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MULTIPLICATION OF CERTAIN NON-COMMUTING RANDOM VARIABLES

(preliminary version)

by

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The problem considered in the present paper can be illustrated by the following example.

Let $G=G_1 * G_3$ be the free product of two groups and let X_1, X_2 be bounded left convolution operators on $\ell^2(G)$ such that $X_j \xi \in \ell^2(G_j) \subset \ell^2(G)$ $(j=1,2)$ where $\xi \in \ell^2(G)$ is the function $\xi(g)=\delta_{g,e}$. The operators X_j may be viewed as "random variables" with moments $\langle X_j^k \xi, \xi \rangle$ or equivalently with distributions given by the analytic functionals μ_j where $\mu_j(f)= \langle f(X_j)\xi, \xi \rangle$. With these conventions the distribution of $X_1 X_2$ depends only on the distributions of X_1 and X_2 (independently of the order, see [8]) and the aim of the present paper is to explicitate this relationship. This may be viewed as a non-commutative analogue of the multiplication of independent random variables. Indeed if $G_1 * G_2$ is replaced by $G_1 \times G_2$ then we have precisely the situation of independent random variables for which multiplication corresponds to multiplicative convolution of their distributions.

We began studying this kind of non-commutative independence of random variables in [8]. The addition problem for such random variables was solved in [9]. A general frame-work in which these operations appear as natural convolutions on state spaces of operator algebras having a certain dual group structure has been given in [10].

In the commutative situation, multiplication of random variables corresponds to multiplicative convolution of their distributions which in turn amounts to multiplication of their

Mellin-Fourier transforms.In our non-commutative situation we find certain series playing the same role as the Mellin-Fourier transform.If μ is the distribution of a random variable we consider the series

$$\psi(z)=\sum_{n\geq 1} m_n z^n$$

where $m_n=\int t^n d\mu(t)$ are the moments of μ .Assume $m_1\neq 0$ and let S_μ be defined by

$$\chi(\psi(z))=z$$

$$S_\mu(z)=\chi(z)z^{-1}(1+z)$$

If μ_1, μ_2, μ_3 are the distributions of X_1, X_2 and X_1X_2 in the example given at the beginning and if $\langle X_1\xi,\xi\rangle\neq 0$, $\langle X_2\xi,\xi\rangle\neq 0$ then we have (Thm.2.6 below):

$$S_{\mu_3}(z)=S_{\mu_1}(z)S_{\mu_2}(z).$$

This shows that S_μ is the analogue of the Mellin-Fourier transform and gives a way for computing μ_3.

Note that if $X_1\geqslant 0, X_2\geqslant 0$ in our example then X_1X_2 has the same distribution as $X_1^{1/2}X_2X_1^{1/2}$. Moreover μ_3 is precisely the trace of the spectral measure of $X_1^{1/2}X_2X_1^{1/2}$ and the support of μ_3 is the spectrum of $X_1^{1/2}X_2X_1^{1/2}$.

The paper has two sections. The first section deals with preliminaries.The second section contains our main result about the solution of the multiplication problem.

§ 1

This section is devoted to preliminaries about free families of non-commutative random variables and their multiplication.

We begin by recalling, in a slightly adapted version facts from § 4 of [8].The operation $\boxtimes$ introduced in [8] and which ... object to be studied below,will be denoted throughout the present paper by $\boxtimes$.

Let (A,φ) be a unital algebra over $\mathbb{C}$ with a specified state φ (i.e.a linear functional $\varphi : A \longrightarrow \mathbb{C}$ such that $\varphi(1)=1$). An element $a \in A$ will be viewed as a "random variable" the "distribution" of which is the functional $\mu_a : \mathbb{C}[X] \longrightarrow \mathbb{C}$ given by $\mu_a(1)=1$, $\mu_a(X^n)= \varphi(a^n)$. If A is a Banach algebra and φ is continuous then μ_a extends to an analytic functional such that $\mu_a(f)= \varphi(f(a))$ where f is a holomorphic function on $\mathbb{C}$. In case A is a C^*-algebra and φ is a C^*-algebraic state (i.e. φ is also positive) and if a is selfadjoint (unitary) then the distribution μ_a has an unique extension to a probability measure on $\mathbb{R}$ (respectively $\mathbb{T} = \{ z \in \mathbb{C} \mid |z| =1 \}$) with compact support.

1.1. Definition ([8]). Let (A,φ) be a unital algebra over $\mathbb{C}$ with specified state φ and let $1 \in A_i \subset A$ $(i \in I)$ be subalgebras. The family $(A_i)_{i \in I}$ will be called free if

$$\varphi(a_1 a_2 \ldots a_n) =0$$

whenever $a_j \in A_{i_j}$ with $i_1 \neq i_2 \neq \ldots \neq i_n$ and $\varphi(a_j)=0$ for $1 \leq j \leq n$. A family of subsets $X_i \subset A$ (elements $a_i \in A$) where $i \in I$ will be called free if the family of subalgebras A_i generated by $\{1\} \cup X_i$ (respectively $\{1, a_i\}$) is free.

1.2. Proposition ([8]). If $\{a,b\}$ is a free pair of elements of (A,φ) then μ_{ab} depends only on μ_a and μ_b. There are universal polynomials with integer coefficients $Q_n(x_1,\ldots,x_n,y_1,\ldots,y_n)$ such that assigning degree j to x_j and y_j, we have:

(i) Q_n is homogeneous of degree n both in the x- and y-variables,

(ii) $\mu_{ab}(X^n)=Q_n(\mu_a(X),\ldots,\mu_a(X^n),\mu_b(X),\ldots\mu_b(X^n))$,

(iii) $Q_n(x_1,\ldots,x_n,y_1,\ldots,y_n)=Q_n(y_1,\ldots,y_n,x_1,\ldots,x_n)$

(iv) $\Sigma = \{ \xi \in \mathbb{C}[X] \longrightarrow \mathbb{C} \mid \xi(1)=1, \xi \text{ linear} \}$ is a commutative semigroup for the operation

$(\xi \boxtimes \eta)(X^n) = Q_n(\xi(X), \ldots, \xi(X^n), \eta(X), \ldots, \eta(X^n))$.

Note that if ξ, η extend to analytic functionals, then $\xi \boxtimes \eta$ also extends to an analytic functional. Indeed it is easily seen that we can find C^*-algebras with specified states (A_j, φ_j) and $a_j \in A_j$ such that $\xi = \mu_{a_1}$, $\eta = \mu_{a_2}$ and forming the reduced free product $(A_1, \varphi_1) * (A_2, \varphi_2)$ (see [8] § 1) we have $\xi \boxtimes \eta = \mu_{\sigma_1(a_1)\sigma_2(a_2)}$ which is an analytic functional.

Also, if ξ, η extend to probability measures on $\mathbb{T}$ then $\xi \boxtimes \eta$ also extends to a probability measure on $\mathbb{T}$. Indeed in this case the $a_j \in A_j$ can be chosen to be unitaries and $\sigma_1(a_1)\sigma_2(a_2)$ will also be unitary which gives the desired conclusion.

Similarly if ξ, η extend to compactly supported probability measures on the positive half-line $\mathbb{R}_{\geq 0}$ then $\xi \boxtimes \eta$ also extends to a compactly supported probability measure on $\mathbb{R}_{\geq 0}$. In this case the elements a_j may be chosen ≥ 0 and the C^*-algebras A_j commutative. Then the states φ_1, φ_2 are traces and the free-product state $\varphi = \varphi_1 * \varphi_2$ is also a trace state. This implies $\varphi((\sigma_1(a_1)\ \sigma_2(a_2))^n) = \varphi((\sigma_2(a_2)^{1/2}\ \sigma_1(a_1)\ \sigma_2(a_2)^{1/2})^n)$ so that $\mu_{\sigma_1(a_1)\ \sigma_2(a_2)} = \mu_b$ where $b = \sigma_2(a_2)^{1/2}\ \sigma_1(a_1)\ \sigma_2(a_2)^{1/2} \geq 0$ which gives the desired conclusion.

Let us also recall some facts about a certain extension of the Cuntz-algebra O_n (see [4],[7],[6],[5]). As in ([5]) this extension may be realized on the full Fock-space

$$\mathcal{T}(H_n) = \mathbb{C}1 \oplus \bigoplus_{n \geq 1} H_n^{\otimes m}$$

where H_n is an n-dimensional complex Hilbert space with orthonormal basis $e_1, \ldots, e_n$. The C^*-algebra $\mathcal{E}_n$ is the C^*-algebra generated by the isometries ℓ_j $(1 \leq j \leq n)$

$$\ell_j h = e_j \otimes h, \quad h \in \mathcal{T}(H_n).$$

Consider also the state ε_n on $\mathcal{E}$ given by $\varepsilon_n(X) = \langle X1, 1 \rangle$. As explained in ([8] § 2 or [3]) the pair $(\mathcal{E}_n, \varepsilon_n)$ may be viewed as the reduced free product of n copies of $(\mathcal{E}_1, \varepsilon_1)$. This implies that a family $a_1,\dots,a_n$ where $a_j \in C^*(\ell_j) \subset \mathcal{E}$ is free in $(\mathcal{E}_n, \varepsilon_n)$.

As in [9] it will be useful to consider random variables of the form

$$X = \ell_1^* + \sum_{k=0}^{\infty} \alpha_{k+1} \ell_1^k$$

in $(\mathcal{E}_1, \varepsilon_1)$. It is easily seen that there are polynomials $E_n(x_1,\dots,x_n)$, homogeneous of degree n when x_j is assigned degree j, such that

$$\varepsilon_1(X^n) = E_n(\alpha_1,\dots,\alpha_n)$$

and

$$E_n(x_1,\dots,x_n) = x_n + \widetilde{E}_n(x_1,\dots,x_{n-1}).$$

(These polynomials have been explicitely determined in [9], but this will not be used here).

§ 2.

In this section we obtain the explcit formulae for the computation of the operation

2.1. Lemma. Let Q_n be the polynomials defined in Proposition 1.2. We have :

$$Q_1(x_1,y_1)=x_1y_1$$

$$Q_n(x_1,\ldots,x_n,y_1,\ldots,y_n)=$$

$$=x_ny_1^n+x_1^ny_n+\tilde{Q}_n(x_1,\ldots,x_{n-1},y_1,\ldots,y_{n-1})$$

where $n\geqslant 2$ and $\tilde{Q}_n$ are polynomials.

Proof. If $\{a,b\}$ is a free pair in (A,φ) then $\varphi(ab)=\varphi(a)\varphi(b)$ which yields the assertion concerning Q_1.

For the second assertion consider a free pair $\{a,b\}$ in $(\mathcal{E}_2,\varepsilon_2)$ of the form

$$a=\ell_1^*+\alpha_1 I+\alpha_2\ell_1+\ldots+\alpha_n\ell_1^{n-1}$$

$$b=\ell_2^*+\beta_1 I+\beta_2\ell_1+\ldots+\beta_n\ell_2^{n-1}$$

and remark that this imposes no restriction on the moments $\varepsilon_2(a^j)$, $\varepsilon_2(b^j)$ for $1\leq j\leq n$.

Remark further that if $\omega_j,\omega'_j\in\{*,0,1,2,\ldots,n-1\}$ $(1\leq j\leq n)$ and at least one of the ω_j equals $n-1$ then

$$\langle \ell_1^{\omega_1}\ell_2^{\omega'_1}\ell_1^{\omega_2}\ell_2^{\omega'_2}\ldots\ell_1^{\omega_n}\ell_2^{\omega'_n}1,1\rangle\neq 0$$

if and only if $j=n$, $\omega_1=\cdots=\omega_{n-1}=*$ and $\omega'_1=\ldots=\omega'_n=0$. Similarly the same expression is $\neq 0$ assuming at least one of the ω'_j equals $n-1$ if and only if $j=n$, $\omega'_1=\ldots=\omega'_{n-1}=*$ and $\omega_1=\ldots=\omega_n=0$.

This shows that

$$\varepsilon_2((ab)^n)=\alpha_n\beta_1^n+\alpha_1^n\beta_n+\tilde{q}_n(\alpha_1,\ldots,\alpha_{n-1},\beta_1,\ldots,\beta_{n-1})$$

where $\tilde{q}_n$ are polynomials. Since

$$\varepsilon_2(a^j)=\alpha_j+\tilde{E}_n(\alpha_1,\dots,\alpha_{n-1})$$

$$\varepsilon_2(b^j)=\beta_j+\tilde{E}_n(\beta_1,\dots,\beta_{n-1})$$

$(1\leq j\leq n)$, our assertion follows.

Q.E.D.

2.2. Lemma. The functional $\xi\in\Sigma$ such that $\xi(X^n)=1$ for all $n\geqslant 1$, is a neutral element with respect to $\boxtimes$. An element $\eta\in\Sigma$ is invertible with respect to $\boxtimes$ if and only if $\eta(X)\neq 0$. The sets $\Sigma^*=\{\xi\in\Sigma \mid \xi(X)\neq 0\}$ and $\Sigma_1=\{\xi\in\Sigma \mid \xi(X)=1\}$ are abelian groups with respect to the operation $\boxtimes$.

Proof. Remark that $\xi(X^n)=1$ for all $n\geqslant 1$ means that ξ is the distribution of the unit of an algebra and hence clearly ξ is neutral in Σ.

The assertions concerning invertible elements are easil obtained using the preceding lemma.

Q.E.D.

2.3. Lemma. In $(\mathcal{E}_2,\varepsilon_2)$ consider random variables $T\in C^*(\ell_1)\subset\mathcal{E}_2$ and

$$Y(\tau)=\ell_2^*+I+\tau\sum_{p=0}^{\infty}\alpha_{p+1}\,\ell_2^p\in C^*(\ell_2)\subset\mathcal{E}_2$$

where only finitely many α_j are non-zero. We have

$$\frac{d}{d\tau}\,\varepsilon_2((Y(\tau)T)^n)\Big|_{\tau=0}=$$

$$=\sum_{p=1}^{n}\alpha_p\sum_{\substack{j_1+\dots+j_p=n\\ j_1\geqslant 1,\dots,j_p\geqslant 1}} j_1\,\varepsilon_2(T^{j_1})\dots\varepsilon_2(T^{j_p})$$

Proof. We have

$$\varepsilon_2((Y(\tau)T)^n)-\varepsilon_2((Y(0)T)^n)=$$

$$= \tau \sum_{p=0}^{n-1} \alpha_{p+1} \Big(\sum_{k=1}^{n} \varepsilon_2(((I+\ell_2^*)T)^{n-k} \ell_2^p T((I+\ell_2^*)T)^{k-1}) + O(\tau^2).$$

Since $((I+\ell_2^*)T)^{k-1}1 = T^{k-1}1$ we infer that

$$\varepsilon_2(((I+\ell_2^*)T)^{n-k} \ell_2^p T((I+\ell_2^*)T)^{k-1}) =$$

$$= \varepsilon_2(((I+\ell_2^*)T)^{n-k} \ell_2^p T^k).$$

Similarly taking into account that $T \in C^*(\ell_2)$ and $\varepsilon_2(S) = \langle S1,1 \rangle$ it is easily seen that

$$\varepsilon_2(((I+\ell_2^*)T)^{n-k} \ell_2^p T^k) =$$

$$= \sum_{\substack{k+j_1+\dots+j_p \leq n \\ j_1 \geq 1,\dots,j_p \geq 1}} \varepsilon_2(T^{n-k-j_1-\dots-j_p} \ell_2^* T^{j_1} \ell_2^* T^{j_2} \dots \ell_2^* T^{j_p} \ell_2^p T^k) =$$

$$= \sum_{\substack{k+j_1+\dots+j_p \leq n \\ j_1 \geq 1,\dots,j_p \geq 1}} \varepsilon_2(T^{n-j_1-\dots-j_p}) \, \varepsilon_2(T^{j_1}) \dots \varepsilon_2(T^{j_p}).$$

Hence

$$\varepsilon_2((Y(\tau)T)^n) - \varepsilon_2((Y(0)T)^n) =$$

$$= \tau \sum_{p=0}^{n-1} \alpha_{p+1} \Big(\sum_{k=1}^{n} \sum_{\substack{k+j_1+\dots+j_p \\ j_1 \geq 1,\dots,j_p \geq 1}} \varepsilon_2(T^{n-j_1-\dots-j_p}) \, \varepsilon_2(T^{j_1}).$$

$$.. \ \varepsilon_2(T^{j_p}) + O(\tau^2) =$$

$$= \tau \sum_{p=0}^{n-1} \alpha_{p+1} \Big(\sum_{\substack{j_1+\dots+j_p \leq n \\ j_1 \geq 1,\dots,j_p \geq 1}} (n-j_1-\dots-j_p) \, \varepsilon_2(T^{n-j_1-\dots-j_p}).$$

$$. \ \varepsilon_2(T^{j_1}) \dots \varepsilon_2(T^{j_p}) + O(\tau^2) =$$

$$= \tau \sum_{p=0}^{n-1} \alpha_{p+1} \Big(\sum_{\substack{j_1+\dots+j_{p+1}=n \\ j_1\geq 1,\dots,j_{p+1}\geq 1}} j_{p+1}\, \varepsilon_2(T^{j_1})\dots\varepsilon_2(T^{j_{p+1}}) + O(\tau^2)\Big)$$

which is the desired result.

Q.E.D.

2.4. Lemma. In $(\mathcal{E}_2, \varepsilon_2)$ consider random variables $T \in C^*(\ell_1) \subset \mathcal{E}_2$ and

$$Y(\tau) = \ell_2^* + I + \tau \sum_{p=0}^{\infty} \alpha_{p+1} \ell_2^p \in C^*(\ell_2) \subset \mathcal{E}_2$$

where only finitely many α_j are non-zero. Let ψ and φ be the formal power series

$$\psi(z) = \sum_{p=1}^{\infty} \varepsilon_2(T^p) z^p$$

$$\varphi(z) = \sum_{p=0}^{\infty} \alpha_{p+1} z^p$$

We have

$$\varepsilon_2((Y(0)T)^p) = \varepsilon_2(T^p)$$

and

$$\varphi(\psi(z)) z \frac{d}{dz} \psi(z) =$$

$$= \sum_{n=1}^{\infty} \Big(\frac{d}{d\tau} \varepsilon_2((Y(\tau)T)^n)\Big|_{\tau=0} \Big) z^n.$$

Proof. It is immediate that the distribution of the random variable $Y(0) = I + \ell_2^*$ is the neutral element in Σ and this implies that $Y(0)T$ and T have the same distribution.

Using the preceding lemma we have $\varphi(\psi(z)) z \frac{d}{dz} \psi(z) =$

$$=\Big(\sum_{p=0}^{\infty}\alpha_{p+1}(\psi(z))^p\Big)\Big(\sum_{j=1}^{\infty}j\,\varepsilon_2(T^j)z^j\Big)=$$

$$=\sum_{p=1}^{\infty}\alpha_p\sum_{j_1\geq 1,\dots,j_p\geq 1} j_1\,\varepsilon_2(T^{j_1})\dots\varepsilon_2(T^{j_p})z^{j_1+\dots+j_p}=$$

$$=\sum_{n=1}^{\infty}\Big(\frac{d}{d\tau}\varepsilon_2((Y(\tau)T)^n)\Big|_{\tau=0}\Big)z^n.$$

Q.E.D.

2.5. *Lemma. Given a formal power series* $\varphi\in\mathbb{C}[[z]]$ *consider the formal power series without constant term* χ *and* ψ *such that*

$$\chi(z)=z(1+z)^{-1}\exp(\varphi(z))$$

$$\psi(\chi(z))=z$$

Let further $\mu\in\Sigma^*$ *be the functional such that* $\psi(z)=$ $=\sum_{n\geq 1}\mu(X^n)z^n$. *Then the map*

$$\mathbb{C}[[z]]\ni\varphi\rightsquigarrow\mu\in\Sigma^*$$

is a surjective homomorphism of the group $(\mathbb{C}[[z]],+)$ *onto* $(\Sigma^*,\boxtimes)$

Proof. If $\mu\in\Sigma^*$ then $\mu(X)\neq 0$ so for $\psi(z)=\sum_{n\geq 1}\mu(X^n)z^n$ there is a power series without constant term χ such that $\psi(\chi(z))=z$ and then $z^{-1}(1+z)\chi(z)$ has non-zero constant term, so there is $\varphi(z)$ such that $z^{-1}(1+z)\chi(z)=\exp(\varphi(z))$. Thus the considered map is surjective.

To prove that $\varphi\rightsquigarrow\mu$ is a homomorphism we begin by studying the restriction of this map to $\mathbb{C}\varphi$ for some $\varphi\in\mathbb{C}[[z]]\subset\mathbb{C}[[z]]$.

Consider

$$\chi(z,t)=z(1+z)^{-1}\exp(t\varphi(z))$$

and $\psi(z,t)$ such that $\psi(\chi(z,t),t)=z$. Then $\psi(z,t)$ has a holomorphic extension to some neighborhood of $\{0\}\times\mathbb{C}$ in $\mathbb{C}^2$.

We have

$$0=\frac{\partial \psi}{\partial z}(\chi(z,t),t)\,\frac{\partial \chi}{\partial t}(z,t)+\frac{\partial \psi}{\partial t}(\chi(z,t),t)=$$

$$=\frac{\partial \psi}{\partial z}(\chi(z,t),t)\,\chi(z,t)\,\varphi(z)+\frac{\partial \psi}{\partial t}(\chi(z,t),t).$$

Replacing z by $\psi(z,t)$ we obtain

$$-z\,\frac{\partial \psi}{\partial z}(z,t)\ \varphi(\psi(z,t))=\frac{\partial \psi}{\partial t}(z,t)$$

and $\psi(z,0)=z(1+z)^{-1}$.

Define $\mu_t\in\Sigma^*$ by $\sum_{n\geqslant 1}\mu_t(X^n)z^n=\psi(z,t)$.

Let also

$$Y(\tau)=\ell_2^*+I-\tau\varphi(\ell_2)\in C^*(\ell_2)\subset\Sigma_2$$

Then in view of Lemma 2.4 denoting by ξ_τ the distribution of $Y(\tau)$, we have

$$\sum_{n\geqslant 1}\left(\frac{d}{d\tau}(\xi_\tau\boxtimes\mu_\tau)(X^n)\Big|_{\tau=0}\right)z^n=$$

$$=-z\,\frac{\partial \psi}{\partial z}(z,t)\ \varphi(\psi(z,t))=\frac{\partial \psi}{\partial t}(z,t)\qquad(*)$$

For each $n\geqslant 1$ the set

$$\Sigma^*(n)=\{\xi:\ \mathbb{C}X\oplus\ldots\oplus\mathbb{C}X^n\to\mathbb{C}\mid \xi(X)\neq 0,\ \xi\text{ linear}\}\simeq$$

$$\simeq(\mathbb{C}\setminus\{0\})\times\mathbb{C}^{n-1}$$

endowed with the operation $\boxtimes_n$ defined by $(\xi_1,\ldots,\xi_n)\boxtimes_n(\eta_1,\ldots,\eta_n)=(Q_1(\xi_1,\eta_1),\ldots,Q_n(\xi_1,\ldots,\xi_n,\eta_1,\ldots,\eta_n))$ is a commutative complex Lie group in view of Proposition 1.2 and Lemma 2.1. The group $(\Sigma^*,\boxtimes)$ is the inverse limit of the groups $\Sigma^*(n)$.

The equality $(*)$ shows that $(\mu_t(X),\ldots,\mu_t(X^n))$ as a function of $t\in\mathbb{C}$ is an integral curve for an invariant vector field on $\Sigma^*(n)$. Since $\psi(z,0)=z(1-z)^{-1}$ we have that $\mu_0(X^k)=1$ for all $k\geqslant 1$ so that this integral curve starts at the identity of $\Sigma^*(n)$. Thus $t \rightsquigarrow (\mu_t(X), \ldots, \mu_t(X^n))$ is a one-parameter subgroup in $\Sigma^*(n)$.

Consider

$$\varphi(z) = \sum_{k \geq 0} c_k z^k$$

then

$$\mu^{(n)} = (\mu(X), \ldots, \mu(X^n))$$

is a function of $(c_1, \ldots, c_n)$, which will be denoted by $\mu^{(n)}(c_1, \ldots, c_n)$. Since $\mu^{(n)}: \mathbb{C}^n \longrightarrow \Sigma^*(n)$ is a holomorphic function, such that $\mathbb{C} \ni t \longrightarrow \mu^{(n)}(tc_1, \ldots, tc_n)$ is a homomorphism, we infer that $\mu^{(n)}$ is the composition of a linear map of $\mathbb{C}^n$ to the Lie algebra of $\Sigma^*(n)$ and the exponential map of $\Sigma^*(n)$. Hence, $\Sigma^*(n)$ being commutative $\mu^{(n)}$ is a homomorphism. Since n is arbitrary this implies $\varphi \rightsquigarrow \mu$ is a homomorphism.

Q.E.D.

2.6. Theorem. Given $\mu \in \Sigma^*$ consider $\psi, \chi, S_\mu \in \mathbb{C}[[z]]$ such that

$$\psi(z) = \sum_{n \geqslant 1} \mu(X^n) z^n$$

$$\chi(\psi(z)) = z$$

$$S_\mu(z) = \chi(z) z^{-1}(1+z).$$

Then we have

$$S_{\mu_1 \boxtimes \mu_2} = S_{\mu_1} S_{\mu_2}$$

and $\mu \rightsquigarrow S_\mu$ is a bijection of Σ^* onto the invertible elements of $\mathbb{C}[[z]]$.

Proof. It is obvious that $\mu \rightsquigarrow S_\mu$ is a bijection. That $\mu \rightsquigarrow S_\mu$ is a homomorphism follows from Lemma 2.5. Indeed associate with $\mu_j (j=1,2)$ the series φ_j as in Lemma 2.5. Then $\varphi_1 + \varphi_2$ corresponds to $\mu_1 \boxtimes \mu_2$ and it is obvious that $S_{\mu_j} = \exp(\varphi_j)$ and $S_{\mu_1 \boxtimes \mu_2} = \exp(\varphi_1 + \varphi_2)$.

Q.E.D.

Theorem 2.6 provides a method for computing $\mu_1 \boxtimes \mu_2$ if $\mu_1, \mu_2 \in \Sigma^*$.

If $\mu_1, \mu_2 \in \Sigma \setminus \Sigma^*$ then $(\mu_1 \boxtimes \mu_2)(X^n)=0$ for all $n \geq 1$. Indeed if (T_1,T_2) is a free pair of random variables with zero first order moments then by the definition of free families of random variables all moments of T_1T_2 will be zero.

Thus, we still have to see how one computes $\mu_1 \boxtimes \mu_2$ if $\mu_1 \in \Sigma \setminus \Sigma^*$ and $\mu_2 \in \Sigma^*$. This case will be covered by the following somewhat more general variant of Theorem 2.6.

2.7. Theorem. Let $\mu_1 \in \Sigma$, $\mu_2 \in \Sigma^*$ and consider $\psi_j(z) = \sum_{n \geq 1} \mu_j(X^n) z^n$. Let further $\chi, \varphi \in \mathbb{C}[[z]]$ be such that

$$\psi_2(\chi(z))=z$$

$$\chi(z)=z(1+z)^{-1}\exp(\varphi(z))$$

where $\chi(z)$ has zero constant term. The differential equation

$$\varphi(\psi(z,t))z\frac{\partial}{\partial z}\psi(z,t)+\frac{\partial}{\partial t}\psi(z,t)=0$$

with initial condition

$$\psi(z,0)=\psi_1(z)$$

determines a unique formal power series in z with coefficients differentiable functions of t

$$\psi(z,t)=\sum_{n\geq 1} c_n(t)z^n$$

Then we have

$$\psi(z,1)=\sum_{n\geq 1}(\mu_1 \boxtimes \mu_2)(X^n)z^n$$

Proof. The differential equation for $\psi(z,t)$ amounts to a system of ordinary differential equations

$$\frac{d}{dt}c_n(t)+n\alpha c_n+D_n(c_1,\ldots,c_{n-1})=0$$

with initial condition $c_n(0)=\mu_1(X^n)$, where α is the constant term of φ and the D_n's are polynomials and $D_1=0$. Moreover the coefficients of D_n depend only on $\mu_2(X^j)$, $1 \leq j \leq n$. Thus it will be sufficient to prove the theorem for the case when ψ_1 and ψ_2 are polynomials. Moreover by continuity it will be sufficient to prove the theorem only for the case when $\mu_1 \in \Sigma^*$. With $S_{\mu_1}(z)$ defined as in Theorem 2.6 consider

$$\chi(z,t)=S_{\mu_1}(z)\ \exp\ (t\ \varphi(z))z(1+z)^{-1}\ .$$

and define $\tilde{\psi}$ by

$$\tilde{\psi}(\chi(z,t),t)=z.$$

It is immediate that $\tilde{\psi}(z,t)$ satisfies the differential equation for $\psi(z,t)$ and since $\tilde{\psi}(z,0)=\psi_1(z)$ we infer $\tilde{\psi}(z,t)=\psi(z,t)$. But then $\chi(z,1)=S_{\mu_1}(z)S_{\mu_2}(z)z(1+z)^{-1}$ and the assertion of the theorem follows from Theorem 2.6.

Q.E.D.

We would like to conclude the paper with an example of a computation done using Theorem 2.6.

We shall compute $\nu=((1-\alpha)\delta_0+\alpha\,\delta_1)\boxtimes((1-\beta)\delta_0+\beta\delta_1)$ $(0\le\alpha,\beta\le 1)$. If $\{P,Q\}$ is a free pair of self-adjoint projections in (A,φ) where A is a C^*-algebra with faithful trace φ, then ν coincides with the trace of the spectral measure of PQP and in particular the support of ν is the spectrum of PQP. The trace of the spectral measure of PQP can also be computed from the trace of the spectral measure of P+Q which can be computed using our results for the addition of free random variables [9] Also the author jointly with M.Pimsner have computed with ad-hoc methods the generating function of the moments of PQP (unpublished On the other hand we learned from Joel Anderson about the paper [2], where computations of spectral data of certain PQP with different methods have been carried out in order to obtain results about the range of the trace on projections in the reduced C^*-algebra of $(\mathbb{Z}/2\mathbb{Z})*(\mathbb{Z}/3\mathbb{Z})$.

2.8. Example: the computation of $\nu = ((1-\alpha)\delta_0 + \alpha\delta_1) \boxtimes ((1-\beta)\delta_0 + \beta\delta_1)$. If $\mu = (1-\alpha)\delta_0 + \alpha\delta_1$ we have

$$\psi(z) = \alpha z(1-z)^{-1}$$
$$\chi(z) = z(\alpha + z)^{-1}$$
$$S_\mu(z) = (1+z)(\alpha+z)^{-1}.$$

It follows that

$$S_\nu(z) = (1+z)^2(\alpha+z)^{-1}(\beta+z)^{-1}$$
$$\chi(z) = z(1+z)(\alpha+z)^{-1}(\beta+z)^{-1}$$

and $\psi(z)$ satisfies the equation

$$\psi(1+\psi)(\alpha+\psi)^{-1}(\beta+\psi)^{-1} = z$$

so that

$$\psi^2(z-1) + \psi(\alpha z + \beta z - 1) + \alpha\beta z = 0$$

and

$$\psi = \frac{1-\alpha z - \beta z + \sqrt{(\alpha z + \beta z - 1)^2 - 4\alpha\beta(z-1)z}}{2(z-1)} =$$

$$= \frac{1 - z(\alpha+\beta) + \sqrt{(az-1)(bz-1)}}{2(z-1)}$$

where the choice of the branch of the square-root will be made precise later and where

$$a,b = \alpha+\beta - 2\alpha\beta \pm \sqrt{4\alpha\beta(1-\alpha)(1-\beta)}$$

The Cauchy transform of ν is

$$G(z) = \int \frac{d\nu(\zeta)}{z-\zeta} = z^{-1}(1+\psi(z^{-1})$$

and the branch of the square root is determined by the condition

$$\operatorname{Im} z > 0 \Longrightarrow \operatorname{Im} G(z) \leq 0.$$

We have

$$G(z) = \frac{1}{z} + \frac{z - (\alpha+\beta) + \sqrt{(z-a)(z-b)}}{2(1-z)z}.$$

Using [1] and taking into account the algebraic nature of G, the measure ν is given by

$$\gamma = \left(-\frac{1}{\pi}\,\mathrm{Im}\,G\right)\lambda + c_0\,\delta_0 + c_1\,\delta_1$$

where λ denotes Lebesgue measure and c_t is the residue of G at t in case t is a simple pole. This gives $c_0 = 1 - \min(\alpha,\beta)$, $c_1 = \max(\alpha+\beta-1, 0)$. The support of the absolutely continuous part of γ is the interval $[a,b]$.

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