\cap \cap(M_{zp}\otimes M_{n\times n})$. Hence $f'\in N_z'\cap M_z=z(N'\cap M)z\subset N'\cap M$ and if f is finite then f' is finite, contradicting the hypothesis. Now by 1^o it follows that for each $p=e_i^n$ there exists a unitary element $u\in N_p$ such that if e is the support of $x=e_i^n f_{n+1} e_i^n$ then $\varphi(eueu^*)=\|eue\|_\varphi^2<1/2\,\|x\|_\varphi^2$. Aproximating u in the uniform norm we may assume it has finite spectrum so that $u=\sum\lambda_i e_i$ with $\sum e_i=p$ and $|\lambda_i|=1$. Then, since $\varphi(xuxu^*)\leq\varphi(eueu^*)$, we have: $\|x\|_\varphi^2=2\,\|x\|_\varphi^2-\|x\|_\varphi^2\leq\|x\|_\varphi^2+\|uxu^*\|_\varphi^2-2\varphi(xuxu^*)=$

$$=\|x-uxu^*\|_\varphi^2=\|\sum_{i\neq j}(\lambda_i\bar\lambda_j-1)e_ixe_j\|_\varphi^2\leq 4\sum_{i\neq j}\|e_jxe_j\|_\varphi^2=4\,\|x\|_\varphi^2-$$

$$-4\sum_i\|e_ixe_i\|_\varphi^2.$$

Thus $\sum_i \|e_i x e_i\|_\varphi^2 \le 3/4 \|x\|_\varphi^2$. Let $A_n^1 \supset A_n$ be defined by $A_n^1 e_i^n = \mathrm{span}\{e_j\}_j$. Then $\|E_{(A_n^1)'\cap M}(f_{n+1})\|_\varphi^2 \le 3/4 \|\sum_i e_i^n f_{n+1} e_i^n\|_\varphi^2 =$
$= 3/4 \|E_{A_n'\cap M}(f_{n+1})\|_\varphi^2$.

Applying this trick m-times, where $(3/4)^m \|f_{n+1}\|_\varphi^2 \le (3/4)^{n+1}$ we get finite dimensional abelian algebras $A_n = A_n^o \subset A_n^1 \subset A_n^2 \subset \ldots \subset A_n^m$ with

$$\|E_{(A_{n+1}^k)'\cap M}(f_{n+1})\|_\varphi^2 < 3/4 \|E_{(A_n^{k+1})'\cap M}(f_{n+1})\|_\varphi^2$$

so that if we define $A_{n+1} = A_n^m$ then

$$\|E_{A_{n+1}'\cap M}(f_{n+1})\|_\varphi^2 < (3/4)^m \|f_{n+1}\|_\varphi^2 \le (3/4)^{n+1} .$$

Let $A = \overline{\bigcup_n A_n}^w$. Suppose $e \in A'\cap M$, $e \neq 0$, is a finite projection of M. Since $f_n \uparrow 1$, there exists n such that $\|f_n e f_n - e\|_\varphi < 1/2 \|e\|_\varphi$ By the construction of $A_n \subset A$ there exists a partition of the unity $e_1, \ldots, e_m$ with projections in A such that $\|\sum_i e_i f_n e_i\|_\varphi < 1/2 \|e\|_\varphi$. But then

$$\|\sum_i e_i f_n e f_n e_i\|_\varphi \le \|\sum_i e_i f_n e_i\|_\varphi < 1/2 \|e\|_\varphi$$

so that, since $e = \sum_i e_i e e_i$,

$$\|e\|_\varphi = \|\sum_i e_i e e_i\|_\varphi < \|\sum_i e_i (f_n e f_n - e) e_i\|_\varphi +$$

$$+ \|\sum_i e_i f_n e f_n e_i\|_\varphi < \|e\|_\varphi ,$$

which is a contradiction.

Q.E.D.

In the rest of this section $N \subset M$ will be a type II_1 von Neumann subalgebra with a fixed normal finite faithful trace τ, $\tau(1)=1$. The norm on N given by τ is denoted $\|x\|_2 = \tau(x^*x)^{1/2}$, $x \in M$. If $B \subset N$ is a von Neumann subalgebra then E_B denotes the unique normal τ-preserving conditional expectation onto B (cf. [11]).

6.3. LEMMA. Assume $A \subset N$ is an abelian von Neumann subalgebra of N such that $A' \cap M$ contains no finite projections of M. Let $\varepsilon > 0$, $n \geq 1$, e and f finite projections of M and v a unitary element in N. Then there exists a unitary element $u \in A$ such that $\| f(uv)^k e \|_\varphi^2$ for any $k \neq 0$, $|k| \leq n$.

PROOF. Since $e \vee f$ is a finite projection in M and $\|(e\vee f)(uv)^k(e \vee f)\|_\varphi \geq \|f(uv)^k e\|$, it is sufficient to prove the statement when e=f. Since $\|e(uv)^k e\|_\varphi = \|e(uv)^{-k}e\|_\varphi$ we only need to prove the estimates for $k>0$. We'll actually prove the following more general result:

(*) If $\varepsilon > 0$, $n \geq 1$, $\mathcal{F} \subset N$ is a finite selfadjoint set of norm one elements containing the identity and e, f are finite projections in M then there exists a unitary element $u \in A$ such that

$$\| f x_o \prod_{i=1}^{k} (u x_i) e \|_\varphi < \varepsilon$$

for any $1 \leq k \leq n$ and $x_o, x_1, \ldots, x_k \in \mathcal{F}$.

We first prove (*) in the case $\varphi(xe) \leq c\tau(x)$, $\varphi(fx) \leq c\tau(x)$, $x \in N_+$, for some constant $c>0$. Let $\mathcal{W} = \{w$ partial isometry in $A \mid \| f x_o \prod_{i=1}^{k} (w x_i) e \|_\varphi^2 \leq \varepsilon \tau(w^*w)$ for any $1 \leq k \leq n$, $x_o, x_1, \ldots, x_k \in \mathcal{F}\}$ and consider on $\mathcal{W}$ the usual order: $w_o \leq w_1$ if w_o is a restriction of w_1 , i.e. $w_o=w_1w_o^*w_o$. The set $\mathcal{W}$ is clearly inductively ordered. Let u be a maximal element of it and suppose $u^*u\neq 1$. Denote by $A_o=(1-u^*u)A(1-u^*u)$, $N_o=(1-u^*u)N(1-u^*u)$ and $\mathcal{F}_o=$ $=\left\{(1-u^*u)x_o(\prod_{i=1}^{k}(ux_i))(1-u^*u) \mid 1\leq k\leq n,\ x_o,x_1,\ldots,x_n\in\mathcal{F}\right\}$. By 1.2 in [6] given any $\delta>0$ there exists a partition of the unity $e_1,\ldots,e_m$ in A_o such that $\sum_i\|e_iye_i-E_{A_o}(y)e_i\|_2^2=$ $=\|\sum_i e_iye_i-E_A(y)\|_2^2\leq\delta\tau(1-u^*u)=\delta\sum_i\tau(e_i)$ for all $y\in\mathcal{F}_o$. It follows that for some $e_o=e_i$ we have

$$(**)\qquad \|e_oye_o-E_{A_o}(y)e_o\|_2^2<\delta\,\tau(e_o)\ ,\qquad y\in\mathcal{F}_o\ .$$

Let $n\geq r,s\geq 0$, $x\in\mathcal{F}$, $y_1,\ldots,y_s\in\mathcal{F}_o$, $x'\in\mathcal{F}^*$, $y_1',\ldots,y_r'\in\mathcal{F}_o^*$ and $w\in A_oe_o$, $\|w\|\leq 1$ and denote $\alpha=|\varphi(ex'\prod_{i=1}^{r}(y_i'w^*)y'fy\prod_{j=1}^{s}(wy_j)xe)|$, with the convention that a product over a void set equals 1.

If s=1 then by the Cauchy-Schwartz inequality we have:

$$\alpha\leq\|fywy_1xe\|_\varphi\|ex'(\prod y_i'w^*)y'f\|_\varphi\leq\|fywy_1xe\|_\varphi\|e\|_\varphi$$
$$\leq\|\bar{f}w\bar{e}\|_\varphi\ \|e\|_\varphi\ ,$$

where $\bar{e}$ is the supremum of the left supports of all the elements of the form zy_1x, with $x\in\mathcal{F}$, $y_1\in\mathcal{F}_o$ and $z\in\mathcal{F}_1=\left\{\prod_{i=1}^{k}E_A(y_i)e_o \mid 0\leq k\leq n,\ y_1,\ldots,y_k\in\mathcal{F}_o\right\}$, and $\bar{f}$ is the supremum of the elements fy with $y\in\mathcal{F}_o$.

If $s\geq 2$ then we have

$$\|\prod_{i=1}^{s}(wy_i)xe\|_\varphi\leq\sum_{j=1}^{s-1}\|\prod_{i=1}^{j-1}(wy_i)w(E_A(y_j)e_o-e_oy_je_o)$$

$$w^{s-j}\left(\prod_{t=j+1}^{s-1} E_A(y_t)\right)y_s xe\|_\varphi + \|fyw^s\sum_{j=1}^{s-1} E_A(y_j)y_s x_e\|_\varphi \leq$$

$$\leq \sum\{\|(E_A(y_o)e_o-e_oy_oe_o)w^j zyxe\|_\varphi \mid 1\leq j\leq s,\ x\in\mathcal{F},\ y_o,y\in\mathcal{F}_o,$$

$$z\in\mathcal{F}_1\}+\|\bar{f}w^{s}\bar{e}\|_\varphi \quad ,$$

where $\bar{e}$, $\bar{f}$ are as before. Thus if β denotes the sum in the right hand side of the above inequalities then by(**) we get $\beta\leq sNN_o^2N_1C^{1/2}\delta^{1/2}\|e_o\|_2$, where N, N_o and N_1 are the number of elements in $\mathcal{F}$, $\mathcal{F}_o$ and respectively $\mathcal{F}_1$.

Thus, by the Cauchy-Schwartz inequality we obtain:

$$\alpha\leq \|ex'\prod_{i=1}^{r}(y_i'w^*)y'fye_o\|_\varphi\ (\beta+\|\bar{f}w^s\bar{e}\|_\varphi)\leq$$

$$\leq \|fye_o\|_\varphi\ (\beta+\|\bar{f}w^s\bar{e}\|_\varphi)\leq C^{1/2}\|e_o\|_2(\beta+\|\bar{f}w^s\bar{e}\|_\varphi)\leq$$

$$\leq sNN_o^2N_1C\delta^{1/2}\|e_o\|_2^2+C^{1/2}\|e_o\|_2\|\bar{f}w^s\bar{e}\|_\varphi\ .$$

Thus if δ is so that $nNN_o^2N_1C\,\delta^{1/2}<\varepsilon 2^{-2n-1}$ and if using 6.2 we choose w to be a unitary element in $A_oe_o=Ae_o\subset e_oMe_o$ such that $C^{1/2}\|\bar{f}w^s\bar{e}\|_\varphi<\varepsilon 2^{-2n-1}\|e_o\|_2$, then we get $\alpha<2^{-2n}\varepsilon\tau(e_o)$.

We now show that if w is chosen like this then $u_o=u+w$ contradicts the maximality of u. Indeed we have for any $1\leq k\leq n$ and $x_o,x_1,\dots,x_k\in\mathcal{F}$:

$$\|fx_o\left(\prod_{i=1}^{k}(u+w)x_i\right)e\|_\varphi^2\leq\|fx_o\prod_{i=1}^{k}(ux_i)e\|_\varphi^2+\sum\alpha$$

where the α's appearing in the sum are of the form estimated above and there are $2^{2k}-1$ terms in that sum. It follows that $\sum\alpha\leq\varepsilon\tau(e_o)$ so that

$$\| fx_o(\prod_{i=1}^{k}(u+w)x_i)e\|_{\varphi}^2\leq\varepsilon(\tau(u^*u)+\tau(w^*w))=\varepsilon\tau((u+w)^*(u+w))$$

This ends the proof in the case $\varphi(xe)\leq c\tau(x)$, $\varphi(fx)\leq c\tau(x)$, for $x\in N_+$.

To prove the general case, i.e. for arbitrary e, f, note that given any $\varepsilon>0$ there exist finite projections $e',f'\in M$ with $\|e-e'\|_{\varphi}\leq\varepsilon/3$, $\|f-f'\|_{\varphi}\leq\varepsilon/3$ and such that $\varphi(xe')\leq c\tau(x)$, $\varphi(f'x)\leq c\tau(x)$ for some constant $c>0$. Indeed, since $\varphi(.e)$, $\varphi(f.)\in N_*$, there exist $X,Y\in L^1(N,\tau)_+$ such that $\varphi(xe)=\tau(xX)$, $\varphi(fx)=\tau(xY)$, for $x\in N$. Thus if E_n,F_n are the spectral projections of X and respectively Y corresponding to the interval $[0,n]$ then $E_n\uparrow 1$, $F_n\uparrow 1$ and $\varphi(xE_neE_n)=\varphi(E_nxE_ne)=\tau(E_nxE_nX)=\tau(xE_nX)\leq n\tau(x)$ and similary $\varphi(F_nfF_nx)\leq n\tau(x)$. It follows that $\|E_neE_n-e\|_{\varphi}\longrightarrow 0$, $\|F_nfF_n-f\|_{\varphi}\longmapsto 0$ so that if e_n', f_n' are the spectral projections of E_neE_n and respectively F_nfF_n corresponding to the interval $[1/2,\infty)$ then an easy computation shows that $\|e_n'-e\|_{\varphi}\longrightarrow 0$, $\|f_n'-f\|_{\varphi}\longrightarrow 0$ and $\varphi(xe_n')\leq 2\varphi(xE_neE_n)\leq 2n\tau(x)$, $\varphi(f_n'x)\leq 2\varphi(F_nfF_nx)\leq$ $\leq 2n\tau(x)$ (see e.g. 1.4 in [8]). Now by the first part of the proof given $\varepsilon>0$ and $n\geq 1$ there exists a unitary element $u\in A$ such that $\|f'x_o\prod_{i=1}^{k}(ux_i)e'\|_{\varphi}<\varepsilon/3$ for any $1\leq k\leq n$, $x_o,x_1,\dots,x_k\in\mathcal{F}$. But then

$$\| fx_o\prod_{i=1}^{k}(ux_i)e\|_{\varphi}\leq 2\varepsilon/3+\|f'x_o\prod_{i=1}^{k}(ux_i)e'\|_{\varphi}\leq 2\varepsilon/3+\varepsilon/3=\varepsilon\ .$$

Q.E.D.

6.4. COROLLARY. Let $\varepsilon>0$, $n\geq 1$, e,f two finite projections of M and $v\in N$ a unitary element. There exist a finite projection $e_n\in M$ and a unitary element $w\in N$ such that:

1°. $\varphi(e_nw^ke_n)=0$ for any $k\neq 0$;

2°. $e_n \leq e$, $\varphi(e-e_n) < \varepsilon$;

3°. $\|fw^k e_n\| < \varepsilon$, for $k \neq 0$, $|k| \leq n$;

4°. $\|w-uv\| < \varepsilon$ for some unitary element $u \in A$.

PROOF. First we prove that given any $\varepsilon' > 0$ there exist unitary elements $u \in A$ and $w' \in M$ and a finite projection $e_n \in M$ such that:

a) $e_n \leq e$, $\varphi(e-e_n) < \varepsilon'$;

(*) b) $\|fw'^k e_n\| < \varepsilon'$, for $k \neq 0$, $|k| \leq n$;

c) $\|w'-uv\| < \varepsilon'$;

d) $\varphi(ew'^k e) = 0$ for all $k \neq 0$.

Then it follows by a) and d) that $|\varphi(w'^k e_n)| \leq \varepsilon'$ for any $k \neq 0$ and thus if ε' is small enough and $\varepsilon' \leq \varepsilon/2$ by 6.1 there exists a unitary element $w \in N$ such that $\|w-w'\| \leq \varepsilon/2n$ and $\varphi(w^k e_n) = 0$ for any $k \neq 0$. But then $\|fw^k e_n\| \leq \|fw'^k e_n\| + n\|w-w'\| < \varepsilon$ for $k \neq 0$, $|k| \leq n$ and $\|w-uv\| \leq \|w-w'\| + \|w'-uv\| \leq \varepsilon/2n + \varepsilon/2 \leq \varepsilon$.

Now to prove (*) we let $\varepsilon'' > 0$, $n' \geq 1$. By the preceding lemma there exists a unitary element $u \in A$ such that $\|(e \vee f)(uv)^k e\|_\varphi < \varepsilon''$ for $k \neq 0$, $|k| \leq n'$. It follows that $|\varphi(e(uv)^k e)| \leq \|e\|_\varphi \|e(uv)^k e\|_\varphi < \varepsilon'' \|e\|_\varphi$, for all $k \neq 0$, $|k| \leq n'$, and $\varphi(e(uv)^{-k} f(uv)^k e) = \|f(uv)^k e\|_\varphi^2 < \varepsilon''^2$. If e'_k is the spectral projection of $e(uv)^{-k} f(uv)^k e$ corresponding to the interval $(0, \varepsilon'']$ then $e'_k \leq e$, $e'_k (uv)^{-k} f(uv)^k e'_k \leq \varepsilon''^2$ and $e - e'_k \leq \varepsilon''^{-1} e(uv)^{-k} f(uv)^k e$ so that $\varphi(e-e'_k) \leq \varepsilon''^{-1} \varepsilon''^2 = \varepsilon''$. Let $e_n = \bigwedge\{e'_k \mid k \neq 0, |k| \leq n\}$. Then $e_n \leq e$, $\varphi(e_n) \geq \varphi(e) - 2n\varepsilon''$ and $\|f(uv)^k e_n\| \leq \|f(uv)^k e'_k\| \leq \varepsilon''$.

Lemma 6.1 shows that if n' is large enough and ε'' is small enough with $\varepsilon'' < \varepsilon'/(n+1)$ then there exists a unitary element $w' \in N$ such that $\varphi(w'^k e) = 0$ for all $k \neq 0$ and $\|w'-uv\| < \varepsilon'/n+1$.

But then $\|fw'^k e_n\| \leq \sum_{p=0}^{k-1} \|f(uv)^p (w'-uv)(w)^{k-p-1} e\| + \|f(uv)^k e\| \leq$

$k\varepsilon'/(n+1)+\varepsilon'/(n+1)=(k+1)\varepsilon'/(n+1) \leq \varepsilon'$ which proves (*).

Q.E.D.

7. END OF THE PROOF OF THE THEOREM: the type II_1 case

In this section we prove 1.1 in the case N is of type II_1. By 2.7 and §5 this will end the proof of the theorem. We begin the section by reducing the problem in several steps to the case when the type II_1 von Neumann algebra N is separable, M is countably decomposable and $N' \cap M$ contains no finite projections of M.

7.1. First reduction: If is sufficient to prove the theorem for separable N (i.e. N with separable predual).

To show this let $R \subset N$ be a copy of the hyperfinite type II_1 factor (cf [5]). There exists an increasing net of separable von Neumann subalgebras $\{N_i\}_i$ of N with $R \subset N_i$ and $\overline{\bigcup_i N_i}^w = N$. Indeed, if $\{p_j\}_{j \in J}$ is a partition of the unity in the center of N such that Np_j is countably decomposable for each j, then any countably generated von Neumann subalgebra of Np_j is separable, so that if N_i are such that $N_i p_j$ is countably generated and contains Rp_j for a finite number J_o of $j \in J$ and if $N_i \sum_{j \notin J_o} p_j = R \sum_{j \notin J_o} p_j$ then N_i will do. Since $R \subset N_i$, each N_i is of type II_1 and if $K_i \in J(M)$ is such that $\delta|_{N_i} = \mathrm{ad}\, K_i$ then by 4.4 there exists $T_i \in K_\delta$ (in fact in $\overline{co}^w\{\delta(u)u^* \mid u$ unitary element of $M_i\} \subseteq K_\delta$) such that $\mathrm{ad}\, T_i = \mathrm{ad}\, K_i = \delta|_{N_i}$. Let T be a weak limit point of $\{T_i\}_i$. Then $\mathrm{ad}\, T = \delta$ on $\bigcup N_i$, so that by the weak continuity of ad T and δ, ad T= δ on $N=\overline{\bigcup N_i}^w$. Since N is of type II_1 it has no minimal projections so that by 4.3, $T\in J(M)$.

7.2. Second reduction: It is sufficient to prove the theorem when N is separable and M is countably decomposable.

Indeed, by the preceding reduction we may assume N is separable. Let $\mathcal{U}_o$ be a countable subset in the unitary group $\mathcal{U}$ of N, dense in $\mathcal{U}$ in the *-strong operator topology. Let $\{p_i\}_{i\in J}$ be an increasing net of countably decomposable projections of M with $p_i\uparrow 1$. By the density of $\mathcal{U}_o$ in $\mathcal{U}$ it follows that for each i, $\bigvee\{up_iu^* | u\in\mathcal{U}\}=\bigvee\{up_iu^* | u\in\mathcal{U}_o\}$ so that if we denote this projection by s_i then it is countably decomposable (being a supremum of a countable set of countably decomposable projections) and moreover $s_i\in N'\cap M$, $s_i\uparrow 1$. Define $\delta_i : N_{s_i} \longmapsto \rightarrow s_iJ(M)s_i=J(M_{s_i})$ by $\delta_i(xs_i)=s_i\delta(x)s_i$. Since $s_i\in N'\cap M$, δ_i are well defined derivation. If for each i there exists an element $K_i\in J(M_{s_i})$ such that $\delta_i=\text{ad } K_i$ then by 4.4 there exists $T_i\in K_\delta$ such that $s_iT_is_i\in K_{\delta_i}\subset s_iK_\delta s_i$ satisfies $\delta_i=\text{ad}(s_iT_is_i)$. Let T be a weak limit point in M of the net $\{T_i\}_i (\subset M)$. Since $\{s_i\}_i$ converges strongly to the identity, $T\in K_\delta$ and $\text{ad}T=\delta$ on N. By 4.3, since N has no minimal projections $T\in J(M)$.

7.3. Third reduction: it is sufficient to prove the theorem when N is separable, M is countable decomposable and $N'\cap M$ contains no finite projections of M.

Let $p_o=\bigvee\{e'\in N'\cap M | e' \text{ finite projections of } M\}$ and assume $\delta(x)=\delta(x)p_o$, $x\in N$. Then $K_\delta=K_\delta p_o$. For each unitary element $u\in N$ define on K_δ the weakly continuous affine transformation $T_u(x)=uxu^*+\delta(u)u^*$. Then $T_uT_v=T_{uv}$ and since $T_u(\delta(v)v^*)=$

$=u\delta(v)v^*u^*+\delta(u)u^*=\delta(uv)v^*u^*$, it follows that $T_u(K_\delta)\subset K_\delta$. Consider on M the seminorms $\mathcal{S}=\{\varphi(x^*xe')^{1/2}$ for $x\in M|e'$ finite projection in $N'\cap M\}$. Then the semigroup of transformation T_u on K_δ is noncontractive, because if $x,y\in K_\delta$, $x\neq y$, then $\inf_u \varphi(u(x-y)^*(x-y)u^*e')=\varphi((x-y)^*(x-y)e')$ and if $\varphi((x-y)^*(x-y)e')=$ $=0$ then $x-y=(x-y)p_o=(x-y)(\bigvee e')=0$ (by the faithfulness of φ). Thus by the Ryll-Navdjewski fixed point theorem (see A.3 in [10]) there exists an element $X\in K_\delta$ with $T_u(X)=X$ for all unitary elements $u\in N$. But then $uXu^*+\delta(u)u^*=X$ and thus $\delta(u)=Xu-uX$ and by linearity, $\delta(x)=Xx-xX$ for all $x\in N$. Since N is of type II_1 it has no minimal projections so that by 4.3 $X\in J(M)$. Similary, if $\delta(x)=p_o\delta(x)$ for any $x\in N$ we obtain that δ is implemented by an element in J(M). It follows that there exists $K\in J(M)$ such that $(\delta-\text{ad }K)(x)=(1-p_o)(\delta-\text{ad }K)(x)(1-p_o)$. Thus, if we define $\delta_o:N_{1-p_o}\longrightarrow M_{1-p_o}$ by $\delta_o(x(1-p_o))=(\delta-\text{ad }K)(x)(1-p_o)$ then δ_o is a well defined derivation taking values into $(1-p_o)J(M)(1-p_o)=$ $=J(M_{1-p_o})$. Since $N'_{1-p_o}\cap M_{1-p_o}$ contains no finite projections of M_{1-p_o}, this shows that in order to prove the theorem for N separable of type II_1 and M countable decomposable, we may in addition assume that $N'\cap M$ contains no finite projections of M.

7.4. In the rest of this section we may therefore assume N is separable, M is of countable type and $N'\cap M$ contains no finite projections of M. By 6.2 there exists a maximal abelian *-subalgebra A of N such that $A'\cap M$ contains no finite projections of M. By § 5, there exists $K\in J(M)$ such that $\delta|_A=\text{ad }K|_A$. Thus, by taking $\delta-\text{ad }K$ instead of δ, we may suppose $\delta|_A=0$. We show that from this it follows that δ vanishes on all N which will end the proof of the theorem.

Assume $\delta \neq 0$. Then there exists a unitary element $v \in M$ such that $\delta(v) \neq 0$. Moreover there exists a finite projection $e \in M$ such that $\varphi(ev^*\delta(v)e) \neq 0$. Indeed, because otherwise $\varphi(v^*\delta(v)x) = 0$ for any linear conbination of finite projections e, and thus, by taking norm limits, for any $x \in M$ with $\varphi(x^*x) < \infty$, which implies $v^*\delta(v) = 0$, a contradiction.

Fix $e \in M$ to be a finite projection with $\varphi(ev^*\delta(v)e) \neq 0$. By replacing if necessary δ with a scalar multiple of it we may then assume $\varphi(ev^*\delta(v)e) = 1$. Moreover we may suppose from now on that the trace φ satisfies $\varphi(e) = 1$.

We now prove that for any n there exist a finite projection $e_n \in M$ and a unitary element $w_n \in N$ such that:

1) $e_n \leq e$, $\varphi(e - e_n) \leq 2^{-n}$.

2) $\|e_n w_n^k e_n\| < 2^{-n}$, for $k \neq 0$, $|k| \leq n$.

3) $\varphi(e_n w_n^k e_n) = 0$, for $k \neq 0$.

4) $|\varphi(e_n w_n^{-p} \delta(w_n^p) e_n) - 1| < 2^{-n}$ if $n \geq p > 0$ and

$|\varphi(e_n w_n^{-s} \delta(w_n^p) e_n)| < 2^{-n}$ if $p \neq s$ or $p \leq 0$ or $s \leq 0$.

To do this let $f \in M$ be a finite projection such that $\|\delta(v)(1-f)\| < (4n)^{-1} 2^{-n-1}$, $\|(1-f)v^{-1}\delta(v)\| < (4n)^{-1} 2^{-n-1}$, $\|\delta(v^{-1})v(1-f)\| < (4n)^{-1} 2^{-n-1}$ and $f \geq e$. Then by the preceding corollary there exist unitary elements $w_n \in N$ and $u_n \in A$ and a projection $e_n \in M$ such that

a) $e_n \leq e$, $\varphi(e - e_n) \leq 2^{-n}$.

b) $\|f w_n^k e_n\| \leq (4n \|\delta\|)^{-1}$ for $k \neq 0$, $|k| \leq n$.

c) $\|w_n - u_n v\| \leq (4n \|\delta\|)^{-1} 2^{-n-1}$ and $\varphi(e_n w_n^k e_n) = 0$ for $k \neq 0$.

It follows that if $n \geqslant k > 0$ then:

(i) $\| \delta(w_n^k)e_n - w_n^{k-1}\delta(w_n)e_n \|_\varphi = \| \sum_{s=0}^{k-2} w_n^s \delta(w_n) w_n^{k-s-1} e_n \|_\varphi \leq$

$$\leq \sum_{s=0}^{k-2} \| \delta(w_n) w_n^{k-s-1} e_n \|_\varphi \leq \sum_{s=0}^{k-2} \| \delta(u_n v) w_n^{k-s-1} e_n \| +$$

$$+ (k-1) \| \delta \| \, \| e \|_\varphi \, \| w_n - u_n v \| = \sum_{s=0}^{k-2} \| \delta(v) w_n^{k-s-1} e_n \|_\varphi +$$

$$+ (k-1) \| \delta \| \, \| w_n - u_n v \| \leq \sum_{s=0}^{k-2} \| \delta(v) f w_n^{k-s-1} e_n \|_\varphi +$$

$$+ (k-1) \| \delta(v)(1-f) \| + 4^{-1} \cdot 2^{-n-1} \leq 2^{-n-1} ;$$

(ii) $\| \delta(w_n^{-k}) e_n \|_\varphi \leq \sum_{s=0}^{k-1} \| (w_n^{-1})^s \delta(w_n^{-1}) (w_n^{-1})^{k-s-1} e_n \|_\varphi \leq$

$$\leq \sum_{s=0}^{k-1} \| (\delta(v^{-1})v)(u_n v)^{-1} (w_n^{-1})^{k-s-1} e_n \|_\varphi +$$

$$+ (k-1) \| \delta \| \, \| u_n v - w_n \| \leq \sum_{s=0}^{k-1} \| \delta(v^{-1}) v (w_n^{-1})^{k-s} e_n \|_\varphi + 2(k-1) \| \delta \| \, \| u_n v - w_n \| \leq$$

$$\leq \sum_{s=0}^{k-1} \| \delta(v^{-1}) v (w_n^{-1})^{k-s} e_n \|_\varphi + 2(k-1) \| \delta \| \, \| u_n v - w_n \| \leq$$

$$\leq \sum_{s=0}^{k-1} \| \delta(v^{-1}) v f (w_n^{-1})^{k-s} e_n \|_\varphi + (k-1)(4n)^{-1} 2^{-n-1} + 2(k-1)(4n)^{-1} 2^{-n-1} \leq$$

$$\leq \| \delta \| \sum_{s=0}^{k-1} \| f (w_n^{-1})^{k-s} e_n \|_\varphi + (3/4) 2^{-n-1} \leq (n \| \delta \| (4n \| \delta \|)^{-1} + 3/4) 2^{-n-1} =$$

$$= 2^{-n-1} .$$

Thus for $p > 0$ we have by (i), (c) and the equality $\delta(u_n v) = u_n \delta(v)$:

$$|\varphi(e_n w_n^{-p}\delta(w_n^p)e_n)-1| \leq |\varphi(e_n w_n^{-1}\delta(w_n)e_n)-1|+2^{-n-1} \leq$$

$$\leq |\varphi(e_n v^{-1}u_n^{-1}\delta(u_n v)e_n)-1| +2\,\|\delta\|\,\|w_n-u_n v\|+2^{-n-1} \leq$$

$$\leq |\varphi(e_n v^{-1}\delta(v)e_n)-1| +2^{-n}=2^{-n}.$$

If $p>0$ and $s\neq p$ then by (i), (c) and (b) we have:

$$|\varphi(e_n w_n^{-s}\delta(w_n^p)e_n)| \leq |\varphi(e_n w_n^{-s+p-1}\delta(w_n)e_n)| +2^{-n-1} \leq$$

$$\leq |\varphi(e_n w_n^{-s+p-1}u_n v v^{-1}\delta(v)e_n)| +4^{-1}2^{-n-1}+2^{-n-1} \leq$$

$$\leq |\varphi(e_n w_n^{-s+p}v^{-1}\delta(v)e_n)| +2\cdot 4^{-1}2^{-n-1}+2^{-n-1} \leq$$

$$\leq |\varphi(e_n w_n^{-s+p}fv^{-1}\delta(v)e_n)| +3\cdot 4^{-1}\cdot 2^{-n-1}+2^{-n-1} \leq$$

$$\leq (4^{-1}+3\cdot 4^{-1}+1)\,2^{-n-1}=2^{-n}.$$

Finally if $p<0$ then by (ii) and the Cauchy-Schwartz inequality we have for any s:

$$|\varphi(e_n w_n^{-s}\delta(w_n^p)e_n)| \leq \|\delta(w_n^p)e_n\|_\varphi \leq 2^{-n}$$

This shows that e_n and w_n as defined before fulfill conditions (1)-(4).

We now define $A_n\subset M$ to be the von Neumann algebra generated by w_n, $p_n\in\mathcal{B}(L^2(M,\varphi))$ the orthogonal projections onto $\overline{A_n e_n}$, the isometries $u_n:L^2(\mathbb{T},\mu)\longmapsto L^2(M,\varphi)$ (where μ is the normalized Lebesgue measure on the thorus $\mathbb{T}$) by $u_n(z^k)=\varphi(e_n)^{-1/2}w_n^k e_n$ and the measure preserving isomorphism $\Psi_n:L^\infty(\mathbb{T},\mu)\longmapsto(A_n,\varphi(e_n)^{-1}\varphi(.e_n))$ by $\Psi_n(z^k)=w_n^k$. Moreover we define $\delta_n:L^\infty(\mathbb{T},\mu)\longmapsto\mathcal{B}(L^2(\mathbb{T},\mu))$ by

$\delta_n(f)=u_n^*\delta(\varphi_n(f))u_n$ for $f\in L^\infty(\mathbb{T},\mu)$. Since $p_n=u_nu_n^*\in A_n'$, an easy computation shows that all δ_n are derivations and clearly $\|\delta_n\|\leq\|\delta\|$.

Let ω be a free ultrafilter on $\mathbb{N}$ and denote $\Delta: L^\infty(\mathbb{T},\mu)\longmapsto \mathcal{B}(L^2(\mathbb{T},\mu))$ by $\Delta(f)=\text{w-}\lim_{n\to\omega}\delta_n(f)$. Then Δ is also a derivation and $\|\Delta\|\leq\|\delta\|$. We show that if P denotes the orthogonal projection onto the Hardy space $H^2(\mathbb{T},\mu)=\overline{\text{span}}\{z^k \mid k\geq 0\}\subseteq L^2(\mathbb{T},\mu)$ then $\Delta=\text{ad } P$ and that Δ is continuous on the unit ball of $L^\infty(\mathbb{T},\mu)$ with the norm $\|\ \|_2$ into $(L^2(\mathbb{T},\mu))$ with the uniform norm.
To prove the first assertion note that by (4) $\langle\delta_n(z^p)1,z^s\rangle = =\varphi(e_nw_n^{-s}\delta(w_n^p)e_n)$ tend to 1 for $p=s\geq 0$ and to 0 otherwise so that $\langle\Delta(z^p)1, z^s\rangle$ is equal to 1 if $p=s\geq 0$ and to 0 otherwise. Since ad P also satisfies these equalities and Δ, ad P are derivations it follows that $\langle\Delta(z^p)z^k,z^s\rangle=\langle\text{ad } P(z^p)z^k,z^s\rangle$ for all $k,p,s\in\mathbb{Z}$ and thus, by linearity and weak continuity of Δ and ad P, $\Delta=\text{ad } P$.

To prove the second assertion we first prove the following:

(*) Given $\beta>0$ there exists $n_o\geq 1$ and $\alpha>0$ such that for any $n\geq n_o$ and $a\in A_n$, with $\|a\|\leq 1$, $\varphi(e_na^*ae_n)<\alpha$, we have $|||\delta(a)|||<\beta$.

Indeed, by 4.1 there exists $\alpha'>0$ such that if $\varphi(ea^*ae)<\alpha'$ then $|||\delta(a)|||<\beta$. Let n_o be such that if $n\geq n_o$ then $\varphi(e-e_n)<\alpha'/2$. If we take $\alpha=\alpha'/2$ and if $\varphi(e_na^*ae_n)\leq\alpha$ then we get $\varphi(ea^*ae)\leq \leq\varphi(e-e_n)\ \|a^*a\|+\alpha\leq\alpha'/2+\alpha'/2=\alpha'$, so that $|||\delta(a)|||<\beta$.

We have to show that given any $\beta>0$ there exists $\alpha>0$ such that if $f\in L^\infty(\mathbb{T},\mu)$, $\|f\|\leq 1$ and $\|f\|_2<\alpha$ then $\|\Delta(f)\xi\|<\beta$ for any $\xi\in L^2(\mathbb{T},\mu)$, $\|\xi\|_2\leq 1$. In fact it is sufficient to check this for ξ Laureant polynomials, $\xi=\sum_{|k|\leq m}\alpha_kz^k$ (with $\sum|\alpha_k|^2\leq 1$).

Let α be the one given by (*). Then if $a_n=\psi_n(f)$ we have $\|\Delta(f)\xi\| \leq \limsup_n \|\delta_n(f)\xi\| = \limsup_n \|p_n\delta(a_n)p_n(\sum_{|k|\leq m}\alpha_k w_n^k)e_n\|_\varphi \leq$
$\leq \limsup_n \|\delta(a_n)(\sum_{|k|\leq m}\alpha_k w_n^k)e_n\|_\varphi$.

But $\|(\sum_{|k|\leq m}\alpha_k w_n^k)e_n\|^2 = \|\sum_{i,j}\bar\alpha_i\alpha_j e_n w_n^{j-i}e_n\| \leq$
$\sum_i |\alpha_i|^2 + \sum_{i\neq j}|\alpha_i||\alpha_j|\,\|e_n w_n^{j-i}e_n\|$ and since $\sum_{i,j}|\alpha_i||\alpha_j| = (\Sigma|\alpha_i|)^2 \leq$
$\leq (2m+1)\sum|\alpha_i|^2 = 2m+1$ by (2) we get $\|(\Sigma\alpha_k w_n^k)e_n\|^2 \leq 1+(2m+1)2^{-n}$.
Thus, since for $n \geq n_o$ we have $\|\delta(a_n)\| < \beta$ it follows that if $n \geq n_o$, $\|\delta(a_n)(\sum_{|k|\leq m}\alpha_k w_n)e_n\|_\varphi \leq (1+(2m+1)2^{-n})$. Hence
$\limsup_n \|\delta(a_n)(\Sigma\alpha_k w_n^k)e_n\|_\varphi \leq \beta$ and thus $\|\Delta(f)\xi\| \leq \beta$.

We have thus proved that ad P is continuous from the unit ball of $L^\infty(\mathbb{T},\mu)$ with the two-norm into $\mathcal{B}(L^2(\mathbb{T},\mu))$ with the uniform norm. But ad P takes values into the finite rank operators on polynomials so that by the above continuity it follows that ad P takes values into $\mathcal{K}(L^2(\mathbb{T},\mu))$ on all $L^\infty(\mathbb{T},\mu)$. But then by §5 (the abelian case of the theorem) ad P is equal to ad K for some $K\in\mathcal{K}(L^2)$. It follows that $P-K\in L^\infty(\mathbb{T},\mu)$ and thus $P-K=f$ for some function $f\in L^\infty(\mathbb{T},\mu)$ (since $L^\infty(\mathbb{T},\mu)$ is maximal abelian in $\mathcal{B}(L^2(\mathbb{T},\mu))$). But $1=\lim_{n\to\infty}\langle(P-K)z^n,z^n\rangle = \int z^{-n}fz^n d\mu(z) = \int f d\mu(z) =$
$= \lim_{n\to-\infty}\langle(P-K)z^n,z^n\rangle = 0$, which is a contradiction.

The initial assumption $\delta\neq 0$ is therefore false and so the theorem is completely proved.

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