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A COMPLETION OF "ON FLOWCHART THEORIES (I)"

by

Gh. STEFANESCU

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Gh.ŞTEFĂNESCU*)

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*) Department of Mathematics, The National Institute for Scientific and Technical Creation, Bd.Pacii 220, 79622 Bucharest, Romania.

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The main result of [2], i.e. if T is a theory with strong iterate then the theory of reduced Σ-flowcharts $RFl_{\Sigma,T}$ is the theory with strong iterate freely generated by adding Σ to T, was proved only when T is an "almost syntactical" theory. Here we show that this technical condition is superfluous.

We shall use the notations from [2]. Suppose T is an S-sorted theory with strong iterate and Σ a double ranked S-sorted set. If one re-reads [2] he can see that in the definition of a reduction $f \underset{y}{\longmapsto} f'$ the condition wac): $Ac(f) \subseteq Dom(f)$ was wrote in this way only for the sake of generality. In fact he can use only reductions $f \underset{y}{\longmapsto} f'$ which fulfils a strong condition <u>sac): Dom(y) fulfils ac) in f</u> .

We shall distinguish two types of reductions, one by functions, denoted by $\underset{y}{\longrightarrow}$ and one by injective partial functions, denoted by $\underset{y}{\vdash\!\!\relbar\!\!\relbar}$. The reductions by bijective functions are of both types. Obviously $\longmapsto \;\subseteq\; \vdash\!\!\relbar\!\!\relbar \circ \longrightarrow$.

Somehow as in the proof of Church-Rosser theorem [1] we shall show that $\longrightarrow$-reductions can pass over $\vdash\!\!\relbar\!\!\relbar$-reductions.

<u>Lemma 1.</u> If $f \underset{y_1}{\vdash\!\!\relbar\!\!\relbar} \tilde{f} \underset{y}{\longrightarrow} f'$ then there exist $\bar{f}$, z_1, z_2 such that $f \underset{z_1}{\longrightarrow} \bar{f} \underset{z_2}{\vdash\!\!\relbar\!\!\relbar} f'$.

<u>Proof.</u> Suppose the flowcharts are from a to b and $f = (i,t,e)$ with $e = e_1 \dots e_{|e|}$. Using a top reduction given by a bijective function we can restrict our analysis to the case when $[\![Dom(y_1)]\!] = \{1,\dots,k\}$, for one $k \leq |e|$. Now $y_1 = (1_{e_1 \dots e_k} + \perp_{e_{k+1} \dots e_{|e|}}) \cdot u$ for one bijective function u , which can be moved in y . Hence we can suppose

$y_1 = 1_{e_1 \dots e_k} + \perp_{e_{k+1} \dots e_{|e|}}$.

The $\bar{f}$-flowchart is obtained by adding to f' an inaccessible copy of the $e_{k+1} \dots e_{|e|}$-part of f . More precisely $\bar{f} = (\bar{i}, \bar{t}, \bar{e})$ is given by

$$\bar{e} = e' e_{k+1} \dots e_{|e|} ,$$

$$\bar{i} = i'(1_{p'} + 0_s + 1_b) ,$$

$$\bar{t} = \langle t'(1_{p'} + 0_s + 1_b) , (0_{e_1 \dots e_k} + 1_{e_{k+1} \dots e_{|e|}})_{out}\, t\, (y_{in} + 1_{sb}) \rangle$$

where $s = r^*_{in}(e_{k+1} \dots e_{|e|})$. Now we shall show that

$$f \xrightarrow[y+1_{e_{k+1} \dots e_{|e|}}]{} \bar{f} \xleftarrow[1_{e'} + \perp_{e_{k+1} \dots e_{|e|}}]{} f' .$$

One can easy see that $1_{e'} + \perp_{e_{k+1} \dots e_{|e|}}$ gives a reduction of desirated type. A bit more difficult is to show that the total surjective function $y + 1_{e_{k+1} \dots e_{|e|}}$ reduces f to $\bar{f}$. By our hypothesis

$$\bar{i} = i'(1_{p'} + 0_s + 1_b) = i((y_1 y)_{in} + 1_b)(1_{p'} + 0_s + 1_b) = i(y + \perp_s 0_s + 1_b) =$$

$$= i(y + 1_s + 1_b)$$

where the last equality is based on $Im(i) \subseteq Dom(y_1)$. If $j \in [k]$ then as $[\![Dom(y_1)]\!] = \{1, \dots, k\}$ fulfils ac) in f , we have

$$\bar{t}_{y(j)} = t'_{y(j)}(1_{p'} + 0_s + 1_b) = t_j(y + \perp_s 0_s + 1_b) = t_j(y + 1_s + 1_b).$$

In the case $j \in \{k+1, \dots, |e|\}$ we have even an identity. □

<u>Lemma 2.</u> Every chain $f = f^0 \xmapsto[y_1]{} f^1 \xmapsto[y_2]{} \dots \xmapsto[y_n]{} f^n = f'$ may be replaced with a two-step reduction $f \longrightarrow f'' \longleftarrow f'$ (hence we have $\overset{*}{\longmapsto} = \longrightarrow \circ \longleftarrow$).

<u>Proof.</u> By lemma 1 we can suppose that in $f^0 \xmapsto[y_1]{} \dots \xmapsto[y_n]{} f^n$ the first k reductions are $\longrightarrow$-reductions snd the last $n-k$ reductions are $\longleftarrow$-reductions. An easy computation shows that every chain of

$\longrightarrow$-reductions can be replaced with one $\longrightarrow$-reduction. The same is true for $\longmapsfrom$ -reductions because if $\bar{f} \underset{y}{\longmapsfrom} \tilde{f}$ then

$[\![x]\!]$ fulfils ac) in $\bar{f}$ iff $y([\![x]\!])$ fulfils ac) in $\tilde{f}$

(the reverse implication is based on $[\![\mathrm{Im}(gz)]\!] = z([\![\mathrm{Im}(g)]\!])$, if z is an injective partial function). □

Lemma 3. If for $f ; a \to ac$, $f' : b \to bc$ and $y \in \mathrm{Str}(a,b)$ surjective function, we have $f(y+1_c) \underset{z}{\longrightarrow} yf'$ then $f^{\dagger} \underset{z}{\longrightarrow} y(f')^{\dagger}$.

Proof. See the last part of the proof of theorem 11.2 in [2] . □

Theorem 4. If T is a theory with strong iterate then $\mathrm{RFl}_{\Sigma,T}$ is a theory with strong iterate.

Proof. We have only to show I4-S :

if $f;a \to ac$, $f':b \to bc$ and $y \in \mathrm{Str}(a,b)$ is a surjective function such that $f(y+1_c) \equiv yf'$ then $f^{\dagger} \equiv y(f')^{\dagger}$.

Let us suppose that f, f' are minimal flowcharts. Then yf' is also a minimal flowchart. Hence the equivalence $f(y+1_c) \equiv yf'$ is, in fact, given by a chain of reductions $f(y+1_c) \mapsto \ldots \mapsto yf'$. By lemma 2 this chain can be replaced with $f(y+1_c) \underset{z}{\longrightarrow} f'' \underset{u}{\longmapsfrom} yf'$. As $i''(u_{in}+1_{bc}) = yi'$ and u has a right inverse v with $uv = \mathrm{Dom}(u)$ and $\mathrm{Im}(i'') \subseteq \mathrm{Dom}(u)_{in}+1_{bc}$ we have

$$i'' = i''(\mathrm{Dom}(u)_{in}+1_{bc}) = i''(u_{in}+1_{bc})(v_{in}+1_{bc}) = yi'(v_{in}+1_{bc}),$$

which shows that $f'' = y\bar{f}$. By lemma 3, $f^{\dagger} \longrightarrow y\bar{f}^{\dagger}$. The theorem is concluded if we show that $\bar{f}^{\dagger} \mapsto (f')^{\dagger}$. By lemma 10.1 in [2] , $y\bar{f} \underset{u}{\longmapsfrom} yf'$ implies $\bar{f} = wy\bar{f} \mapsto wyf' = f'$, where w is a left inverse of y $(wy=1_b)$ and $\bar{f}^{\dagger} \mapsto (f')^{\dagger}$. □

As a corollary we rewrite the main theorem from [2].

Main theorem. If T is an S-sorted theory with strong iterate and Σ a double ranked S-sorted set, then the theory of reduced flowcharts $RFl_{\Sigma,T}$ is the theory with strong iterate freely generated by adding Σ to T . □

By lemma 2, $\overset{*}{\longmapsto} = \longrightarrow \circ \longleftarrow$. By Lemma 11.1 in [2] , if T is an almost syntactical theory then $\overset{*}{\longmapsto} = \longmapsto$. We conclude this note with an example, in the one-sorted case, which shows that generally $\overset{*}{\longmapsto} \neq \longmapsto$. As T we chose the quotient of the thoery of rational Γ-trees, RT_Γ (Γ is a one ranked set, with one γ of arity two) by the congruence $\equiv$ generated by

$$\gamma\langle x_1^1, x_1^1\rangle = \perp_1 0_1 .$$

Now it is easy to see that $RT_\Gamma/\equiv$ is, as RT_Γ , a theory with strong iterate. Then the flowchart

$$(\gamma(\langle x_1^2, x_2^2\rangle + 0_1) , 0_2 + 1_1 , \sigma\sigma)$$

where $r_{in}(\sigma) = r_{out}(\sigma) = 1$, can be reduced (in two steps) to $(\perp_1 0_1, 0_1, \lambda)$:

$$(\gamma(\langle x_1^1, x_1^1\rangle + 0_1), 0_2 + 1_1, \sigma\sigma) \xrightarrow[\langle 1_1, 1_1\rangle]{} (\perp_1 0_1 + 0_1, 0_1 + 1_1, \sigma) \underset{\perp_1}{\longmapsto} (\perp_1 0_1, 0_1, \lambda)$$

but not in a single step.

References.

[1] J.R.HINDLEY, B.LERCHER and J.P.SELDIN, "Introduction to Combinatory Logic", Cambridge University Press, 1972.

[2] GH.ŞTEFĂNESCU, "On flowchart theories (I)", Preprint INCREST, No.39, 1984.