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by

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This paper is divided into two sections. In the first section it is shown that some recent module-theoretical notions as the notion of F-semicocritical module and the notion of F-Loewy series of a module, where F is a right Gabriel topology on a ring R, are in fact latticial. This allows to establish their basic properties in a latticial setting, from which may be derived in a unified and simplified manner a series of results previously proved by other authors in the framework of module theory.

The second section of this paper is concerned with the investigation of the relationship between the F-primitive ideals and the prime ideals of an F-artinian ring R. This makes more transparent the interconnection between the process of associating minimal F-primitive ideals with F-torsion-free indecomposable injective right R-modules, due to Teply [1985], and the process of associating prime ideals in $Spec_F(R)$ with the same kind of modules, due to Năstăsescu [1980].

This paper is a sequel of our previous papers Albu [1984a, 1984b], where it is shown how certain results concerning chain conditions for modules relative to Gabriel topologies can be much easier and more naturally proved by placing these conditions in a latticial setting.

Throughout this paper R will denote an associative ring with nonzero unit element, Mod-R the category of all unitary right R-modules, F a right Gabriel topology on R, $\tau = (\mathcal{T}, \mathcal{F})$ the corresponding hereditary torsion theory, and t the torsion radical associated to τ. If $M \in$ Mod-R, we shall use the following notation:

$$C_F(M) = \{N \mid N \leq M,\ M/N \in \mathcal{F}\}.$$

For each P, $P \leq M$, P^c will denote the F-saturation of P in M, i.e., $P^c/P = t(M/P)$; thus, $P \in C_F(M)$ iff $P = P^c$, i.e. P is F-saturated. If $(N_i)_{i\in I}$ is a family of elements of $C_F(M)$, then $\bigvee_{i\in I} N_i = (\sum_{i\in I} N_i)^c$ and $\bigwedge_{i\in I} N_i = \bigcap_{i\in I} N_i$ are elements of $C_F(M)$. Moreover, $C_F(M)$ is an upper continuous and modular lattice with respect to the partial ordering given by "$\subseteq$" (inclusion) and with respect to the operations "$\vee$" and "$\wedge$", having $t(M)$ as the least element and M as the greatest one.

For all undefined terms or notations the reader is referred to Stenström [1975] or Albu and Năstăsescu [1984].

§1. F-FULL MODULES, F-SEMICOCRITICAL MODULES, AND F-LOEWY SERIES

Let $M \in$ Mod-R and $N \leq M$. If $M \in \mathcal{F}$ and $M/N \in \mathcal{T}$ then it is well-known that N is essential in M. We say that M is F-full if the converse is true. More precisely, we give the following

DEFINITION M is said to be F-full if $M/N \in \mathcal{T}$ for each essential submodule N of M. ∎

This notion was first considered by Boyle [1978], Boyle and Feller [1979, 1980] under the name of "the large condition" in connection with their investigation of modules with Krull dimension; it was extended to arbitrary torsion theories by Lau [1980], and used among others by Golan [1983, 1985] and Teply [1983, 1985].

The basic properties of F-full modules are given by the following

1.1. PROPOSITION (LAU [1980]). Let $M \in$ Mod-R and $N \leq M$. Then

(1) If M is F-full, so are N and M/N.

(2) If N is F-full and $M/N \in \mathcal{T}$, them M is F-full.

(3) If $(M_i)_{i\in I}$ is a family of F-full modules, then $\bigoplus_{i\in I} M_i$ is also F-full.

(4) If $(N_i)_{i\in I}$ is a family of F-full submodules of M, then so is $\sum_{i\in I} N_i$. ■

1.2. PROPOSITION. The following assertions are equivalent for $M \in$ Mod-R:

(1) M is F-full.

(2) For each $N \in C_F(M) \setminus \{M\}$, N is not an essential submodule of M.

(3) Whenever N is an essential submodule of M with $N \in C_F(M)$, it follows that $N = M$.

Proof. (1)⇒(2). Let $N \in C_F(M) \setminus \{M\}$; then N is not an essential submodule of M, for otherwise it would follow $M/N \in \mathcal{T} \cap \mathcal{F} = \{0\}$, i.e., $N = M$, a contradiction.

(2)⇒(3). Obviously.

(3)⇒(1). Let N be an essential submodule of M. Then N^c is an essential submodule of M, and $N^c \in C_F(M)$, hence $N^c = M$, and so $M/N = N^c/N \in \mathcal{T}$. ■

1.3. COROLLARY (STENSTRÖM [1971]). Let $M \in \mathcal{F}$. Then M is F-full if and only if $C_F(M)$ is a complemented lattice.

Proof. It is well-known that an upper continuous modular lattice is complemented iff 1 is the only essential element of the lattice. By 1.2, it is sufficient to show that $N \in C_F(M)$ is an essential element of the lattice $C_F(M)$ iff N is an essential submodule of M. Suppose that N is an essential element of $C_F(M)$, and let $X \leq M$, $X \neq 0$. If $X^c = t(M)$, then $X^c \cap N = t(M)$, hence $X \cap N = X \neq 0$. If $X^c \neq t(M)$, then $X^c \cap N \neq t(M)$, and consequently there exists $x \in X^c \cap N$, $x \notin t(M)$. Thus $xI \subseteq X$ for some $I \in F$. But $xI \neq 0$, for otherwise it would follow $I \subseteq Ann_R(x)$, and so $Ann_R(x) \in F$, a contradiction. Therefore, there exists $a \in I$ such that $0 \neq xa \in X \cap N$, and consequently N is an essential submodule of M. Let us note that the implication proved above is true without the condition $M \in \mathcal{F}$. Conversely, if $N \in C_F(M)$ is an essential submodule of M, then for each $0 \neq Y \in C_F(M)$, $Y \wedge N = Y \cap N \neq t(M) = 0$, hence N is an essential element of $C_F(M)$. ■

1.4. COROLLARY. If M is F-full, then $C_F(M)$ is a complemented lattice.

Proof. According to 1.1, $M/t(M)$ is F-full, hence $C_F(M/t(M))$ is a complemented lattice. But the lattices $C_F(M)$ and $C_F(M/t(M))$ are canonical isomorphic, and so, we are done. ■

1.5. REMARK. Example 1.16 shows that the converse implication from 1.4 does not hold, and in the same time shows that an F-saturated submodule of M, which is an essential submodule of M is not necessarily an essential element of the lattice $C_F(M)$. ■

An other notion, also first introduced by Boyle and Feller [1979, 1980] in connection with their work on modules with Krull dimension, then generalized by Lau [1980] to arbitrary torsion theories and subsequently used by Golan [1983, 1985], Teply [1983, 1985], etc., is that of F-semicocritical module.

DEFINITION. An R-module M is called F-semicocritical if one of the following equivalent conditions is satisfied:

(1) There exists a finite family $(C_i)_{1 \leq i \leq n}$ of F-cocritical modules and a monomorphism $M \longrightarrow \bigoplus_{i=1}^{n} C_i$.

(2) There exists a finite family $(K_i)_{1 \leq i \leq n}$ of submodules of M such that $\bigcap_{i=1}^{n} K_i = 0$ and M/K_i are F-cocritical modules for each i, $1 \leq i \leq n$. ■

Since any F-cocritical module is F-artinian and coirreducible, using 1.1 one deduces that any F-semicocritical module is F-full, F-artinian, and has finite Goldie dimension.

Our next aim is to show that the notion of F-semicocritical module is of latticial nature, by expliciting Remark 2.4 from Albu [1984b]. For this we need some preparatory results.

Recall that $S \in$ Mod-R is called <u>F-simple</u> if the lattice $C_F(S)$ has exactly two elements: t(M) and M, $t(M) \neq M$; thus $C \in$ Mod-R is F-cocritical iff $C \in \mathcal{F}$ and C is F-simple.

<u>1.6. LEMMA</u>. Let $L, N \in C_F(M)$ with $L \leq N$. Then, the assignment $X/L \mapsto X$ defines a lattice isomorphism from the lattice $C_F(N/L)$ on the interval $[L,N]$ of the lattice $C_F(M)$.

<u>Proof</u>. Let $X/L \in C_F(N/L)$; then $L \leq X \leq N$ and $(N/L)/(X/L) \simeq$ $\simeq N/X \in \mathcal{F}$. The exact sequence

$$0 \to N/X \to M/X \to M/N \to 0$$

with $N/X \in \mathcal{F}$ and $M/N \in \mathcal{F}$ yields $M/X \in \mathcal{F}$, i.e., $X \in [L,N]$.

Conversely, if $X \in [L,N]$, then $M/X \in \mathcal{F}$, and so $(N/L)/(X/L) \simeq$ $\simeq N/X \in \mathcal{F}$; hence $X/L \in C_F(N/L)$. It follows that $C_F(N/L)$ and $[L,N]$ are isomorphic as posets, and consequently, also as lattices.■

<u>1.7. COROLLARY</u>. The atoms of the lattice $C_F(M)$ are exactly the F-saturated F-simple submodules of M.

<u>Proof</u>. Let $X \in C_F(M)$; then $t(M) \leq X$, and consequently $t(X) =$ $= t(M)$. According to 1.6, the interval $[t(M),X]$ of $C_F(M)$ is isomorphic to the lattice $C_F(X/t(X))$, and so, to the lattice $C_F(X)$. It follows that $[t(M),X]$ has exactly two elements iff $C_F(X)$ has exactly two elements, i.e., X is an atom of $C_F(M)$ iff X is F-simple.■

Recall that an upper continuous and modular lattice $\mathcal{L}$ is called <u>semi-atomic</u> if its greatest element 1 is a join of atoms; in this case, as in the case of modules, it can be shown (see e.g. Năstăsescu [1983]) that $\mathcal{L}$ is complemented, and for every $x, y \in \mathcal{L}$ with $x \leq y$, the interval $[x,y]$ of $\mathcal{L}$ is also a semi-atomic lattice.

A family $(x_i)_{i\in I}$ of elements of $\mathcal{L}$ is called <u>$\vee$-independent</u> if $x_i \wedge (\bigvee_{j\in I\setminus\{i\}} x_j) = 0$ for each $i \in I$, where 0 is the least element of $\mathcal{L}$; in this case, the join $x = \bigvee_{i\in I} x_i$ is said to be <u>direct</u>, and we write $x = \dot{\bigvee}_{i\in I} x_i$. If $\mathcal{L}$ is a semi-atomic lattice, then 1 is a direct join of atoms, cf. Năstăsescu [1983], and consequently $\mathcal{L}$ is of finite length iff 1 is a finite (direct) join of atoms.

<u>1.8. PROPOSITION</u>. The following assertions are equivalent for an R-module M:

(1) M is F-semicocritical.

(2) $M \in \mathcal{F}$ and $C_F(M)$ is a semi-atomic lattice of finite length.

(3) $M \in \mathcal{F}$ and there exists a finite (independent) family $(C_i)_{1\leq i\leq n}$ of F-cocritical submodules of M such that $M/\sum_{i=1}^{n} C_i \in \mathcal{T}$.

<u>Proof</u>. (1)$\Rightarrow$(2). Let $(D_i)_{i\in I}$ be a family of F-cocritical modules such that M is isomorphic to a submodule Y of $X = \bigoplus_{i=1}^{m} D_i$. For each i, $1 \leq i \leq m$ denote by X_i the canonical image of D_i in X. Since $X/X_i \simeq \bigoplus_{j\neq i} D_j \in \mathcal{F}$, it follows that $X_i \in C_F(X)$ for all i. Consequently

$$X = \sum_{i=1}^{m} X_i = \bigvee_{1\leq i\leq m} X_i$$

and each X_i is an atom of the lattice $C_F(X)$ by 1.7. Thus $C_F(X)$ is a semi-atomic lattice of finite length. According to 1.6, $C_F(Y^c)$ is isomorphic to the interval $[0, Y^c]$ of the lattice $C_F(X)$, hence $C_F(Y^c)$ is also a semi-atomic lattice of finite length; since $C_F(M) \simeq C_F(Y) \simeq C_F(Y^c)$, we are done.

(2)$\Rightarrow$(3). According to the statement made just before the above proposition, there exists an independent family $(C_i)_{1\leq i\leq n}$

of atoms of the lattice $C_F(M)$, i.e. of F-cocritical submodules of M such that $M = \dot{\bigvee}_{1\leq i\leq n} C_i = (\sum_{i=1}^{n} C_i)^c$. Then $M/\sum_{i=1}^{n} C_i \in \mathcal{T}$ and $C_i \cap (\sum_{j\neq i} C_j) \subseteq C_i \wedge (\bigvee_{j\neq i} C_j) = 0$ for each i, $1\leq i\leq n$, hence the family $(C_i)_{1\leq i\leq n}$ of submodules of M is independent.

(3)$\Rightarrow$(1). Let $(C_i)_{1\leq i\leq n}$ be a family of F-cocritical submodules of M such that $M/\sum_{i=1}^{n} C_i \in \mathcal{T}$. Then $M = (\sum_{i=1}^{n} C_i)^c = \bigvee_{1\leq i\leq n} C_i$, hence $C_F(M)$ is a semi-atomic lattice of finite length; it follows that M is a finite direct join of atoms of $C_F(M)$, and consequently, using the proof of the implication (2)$\Rightarrow$(3) we may suppose that the family $(C_i)_{1\leq i\leq n}$ is independent. If M_F denotes the modules of quotients of M with respect to F, then $M_F \simeq \bigoplus_{i=1}^{n} (C_i)_F$ and the canonical morphism $M \to M_F$ is injective since $M \in \mathcal{F}$. But $(C_i)_F$ considered as an R-module is F-cocritical for all i by Proposition 18.2.(3) from Golan [1975]. ∎

1.9. COROLLARY. (LAU [1980], GOLAN [1985]). Let $M \in$ Mod-R and $N \leq M$. Then

(1) If N is F-semicocritical and $N^c \in \mathcal{F}$ then N^c is so.

(2) If M is F-semicocritical and $N \in C_F(M)$ then M/N is F-semicocritical.

(3) If $(N_i)_{1\leq i\leq n}$ is a finite family of F-semicocritical submodules of M, then $\sum_{i=1}^{n} N_i$ is F-semicocritical iff $\sum_{i=1}^{n} N_i \in \mathcal{F}$.

Proof. (1) If N is F-semicocritical, then $C_F(N) \simeq C_F(N^c)$ is a semi-atomic lattice of finite length, hence N^c is F-semicocritical by 1.8.

(2) By 1.6, $C_F(M/N)$ is isomorphic to the interval $[N,M]$ of the semi-atomic lattice of finite length $C_F(M)$, and so $C_F(M/N)$ is a semi-atomic lattice of finite length. Apply again 1.8.

(3) Denote $P = \sum_{i=1}^{n} N_i$, and consider the lattice $C_F(P)$; Since $P \in \mathcal{F}$, it follows that $N_i^c \in C_F(P)$ is F-semicocritical for

all i; then N_i^c is a finite join of atoms of the interval $[0, N_i^c]$ of the lattice $C_F(P)$, which are also atoms of $C_F(P)$. Hence $P = \bigvee_{1 \leq i \leq n} N_i^c$ is a finite join of atoms of $C_F(P)$, and consequently P is F-semicocritical by 1.8. ∎

Note that directly from definition one deduces that the class of F-semicocritical modules is closed under subobjects and finite direct sums; in general, this class is not closed under extensions, quotient objects and arbitrary direct sums.

1.10. REMARK. A part of our Proposition 1.8, namely the equivalence (1)⇔(3) is proved in Golan [1985] (see Corollary 16.9) under the superfluous condition "M has finite Goldie dimension". The statement (1) of our Corollary 1.9 appears also in Golan [1985] (see Proposition 16.7) under the same superfluous condition. ∎

We shall present now a general result on lattices from which may be derived in a unified and simplified manner a series of known results. The terminology on lattices will follow Năstăsescu [1983].

1.11. PROPOSITION. The following assertions are equivalent for an upper continuous and modular lattice $\mathcal{L}$:

(1) $\mathcal{L}$ is a semi-atomic lattice of finite length.

(2) $\mathcal{L}$ is an artinian and complemented lattice.

(3) $\mathcal{L}$ is a noetherian and complemented lattice.

(4) $\mathcal{L}$ is a complemented lattice having finite Goldie dimension.

Proof. (1)⇒(2) and (1)⇒(3) since any semi-atomic lattice is complemented.

(2)⇒(1) and (3)⇒(1). It is known that a complemented and modular lattice is artinian iff it is noetherian (see e.g. Albu [1984a, Corollary 0.4]). Hence (2) or (3) implies that $\mathcal{L}$ is is a complemented lattice of finite length. This implies that each element $a \in \mathcal{L}$, $a \neq 0$ contains an atom. Let a_o be the join of all atoms of $\mathcal{L}$, i.e., a_o is the socle of $\mathcal{L}$. If $a_o \neq 1$, let b_o be a complement of a_o; then $b_o \neq 0$, hence b_o contains an atom s. But $s \leq a_o \wedge b_o = 0$, a contradiction. Consequently $a_o = 1$, i.e., $\mathcal{L}$ is semi-atomic.

(1)$\Rightarrow$(4). Obviously.

(4)$\Rightarrow$(1). By Năstăsescu [1983 , Teorema 5.5, p.21], or by Grzeszczuk and Puczyłowski [1984, Theorem 5], there exists a finite $\vee$-independent family $(c_i)_{1 \leq i \leq n}$ of coirreducible elements of $\mathcal{L}$ such that $\dot{\bigvee}_{1 \leq i \leq n} c_i$ is an essential element of $\mathcal{L}$. Since $\mathcal{L}$ is complemented, $1 = \dot{\bigvee}_{1 \leq i \leq n} c_i$. For each i, $1 \leq i \leq n$, the interval $[0, c_i]$ of $\mathcal{L}$ is a coirreducible (or uniform) and complemented lattice, consequently c_i is an atom, and so, we are done.∎

<u>1.12. LEMMA</u>. Let $M \in \mathcal{F}$. Then M has finite Goldie dimension if and only if the lattice $C_F(M)$ has finite Goldie dimension.

<u>Proof</u>. Suppose that M has an infinite independent family $(X_n)_{n \in \mathbb{N}}$ of nonzero submodules. Then $(X_n^c)_{n \in \mathbb{N}}$ is an infinite $\vee$-independent family of nonzero elements of the lattice $C_F(M)$, because, for each $n \in \mathbb{N}$

$$\left(\bigvee_{0 \leq i \leq n} X_i^c\right) \wedge X_{n+1}^c = \left(\sum_{i=0}^{n} X_i\right)^c \wedge X_{n+1}^c = \left(\left(\sum_{i=0}^{n} X_i\right) \cap X_{n+1}\right)^c = 0^c = 0.$$

Conversely, if $C_F(M)$ has an infinite $\vee$-independent family $(Y_n)_{n \in \mathbb{N}}$ of nonzero elements, then clearly $(Y_n)_{n \in \mathbb{N}}$ is an independent family of nonzero submodules of M. We shall see in Example 1.16 that the condition $M \in \mathcal{F}$ is essential.∎

<u>1.13. COROLLARY</u> (LAU [1980], GOLAN [1983, 1985], TEPLY [1985]). The following conditions are equivalent for an R-module M:

(1) M is F-semicocritical.

(2) $M \in \mathcal{F}$, M is F-artinian, and M is F-full.

(3) $M \in \mathcal{F}$, M is F-noetherian, and M is F-full.

(4) $M \in \mathcal{F}$, M has finite Goldie dimension, and M is F-full.

Proof. Apply 1.3, 1.8, 1.11, and 1.12. ∎

If $M \in \text{Mod-}R$ we shall consider the following two F-saturated submodules of M:

$$s^F(M) = (\sum\{S \mid S \leq M,\ S \text{ is F-simple}\})^c$$

$$\bar{s}^F(M) = (\sum\{C \mid C \leq M,\ C \text{ is F-cocritical}\})^c,$$

and, starting with these, we shall define inductively two ascending chain of F-saturated submodules of M, indexed by ordinals:

(*) $s_0^F(M) \leq s_1^F(M) \leq \dots \leq s_\alpha^F(M) \leq \dots \leq M$

(**) $\bar{s}_0^F(M) \leq \bar{s}_1^F(M) \leq \dots \leq \bar{s}_\alpha^F(M) \leq \dots \leq M$

as follows:

$$s_0^F(M) = \bar{s}_0^F(M) = t(M),$$

$$s_1^F(M) = s^F(M),\ \bar{s}_1^F(M)/\bar{s}_0^F(M) = \bar{s}^F(M/\bar{s}_0^F(M)),$$

and if the elements $s_\beta^F(M)$, resp. $\bar{s}_\beta^F(M)$ of $C_F(M)$ have been defined for all ordinals $\beta < \alpha$, then

$$s_\alpha^F(M) = (\sum_{\beta<\alpha} s_\beta^F(M))^c \text{ , resp. } \bar{s}_\alpha^F(M) = (\sum_{\beta<\alpha} \bar{s}_\beta^F(M))^c$$

if α is a limit ordinal, and

$$s_\alpha^F(M)/s_\gamma^F(M) = s^F(M/s_\gamma^F(M)), \text{ resp. } \bar{s}_\alpha^F(M)/\bar{s}_\gamma^F(M) = \bar{s}^F(M/\bar{s}_\gamma^F(M))$$

if $\alpha = \gamma + 1$.

1.14. LEMMA. Let $M \in \text{Mod-}R$ and $N \leq X \leq M$. Then $(X/N)^c = X^c/N$.

Proof. Denote $(X/N)^c = Y/N$, and let $y \in X^c$. Then $yI \subseteq X$ for some $I \in F$, hence $\hat{y}I \subseteq X/N$, where $\hat{y}$ denotes the congruence class of y modulo N. Hence $\hat{y} \in Y/N$, and consequently $\hat{y} = \hat{y}_1$ for some $y_1 \in Y$. Then $y - y_1 \in N \subseteq Y$, and therefore $y \in N$. Thus $X^c \subseteq Y$.

Conversely, let $y \in Y$. Then $\hat{y} \in Y/N$, hence $\hat{y}J \subseteq X/N$ for some $J \in F$, from which follows that for each $a \in J$ there exists $x_a \in X$ such that $\widehat{ya} = \hat{x}_a$. Thus $ya \in N + x_a \subseteq X$ for each $a \in J$, i.e., $yJ \subseteq X$. Hence $y \in X^c$, and so $Y \subseteq X^c$. ∎

As in the case of modules, for each upper continuous and modular lattice $\mathcal{L}$ can be defined its ascending Loewy series $(s_\alpha(\mathcal{L}))_{\alpha\geqslant 0}$ cf. Năstăsescu [1983 , p.26] (see also Albu [1984a] for further properties of this series). The next result shows that the two ascending series of F-saturated submodules of M defined above are nothing else than the ascending Loewy series of the lattice $C_F(M)$; from this result may be derived very quickly, in a unified and simplified manner a main part of the facts established in Bueso and Jara [1985].

1.15. PROPOSITION. For each R-module M and for each ordinal $\alpha \geqslant 0$, one has $s^F_\alpha(M) = \bar{s}^F_\alpha(M) = s_\alpha(C_F(M))$.

Proof. First of all note that $S \leqslant M$ is F-simple iff S^c is F-simple, because the lattices $C_F(S)$ and $C_F(S^c)$ are isomorphic. Now, using this fact, as well as 1.6, 1.7, and 1.14, the proof can be achieved by means of a straightforward transfinite induction, and so is left to the reader. ∎

We end this section with an example inspired by García Hernández [1983 , 2.1.3] .

1.16. EXAMPLE. Let R be an infinite direct product of copies of a commutative field and $\underline{m}$ a maximal ideal of R which is not principal (e.g. $R = \mathbb{Q}^{\mathbb{N}}$ and $\underline{m}$ is a maximal ideal of $\mathbb{Q}^{\mathbb{N}}$ including the ideal $\mathbb{Q}^{(\mathbb{N})}$). Denote by $F_{\underline{m}}$ the Gabriel topology on R defined by $\underline{m}$, i.e., $F_{\underline{m}} = \{I \leqslant R \mid I \not\subseteq \underline{m}\}$. Then, denoting by $t_{\underline{m}}$ the corresponding torsion radical, one has:

(1) $t_{\underline{m}}(R) = \underline{m}$ and $\mathrm{Max}_{F_{\underline{m}}}(R) = \{\underline{m}\}$.

(2) R is $F_{\underline{m}}$-simple, but R is not $F_{\underline{m}}$-cocritical. Moreover, R does not contain any $F_{\underline{m}}$-cocritical submodule.

(3) $t_{\underline{m}}(R)$ is an essential submodule of R, but $t_{\underline{m}}(R)$ is not an essential element of the lattice $C_{F_{\underline{m}}}(R)$. Consequently, $C_{F_{\underline{m}}}(R)$

is a complemented lattice but R is not $F_{\underline{m}}$-full.

(4) R has infinite Goldie dimension but the lattice $C_{F_{\underline{m}}}(R)$ has Goldie dimension 1.

(5) $\bar{s}^{F_{\underline{m}}}(R) \subsetneq s^{F_{\underline{m}}}(R)$. ■

§2. F-PRIMITIVE IDEALS AND PRIME IDEALS

Following Teply [1985], for each $M \in$ Mod-R we shall denote

$$S_F(M) = \sum\{C \mid C \leq M,\ C \text{ is F-cocritical}\}$$

$$Sc_F(M) = \sum\{N \mid N \leq M,\ N \text{ is F-semicocritical}\}.$$

The submodules $S_F(M)$ and $Sc_F(M)$ are called the F-cocritical socle and respectively the F-semicocritical socle of M. The relationship between them is explored in Teply [1985].

DEFINITION. A two-sided ideal D of R is called F-primitive if $D = \mathrm{Ann}_R(C)$ for some F-cocritical module C. ■

By Spec(R) we shall denote the set of all two-sided prime ideals of R, by $\mathrm{Spec}_F(R)$ the set $C_F(R) \cap \mathrm{Spec}(R)$, and by $\mathrm{Primit}_F(R)$ the set of all F-primitive ideals of R. If $\mathfrak{X}$ is a nonempty subset of Spec(R), then $\mathrm{Min}(\mathfrak{X})$ (resp. $\mathrm{Max}(\mathfrak{X})$) will denote the set of all minimal (resp. maximal) elements of $\mathfrak{X}$, with respect to inclusion.

If R is F-artinian, then Teply [1985] has established a bijective map

$$\mathrm{Spec}(\text{Mod-}R/\mathcal{T}) \longrightarrow \mathrm{Min}(\mathrm{Primit}_F(R))$$

$$Q \longmapsto \mathrm{Ann}_R(S_F(Q))$$

where $\mathrm{Spec}(\text{Mod-}R/\mathcal{T})$ is the spectrum of the quotient category Mod-$R/\mathcal{T}$, or equivalently, the set of isomorphism classes of all F-torsion-free indecomposable injective R-modules.

An other bijective map

$$\begin{array}{ccc} \mathrm{Spec}(\mathrm{Mod}\text{-}R/\mathcal{T}) & \longrightarrow & \mathrm{Spec}_F(R) \\ Q & \longmapsto & P, \text{ where } \mathrm{Ass}(Q) = \{P\} \end{array}$$

was found previously by Năstăsescu [1980](see also Albu and Năstăsescu [1984, 11.26]).

The aim of this section is to explore more closely the relationship between these two bijections.

<u>2.1. LEMMA</u>. If R is F-artinian and D is an F-primitive ideal of R, then there exists $P \in \mathrm{Spec}_F(R)$ such that $\mathrm{Ass}(R/D) = \{P\}$, i.e., D is a P-tertiary ideal of R.

<u>Proof</u>. By definition, there exists an F-cocritical module C such that $D = \mathrm{Ann}_R(C)$. Since $C \in \mathcal{F}$ and R is F-artinian, it follows that there exists a finite family $(x_i)_{1 \leq i \leq n}$ of elements of C such that $D = \bigcap_{i=1}^{n} \mathrm{Ann}_R(x_i)$. Hence, there exists a monomorphism of R-modules $R/D \longrightarrow \bigoplus_{i=1}^{n} x_iR$, and so, a monomorphism $R/D \to C^n$. By the Miller-Teply theorem (see e.g. Albu and Năstăsescu [1984, 7.11]), R is also F-noetherian, and by Albu and Năstăsescu [1984, 9.1]) $\emptyset \neq \mathrm{Ass}(R/D) \subseteq \mathrm{Ass}(C^n) = \mathrm{Ass}(C) \subseteq \mathrm{Spec}_F(R)$. But C is a nonzero coirreducible module, hence $\mathrm{Ass}(C)$ has exactly one element. ■

The next result was established with other methods by Teply [1985].

<u>2.2. LEMMA</u> (TEPLY [1985, 2.4]). Let C_1 and C_2 be two F-cocritical modules such that $\mathrm{Ann}_R(C_1) \subseteq \mathrm{Ann}_R(C_2)$. If R is F-artinian, then $E(C_1) \simeq E(C_2)$.

<u>Proof</u>. Denote $D_i = \mathrm{Ann}_R(C_i)$ for $i = 1,2$. Since $D_1 \subseteq D_2$ consider the canonical epimorphism $R/D_1 \to R/D_2$. Then, using 2.1 one gets $\mathrm{Ass}(R/D_1) = \mathrm{Ass}(R/D_2) = \{P\}$. Denote $Q_i = E(C_i)$ for $i = 1,2$. Then $\mathrm{Ass}(Q_1) = \mathrm{Ass}(Q_2) = \{P\}$, hence there exists $0 \neq X_i \leq Q_i$ such that $P = \mathrm{Ann}_R(X_i)$ for $i = 1,2$. As in the proof of 2.1, there exist natural numbers n_i and monomorphisms $R/P \to Q_i^{n_i}$ for $i=1,2$.

It follows that $E(R/P)$ is isomorphic to a submodule of $Q_i^{n_i}$ for $i=1,2$. Since Q_i are indecomposable injective modules, we conclude that $Q_1 \simeq Q_2$. ∎

2.3. PROPOSITION. If R is F-artinian, then

$$\mathrm{Spec}_F(R) = \mathrm{Max}(\mathrm{Primit}_F(R)).$$

Proof. Let $P \in \mathrm{Spec}_F(R)$; then $\mathrm{Ass}(R/P) = \{P\}$, hence there exists $X \leq R/P$ such that $\mathrm{Ann}_R(X) = \mathrm{Ann}_R(X') = P$ for all $0 \neq X' \leq$ Since $R/P \in \mathcal{F}$, by Albu and Năstăsescu [1984,7.17], X has an F-cocritical submodule X_o. It follows that $P = \mathrm{Ann}_R(X_o)$, i.e., P is F-primitive. Let $D \in \mathrm{Primit}_F(R)$ with $P \subseteq D$; then $D = \mathrm{Ann}_R(C)$ for some F-cocritical module C. Then $E(X_o) \simeq E(C)$ by 2.2, hence there exist nonzero submodules $X_o' \leq X_o$ and $C' \leq C$ such that $X_o' \simeq C'$; this implies $P = \mathrm{Ann}_R(X) = \mathrm{Ann}_R(X_o') = \mathrm{Ann}_R(C') \supseteq \mathrm{Ann}_R(C) = D$, in other words $P = D$, i.e., $P \in \mathrm{Max}(\mathrm{Primit}_F(R))$.

Consider now an arbitrary element $D \in \mathrm{Primit}_F(R)$ and suppose that $D \subseteq P'$ for some $P' \in \mathrm{Spec}_F(R)$; then, as in the proof of 2.2, one gets $\mathrm{Ass}(R/D) = \mathrm{Ass}(R/P') = \{P'\}$; on the other hand, if $\mathrm{Ass}(R/D) = \{P\}$, then clearly $D \subseteq P$ because D is a two-sided ideal of R. Thus, any F-primitive ideal of R is included in a unique prime ideal of $\mathrm{Spec}_F(R)$. If now $D \in \mathrm{Max}(\mathrm{Primit}_F(R))$ and $P = \mathrm{Ass}(R/D)$, then $D \subseteq P$ and $P \in \mathrm{Primit}_F(R)$, hence $D = P$, and so $D \in \mathrm{Spec}_F(R)$. ∎

2.4. PROPOSITION. Suppose that the ring R is F-artinian. Let D be an F-primitive ideal of R and $\{P\} = \mathrm{Ass}(R/D)$. If C is an F-cocritical module such that $\mathrm{Ann}_R(C) = D$, then $\mathrm{Ann}_R(S_F(E(C)))$ is the unique minimal F-primitive ideal of R contained in D, and P is the unique maximal F-primitive ideal of R containing D.

Proof. The last assertion of the Proposition was proved in 2.3. Denote by $\mathcal{C}$ the set of all F-cocritical submodules of $Q = E(C)$. Then $\mathrm{Ann}_R(S_F(Q)) = \mathrm{Ann}_R(\sum_{C \in \mathcal{C}} C) = \bigcap_{C \in \mathcal{C}} \mathrm{Ann}_R(C)$. Since R is F-artinian, there exists a finite subset $\{C_i \mid 1 \leq i \leq n\}$ of $\mathcal{C}$ such that $\mathrm{Ann}_R(S_F(Q)) = \bigcap_{i=1}^{n} \mathrm{Ann}_R(C_i) = \mathrm{Ann}_R(\sum_{i=1}^{n} C_i)$. By 1.9.(3), $\sum_{i=1}^{n} C_i$ is an F-semicocritical submodule of the coirreducible module Q, hence it is necessarily F-cocritical, and consequently $\mathrm{Ann}_R(S_F(Q))$ is F-primitive. Since $C \leqslant S_F(E(C))$, then clearly $\mathrm{Ann}_R(S_F(Q)) \subseteq D$. Let now $D' \in \mathrm{Primit}_F(R)$ with $D' \subseteq D$; then $D' = \mathrm{Ann}_R(C')$ for some F-cocritical module C'. By 2.2, $E(C') \simeq E(C)$ hence C' is isomorphic to a cocritical submodule C'_o of Q. Then clearly $D' = \mathrm{Ann}_R(C') = \mathrm{Ann}_R(C'_o) \supseteq \mathrm{Ann}_R(S_F(Q))$. Taking $D' \in \mathrm{Min}(\mathrm{Primit}_F(R))$ with $D' \subseteq D$, possible since R is F-artinian, it follows that $\mathrm{Ann}_R(S_F(E(C))) \in \mathrm{Min}(\mathrm{Primit}_F(R))$ and this ideal is the unique minimal F-primitive ideal of R contained in D.∎

2.5. REMARKS. (1) Using the previous results it can be seen easily that any finite intersection of distinct minimal F-primitive ideals of R is irredundant, the ring R being supposed F-artinian.

(2) If R is F-artinian, then the Jacobson radical $J_F(R)$ of R with respect to the Gabriel topology F, i.e., the intersection of all F-primitive ideals of R is a finite intersection of distinct minimal F-primitive ideals of R:

$$J_F(R) = D_1 \cap D_2 \cap \ldots \cap D_n .$$

By 2.1, this is exactly an irredundant tertiary decomposition of $J_F(R)$ in R. Moreover, as was proved in Teply [1985,3.3],

$$\mathrm{Min}(\mathrm{Primit}_F(R)) = \{D_1, D_2, \ldots, D_n\} .∎$$

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