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MEASURE-THEORETIC PROPERTIES OF THE CENTRAL TOPOLOGY ON THE SET OF THE FACTORIAL STATES OF A C^*-ALGEBRA

CENTRAL REDUCTION

by Silviu TELEMAN

On the set of the extreme points of a compact convex set various topologies have been introduced, having relevance to the boundary measures (see [1], [4], [6], [8], [9], [13], [14], [15], [16], [21], [28], [29], [30] [31], [33], [35]). As a result, a general topological measure theory has been obtained, which contains, as particular cases, both the Borel measure theory in Polish spaces, as well as the Radon measure theory in arbitrary compact spaces; and also, this allowed the development of a "non-commutative Borel analysis" over arbitrary C^*-algebras (see [36]).

In the theory of C^*-algebras an important role is also played by the set of the factorial (primary) states of a given C^*-algebra.

In this paper we shall introduce a natural topology in the set $F(A)$ of the factorial states of an arbitrary C^*-algebra A, and we shall show that the central measures on the set $E_o(A)$ of the quasi-states of A, which represent states of A, induce regular Borel measures on $F(A)$, with whose help a spatial central reduction theory for the representations of A can be developed.

We refer to [10], [11], [12], [22], [24], and [27] for general facts concerning the theory of C^*-algebras. In ([22], Ch. IV) one can find a deep analysis of Reduction Theory for the separable case.

In the present paper no separability conditions are assumed. Theorem 33 is the main result of the paper.

§1. INTRODUCTION

We shall denote by $E_o(A)$ the convex set of all the <u>quasi-states</u> of A, endowed with the topology $\sigma(A^*;A)$, for which it is compact; $E(A)= \{f \in E_o(A);\ \|f\|=1\}$ is the set of the <u>states</u> of A, which is a convex G_δ-subset of $E_o(A)$. The set $E(A)$ is compact if, and only if, A possesses the unit element, which we shall denote by 1.

The set of the extreme points of $E_o(A)$ is given by the equality

$$\mathrm{ex}\, E_o(A) = P(A)\cup\{0\},$$

where $P(A)$ is the set of the pure states of A (see [12], Proposition 2.5.5.).

For any positive $f\in A^*_+ = \{f\in A^*;\ f\geq 0\}$, we shall denote by $\pi_f: A\to \mathcal{L}(H_f$

the representation of A on the associated Hilbert space H_f, by $\theta_f : A \to H_f$ the associated linear mapping, and by ξ_f^o the associated cyclic vector, according to the GNS-construction. We have

$$f(a) = (\pi_f(a)\xi_f^o \mid \xi_f^o), \quad \theta_f(a) = \pi_f(a)\xi_f^o, \qquad a \in A,$$

$$\|f\| = \|\xi_f^o\|^2, \qquad\qquad f \in A_+^*.$$

The state $f \in E(A)$ is said to be <u>factorial</u> (or <u>primary</u>) if $\pi_f(A)''$ is a factor. We denote by F(A) the set of all factorial states of A. We obviously have that

$$F(A) \supset P(A).$$

We refer to [3],[7] for recent results relating topological properties of F(A) to structural properties of A.

In order to develop a working topological theory for the boundary measures induced on P(A) by the maximal (orthogonal) measures on $E_o(A)$ one has first to find a suitable topology on P(A). In this respect, we refer to [5],[6],[28],[29],[30],[31],[32],[33],[36] and [37] for the corresponding definitions, results and applications obtained. In particular, an irreducible spatial disintegration theory has been obtained for the (cyclic) representations of A, and for their extensions to the Baire and the Borel enveloping C^*-algebras of A. We point out here only the fact that several topologies have been intrduced on P(A), compatible with the maximal (orthogonal) measures, thus allowing several regular Borel extensions of the induced boundary measures, the maximal orthogonal topology being, so far, the strongest in this family (see [6], [33]).

We shall define below the <u>central topology</u> on F(A) and we shall show that any central Radon probability measure on $E_o(A)$, whose barycenter is in E(A), induces a regular Borel probability measure on F(A).

For the central topology, the space F(A) satisfies the (T_1) separation axiom; and it is quasi-compact if, and only if, A possesses the unit element.

For any topological space X we shall denote by $\mathcal{B}_o(X)$ (or by $\mathcal{B}_o(X;\tau)$ when the topology τ of X is to be emphasized) the σ-algebra of the <u>Baire</u> subsets of X: it is the smallest σ-algebra of subsets of X, containing all closed G_δ-subsets of X; by $\mathcal{B}(X)$ (or by $\mathcal{B}(X;\tau)$, when the topology τ of X is to be emphasized) we shall denote the σ-algebra of the <u>Borel</u> subsets of X: it is the smallest σ-algebra of subsets of

X, containing all closed subsets of X. Of course, we have the inclusion

$$\mathcal{B}_0(X) \subset \mathcal{B}(X).$$

If X is metrizable, then

$$\mathcal{B}_0(X) = \mathcal{B}(X),$$

but for most compact spaces X we have $\mathcal{B}_0(X) \neq \mathcal{B}(X)$.

By C(X) we shall denote the algebra of all continuous complex functions on X, whereas $C^b(X)$ ($\subset C(X)$) will stand for the C^*-algebra of all bounded continuous complex functions on X, endowed with the sup-norm.

For any compact (Hausdorff) topological space K we shall denote by $\mathcal{M}_+(K)$ the space of all positive Radon measures on K, endowed with the vague topology; i.e., the topology induced by $\sigma(C(K)^*; C(K))$, since $\mathcal{M}_+(K)$ can be identified with a subset of the dual Banach space $C(K)^*$ of C(K). By $\mathcal{M}_+^1(K)$ we shall denote the compact convex subset of $\mathcal{M}_+(K)$, consisting of all Radon probability measures on K.

If K is a convex compact subset of a Hausdorff locally convex topological real vector space X, then for any $\mu \in \mathcal{M}_+(K)$ we can define its <u>resultant</u> $r(\mu) \in X$ by the formula

$$x^*(r(\mu)) = \int_K x^*(x)\, d\mu(x), \qquad x^* \in X^*;$$

it always exists and it is unique. If $\mu \in \mathcal{M}_+^1(K)$, then $r(\mu) \in K$; in this case it is called the <u>barycenter</u> of μ and it is denoted by $b(\mu)$.

If we denote by A(K) the real Banach space of all continuous affine real functions on K, then, by taking into account the fact that the space of all the restrictions to K of the functions $x^* \in X^*$ is uniformly dense in A(K), we immediately infer that we have the formula

$$h(b(\mu)) = \int_K h(x)\, d\mu(x), \qquad h \in A(K).$$

We shall also denote $\mathcal{M}_+^1(K;x) = \{\mu \in \mathcal{M}_+^1(K);\ b(\mu) = x\}$. By S(K) we shall denote the sup-cone of all convex continuous real functions on the compact convex set K.

§2. SUBCENTRAL MEASURES

I. For any $f_o \in E_o(A)$, and any Radon measure μ on $E_o(A)$, whose resultant $r(\mu) = f_o$, one has the Tomita mapping $K_\mu : L^\infty(\mu) \to \mathcal{L}(H_{f_o})$, which is linear, positive, $(\sigma(L^\infty(\mu); L^1(\mu))$, weak operator)-continuous and $\operatorname{im} K_\mu \subset \pi_{f_o}(A)'$; it is defined by the equality

$$\int_{E_o(A)} \varphi \lambda_A(a)\, d\mu = (K_\mu([\varphi])\pi_{f_o}(a)\xi^o_{f_o} | \xi^o_{f_o}), \quad a \in A,$$

where $[\varphi]$ is the class in $L^\infty(\mu)$ of the bounded Borel measurable function $\varphi : E_o(A) \to \mathbb{C}$, and $\lambda_A(a) : E_o(A) \to \mathbb{C}$ is given by $\lambda_A(a)(f) = f(a)$, $f \in E_o(A)$.

We refer to [28] for the basic properties of K_μ, which we shall use below.

The measure μ is said to be orthogonal if for any $M \in \mathcal{B}(E_o(A))$ we have that

$$r(\chi_M \mu) \perp r(\chi_{CM} \mu);$$

i.e., for any $f \in A^*_+$, such that

$$f \leq r(\chi_M \mu) \text{ and } f \leq r(\chi_{CM} \mu),$$

we have $f = o$.

The Radon measure μ is orthogonal if, and only if, the mapping K_μ is multiplicative. In this case $\mathcal{C}_\mu \overset{def}{=} \operatorname{im} K_\mu$ is an abelian von Neumann subalgebra of $\pi_{f_o}(A)'$. We denote by e_μ the projection onto $\overline{\mathcal{C}_\mu \xi^o_{b(\mu)}}$.

Conversely, for any state $f_o \in E(A)$ and any abelian von Neumann subalgebra $\mathcal{C}$ of $\pi_{f_o}(A)'$ there exists a unique orthogonal Radon probability measure $\mu \in \mathcal{M}^1_+(K)$, such that $b(\mu) = f_o$ and $\mathcal{C}_\mu = \mathcal{C}$.

We shall denote by $\Omega(E_o(A); f_o)$ the set of all orthogonal Radon probability measures μ on $E_o(A)$, such that $b(\mu) = f_o$; also, it will be convenient to denote by $\Omega(E_o(A))$ the set $\cup\{\Omega(E_o(A); f_o);\ f_o \in E_o(A)\}$ of all orthogonal Radon probability measures on $E_o(A)$.

If A possesses the unit element ($1 \in A$), then $\Omega(E(A); f_o)$ will stand for the set of all orthogonal Radon probability measures μ on $E(A)$, such that $b(\mu) = f_o \in E(A)$, whereas $\Omega(E(A))$ will denote the set $\cup\{\Omega(E(A); f_o);\ f_o \in E(A)\}$ of all orthogonal Radon probability measures on $E(A)$.

In particular, to any state $f_o \in E(A)$ we can associate its corresponding central measure $\mu_{f_o} \in \mathcal{M}^1_+(E_o(A))$, which is the orthogonal Radon probability measure corresponding to the center $\pi_{f_o}(A)'' \cap \pi_{f_o}(A)'$ of $\pi_{f_o}(A)'$, and whose barycenter is at f_o.

An orthogonal measure $\mu \in \mathcal{M}_+(E_o(A))$ is said to be subcentral if $\mathcal{C}_\mu \subset \pi_{f_o}(A)'' \cap \pi_{f_o}(A)'$, where $f_o = r(\mu)$.

LEMMA 1. For any measure $\mu \in \mathcal{M}_+(E_o(A))$ the following are equivalent

a) μ is subcentral;

b) for any $M \in \mathcal{B}(E_o(A))$ the representations $\pi_{r(\chi_M \mu)}$ and $\pi_{r(\chi_{\complement M}\mu)}$ are disjoint.

Proof. a) $\Longrightarrow$ b). We have $f_o \overset{def}{=} r(\mu) \in A^*_+$ and

$$(K_\mu([\varphi])\pi_{f_o}(a)\xi^o_{f_o} | \xi^o_{f_o}) = \int_{E_o(A)} \varphi\lambda_A(a)\, d\mu, \quad \varphi \in \mathcal{L}^\infty(E_o(A), \mathcal{B}(E_o(A)))$$

Therefore, we have

$$r(\chi_M \mu)(a) = (K_\mu([\chi_M])\, \pi_{f_o}(a)\xi^o_{f_o} | \xi^o_{f_o}), \qquad a \in A,$$

$$r(\chi_{\complement M}\mu)(a) = (K_\mu(1-[\chi_M])\, \pi_{f_o}(a)\xi^o_{f_o} | \xi^o_{f_o}), \qquad a \in A,$$

for any $M \in \mathcal{B}(E_o(A))$. By hypothesis, for $e \overset{def}{=} K_\mu([\chi_M])$, we have $e \in \pi_{f_o}(A)'' \cap \pi_{f_o}(A)'$, and the representations $\pi_{r(\chi_M\mu)}$, resp., $\pi_{r(\chi_{\complement M}\mu)}$, are unitarily equivalent to the representations $\pi_1 : A \ni a \mapsto \pi_{f_o}(a)e$, in eH_{f_o}, resp. $\pi_2 : A \ni a \mapsto \pi_{f_o}(a)(1-e)$, in $(1-e)H_{f_o}$. Assume that there exists a partial isometry $v : eH_{f_o} \to (1-e)H_{f_o}$, such that $v^*v \leq e$, $vv^* \leq 1-e$ and

$$v\pi_{f_o}(a) = \pi_{f_o}(a)v, \qquad a \in A.$$

Then we have $ve = v$, $(1-e)v = v$ and, by virtue of Kaplansky's Theorem, there exists a net $a_\alpha \in A$, such that $\pi_{f_o}(a_\alpha) \to e$. We infer that

$$v = ve = \lim_\alpha v\pi_{f_o}(a_\alpha) = \lim_\alpha \pi_{f_o}(a_\alpha)v = ev = o,$$

and this shows that $\pi_{r(\chi_M\mu)}$ and $\pi_{r(\chi_{\complement M}\mu)}$ are disjoint.

b) $\Longrightarrow$ a). First of all, from b) one immediately infers that μ is orthogonal. If μ were not subcentral, then one would have that $\mathcal{C}_\mu \not\subset \pi_{f_o}(A)''$, and then one could find a set $M \in \mathcal{B}(E_o(A))$, such that $e \overset{def}{=} K_\mu([\chi_M]) \notin \pi_{f_o}(A)''$. Of course, e is a projection in $\pi_{f_o}(A)'$, and we have that

$$e\,\pi_{f_o}(A)'(1-e) \neq \{o\}$$

(otherwise, e would be in $\pi_{f_o}(A)''$). From the Comparison Theorem we infer that there exists a central projection $g \in \pi_{f_o}(A)'' \cap \pi_{f_o}(A)'$, such that $ge \prec g(1-e)$ and $(1-g)e \succ (1-g)(1-e)$. Then either $ge \neq o$, or $(1-g).(1-e) \neq o$. In the first case, there exists a partial isometry $u \neq o$, such that $u^*u = ge$, $uu^* \leq g(1-e)$, and the mapping

$$\pi_{f_o}(a)ge \mapsto u\pi_{f_o}(a)u^* , \qquad a \in A,$$

is a non-zero correctly defined unitary equivalence of a subrepresentation of π_1 with a subrepresentation of π_2. These are, therefore, not disjoint. The second case can be treated similarly.

Remark 1. The fact that two representations π_1, π_2 of A are disjoint is usually denoted by $\pi_1 \natural \pi_2$. If $f_1, f_2 \in A_+^*$, then they are said to be disjoint, and one writes $f_1 \natural f_2$, if $\pi_{f_1} \natural \pi_{f_2}$. (see [12], 5.2.2.; [22], p.65).

Remark 2. The preceding Lemma extends Proposition 2o from [25] to the slightly more general case of a C^*-algebra which is not assumed to possess a unit element. See, also, ([1o], Proposition 4.2.9).

II. The following theorem extends to the general, possibly non-separable case, a theorem belonging to W. Wils (see [4o], [41]; and also [25], Theorem 27). Also, we do not assume the existence of the unit element.

THEOREM 1. *For any state $f_o \in E(A)$ the corresponding central measure μ_{f_o} is the greatest (with respect to the Choquet-Meyer order relation) Radon probability measure on $E_o(A)$, whose barycenter is at f_o, and which is dominated (with respect to the Choquet-Meyer order relation) by any Choquet maximal Radon probability measure, whose barycenter is at f_o.*

For the proof we need some lemmas, which are interesting in themselves. In presenting them, we follow C.F. Skau, except that we do not assume A to possess the unit element (see [25], Lemmas 22, 23, 24 and 25). Lemma 2 below is a slight improvement of a Theorem of G. Choquet (see [2o], Ch. XI, §1.8; [1o], Proposition 4.1.1.; [28], Proposition 1.15).

LEMMA 2. *Let K be any compact convex set in a Hausdorff locally convex topological real vector space. Then for any $x_o \in K$ and any $\mu \in \mathcal{M}_+^1(K;x_o)$ there exists an increasing net $(\nu_{\mathcal{L}})$ in the set $\mathcal{F}_+^1(K;x_o)$ of all Radon probability measures on K, having finite supports, and whose barycenters are at x_o, such that*

a) $\sup\{\nu_{\mathcal{L}}(f); \mathcal{L}\} = \mu(f)$, $\quad f \in S(K)$;

b) $\lim_{\mathcal{L}} \nu_{\mathcal{L}} = \mu$ in $\sigma(C(K)^*; C(K))$.

Proof.i)Let $f \in C(K)$ and $\varepsilon > o$ be given.Then there exists a cover K_i, $1 \leq i \leq n$, of K, by compact convex subsets of K, such that

$$\bigcup_{i=1}^{n} K_i = K \quad \text{and} \quad x', x'' \in K_i \Rightarrow |f(x') - f(x'')| < \varepsilon .$$

Let $L_1 = K_1$ and $L_i = K_i \setminus (L_1 \cup \ldots \cup L_{i-1})$, $2 \leq i \leq n$. Let $J = \{i; 1 \leq i \leq n, \mu(L_i) > o\}$ and define $\nu = \sum_{i \in J} \mu(L_i)\varepsilon_{x_i}$, where $x_i = b(\mu(L_i)^{-1}\chi_{L_i}\mu)$, $i \in J$. We obviously have $x_i \in K_i$, $i \in J$. It follows that

$$\left|\int_K f\,d\nu - \int_K f\,d\mu\right| = \left|\sum_{i \in J}\mu(L_i)f(x_i) - \sum_{i \in J}\int_{L_i} f\,d\mu\right| \leq$$
$$\leq \sum_{i \in J}\int_{L_i} |f(x_i) - f(x)|\,d\mu(x) < \varepsilon \sum_{i=1}^{n}\mu(L_i) = \varepsilon .$$

ii)Let us now consider any cover $\mathcal{L} = \{L_1, L_2, \ldots, L_m\}$ of K, by Borel subsets $L_i \subset K$, such that $L_i \cap L_j = \emptyset$, for $i \neq j$, $i,j = 1,2,\ldots,n$. For $i \in J = \{i;\ 1 \leq i \leq m,\ \mu(L_i) > o\}$, let us define $x_i = b(\mu(L_i)^{-1}\chi_{L_i}\mu)$ and $\nu_{\mathcal{L}} = \sum_{i \in J}\mu(L_i)\varepsilon_{x_i}$. For any convex continuous real function f on K we shall have

$$\nu_{\mathcal{L}}(f) = \sum_{i \in J}\mu(L_i)f(x_i) \leq \sum_{i \in J}\mu(L_i)\mu(L_i)^{-1}\int_{L_i} f\,d\mu = \int_K f\,d\mu,$$

and this shows that $\nu_{\mathcal{L}} < \mu$. Of course, this implies that $\nu_{\mathcal{L}} \in \mathcal{M}_+^1(K; x_o)$.
iii)Let $\mathcal{L}'$, $\mathcal{L}''$ be two such covers of K, as above:

$$\mathcal{L}' = \{L_1', L_2', \ldots, L_m'\} \quad , \quad \mathcal{L}'' = \{L_1'', L_2'', \ldots, L_n''\} .$$

We shall now consider the cover

$$\mathcal{L} = \{L_i' \cap L_j'' ;\ 1 \leq i \leq m,\ \ 1 \leq j \leq n\} .$$

Let $J = \{(i,j);\ \mu(L_i' \cap L_j'') > o\}$ and define

$$\mu_{ij} = \mu(L_i' \cap L_j'')^{-1}\chi_{L_i \cap L_j}\mu \quad , \quad x_{ij} = b(\mu_{ij}),$$

for any $(i,j) \in J$. We shall then have

$$\nu_{\mathcal{L}} = \sum_{(i,j) \in J}\mu(L_i' \cap L_j'')\,\varepsilon_{x_{ij}} .$$

We shall prove that $\nu_{\mathcal{L}'} < \nu_{\mathcal{L}}$ and $\nu_{\mathcal{L}''} < \nu_{\mathcal{L}}$. Indeed, for $J' = \{i;\ 1 \leq i \leq m,\ \mu(L_i') > o\}$, let us define

$$\nu_i = \mu(L_i')^{-1}\sum_j \mu(L_i' \cap L_j'')\,\varepsilon_{x_{ij}} \qquad i \in J',$$

where the sum extends over all j, $1 \leq j \leq n$, such that $\mu(L'_i \wedge L''_j) > 0$. We have $b(\nu_i) = b(\mu(L'_i)^{-1} \chi_{L_i}\mu)$, $i \in J'$. Indeed, we have

$$b(\nu_i) = \mu(L'_i)^{-1} \sum_j \mu(L'_i \wedge L''_j)\, x_{ij}$$

and, therefore, for any $h \in A(K)$, we have

$$h(b(\nu_i)) = \mu(L'_i)^{-1} \sum_j \mu(L'_i \wedge L''_j)\, h(x_{ij}) =$$

$$= \mu(L'_i)^{-1} \sum_j \mu(L'_i \wedge L''_j)\,\mu(L'_i \wedge L''_j)^{-1} \int_{L'_i \wedge L''_j} h\, d\mu =$$

$$= \mu(L'_i)^{-1} \int_{L'_i} h\, d\mu = (\mu(L'_i)^{-1} \chi_{L'_i}\mu)(h) = h(b(\mu(L'_i)^{-1}\chi_{L'_i}\mu)),$$

and this shows that

$$b(\nu_i) = b(\mu(L'_i)^{-1} \chi_{L'_i}\mu), \qquad i \in J'.$$

For any $f \in S(K)$ we shall have $(x'_i \overset{def}{=} b(\nu_i),\ i \in J')$:

$$\nu_{\mathcal{L}'}(f) = \sum_{i \in J'} \mu(L'_i) f(x'_i) = \sum_{i \in J'} \mu(L'_i)\, f(b(\nu_i)) \leq$$

$$\leq \sum_{i \in J'} \mu(L'_i)\,\mu(L'_i)^{-1} \sum_j \mu(L'_i \wedge L''_j)\, f(x_{ij}) = \sum_{(i,j) \in J} \mu(L'_i \wedge L''_j)\, f(x_{ij}) =$$

$$= \nu_{\mathcal{L}}(f),$$

and this shows that $\nu_{\mathcal{L}'} \prec \nu_{\mathcal{L}}$. Similarly, we obtain that $\nu_{\mathcal{L}''} \prec \nu_{\mathcal{L}}$.

iv) From ii) and iii) we infer that $(\nu_{\mathcal{L}})$ is an increasing net in $\mathcal{F}^1_+(K;x_0)$, such that $\nu_{\mathcal{L}}(f) \leq \mu(f)$, for any $\mathcal{L}$ and any $f \in S(K)$. From i) we infer that

$$\sup\{\nu_{\mathcal{L}}(f); \mathcal{L}\} = \mu(f), \qquad f \in S(K).$$

It follows that we have $\lim_{\mathcal{L}} \nu_{\mathcal{L}}(f) = \mu(f)$, for any $f \in S(K)$. Since the vector subspace $S(K) - S(K) \subset C(K;\mathbb{R})$ is uniformly dense in $C(K;\mathbb{R})$, we infer that

$$\lim_{\mathcal{L}} \nu_{\mathcal{L}}(f) = \mu(f),$$

for any $f \in C(K;\mathbb{R})$ and, therefore, $\lim_{\mathcal{L}} \nu_{\mathcal{L}} = \mu$ in the topology $\sigma(C(K)^*; C(K))$. The Lemma is proved.

Remark 1. Measures having finite supports are also called simple.

Remark 2. Part a) of the Lemma shows that $\mu = \sup\{\nu_{\mathcal{L}}; \mathcal{L}\}$ in the set $\mathcal{M}_+^1(K;x_0)$, partially ordered by the Choquet-Meyer order relation.

The following Lemma is an adaptation of Lemma 2 to the case of the orthogonal measures. It is a slight improvement of Lemma 23 from [25].

LEMMA 3. Let μ be an orthogonal Radon probability measure on $E_0(A)$, whose barycenter is a state ($b(\mu) \in E(A)$). Then there exists an increasing net $(\mu_{\mathcal{C}})$ in the set $\Omega(E_0(A); b(\mu))$, consisting of orthogonal Radon probability measures on $E_0(A)$, and having finite supports, such that

a) $\sup\{\mu_{\mathcal{C}}(f); \mathcal{C}\} = \mu(f), \qquad f \in S(E_0(A))$;

b) $\lim_{\mathcal{C}} \mu_{\mathcal{C}} = \mu$ in $\sigma(C(E_0(A))^*; C(E_0(A)))$.

(as above, $\Omega(E_0(A), b(\mu))$ is equipped with the Choquet-Meyer order relation; or, equivalently, with the Bishop-de Leeuw order relation (see [28], Theorem 3.5)).

Proof. Let $\mathcal{C}_\mu$ be the abelian von Neumann subalgebra of $\pi_{b(\mu)}(A)'$, corresponding to μ, and denote by $e_\mu \in \mathcal{C}_\mu'$ the projection onto the subspace $\overline{\mathcal{C}_\mu \xi^0_{b(\mu)}} \subset H_{b(\mu)}$. Let $\mathcal{J}$ be the set of all finite dimensional von Neumann subalgebras of $\mathcal{C}_\mu$, partially ordered by inclusion. For any $\mathcal{C} \in \mathcal{J}$, let us denote by $e_{\mathcal{C}} \in \mathcal{C}'$ the projection onto $\overline{\mathcal{C}\xi^0_{b(\mu)}}$ and let $\mu_{\mathcal{C}}$ be the unique orthogonal Radon probability measure on $E_0(A)$, such that $b(\mu_{\mathcal{C}}) = b(\mu)$ and $\operatorname{im} K_{\mu_{\mathcal{C}}} = \mathcal{C}$ (see [28], Theorem 3.2). It is obvious that $\mathcal{J}$ is an increasing net and that the net $\mathcal{J} \ni \mathcal{C} \to e_{\mathcal{C}}$ converges, in the strong operator topology, to e_μ.

On the other hand, from $\mathcal{C} \subset \mathcal{C}_\mu$, $\mathcal{C} \in \mathcal{J}$, we infer that $\mu_{\mathcal{C}} \ll \mu$; whereas, from $\mathcal{C}_1 \subset \mathcal{C}_2$, $\mathcal{C}_1, \mathcal{C}_2 \in \mathcal{J}$, we infer that $\mu_{\mathcal{C}_1} \ll \mu_{\mathcal{C}_2}$ (or, equivalently, $\mu_{\mathcal{C}} < \mu$; resp., $\mu_{\mathcal{C}_1} < \mu_{\mathcal{C}_2}$; see [28], Theorem 3.5).

From the fact that any $\mathcal{C} \in \mathcal{J}$ is finite dimensional, it is easy to infer that the measures $\mu_{\mathcal{C}}$ have finite supports.

By taking into account ([28], Corollary 1 to Lemma 3.2, Lemma 3.3 and Lemma 3.6), we infer that for any $a_1, a_2, \ldots, a_n \in A$, $n \geq 1$, we have

$$\int_{E_0(A)} \lambda_A(a_1)\lambda_A(a_2)\ldots\lambda_A(a_n)\, d\mu_{\mathcal{C}} = \int_{E_0(A)} \lambda_A(a_1)\cdot\lambda_A(a_2)\ldots\lambda_A(a_n)\, d\mu_{\mathcal{C}} =$$

$$= (K_{\mu_{\mathcal{C}}}(\lambda_A(a_1)\lambda_A(a_2)\ldots\lambda_A(a_n))\xi^0_{b(\mu)} | \xi^0_{b(\mu)}) =$$

$$= (K_{\mu_{\mathcal{C}}}(\lambda_A(a_1))K_{\mu_{\mathcal{C}}}(\lambda_A(a_2))\ldots K_{\mu_{\mathcal{C}}}(\lambda_A(a_n))\xi^0_{b(\mu)} | \xi^0_{b(\mu)}) =$$

$$= (e_{\mathcal{C}}\pi_{b(\mu)}(a_1)e_{\mathcal{C}}\pi_{b(\mu)}(a_2)e_{\mathcal{C}}\ldots e_{\mathcal{C}}\pi_{b(\mu)}(a_n)e_{\mathcal{C}}\xi^0_{b(\mu)} | \xi^0_{b(\mu)}) \to$$

$$\to (e_\mu\pi_{b(\mu)}(a_1)e_\mu\pi_{b(\mu)}(a_2)e_\mu\ldots e_\mu\pi_{b(\mu)}(a_n)e_\mu\xi^0_{b(\mu)} | \xi^0_{b(\mu)}) =$$

$$= \int_{E_o(A)} \lambda_A(a_1)\lambda_A(a_2)\ldots\lambda_A(a_n)\, d\mu.$$

Since we have $1 = \|\mu_{\mathcal{C}}\| = \mu_{\mathcal{C}}(E(A))$, from the Stone-Weierstrass Theorem we now infer that $\lim_{\mathcal{C}} \mu_{\mathcal{C}} = \mu$ in the w^*-topology on $E_o(A)$. The Lemma is proved.

COROLLARY. *Let μ be a subcentral Radon probability measure on $E_o(A)$, whose barycenter is a state. Then there exists an increasing net $(\mu_{\mathcal{C}})$ in the set $\Omega(E_o(A); b(\mu))$, consisting of subcentral Radon probability measures on $E_o(A)$, and having finite supports, such that*

a) $\sup\{\mu_{\mathcal{C}}(f); \mathcal{C}\} = \mu(f), \quad f \in S(E_o(A))$;

b) $\lim_{\mathcal{C}} \mu_{\mathcal{C}} = \mu$ *in* $\sigma(C(E_o(A))^*; C(E_o(A)))$.

Proof. This is an immediate consequence of the proof of the preceding Lemma: indeed, if μ is subcentral, then $\mathcal{C}_\mu$ is contained in the center of $\pi_{b(\mu)}(A)'$, and, therefore, all the von Neumann algebras $\mathcal{C} \in \mathcal{J}$ are also contained in the center of $\pi_{b(\mu)}(A)'$; hence, the measures $\mu_{\mathcal{C}}$ are subcentral.

The following Lemma is a slight improvement of a Lemma of C.F.Skau (see [25], Lemma 12; [28], Lemma 3.7; see, also, [41], proof of Lemma 3.19, where only subcentral measures are considered).

LEMMA 4. *Let $\mu, \nu \in \mathcal{M}^1_+(E_o(A))$ be such that $\mu \prec \nu$. Then, for any subset $\{\varphi_1, \ldots, \varphi_m\} \subset L^\infty(\mu)^+_1$, such that $\sum_{i=1}^{n} \varphi_i = 1$, there exists a subset $\{\psi_1, \ldots, \psi_m\} \subset L^\infty(\nu)^+_1$, such that $\sum_{i=1}^{m} \psi_i = 1$ and $K_\mu(\varphi_i) = K_\nu(\psi_i)$, $1 \le i \le m$.*

Proof. Let us define $\mu_i = \varphi_i \mu$, $1 \le i \le m$. Then we have $\mu = \sum_{i=1}^{m} \mu_i$. From the Cartier-Fell-Meyer Theorem (see [28], Theorem 1.6), we infer that there exist positive Radon measures ν_i, $1 \le i \le m$, on $E_o(A)$, such that $\nu = \sum_{i=1}^{m} \nu_i$ and $\mu_i \sim \nu_i$, $1 \le i \le m$. We infer that there exist functions $\psi_i \in L^\infty(\nu)^+_1$, such that $\nu_i = \psi_i \nu$, $1 \le i \le m$, and, of course, we have $\sum_{i=1}^{m} \psi_i = 1 (\mathrm{mod}\, \nu)$. We then have

$$(1) \quad (K_\nu(\psi_i)\pi_{f_o}(a)\xi^o_{f_o} | \xi^o_{f_o}) = \int_{E_o(A)} \psi_i \lambda_A(a)\, d\nu = \int_{E_o(A)} \lambda_A(a)\, d\nu_i =$$

$$= \int_{E_o(A)} \lambda_A(a)\, d\mu_i = \int_{E_o(A)} \varphi_i \lambda_A(a)\, d\mu = (K_\mu(\varphi_i)\pi_{f_o}(a)\xi^o_{f_o} | \xi^o_{f_o}), \quad a \in A,$$

for any $i \in \{1, 2, \ldots, m\}$, where we have denoted $f_o \overset{\mathrm{def}}{=} b(\mu) = b(\nu)$, taking into account the fact that $\mu \prec \nu \Rightarrow \mu \sim \nu$. From (1) we infer that $K_\nu(\psi_i) = K_\mu(\varphi_i)$, $1 \le i \le m$. The Lemma is proved.

The following Lemma essentially belongs to W.Wils (see [41], proof of Proposition 3.2o; and also [25], Lemma 24).

LEMMA 5. Let $\mu = \sum_{i=1}^{m} c_i \varepsilon_{f_i}$, $\nu = \sum_{j=1}^{n} d_j \varepsilon_{g_j}$ be two measures on $E_o(A)$, with finite supports, such that

a) $f_i \in E(A)$, $1 \leq i \leq m$; $g_j \in E(A)$, $1 \leq j \leq n$;

b) $c_i \geq o$, $1 \leq i \leq m$; $d_j \geq o$, $1 \leq j \leq n$; $\sum_{i=1}^{m} c_i = \sum_{j=1}^{n} d_j = 1$;

c) μ is subcentral;

d) $b(\mu) = b(\nu)$.

Then there exists the least upper bound of μ and ν in $\mathcal{M}_+^1(E_o(A); f_o)$, where $f_o \overset{def}{=} b(\mu) = b(\nu)$.

Proof. We can assume that $f_i \neq f_j$ and $g_i \neq g_j$ for $i \neq j$; and also that $c_i > o$, $1 \leq i \leq m$; and $d_j > o$, $1 \leq j \leq n$.

By hypothesis, the von Neumann algebra $\mathcal{C}_\mu$, corresponding to μ, is contained in $\pi_{f_o}(A)'' \cap \pi_{f_o}(A)'$.

Let $C_i \overset{def}{=} K_\mu(\chi_{\{f_i\}})$, $1 \leq i \leq m$, and $D_j \overset{def}{=} K_\nu(\chi_{\{g_j\}})$, $1 \leq j \leq n$. Then we have

i) $C_i \in \pi_{f_o}(A)'' \cap \pi_{f_o}(A)'$, $1 \leq i \leq m$;

ii) C_i, $1 \leq i \leq m$, is a central projection and $\sum_{i=1}^{m} C_i = 1$;

iii) $D_j \in \pi_{f_o}(A)'$, $1 \leq j \leq n$;

iv) $D_j \geq o$, $1 \leq j \leq n$; and $\sum_{j=1}^{n} D_j = 1$.

Of course, we have

$$c_i f_i(a) = \int_{E_o(A)} \chi_{\{f_i\}} \lambda_A(a)\, d\mu = (C_i \pi_{f_o}(a) \xi_{f_o} \mid \xi_{f_o}), \quad 1 \leq i \leq m,$$

and

$$d_j g_j(a) = \int_{E_o(A)} \chi_{\{g_j\}} \lambda_A(a)\, d\mu = (D_j \pi_{f_o}(a) \xi_{f_o} \mid \xi_{f_o}), \quad 1 \leq j \leq n,$$

for any $a \in A$. We shall define the positive linear functional $r'_{ij} \in E_o(A)$ by

$$r'_{ij}(a) = (C_i D_j \pi_{f_o}(a) \xi_{f_o} \mid \xi_{f_o}), \qquad a \in A.$$

(since $C_i D_j = D_j C_i$, we have $C_i D_j \geq o$ and, therefore, $r'_{ij} \geq o$, $1 \leq i \leq m$, $1 \leq j \leq n$). If $r'_{ij} \neq o$, define $r_{ij} = \|r'_{ij}\|^{-1} r'_{ij}$; if $r'_{ij} = o$, define $r_{ij} = o$. It is obvious that for

$$\theta \overset{def}{=} \sum_{i,j} \|r'_{ij}\| \varepsilon_{r_{ij}},$$

we have that θ is a simple measure in $\mathcal{M}_+^1(E_o(A); f_o)$ and

$$c_i f_i = \sum_{j=1}^{n} \|r'_{ij}\| r_{ij}, \quad 1 \leq i \leq m;$$

$$d_j g_j = \sum_{i=1}^{m} \| r'_{ij} \| \; r_{ij} \; , \; 1 \leq j \leq n.$$

By simple convexity arguments, we have that $\mu \prec \theta$ and $\nu \prec \theta$.

Conversely, let $\tau \in \mathcal{M}_+^1(E_o(A); f_o)$ be such that $\mu \prec \tau$ and $\nu \prec \tau$. We shall prove that $\theta \prec \tau$.

From Lemma 4 we infer that there exist functions $\varphi_i \in L^\infty(\tau)_1^+$, such that $C_i = K_\tau(\varphi_i)$, $1 \leq i \leq m$; and also, there exist functions $\psi_j \in L^\infty(\tau)_1^+$, such that $D_j = K_\tau(\psi_j)$, $1 \leq j \leq n$; moreover, we can assume that

$$\sum_{i=1}^{m} \varphi_i = \sum_{j=1}^{n} \psi_j = 1 \quad (\bmod \tau).$$

Of course, any decomposition $\theta = \sum_{k=1}^{\ell} \theta_k$ of θ is of the form

$$\theta_k = \sum_{i,j} t_{ijk} \| r'_{ij} \| \varepsilon_{r_{ij}} \; ,$$

where $o \leq t_{ijk}$, $1 \leq k \leq \ell$; and $\sum_{k=1}^{\ell} t_{ijk} = 1$, $1 \leq i \leq m$, $1 \leq j \leq n$.

If we define

$$\tau_k = \Big(\sum_{i,j} t_{ijk} \varphi_i \psi_j \Big) \tau \quad , \qquad 1 \leq k \leq \ell,$$

we shall have

$$(1) \qquad \tau = \sum_{k=1}^{\ell} \tau_k \quad \text{and} \quad \tau_k \sim \theta_k \; , \quad 1 \leq k \leq \ell.$$

With the Cartier-Fell-Meyer Theorem we shall infer that $\theta \prec \tau$.

Indeed, we have

$$\sum_{k=1}^{\ell} \tau_k = \Big(\sum_{i,j,k} t_{ijk} \varphi_i \psi_j \Big) = \Big(\sum_{i,j} \Big(\sum_{k=1}^{\ell} t_{ijk} \Big) \varphi_i \psi_j \Big) =$$
$$= \Big(\sum_{i,j} \varphi_i \psi_j \Big) = \Big(\sum_{i=1}^{m} \varphi_i \Big) \Big(\sum_{j=1}^{n} \psi_j \Big) \tau = \tau \; ,$$

and the equality in (1) is proved.

For the second part of the assertion (1), we shall first prove that

$$(2) \qquad K_\tau(\varphi_i \psi_j) = K_\tau(\varphi_i) K_\tau(\psi_j), \qquad 1 \leq i \leq m, 1 \leq j \leq n.$$

Indeed, from $o \leq \varphi_i \psi_j \leq \varphi_i$, ψ_j, we infer that

$$o \leq K_\tau(\varphi_i \psi_j) \leq K_\tau(\varphi_i), \; K_\tau(\psi_j), \quad 1 \leq i \leq m, \; 1 \leq j \leq n.$$

Since $K_\tau(\varphi_i)$ is a projection, we infer that

$$K_\tau(\varphi_i\psi_j) = K_\tau(\varphi_i)K_\tau(\varphi_i\psi_j) = K_\tau(\varphi_i\psi_j)K_\tau(\varphi_i);$$

whereas from the fact that $K_\tau(\varphi_i)$ is a central projection, we infer that

$$(3) \qquad K_\tau(\varphi_i\psi_j) \leq K_\tau(\varphi_i)K_\tau(\psi_j).$$

By replacing φ_i with $1-\varphi_i$, we similarly get that

$$K_\tau((1-\varphi_i)\psi_j) \leq K_\tau(1-\varphi_i)K_\tau(\psi_j),$$

and this implies that

$$(4) \qquad K_\tau(\varphi_i)K_\tau(\psi_j) \leq K_\tau(\varphi_i\psi_j).$$

From (3) and (4) we get the desired result (1).

Let us now remark that, by virtue of (2), we have

$$\int_{E_o(A)} \lambda_A(a)\, d\tau_k = \int_{E_o(A)} \Big(\sum_{i,j} t_{ijk}\varphi_i\psi_j\Big)\lambda_A(a)\, d\tau =$$

$$= \Big(\sum_{i,j} t_{ijk}K_\tau(\varphi_i\psi_j)\pi_{f_o}(a)\xi^o_{f_o} \mid \xi^o_{f_o}\Big) =$$

$$= \sum_{i,j} t_{ijk}(K_\tau(\varphi_i)K_\tau(\psi_j)\pi_{f_o}(a)\xi^o_{f_o} \mid \xi^o_{f_o}) =$$

$$= \sum_{i,j} t_{ijk}(C_iD_j\pi_{f_o}(a)\xi^o_{f_o} \mid \xi^o_{f_o}) = \sum_{i,j} t_{ijk}\, r'_{ij}(a) =$$

$$= \sum_{i,j} t_{ijk}\|r'_{ij}\|\, r_{ij}(a) = \sum_{i,j} t_{ijk}\|r'_{ij}\|\lambda_A(a)(r_{ij}) =$$

$$= \int_{E_o(A)} \lambda_A(a)\, d\theta_k, \quad 1 \leq k \leq \ell, \quad a \in A.$$

We also have that

$$\tau_k(1) = \int_{E_o(A)} \Big(\sum_{i,j} t_{ijk}\varphi_i\psi_j\Big)\, d\tau = \int_{E_o(A)} \Big(\sum t_{ijk}\varphi_i\psi_j\Big)\|\cdot\|\, d\tau =$$

$$= \Big(K_\tau\Big(\sum_{i,j} t_{ijk}\varphi_i\psi_j\Big)\xi^o_{f_o} \mid \xi^o_{f_o}\Big) = \sum_{i,j} t_{ijk}(C_iD_j\xi^o_{f_o} \mid \xi^o_{f_o}) = \theta_k(1),$$

for $1 \leq k \leq \ell$, by taking into account ([28], Corollary 1 to Lemma 3.2 and Lemma 3.3). From (5) and (6) we infer that $\tau_k \sim \theta_k$, $1 \leq k \leq \ell$, and so the second assertion in (1) is proved, and also the Lemma.

We shall denote $\theta = \mu \vee \tau$.

The following Theorem belongs to W.Wils (see [41], Proposition 3.2o where it is stated under more general conditions). It is stated as Lemma 25 in [25], where it is given for C^*-algebras with a unit element. Here we shall drop this assumption.

THEOREM 2. For any state $f_o \in E(A)$, any measure $\mu \in \mathcal{M}_+^1(E_o(A);f_o)$ and any subcentral measure $\nu \in \mathcal{M}_+^1(E_o(A);f_o)$ there exists the least upper bound of μ and ν in $\mathcal{M}_+^1(E_o(A);f_o)$, with respect to the Choquet-Meyer order relation.

Proof. Let $(\mu_{\mathcal{L}})$ be an increasing net in $\mathcal{F}_+^1(E_o(A);f_o)$, as given by Lemma 2, for the measure μ. Let $(\nu_{\mathcal{C}})$ be an increasing net of simple subcentral measures, as given by the Corollary to Lemma 3, for the measure ν. Let us define $\theta_{\mathcal{L},\mathcal{C}} = \mu_{\mathcal{L}} \vee \nu_{\mathcal{C}}$, as given by Lemma5. Since we have that $\theta_{\mathcal{L}\mathcal{C}} \in \mathcal{M}_+^1(E_o(A);f_o)$, and since this space is w^*-compact, we can select a subnet of $(\theta_{\mathcal{L},\mathcal{C}})$, which converges to $\theta \in \mathcal{M}_+^1(E_o(A);f_o)$. Since for any $\varphi \in S(E_o(A))$ we have

$$\int_{E_o(A)} \varphi \, d\mu_{\mathcal{L}} \leq \int_{E_o(A)} \varphi \, d\theta_{\mathcal{L},\mathcal{C}}, \quad \int_{E_o(A)} \varphi \, d\nu_{\mathcal{C}} \leq \int_{E_o(A)} \varphi \, d\theta_{\mathcal{L},\mathcal{C}}, \quad \forall \mathcal{L},\mathcal{C}$$

we infer that

$$\int_{E_o(A)} \varphi \, d\mu \leq \int_{E_o(A)} \varphi \, d\theta, \quad \int_{E_o(A)} \varphi \, d\nu \leq \int_{E_o(A)} \varphi \, d\theta, \quad \varphi \in S(E_o(A))$$

and this shows that $\mu < \theta$, $\nu < \theta$.

Let now $\tau \in \mathcal{M}_+^1(E_o(A);f_o)$ be such that $\mu < \tau$ and $\nu < \tau$. Then we have $\mu_{\mathcal{L}} < \tau$, $\nu_{\mathcal{C}} < \tau$, for any $\mathcal{L}$ and $\mathcal{C}$; therefore, by Lemma 5, we have $\theta_{\mathcal{L},\mathcal{C}} < \tau$ and this implies that $\theta < \tau$. It follows that $\theta = \mu \vee \nu$.

Remark. Since the preceding argument works for any converging subnet of $(\theta_{\mathcal{L},\mathcal{C}})$, and since the least upper bound, when it exists, is uniqu we infer that we actually have $\theta = \lim \theta_{\mathcal{L},\mathcal{C}}$. Moreover, since

$$\mathcal{C}_1 \leq \mathcal{C}_2 \quad \text{and} \quad \mathcal{L}_1 \leq \mathcal{L}_2$$

implies that

$$\theta_{\mathcal{L}_1,\mathcal{C}_1} < \theta_{\mathcal{L}_2,\mathcal{C}_2},$$

we see that we have

$$\sup\left\{\int_{E_o(A)} \varphi \, d\theta_{\mathcal{L},\mathcal{C}} ; \mathcal{L},\mathcal{C}\right\} = \int_{E_o(A)} \varphi \, d\theta, \quad \varphi \in S(E_o(A)),$$

and so we have $\sup\{\theta_{\mathcal{L},\mathcal{C}} ; \mathcal{L},\mathcal{C}\} = \theta$ in $\mathcal{M}_+^1(E_o(A);f_o)$.

The following Theorem was first proved by W.Wils (see [4o]) for the case of C^*-algebras having a unit element (see, also, [25], Theorem 26)

THEOREM 3. Let $f_o \in E(A)$ be a state of A and let $\mu \in \mathcal{M}_+^1(E_o(A);f_o)$ be any subcentral measure. Then we have $\mu \prec \nu$ for any Choquet maximal measure $\nu \in \mathcal{M}_+^1(E_o(A);f_o)$.

Proof. By Theorem 2, the least upper bound $\mu \vee \nu$ exists in $\mathcal{M}_+^1(E_o(A);f_o)$. Since we have $\nu \prec \mu \vee \nu$, and since ν is maximal, we infer that $\nu = \mu \vee \nu$ and therefore, $\mu \prec \nu$. The Theorem is proved.

Proof of THEOREM 1.

Let $f_o \in E(A)$ and let μ_{f_o} be the central measure corresponding to f_o. Then we have $\mu_{f_o} \prec \nu$, for any Choquet maximal measure $\nu \in \mathcal{M}_+^1(E_o(A);f_o)$, by virtue of Theorem 3.

Conversely, let $\mu \in \mathcal{M}_+^1(E_o(A);f_o)$ be a measure, such that $\mu \prec \nu$, for any Choquet maximal measure $\nu \in \mathcal{M}_+^1(E_o(A);f_o)$. By virtue of Henrichs' Theorem (see [18]; [28], Theorem 3.1o), we shall have $\mu \prec \nu$, for any maximal orthogonal measure $\nu \in \Omega(E_o(A);f_o)$. By ([28], Lemma 3.7), we infer that

$$K_\mu(L^\infty(\mu)_1^+) \subset K_\nu(L^\infty(\nu)_1^+),$$

for any maximal orthogonal measure ν, and, therefore,

$$\text{(1)} \quad K_\mu(L^\infty(\mu)_1^+) \subset \bigcap_\nu K_\nu(L^\infty(\nu)_1^+) = [\pi_{f_o}(A)'' \cap \pi_{f_o}(A)']_1^+ = K_{\mu_{f_o}}(L^\infty(\mu_{f_o})_1^+),$$

where the intersection is taken over all maximal orthogonal measures ν. Since μ_{f_o} is simplicial (see [28], Corollary 1 to Theorem 3.1), from (1) and from ([28], Lemma 3.7, ii)) we infer that $\mu \prec \mu_{f_o}$. The Theorem is proved.

III. The following Theorem essentially belongs to W. Wils (see [41]). We shall denote by $\mathcal{Z}_+^1(E_o(A);f_o)$ the set of all subcentral Radon probability measures on $E_o(A)$, whose barycenter is at f_o.

THEOREM 4. For any state $f_o \in E(A)$, the set $\mathcal{Z}_+^1(E_o(A);f_o)$ is a complete lattice with respect to the Choquet-Meyer order relation.

Proof. The mapping $\mathcal{Z}_+^1(E_o(A);f_o) \ni \mu \mapsto \mathcal{C}_\mu \subset \pi_{f_o}(A)' \cap \pi_{f_o}(A)''$ is an order isomorphism between $\mathcal{Z}_+^1(E_o(A); f_o)$ and the set of all von Neumann subalgebras of the center of $\pi_{f_o}(A)'$ (see [28], Theorem 3.4).

As usually, for any operator $a \in \mathcal{L}(H)$ we shall denote by $r(a)$, respectively $\ell(a)$, its right, respectively left support. If $a^* = a$, then $\ell(a) = r(a)$, and this projection is called the support of a and it is denoted by $s(a)$.

THEOREM 5. Let $\mu\in\mathcal{M}^1_+(E_o(A))$ and $\varphi\in L^\infty(\mu)$, $\varphi\geq o$. Then:
a) If μ is a subcentral measure, then $\varphi\mu$ is a subcentral measure;
b) If μ is a central measure, then $\varphi\mu$ is a central measure.

Proof. It is known that if μ is an orthogonal measure, then $\varphi\mu$ is also orthogonal (see [28], Proposition 3.16).

a) Let $f_o = r(\mu)$. Then the mapping K_μ maps $L^\infty(\mu)$ into the center $\pi_{f_o}(A)'\cap\pi_{f_o}(A)''$ of $\pi_{f_o}(A)'$, and

$$\int_{E_o(A)}\psi\lambda_A(a)\,d\mu = (K_\mu(\psi)\,\pi_{f_o}(a)\,\xi^o_{f_o}|\xi^o_{f_o}),\qquad a\in A,\ \psi\in L^\infty(\mu).$$

Since we have

$$(1)\quad \int_{E_o(A)}\psi\lambda_A(a)\,d(\varphi\mu)= \int_{E_o(A)}\psi\varphi\lambda_A(a)\,d\mu =$$
$$= (K_\mu([\psi])\,\pi_{f_o}(a)K_\mu([\varphi])^{1/2}\xi^o_{f_o}|K_\mu([\varphi])^{1/2}\xi^o_{f_o})$$

for any $a\in A$ and $\psi\in L^\infty(\varphi\mu)$, we infer that $H_{r(\varphi\mu)}$ can be identified with $s(K_\mu(\varphi))H_{r(\mu)}$, $\xi^o_{r(\varphi\mu)}$ can be identified with $K_\mu(\varphi)^{1/2}\,\xi^o_{f_o}$, whereas $\pi_{r(\varphi\mu)}$ can be identified with the representation

$$A\ni a\longmapsto \pi_{f_o}(a)s(K_\mu(\varphi)).$$

From (1) we infer that

$$(2)\qquad K_{\varphi\mu}([\psi]) = K_\mu([\psi])s(K_\mu([\varphi])),\qquad \psi\in\mathcal{L}^\infty(\mathcal{B}(E_o(A)))$$

(we can always use bounded Borel measurable representatives of element in $L^\infty(\mu)$, etc.). Of course, we have used the fact that $s(K_\mu([\varphi]))\in\mathcal{C}_\mu$.

From (2) we infer that $K_{\varphi\mu}$ maps $L^\infty(\varphi\mu)$ into the center of $\pi_{r(\varphi\mu)}(A)'$; hence, $\varphi\mu$ is subcentral.

b) If μ is central, then K_μ maps $L^\infty(\mu)$ onto $\pi_{f_o}(A)'\cap\pi_{f_o}(A)''$, whence we infer that $K_{\varphi\mu}$ maps $L^\infty(\varphi\mu)$ onto $\pi_{g_o}(A)'\cap\pi_{g_o}(A)''$, where $g_o\overset{def}{=}r(\varphi\mu)$. It follows that $\varphi\mu$ is central. The Theorem is proved.

IV. The following Theorem characterizes the central measures. It is implicitly contained in the proof following Definition 3.5.1. in [24], where the C^*-algebra A is assumed to possess the unit element. Since at the head of Section 3.5, p.146, in [24], A is assumed, moreover, to be separable, a superficial reading might leave the impression that Sakai's characterization is established for the separable case only. In fact, it is easy to see that the argument following Definition 3.5.1. in [24] does not require the separability of A, and so the following

Theorem is only the extension of Sakai's result to the slightly more general case of a C^*-algebra A, possibly not possessing the unit element.

THEOREM 6. Let A be any C^*-algebra, $f_o \in E(A)$ a state of A, and denote by $\pi''_{f_o} : A^{**} \to \mathcal{L}(H_{f_o})$ the canonical normal extension of π_{f_o} to A^{**}. Then a measure $\mu \in \mathcal{M}^1_+(E_o(A); f_o)$ is the central measure μ_{f_o} if, and only if, there exists a $*$-homomorphism $\Phi : Z(A^{**}) \to L^\infty(\mu)$, of the center $Z(A^{**})$ of A^{**}, onto $L^\infty(\mu)$, such that

$$(1) \qquad f_o(za) = \int_{E_o(A)} \Phi(z)\lambda_A(a)\, d\mu , \qquad a \in A, z \in Z(A^{**}).$$

In this case, Φ is unique.

(Of course, in (1), for $\Phi(z)$ one should take a function representative of the class $\Phi(z) \in L^\infty(\mu)$; it can always be chosen to be a bounded Baire measurable function on $E_o(A)$, whereas with the help of Lifting Theory, one can even obtain a $*$-homomorphism into $\mathcal{L}^\infty(E_o(A);\mu)$).

Proof. Let us assume that $\mu = \mu_{f_o}$. Then, by its definition, the mapping K_μ maps isomorphically the C^*-algebra $L^\infty(\mu)$ onto the center of $\pi_{f_o}(A)'$ (hence, also of $\pi_{f_o}(A)''$). On the other hand, the restriction $\pi''_{f_o} | Z(A^{**})$ is a normal $*$-homomorphism of $Z(A^{**})$ onto the center of $\pi_{f_o}(A)''$, and we have

$$f_o(za) = (\pi_{f_o}(a)\, \pi''_{f_o}(z)\xi^o_{f_o} | \xi^o_{f_o}), \qquad a \in A, \quad z \in Z(A^{**}).$$

The mapping $\Phi_{f_o} = K^{-1}_{\mu_{f_o}} \circ (\pi''_{f_o} | Z(A^{**}))$ has then the required property (1) (see [28], Lemma 3.3, Proposition 3.1, ii) and Corollary 1 to Theorem 3.1).

Conversely, let us assume that there exists a surjective $*$-homomorphism $\Phi : Z(A^{**}) \to L^\infty(\mu)$, such that equality (1) be satisfied.

a) For any $z \in Z(A^{**})$ we shall have

$$f_o(za) = (\pi_{f_o}(a)\, \pi''_{f_o}(z)\xi^o_{f_o} | \xi^o_{f_o}) = \int_{E_o(A)} \Phi(z)\lambda_A(a)\, d\mu =$$

$$= (K_\mu(\Phi(z))\, \pi_{f_o}(a)\xi^o_{f_o} | \xi^o_{f_o}), \qquad a \in A,$$

and this implies that

$$(2) \qquad K_\mu(\Phi(z)) = \pi''_{f_o}(z), \qquad z \in Z(A^{**}).$$

b) For any $\varphi_1, \varphi_2 \in L^\infty(\mu)$ there exist $z_1, z_2 \in Z(A^{**})$, such that

$\Phi(z_i) = \varphi_i, i = 1,2$. We shall then have

$$K_\mu(\varphi_1\varphi_2) = K_\mu(\Phi(z_1)\Phi(z_2)) = K_\mu(\Phi(z_1z_2)) = \pi''_{f_o}(z_1z_2) =$$

$$= \pi''_{f_o}(z_1)\pi''_{f_o}(z_2) = K_\mu(\Phi(z_1))K_\mu(\Phi(z_2)) = K_\mu(\varphi_1)K_\mu(\varphi_2),$$

and this shows that μ is orthogonal. From (2) we infer that μ is central, whereas $\|b(\mu)\| = 1$ implies that K_μ is injective. From (2) we infer that

$$\Phi(z) = (K_\mu^{-1} \circ \pi''_{f_o})(z), \qquad z \in Z(A^{**}),$$

and so the uniqueness of Φ is established. The Theorem is proved.

We shall denote by Φ_{f_o} the $*$-homomorphism corresponding to the state $f_o \in E(A)$, and whose existence and uniqueness is established above.

LEMMA 6. For any two positive linear functionals $f_1, f_2 \in A_+^*$ consider the relations:

a) $f_1 \smile f_2$;
b) $\|f_1 - f_2\| = \|f_1\| + \|f_2\|$;
c) $f_1 \perp f_2$;
d) $f_1 \neq f_2$, unless $f_1 = f_2 = o$.

Then a) $\Rightarrow$ b) $\Rightarrow$ c) $\Rightarrow$ d), and no implication here can be reversed.

Proof. Indeed, there exists a projection $e_o \in Z(A^{**})$, such that $f_1(a) = f_1(ae_o)$, $f_2(a) = f_2(a(1-e_o))$, $a \in A$. If we denote $f = f_1 + f_2$, then we have

$$f(e_oa) = f_1(a), \qquad a \in A,$$

$$f((1 - e_o)a) = f_2(a), \qquad a \in A,$$

and, therefore,

$$(f_1 - f_2)(a) = f((2e_o - 1)a), \qquad a \in A,$$

whence we get that

$$\|f_1 - f_2\| = \|f \cdot (2e_o - 1)\| = \|f\| = \|f_1\| + \|f_2\|,$$

because $2e_o - 1$ is unitary. Thus, the implication a) $\Rightarrow$ b) is proved.

b) $\Rightarrow$ c). Indeed, assume that $o \leq f \leq f_1$ and $o \leq f \leq f_2$. Then we have

$f_i = f + (f_i - f), i = 1,2,$ and, therefore,

$$\|f_1\| + \|f_2\| = \|f_1 - f_2\| = \|(f_1 - f) - (f_2 - f)\| \leq$$

$$\leq \|f_1 - f\| + \|f_2 - f\| = \|f_1\| + \|f_2\| - 2\|f\|,$$

and this implies that $f = o$. Hence, $f_1 \perp f_2$.

c) $\Rightarrow$ d). If $f_1 \perp f_2$ and $f_1 = f_2$, then, obviously, $f_1 = f_2 = o$. Simple examples can be found to show that the preceding implications cannot be reversed, in general.

LEMMA 7. Let $f_1, f_2 \in E(A)$ be two states, such that

$$2 = \|f_1 - f_2\|$$

and let $\mu_i \in \mathcal{M}^1_+(E_o(A); f_i)$, $i = 1,2$. Then the measures μ_1 and μ_2 are mutually singular.

Proof. Indeed, by ([12], Proposition 12.3.1.) the supports $s(f_1)$, resp. $s(f_2)$, of f_1, resp. f_2, in A^{**} are orthogonal: $s(f_1)s(f_2) = o$. Since $s(f_1)$ and $s(f_2)$ are countably decomposable in A^{**}, by ([42], Ch. I, Lemma 1.2) there exists a Baire element $b \in \mathcal{B}_o(A) \subset A^{**}$ (see [31]), such that $o \leq b \leq \leq 1$ and $f_1(b) = \|f_1\| = 1$, $f_2(b) = o$. If we denote $e = \lim_{n \to \infty} b^n$, then e is a Baire projection and we have

$$f_1(e) = 1, \quad f_2(e) = o.$$

Since the barycentric calculus holds for Baire elements over A (see [31], Theorem 3), we infer that

$$(1) \quad \int_{E_o(A)} \lambda_A(e)\, d\mu_1 = 1, \qquad \int_{E_o(A)} \lambda_A(e)\, d\mu_2 = o.$$

From (1) we infer that

$$\mu_1(\{f \in E_o(A);\ f(e) = 1\}) = 1$$

and

$$\mu_2(\{f \in E_o(A);\ f(e) = 1\}) = o;$$

hence, μ_1 and μ_2 are mutually singular.

THEOREM 7. Let $f_1, f_2 \in E(A)$ be two states, such that $f_1 \delta f_2$. Let $t \in [o,1]$ and define $f = tf_1 + (1 - t)f_2$. Then

a) $\mu = t\mu_{f_1} + (1 - t)\mu_{f_2}$ is the central measure corresponding to f;

b) $\Phi_f = \Phi_{f_1} + \Phi_{f_2}$.

Proof. a) By Lemma 6 and by Godement's Theorem (see [25], p.281; [17]), we can identify $\pi_f : A \to \mathcal{L}(H_f)$ with $\pi = \pi_{f_1} \oplus \pi_{f_2} : A \to \mathcal{L}(H_{f_1} \oplus H_{f_2})$, with the associated cyclic vector

$$\xi_o = (t^{1/2} \xi^o_{f_1}, (1-t)^{1/2} \xi^o_{f_2}).$$

We shall assume from now on that $o < t < 1$. Otherwise, the assertion is trivial.

For any bounded Borel measurable complex function φ on $E_o(A)$ we shall have

$$\int_{E_o(A)} \varphi\lambda_A(a)\, d(t\mu_{f_1} + (1-t)\mu_{f_2}) = t \int_{E_o(A)} \varphi\lambda_A(a)\, d\mu_{f_1} +$$

$$+ (1-t) \int_{E_o(A)} \varphi\lambda_A(a)\, d\mu_{f_2} = t(K_{\mu_{f_1}}(\varphi)\, \pi_{f_1}(a)\, \xi^o_{f_1} | \xi^o_{f_1}) +$$

$$+ (1-t)(K_{\mu_{f_2}}(\varphi)\, \pi_{f_2}(a)\, \xi^o_{f_2} | \xi^o_{f_2}) =$$

$$= ((K_{\mu_{f_1}}(\varphi)\, \pi_{f_1}(a) \oplus K_{\mu_{f_2}}(\varphi)\, \pi_{f_2}(a))\, \xi_o | \xi_o) =$$

$$= ((K_{\mu_{f_1}}(\varphi) \oplus K_{\mu_{f_2}}(\varphi))\, \pi_f(a)\, \xi_o | \xi_o), \qquad a \in A,$$

and this shows that

$$(1) \qquad K_{\mu_{f_1}}(\varphi) \oplus K_{\mu_{f_2}}(\varphi) = K_{t\mu_{f_1} + (1-t)\mu_{f_2}}(\varphi), \quad \varphi \in \mathcal{L}^\infty(E_o(A), \mathcal{B}(E_o(A))).$$

Remark. This equality is established only on the basis of the assumption that $f_1 \perp f_2$. Under this assumption it is easy to infer that μ is subcentral.

Let $e_i : H_{f_1} \oplus H_{f_2} \to H_{f_i}$, $i = 1,2$, be the canonical projections. Of course, we have $e_1, e_2 \in \pi(A)'$. From $f_1 \flat f_2$ we now infer that

$$(2) \qquad \pi(A)' = \pi_{f_1}(A)' \oplus \pi_{f_2}(A)'$$

and

$$(3) \qquad \pi(A)'' = \pi_{f_1}(A)'' \oplus \pi_{f_2}(A)''$$

(see [22], Theorem 3.8.11.). We infer that

$$e_1, e_2 \in \pi(A)' \cap \pi(A)''.$$

From Lemma 6 and Lemma 7 we now infer that there exists a Borel measurable subset $M \subset E_o(A)$, such that

$$\mu_{f_1}(M) = 1,\ \mu_{f_2}(M) = 0.$$

We infer that

(4) $K_{\mu_{f_1}}(\chi_M) = 1_{H_{f_1}}$, $K_{\mu_{f_2}}(\chi_M) = o.$

From (1) we immediately infer that the measure $t\mu_{f_1} + (1-t)\mu_{f_2}$ is orthogonal; whereas from (2) and (3) it follows that this measure is subcentral. Moreover, we have

$$(5)\quad \pi(A)'\cap\pi(A)'' = (\pi_{f_1}(A)'\cap\pi_{f_1}(A)'')\oplus(\pi_{f_2}(A)'\cap\pi_{f_2}(A)'').$$

For any $\varphi_1,\varphi_2 \in \mathcal{L}^{\infty}(E_o(A);\ \mathcal{B}(E_o(A)))$, let us define

$$\varphi = \varphi_1\chi_M + \varphi_2\chi_M\ .$$

From (4) we infer that $K_{\mu_1}(\varphi) = K_{\mu_1}(\varphi_1)$, $K_{\mu_2}(\varphi) = K_{\mu_2}(\varphi_2)$; hence, by taking into account (1), we infer that μ is central (we have denoted $\mu = t\mu_{f_1} + (1-t)\mu_{f_2}$). Of course, $b(\mu) = f$.

b) On account of equalities (4), it is obvious that we can assume that

$$(6)\quad \Phi_{f_1}(z)\chi_M = \Phi_{f_1}(z),\quad \Phi_{f_2}(z)\chi_{\complement M} = \Phi_{f_2}(z),\qquad z\in Z(A^{**}).$$

From the equalities (6) it is obvious that the mapping

$$Z(A^{**})\ni z\mapsto \Phi_1(z) + \Phi_2(z) \in \mathcal{L}^{\infty}(\mu),$$

is a multiplicative $*$-homomorphism, easily shown to be surjective, onto $L^{\infty}(\mu)$. On the other hand it is obvious that

$$f(za) = \int_{E_o(A)}(\Phi_1(z) + \Phi_2(z))\lambda_A(a)\, d\mu,\quad a\in A,\ z\in Z(A^{**}).$$

From Theorem 6 we infer that $\Phi_f = \Phi_{f_1} + \Phi_{f_2}$, and the Theorem is proved.

<u>Remark</u>. Since the measures μ_{f_1} and μ_{f_2} are mutually singular, there is a canonical isomorphism (for $o<t<1$)

$$j: L^{\infty}(t\mu_{f_1} + (1-t)\mu_{f_2}) \longrightarrow L^{\infty}(\mu_{f_1})\oplus L^{\infty}(\mu_{f_2}).$$

Then part b) of the Theorem is better expressed as follows

$$j\circ\Phi_f = \Phi_{f_1}\oplus\Phi_{f_2}.$$

§3. THE CENTRAL TOPOLOGY

In this section we shall introduce the central topology on the set $F(A)$ of the factorial states of any C^*-algebra A.

I. A subset $F \subset E_o(A)$ will be said to be centrally extremal (Z-extremal, for short), if the relations

$$f \in F \cap E(A), f = tf_1 + (1-t)f_2, o < t < 1, f_1 \,\delta\, f_2, \quad f_1, f_2 \in E_o(A)$$

imply $f_1, f_2 \in F$.

Of course, any extremal subset of $E_o(A)$, and any orthogonally extemal subset of $E_o(A)$, are Z-extremal (see [28], p.141, for the definition of the orthogonally extremal subsets; and also, the Remark below). Also, it is obvious that any subset of $F(A)$ is Z-extremal.

THEOREM 8. For any compact subset $F \subset E_o(A)$ the following are equivalent:

a) F is Z-extremal;

b) for any subcentral measure $\mu \in \mathcal{M}_+^1(E_o(A))$, such that $b(\mu) \in F \cap E(A)$, we have $\mu(F) = 1$.

Proof. a) $\Rightarrow$ b). By way of contradiction, we shall assume that μ is a subcentral measure in $\mathcal{M}_+^1(E_o(A))$, such that $b(\mu) \in F \cap E(A)$, and $\mu(F) < 1$. Then we have $\mu \neq \varepsilon_{b(\mu)}$ (the Dirac measure at $b(\mu)$), and therefore, $\mathrm{supp}\,\mu$ has at least two points. If $\mathrm{supp}\,\mu \subset F$, then $\mu(F) = 1$, and this contradicts the assumption. It follows that $\mathrm{supp}\,\mu \not\subset F$. Let then $f_o \in (\mathrm{supp}\,\mu) \setminus F$, and let $K_1 \subset E_o(A)$ be a compact convex neighbourhood of f_o, such that

$$K_1 \cap F = \emptyset \quad \text{and} \quad f_1 \in \mathrm{int}\, K_1.$$

Of course, we have $\mu(K_1) > o$. If $\mu(K_1) = 1$, then we would have $b(\mu) \in K_1$; hence, $b(\mu) \notin F$, a contradiction. Therefore, we have $o < \mu(K_1) < 1$. Let us define

$$\mu_1 = \mu(K_1)^{-1} \chi_{K_1} \mu$$

and

$$\mu_2 = (1-\mu(K_1))^{-1} \chi_{\complement K_1} \mu \quad .$$

Since μ is subcentral, by Lemma 1 we have

$$b(\mu_1) \,\delta\, b(\mu_2)$$

and, of course, from $\mu = \mu(K_1)\mu_1 + (1-\mu(K_1))\mu_2$, we get

$$b(\mu) = \mu(K_1)b(\mu_1) + (1-\mu(K_1))b(\mu_2) \in F \cap E(A),$$

whence, by virtue of a), we infer that $b(\mu_1), b(\mu_2) \in F$. Since it is obvious that $b(\mu_1) \in K_1$, we arrived at a contradiction.

b) $\Rightarrow$ a). Let $f_1, f_2 \in E_o(A)$ be such that $f_1 \flat f_2$ and assume that there exists a $t \in (o,1)$, such that

$$tf_1 + (1-t)f_2 \in F \cap E(A);$$

then the measure $\mu = t\varepsilon_{f_1} + (1-t)\varepsilon_{f_2}$ is subcentral ($f_1 \flat f_2$ implies that $f_1 \perp f_2$; hence, μ is orthogonal. Apply then Godement's Theorem). Since

$$b(\mu) = tf_1 + (1-t)f_2 \in F \cap E(A),$$

we infer that $\mu(F) = 1$, and, therefore, $f_1, f_2 \in F$. The Theorem is proved.

Remark. We can strengthen the concept of Z-extremality of a subset $F \subset E_o(A)$ as follows: we shall say that F is measure Z-extremal (or Z-extremal in measure) if:

a) F is a universally measurable subset of $E_o(A)$;

and

b) For any subcentral measure $\mu \in \mathcal{M}_+^1(E_o(A))$, such that $b(\mu) \in F \cap E(A)$, we have $\mu(F) = 1$.

We recall that in ([28], p.141) we introduced the notion of an orthogonally extremal (ω-extremal, for short) subset $F \subset E_o(A)$; namely, F is said to be ω-extremal if $f_o \in F \cap E(A)$, $f_o = tf_1 + (1-t)f_2$, $o<t<1$, $f_1, f_2 \in E_o(A)$, $f_1 \perp f_2$, implies that $f_1, f_2 \in F$. A subset $F \subset E_o(A)$ is measure ω-extremal if F is universally measurable and if for any orthogonal Radon probability measure μ on $E_o(A)$ we have the implication

$$b(\mu) \in F \cap E(A) \Rightarrow \mu(F) = 1.$$

In ([33], p.8) we introduced the notion of a maximally orthogonally extremal subset $F \subset E_o(A)$ (Ω-extremal, for short), as being any compact subset $F \subset E_o(A)$, such that for any maximal orthogonal Radon probability measure μ on $E_o(A)$ one should have that

$$b(\mu) \in F \cap E(A) \Rightarrow \mu(F) = 1.$$

Of course, this notion can be extended a little, by considering universally measurable subsets $F \subset E_o(A)$, for which the same implication should hold; we obtain then the notion of a measure Ω-extremal subset.

It is easy to see that any ω-extremal subset $F \subset E_o(A)$ is Z-extremal; and also, any measure ω-extremal subset is measure Z-extremal.

We shall denote by $Z(E_o(A))$ the set of all compact Z-extremal subsets of $E_o(A)$.

It is obvious that:

i) any finite union of elements in $Z(E_o(A))$ is again in $Z(E_o(A))$;

ii) any intersection of elements in $Z(E_o(A))$ is again in $Z(E_o(A))$;

iii) for any $f_o \in F(A)$ the set $\{f_o\}$ belongs to $Z(E_o(A))$.

We immediately infer that the set

$$\hat{Z}(F(A)) = \{ F \cap F(A) \ ; \ F \in Z(E_o(A)) \}$$

is the set of all closed subsets of $F(A)$, for a topology on $F(A)$, which we shall call the central topology of $F(A)$.

It is obvious from iii) that $F(A)$, endowed with the central topology, is a (T_1)-space.

II. Let $F \subset E_o(A)$ be any subset. We shall say that $f_o \in F$ is a Z-extremal point of F if there is no decomposition of the form

$$f_o = tf_1 + (1-t)f_2 \ , \ o < t < 1, \ f_1 \between f_2 \ , \ \ f_1, f_2 \in F.$$

It is obvious that any extremal point of F is Z-extremal. (with the exception of 0, if $0 \in F$!).

We shall denote by $ex_Z\, F$ the set of all Z-extremal points of F.

THEOREM 9. a) For any subset $F \subset E_o(A)$ we have $(ex\, F) \setminus \{0\} \subset ex_Z\, F$;
b) For any Z-extremal subset $F \subset E_o(A)$ we have

$$(\ ex_Z\, F\ ) \cap E(A) = F \cap F(A).$$

Proof. a) Obvious.
b) Let $f_o \in (ex_Z\, F) \cap E(A)$; then $f_o \in F$ and $f_o \in E(A)$. If $f_o \notin F(A)$, then there exists a decomposition of the form

$$f_o = tf_1 + (1-t)f_2,$$

where $t \in (o,1)$, $f_1 \between f_2$ and $f_1, f_2 \in E(A)$. It follows that $f_1, f_2 \in F$, from the Z-extremality of F, and this contradicts the Z-extremality of f_o.

Conversely, if $f_o \in F \cap F(A)$, then it is obvious that $f_o \in (ex_Z F) \cap E(A)$. The Theorem is proved.

THEOREM 10. For any compact Z-extremal subset $F \subset E_o(A)$, such that $F \cap E(A) \neq \emptyset$, we have also $F \cap F(A) \neq \emptyset$.

Proof. By the Converse Milman Theorem, we have that $ex\ \overline{co}(F) \subset F$. If we had $\| f \| < 1$, for any $f \in ex\ \overline{co}(F)$, then from the Strict Minimum Principle (see [28], Theorem 1.2) we would infer that $\| f \| < 1$, for any $f \in \overline{co}(F)$; hence, $\| f \| < 1$, for any $f \in F$, a contradiction. We infer that there exists a $f_o \in ex\ \overline{co}(F) \cap E(A)$. But then we have $f_o \in (ex\ F) \cap E(A) \subset (ex_Z F) \cap E(A) = F \cap F(A)$. The Theorem is proved.

THEOREM 11. F(A) is quasi-compact for the central topology if, and only if, A possesses the unit element.

Proof. a) Assume that $1 \in A$, and let $(F_\alpha)_\alpha$ be a decreasing net of compact Z-extremal subsets of $E_o(A)$, such that

$$(1) \qquad F_\alpha \cap F(A) \neq \emptyset, \qquad \forall \alpha .$$

Then $F_o \stackrel{def}{=} \bigcap_\alpha F_\alpha$ is a compact Z-extremal subset of $E_o(A)$, and $F_o \cap E(A) \neq \emptyset$, because E(A) is compact. Since $F_o \cap E(A)$ is a non-empty compact Z-extremal subset of E(A), from Theorem 9 we infer that $F_o \cap F(A) \neq \emptyset$, and this shows that $(F(A); \hat{Z}(F(A)))$ is quasi-compact.

b) Conversely, assume that $(F(A); \hat{Z}(F(A)))$ is quasi-compact. We shall prove that A has the unit element by adapting the proof of ([28], Proposition 3.19).

b') A possesses a strictly positive element. Indeed, for any $a \in A^+$ let us consider the subset $F(a) = \{ f \in E_o(A)\ ;\ f(a) = o \}$. It is obvious that F(a) is a compact face of $E_o(A)$. If A does not possess a strictly positive element, then $F(a) \cap P(A) \neq \emptyset$, for any $a \in A^+$. Since we have

$$F(a_1 + \ldots + a_n) = \bigcap_{i=1}^{n} F(a_i) , \quad a_1, \ldots, a_n \in A^+ ,$$

we infer that

$$\emptyset = \{0\} \cap F(A) = (\bigcap_{a \in A^+} F(a)) \cap F(A) \neq \emptyset,$$

and this is a contradiction. Assuming that $A \neq \{0\}$, let then $a_o \in A^+$, $\| a_o \| = 1$, be a strictly positive element of A.

b") For any $\alpha \in (o,1)$, let us consider the function $\varphi_\alpha : [o,1] \to [o,1]$, given by

$$\varphi_\alpha(t) = \begin{cases} 0, & 0 \le t \le \alpha \\ (1-\alpha)^{-1}(t-\alpha), & \alpha \le t \le 1. \end{cases}$$

Let $A_0 \subset A$ be the C^*-subalgebra generated by a_0; then we have $b \overset{def}{=} \varphi_\alpha(a_0) \in A_0$ and, by the Gelfand-Naimark Representation Theorem, A_0 can be identified to the C^*-algebra $C_0(\mathfrak{M})$ of all continuous complex functions defined on the compact subset $\mathfrak{M} \subset [0,1]$, which vanish at 0 (we can always assume that $0 \in \mathfrak{M}$). For any $\alpha \in (0,1)$ we have

$$(1) \qquad F(b_\alpha) \subset \{ f \in E_0(A) ; \; f(a_0) \le \alpha \| f \| \}.$$

Indeed, if $f \in E(A)$, then $f \backslash A_0$ is a Radon probability measure μ on $A_0 \cong C_0(\mathfrak{M})$ (take into account the fact that since a_0 is strictly positive in A, the sequence $(a_0^{1/n})_{n \ge 1}$ is an approximate unit of A). If $f \in F(b_\alpha) \cap E(A)$, then

$$(2) \qquad 0 = f(b_\alpha) = \int_{\mathfrak{M}} \tilde{b}_\alpha \, d\mu \; ; \qquad \alpha \in (0,1),$$

where $\tilde{b}_\alpha$ is the Gelfand transform of b_α. From (2) we infer that $\tilde{b}_\alpha \backslash \text{supp} = 0$; i.e., we have

$$(3) \qquad \varphi_\alpha(\tilde{a}_0(m)) = 0, \qquad m \in \text{supp}\, \mu.$$

From (3) we infer that

$$\tilde{a}_0(m) \le \alpha, \qquad m \in \text{supp}\, \mu,$$

and, therefore,

$$f(a_0) = \int_{\mathfrak{M}} \tilde{a}_0 \, d\mu \le \alpha,$$

whence (1) immediately follows.

b''') For any $f_0 \in F(A)$ we have $f_0 \notin F(b_{\frac{1}{2}f_0(a_0)})$. Indeed, we have $f_0(a_0) > 0$ and, therefore,

$$f_0 \in \{ f \in E_0(A) ; \; f(a_0) > \tfrac{1}{2} f_0(a_0) \| f \| \}.$$

The assertion now immediately follows from b'').

b'''') $0 < \alpha_1 \le \alpha_2 < 1 \Rightarrow F(b_{\alpha_1}) \subset F(b_{\alpha_2})$. Indeed, we have $\varphi_{\alpha_1} \ge \varphi_{\alpha_2}$.

b'''')Let us now assume that a_o is not invertible in A.Then,for any $\alpha \in (o,1)$ there exists an $m_\alpha \in \mathfrak{M}\cap(o,\alpha)$, such that $\tilde{a}_o(m_\alpha) = m_\alpha > o$, $\tilde{b}_\alpha(m_\alpha) = o$.Let $p_o \in P(A)$ be such that $p_o|A_o$ be the homomorphism $A_o \ni a \to \tilde{a}(m_\alpha)$. We then have $p_o(b_\alpha) = o$;i.e., $p_o \in F(b_\alpha)$.It follows that

$$F(b_\alpha)\cap F(A) \supset F(b_\alpha)\cap P(A) \neq \emptyset, \qquad \alpha \in (o,1).$$

From b'''') we now infer that

$$(4) \qquad \Big(\bigcap_{\alpha\in(o,1)} F(b_\alpha)\Big)\cap F(A) \neq \emptyset,$$

by taking into account the assumption that F(A) is quasi-compact in the central topology;but relation (4) is in contradiction to b''').It follows that a_o is invertible,and the Theorem is proved.

§4.STABLE PROJECTIONS AND LIFTINGS OF MEASURES

The theory developed in this section is inspired by the following observation from ([26],p. 5):

Let X_1,X_2 be compact spaces and $r: X_1 \to X_2$ a continuous mapping. Let $\mu_1 \in \mathcal{M}^1_+(X_1)$ and define $\mu_2 = r_*(\mu_1) \in \mathcal{M}^1_+(X_2)$.Then the mapping

$$\mathcal{L}^1(X_2, \mathcal{B}(X_2)) \ni \varphi \mapsto \varphi \circ r \in \mathcal{L}^1(X_1, \mathcal{B}(X_1))$$

induces a <u>bijective</u> isomorphism

$$L^1(X_2,\mu_2) \ni [\varphi] \mapsto [\varphi \circ r] \in L^1(X_1,\mu_1)$$

if and only if, $\mu_1 \in \mathrm{ex}\ r_*^{-1}(\{\mu_2\})$.

I.Let (X,Σ) be any <u>measurable space</u> (i.e., Σ is a σ-algebra of subsets of the set X).We shall denote by $\mathcal{L}(X,\Sigma)$ the algebra of all Σ-measurable complex functions defined on X,whereas $\mathcal{L}^\infty(X,\Sigma)$ will stand for the commutative C^*-algebra of all <u>bounded</u> functions in $\mathcal{L}(X,\Sigma)$,endowed with the sup-norm.

If μ is any probability measure on Σ,for any given $p\in[1,+\infty]$,we shall denote by $\mathcal{L}^p(X,\Sigma,\mu)$ the <u>semi-normed</u> complex vector space of all functions $f \in \mathcal{L}(X,\Sigma)$,such that $|f|^p$ is μ-integrable,if $1 \le p < +\infty$ in which case the semi-norm is given by

$$\|f\|_p = \left(\int_X |f(x)|^p \, d\mu(x)\right)^{1/p}, \qquad f \in \mathcal{L}^p(X,\Sigma,\mu);$$

for $p = +\infty$, we shall define

$$\mathcal{L}^\infty(X,\Sigma,\mu) = \mathcal{L}^\infty(X,\Sigma)$$

and the semi-norm will be given by

$$\|f\|_\infty = \mu\text{-vrai sup}\{|f(x)| \; ; \; x \in X\} ;$$

i.e., modulo the measure μ.

We shall denote by $L^p(X,\Sigma,\mu)$ the corresponding Banach spaces, obtained by identifying two functions $f_1, f_2 \in \mathcal{L}^p(X,\Sigma,\mu)$, such that $\|f_1 - f_2\|_p = 0$. In this case, we shall also write $f_1 \sim f_2 (\bmod \mu)$.

We shall denote by $[f]$, or $C_p(f)$, the class of $f \in \mathcal{L}^p(X,\Sigma,\mu)$ in $L^p(X,\Sigma,\mu)$, thus obtaining the canonical mapping

$$C_p \; \mathcal{L}^p(X,\Sigma,\mu) \longrightarrow L^p(X,\Sigma,\mu), \qquad p \in [1,+\infty].$$

We shall denote by $\Sigma(\mu)$ the <u>Lebesgue completion</u> of Σ with respect to μ:

$$\Sigma(\mu) = \{X_0 \in \mathcal{P}(X) \; ; \; \exists \, X_1, X_2 \in \Sigma, \text{ such that } X_0 \Delta X_1 \subset X_2 \text{ and } \mu(X_2) = 0\} .$$

We shall denote by $\mathcal{M}^1_+(X,\Sigma)$ the convex set of all probability measures $\mu : \Sigma \to [0,1]$.

II. Let Σ_1, Σ_2 be two σ-algebras of subsets of the set X, such that $\Sigma_1 \subset \Sigma_2$, and let μ_1 be a probability measure on Σ_1. The problem of extending μ_1 to a (probability) measure μ_2 on Σ_2 is very difficult, in most cases, and it depends on deep set-theoretical properties.

Assuming that such an extension exists (we could start with a given probability measure μ_2 on Σ_2 and consider the restriction $\mu_1 \overset{\text{def}}{=} \mu_2 | \Sigma_1$ of μ_2 to Σ_1), we obviously have the inclusions

a) $\qquad \mathcal{L}(X,\Sigma_1) \subset \mathcal{L}(X,\Sigma_2),$

b) $\qquad \mathcal{L}^p(X,\Sigma_1,\mu_1) \subset \mathcal{L}^p(X,\Sigma_2,\mu_2), \qquad p \in [1,+\infty].$

Of course, the semi-norm $\|\cdot\|_p$ on $\mathcal{L}^p(X,\Sigma_1,\mu_1)$ is the restriction to $\mathcal{L}^p(X,\Sigma_1,\mu_1)$ of the semi-norm $\|\cdot\|_p$ on $\mathcal{L}^p(X,\Sigma_2,\mu_2)$, for any $p\in[1,+\infty]$. We then have the commutative diagram

$$\begin{array}{ccc} \mathcal{L}^p(X,\Sigma_1,\mu_1) & \xrightarrow{\;i_p\;} & \mathcal{L}^p(X,\Sigma_2,\mu_2) \\ C'_p \downarrow & & \downarrow C''_p \\ L^p(X,\Sigma_1,\mu_1) & \xrightarrow[\;j_p\;]{} & L^p(X,\Sigma_2,\mu_2) \end{array}$$

where i_p is a proper inclusion, whereas j_p is an isometry <u>into</u> its co-domain, which can be correctly and uniquely determined by the commutativity condition

$$j_p\circ C'_p = C''_p\circ i_p\,, \qquad\qquad p\in[1,+\infty].$$

We shall say that the extension $\mu_1\mapsto\mu_2$, or the restriction $\mu_2\mapsto\mu_1$, are <u>stable</u> iff j_p is <u>onto</u>; i.e., j_p is an isomorphism of Banach spaces.

Standard examples of stable extensions are:

<u>Example 1</u>. The Lebesgue completion of any probability space (X,Σ,μ)

<u>Example 2</u>. The Radon (i.e., regular Borel) extension of any Baire probability space $(X,\mathcal{B}_0(X),\mu)$, on any compact space.

We remark here that the second example is not reducible to the first; more precisely, there exist compact spaces X and Radon probability measures μ on X, such that the Lebesgue completion of the restriction of μ to $\mathcal{B}_0(X)$ does not contain $\mathcal{B}(X)$.

III. Let us now consider a measurable space (X_1,Σ_1), a set X_2 and a mapping $r:X_1\to X_2$. We can define the (full) <u>direct image</u> $r_*(\Sigma_1)$ of Σ_1 by r, given by

$$r_*(\Sigma_1) = \{\, B\subset X_2\ ;\ r^{-1}(B)\in\Sigma_1 \,\};$$

it is obvious that $r_*(\Sigma_1)$ is a σ-algebra of subsets of X_2. On $r_*(\Sigma_1)$ we can define the (full) <u>direct image</u> $r_*(\mu_1)$ of any probability measure μ_1, given on Σ_1, by the formula

$$r_*(\mu_1)(B) = \mu_1(r^{-1}(B)), \qquad\qquad B\in r_*(\Sigma_1).$$

We shall say that $r_*(\mu_1)$ is the <u>projection</u> of μ_1 by r.

If Σ_2 is a given σ-algebra of subsets of X_2, then r is said to be (Σ_1,Σ_2)-<u>measurable</u> if

$$\Sigma_2 \subset r_*(\Sigma).$$

Remark. Given a probability measure μ_1 on Σ_1, and a (Σ_1,Σ_2)-measurable mapping $r:X_1 \to X_2$, one usually considers that $r_*(\mu_1)$ is defined on Σ_2 only, by restricting $r_*(\mu_1)$ to Σ_2, but we prefer this, more general, setting.

Now, given a probability measure μ_2 on Σ_2, and a (Σ_1,Σ_2)-measurable mapping, one could put the problem of finding a probability measure μ_1 on Σ_1, such that $r_*(\mu_1) \setminus \Sigma_2 = \mu_2$. Such a measure μ_1 will be called a lifting of μ_2 by r.

Of course, the restriction of a probability measure to a sub-σ-algebra of sets is a particular case of a projection; whereas the extension of a probability measure, to a larger σ-algebra of sets, is a particular case of a lifting. This shows that, in contrast to the case of a projection, the lifting of a measure is not always possible. Therefore, the following well-known result is quite remarkable :

Let $r : X_1 \to X_2$ be a continuous mapping between compact spaces. Then the following are equivalent :

a) r is surjective;

b) any Radon probability measure on X_2 can be lifted to a Radon probability measure on X_1.

(see [26], p.5; [28], Lemma 2.1).

IV. Given a (Σ_1,Σ_2)-measurable mapping $r : X_1 \to X_2$ between two measurable spaces (X_1,Σ_1) and (X_2,Σ_2), and a probability measure μ_1 on Σ_1, let us consider the probability measure $\mu_2 = r_*(\mu_1) \setminus \Sigma_2$. We can then obviously define the mapping

$$r_p : \mathcal{L}^p(X_2,\Sigma_2,\mu_2) \longrightarrow \mathcal{L}^p(X_1,\Sigma_1,\mu_1), \qquad 1 \le p \le +\infty$$

by the formula $r_p(f) = f \circ r$, $f \in \mathcal{L}^p(X_2,\Sigma_2,\mu_2)$.

It is obvious that we can now obtain the commutative diagram

$$\begin{array}{ccc} \mathcal{L}^p(X_2,\Sigma_2,\mu_2) & \xrightarrow{r_p} & \mathcal{L}^p(X_1,\Sigma_1,\mu_1) \\ C''_p \downarrow & & \downarrow C'_p \\ L^p(X_2,\Sigma_2,\mu_2) & \xrightarrow[q_p]{} & L^p(X_1,\Sigma_1,\mu_1) \end{array}$$

where q_p is uniquely determined by the commutativity condition

$$q_p \circ C''_p = C'_p \circ r_p, \qquad p \in [1,+\infty].$$

It is easy to show that q_p is an isometry into its codomain.

We shall say that μ_2 is a stable projection of μ_1 (induced by the mapping $r : X_1 \to X_2$), or, equivalently, that μ_1 is a stable lifting of μ_2, iff the mappings q_p are onto, for all $p\in[1,+\infty]$.

THEOREM 12. a) For any $\mu_1\in\mathcal{M}_+^1(X_1,\Sigma_1)$ and any $p\in[1,+\infty]$, the mapping

$$r_p:\mathcal{L}^p(X_2,\Sigma_2,\mu_2)\ni f\longmapsto f\circ r\in\mathcal{L}^p(X_1,\Sigma_1,\mu_1)$$

induces an isometry $q_p:L^p(X_2,\Sigma_2,\mu_2)\longrightarrow L^p(X_1,\Sigma_1,\mu_1)$.

b) q_∞ is a C^*-algebra homomorphism.

c) The following statements are equivalent:

i) μ_1 is a stable lifting of μ_2;

ii) there exists a $p\in[1,+\infty]$, such that q_p is surjective;

iii) $\mu_1\in\mathrm{ex}\, r_*^{-1}(\{\mu_2\})$.

(Here $r:X_1\to X_2$ is a given (Σ_1,Σ_2)-measurable mapping and $\mu_2 = r_*(\mu_1)\setminus\Sigma_2$; r_* is the direct image mapping $r_*:\mathcal{M}_+^1(X_1,\Sigma_1)\to\mathcal{M}_+^1(X_2,\Sigma_2)$.

Proof. Statements a) and b) are obvious.

c) It is obvious that i) $\Rightarrow$ ii).

ii) $\Rightarrow$ iii). Indeed, let $p\in[1,+\infty]$ be such that q_p be surjective. Let $\mu_1 = \frac{1}{2}(\mu_1' + \mu_1'')$ be a decomposition of μ_1, such that $\mu_1',\mu_1''\in r_*^{-1}(\{\mu_2\})$; i.e.,

$$r_*(\mu_1')\setminus\Sigma_2 = r_*(\mu_1'')\setminus\Sigma_2 = \mu_2.$$

Let $g\in\mathcal{L}^p(X_1,\Sigma_1,\mu_1)$ be given. Then there exists a $f\in\mathcal{L}^p(X_2,\Sigma_2,\mu_2)$, such that $f\circ r\sim g \pmod{\mu_1}$. We then infer that $f\circ r\sim g \pmod{\mu_1'}$, and also, that $f\circ r\sim g\pmod{\mu_1''}$. We obtain

$$\int_{X_1} g\, d\mu_1' = \int_{X_1}(f\circ r)\, d\mu_1' = \int_{X_2} f\, d\mu_2 = \int_{X_1}(f\circ r)\, d\mu_1'' = \int_{X_1} g\, d\mu_1'',$$

for any $g\in\mathcal{L}^p(X_1,\Sigma_1,\mu_1)$, and this shows that $\mu_1' = \mu_1''$; hence, we have that $\mu_1\in\mathrm{ex}\, r_*^{-1}(\{\mu_2\})$.

iii) $\Rightarrow$ i). We shall first prove that q_1 is surjective. Indeed, if $\mathrm{im}\, q_1 \neq L^1(X_1,\Sigma_1,\mu_1)$, then, by the Hahn-Banach Theorem, there exists a $[g]\in(\mathrm{im}\, q_1)^\perp$, $[g]\neq o$. It is obvious that we can choose $[g]$, such that g be real, and $|g(x)|\le 1, x\in X_1$. We then have

$$\int_{X_1}(f\circ r)g\, d\mu_1 = o, \qquad f\in\mathcal{L}^1(X_2,\Sigma_2,\mu_2),$$

and, in particular,

$$\int_{X_1} g \, d\mu_1 = 0.$$

Let us define $\mu_1' = (1+g)\mu_1$, $\mu_1'' = (1-g)\mu_1$. We have $\mu_1', \mu_1'' \in \mathcal{M}_+^1(X_1, \Sigma_1)$, and for any $f \in \mathcal{L}^1(X_2, \Sigma_2, \mu_2)$ we have

$$\int_{X_2} f \, d\mu_2 = \int_{X_1} (f \circ r) \, d\mu_1 = \int_{X_1} (f \circ r)(1+g) \, d\mu_1 = \\ = \int_{X_1} (f \circ r) \, d\mu_1' ,$$

$$\int_{X_2} f \, d\mu_2 = \int_{X_1} (f \circ r) \, d\mu_1 = \int_{X_1} (f \circ r)(1-g) \, d\mu_1 = \\ = \int_{X_1} (f \circ r) \, d\mu_1'' ,$$

and this shows that $r_*(\mu_1') = \mu_2 = r_*(\mu_1'')$. It follows that μ_1', $\mu_1'' \in r_*^{-1}(\{\mu_2\})$, and since we have $\mu_1 = \frac{1}{2}(\mu_1' + \mu_1'')$, we infer that $\mu_1' = \mu_1 = \mu_1''$. We infer that $[g] = 0$ in $L^\infty(X_1, \Sigma_1, \mu_1)$, and this is a contradiction.

Let us now remark that the mapping q_∞ is (w^*, w^*)-continuous. Indeed, for any $f \in \mathcal{L}^1(X_2, \Sigma_2, \mu_2)$ and any $g \in \mathcal{L}^\infty(X_2, \Sigma_2, \mu_2)$ we have

$$\int_{X_2} fg \, d\mu_2 = \int_{X_1} (f \circ r)(g \circ r) \, d\mu_1$$

and, since the mapping q_1 is surjective, we infer that if $[g] \to 0$ in $\sigma(L^\infty(X_2, \Sigma_2, \mu_2); L^1(X_2, \Sigma_2, \mu_2))$, then $q_\infty([g]) \to 0$ in the topology $\sigma(L^\infty(X_1, \Sigma_1, \mu_1); L^1(X_1, \Sigma_1, \mu_1))$.

Since q_∞ is an isometry, it follows that the closed unit ball in im q_∞ is the image by q_∞ of the closed unit ball in $L^\infty(X_2, \Sigma_2, \mu_2)$; hence, it is w^*-compact, by the Alaoglu-Bourbaki Theorem.

From the Krein-Šmulian Theorem we now infer that im q_∞ is w^*-closed in $L^\infty(X_1, \Sigma_1, \mu_1)$. If im $q_\infty \neq L^\infty(X_1, \Sigma_1, \mu_1)$, then we could find an $f_1 \in \mathcal{L}^1(X_1, \Sigma_1, \mu_1)$, such that $[f_1] \neq 0$, and

$$\int_{X_1} f_1 (g \circ r) \, d\mu_1 = 0, \qquad g \in \mathcal{L}^\infty(X_2, \Sigma_2, \mu_2).$$

Since q_1 is surjective, we infer that there exists an $f_2 \in \mathcal{L}^1(X_2, \Sigma_2, \mu_2)$ such that $q_1([f_2]) = [f_1]$, and, therefore, we had

$$0 = \int_{X_1} f_1 (g \circ r) \, d\mu_1 = \int_{X_1} (f_2 \circ r)(g \circ r) \, d\mu_1 = \int_{X_2} f_2 g \, d\mu_2 ,$$

for any $g\in\mathcal{L}^\infty(X_2,\Sigma_2,\mu_2)$. This implies that $[f_2]=0$ and, therefore, $[f_1]=0$, contrary to the assumption.

We have thus proved that q_∞ is surjective. Let us now remark that for any $A_1\in\Sigma_1$ there exists an $A_2\in\Sigma_2$, such that $\chi_{A_1}\sim\chi_{r^{-1}(A_2)} \pmod{\mu_1}$ Indeed, since q_∞ is surjective, there exists a $g\in\mathcal{L}^\infty(X_2,\Sigma_2,\mu_2)$, such that

$$0=\int_{X_1}|\chi_{A_1}-g\circ r|\,d\mu=\int_{A_1}|1-g\circ r|\,d\mu_1+\int_{\complement A_1}(|g|\circ r)\,d\mu_1 .$$

Let $A_1'=\{x\in A_1;\ (g\circ r)(x)=1\}$ and $A_1''=\{x\in\complement A_1;\ (g\circ r)(x)=0\}$. We then have $\mu_1(A_1)=\mu_1(A_1')$ and $\mu_1(\complement A_1)=\mu_1(A_1'')$. Let $B_1=\{y\in X_2; g(y)=1\}$ and $B_0=\{y\in X_2;\ g(y)=0\}$. We have

$$A_1'=r^{-1}(B_1)\wedge A_1 \quad \text{and} \quad A_1''=r^{-1}(B_0)\wedge\complement A_1.$$

We infer that we have

$$\int_{X_1}|\chi_{r^{-1}(B_1)}-\chi_{A_1}|\,d\mu_1=\int_{A_1}|\chi_{r^{-1}(B_1)}-1|\,d\mu_1+\int_{\complement A_1}\chi_{r^{-1}(B_1)}d\mu_1=$$
$$=\int_{A_1}|\chi_{A_1'}-1|\,d\mu_1+\int_{A_1''}\chi_{r^{-1}(B_1)}\,d\mu_1=0,$$

and this shows that $\chi_A\sim\chi_{r^{-1}(B_1)} \pmod{\mu_1}$.

Let us now prove that q_p is surjective, for any $p\in[1,+\infty)$. Indeed, if $f\in\mathcal{L}^p(X_1,\Sigma_1,\mu_1)$, then there exists a sequence $(A_n)_{n\geq 0}$ of sets $A_i\in\Sigma_1$, $i\in\mathbb{N}$, such that

α) $A_i\wedge A_j=\emptyset$, for $i\neq j$;

β) $\bigcup_{i\geq 0} A_i = X_1$;

γ) $f\chi_{A_i}\in\mathcal{L}^\infty(X_1,\Sigma_1,\mu_1)$, $\quad i\geq 0$.

We infer that for any $i\in\mathbb{N}$ there exists a $g_i\in\mathcal{L}^\infty(X_2,\Sigma_2,\mu_2)$, such that $g_i\circ r\sim f\chi_{A_i}$. It is easy to prove that the series $\sum_{i\geq 0} g_i$ is norm converging in $\mathcal{L}^p(X_2,\Sigma_2,\mu_2)$. If we define $g=\sum_{i=0}^{\infty} g_i$, we have $g\in\mathcal{L}^p(X_2,\Sigma_2,\mu_2)$, and $r_p([g])=[f]$. The Theorem is proved.

In the same setting as above, we can also prove the following criterion of stability.

<u>THEOREM 13</u>. <u>The following statements are equivalent</u>:

a) <u>μ_1 is a stable lifting of μ_2</u>;

b) <u>for any $A_1\in\Sigma_1$ there exists an $A_2\in\Sigma_2$, such that</u> $\mu_1(A_1\Delta r^{-1}(A_2))=0$.

Proof. The implication a) $\Rightarrow$ b) follows from the fact that the stability of μ_1 with respect to μ_2 implies that q_∞ is surjective, whereas b) $\Rightarrow$ a) follows from the fact that b) implies the surjectivity of q_∞, as one could easily see by approximating any function in $\mathcal{L}^\infty(X_1, \Sigma_1, \mu_1)$ by simple functions, uniformly.

V. The well-known method of introducing measures on possibly non-measurable subsets gives another instance of a stable lifting.

Example 3. Let (X_2, Σ_2, μ_2) be any probability space and consider any subset $X_1 \subset X_2$, such that $\mu_2^*(X_1) = 1$. Define

$$\Sigma_1 = \{ M \cap X_1 \ ; \ M \in \Sigma_2 \} ;$$

obviously, Σ_1 is a σ-algebra of subsets of X_1.

Since $\mu_2^*(M \cap X_1) = \mu_2(M)$, $M \in \Sigma_2$, we can define a probability measure $\mu_1 : \Sigma_1 \to [0,1]$ by the formula

$$\mu_1(M \cap X_1) = \mu_2(M), \qquad M \in \Sigma_2 .$$

If $r: X_1 \to X_2$ is the inclusion mapping, then it is easy to see that μ_1 is a stable lifting of μ_2, by r.

Consider now a stable extension (X_2, Σ_3, μ_3) of (X_2, Σ_2, μ_2), where $\Sigma_2 \subset \Sigma_3$ and $\mu_3 | \Sigma_2 = \mu_2$. We can define correctly a canonical bijective isometry $r_p : L^p(X_2, \Sigma_3, \mu_3) \to L^p(X_1, \Sigma_1, \mu_1)$, for any $p \in [1, +\infty]$, in the following manner.

For any class $[f] \in L^p(X_2, \Sigma_3, \mu_3)$ choose a representative $f \in \mathcal{L}^p(X_2, \Sigma_2, \mu_2)$ and restrict it to X_1; then $[f|X_1]$ is the correctly defined image $r_p'([f])$ of $[f]$, and the mapping r_p' is easily seen to be a bijective isometry.

This setting is often encountered in Choquet Theory and, as an application, in Reduction Theory. Namely, let X_2 be any compact convex set in a Hausdorff locally convex topological real vector space, and let $X_1 = \text{ex}\, X_2$ be its extremal boundary. Let μ_3 be any Choquet maximal Radon probability measure, defined on $\Sigma_3 = \mathcal{B}(X_2)$, and let $\Sigma_2 = \mathcal{B}_0(X_2)$, $\mu_2 = \mu_3 | \Sigma_2$. Then $\mu_2^*(X_1) = 1$, by the Choquet-Bishop-de Leeuw Theorem and the preceding considerations can be applied.

Below we shall encounter another instance of this example, by considering the set of the factorial states of a C^*-algebra, and central measures.

Remark. The indiscriminate restriction mapping

$$\mathcal{L}^p(X_2,\Sigma_3,\mu_3) \ni f \mapsto f|X_1$$

is not legitimate, in general, for defining the mapping r'_p.

The following Theorem shows that there exist maximal stable extensions for any probability measure.

THEOREM 14. *For any probability space* (X_0,Σ_0,μ_0) *there exist maximal stable extensions* (X_0,Σ_1,μ_1).

Proof. Let $\mathcal{S}$ be a totally ordered set of stable extensions (X_0,Σ,μ_Σ) of (X_0,Σ_0,μ_0), such that for any $(X_0,\Sigma',\mu_{\Sigma'}),(X_0,\Sigma'',\mu_{\Sigma''})\in\mathcal{S}$, either $\Sigma'\subset\Sigma''$ and $\mu_{\Sigma''}|\Sigma'=\mu_{\Sigma'}$, or $\Sigma''\subset\Sigma'$ and $\mu_{\Sigma'}|\Sigma''=\mu_{\Sigma''}$.

Of course, we can assume that $(X_0,\Sigma_0,\mu_0)\in\mathcal{S}$ and it is easy to prove that if $(X_0,\Sigma',\mu_{\Sigma'}),(X_0,\Sigma'',\mu_{\Sigma''})\in\mathcal{S}$, and $\Sigma'\subset\Sigma''$, then $(X_0,\Sigma'',\mu_{\Sigma''})$ is a stable extension of $(X_0,\Sigma',\mu_{\Sigma'})$.

We can define on the algebra $\widetilde{\Sigma}=\cup\{\Sigma;\Sigma\in\mathcal{S}\}$ the finitely additive set function $\widetilde{\mu}:\widetilde{\Sigma}\to[0,1]$ by

$$\widetilde{\mu}(M)=\mu_\Sigma(M),\ \text{if}\quad M\in\Sigma,$$

where (X_0,Σ,μ_Σ) is a suitably chosen extension in $\mathcal{S}$ (For any $M\in\widetilde{\Sigma}$ there exists such an extension).

Let us now consider the outer measure $\widetilde{\mu}^*$ corresponding to $\widetilde{\mu}$; i.e., we define

$$\widetilde{\mu}^*(M)=\inf\left\{\sum_{i=0}^{\infty}\widetilde{\mu}(M_i);\ M_i\in\widetilde{\Sigma},\quad\bigcup_{i=0}^{\infty}M_i\supset M\right\},$$

for any $M\subset X_0$. Let $\Sigma_1=\{M\in\mathcal{P}(X_0);\exists\, M_0\in\Sigma_0,$ such that $\widetilde{\mu}^*(M\,\Delta M_0)=0\}$, and define $\mu_1=\mu^*|\Sigma_1$. Then (X_0,Σ_1,μ_1) is a probability space, and it is a stable extension of any $(X_0,\Sigma,\mu_\Sigma)\in\mathcal{S}$. The application of Zorn's Lemma finishes the proof.

§5. STABLE LIFTINGS OF ORTHOGONAL MEASURES

In order to prove the topological properties of the central measures, we have in view, we need to establish the possibility of lifting orthogonal measures as orthogonal measures.

I. Let $\pi : A \to B$ a homomorphism of C^*-algebras. We do not assume them to possess the unit element; whereas, if they possess it, we do not assume that $\pi(1) = 1$.

Let $\pi^* : B^* \to A^*$ be the adjoint mapping and denote by $s : E_o(B) \to E_o(A)$ its restriction-corestriction to $E_o(B)$ and $E_o(A)$.

THEOREM 15. a) $s(E_o(B))$ is a compact face of $E_o(A)$.
b) $s(E(B))$ is a face of $E_o(A)$.
c) If $1 \in B$, then $s(E(B))$ is a compact face of $E_o(A)$.
d) If $1 \in A, 1 \in B$ and $\pi(1) = 1$, then $s(E(B))$ is a compact face of $E(A)$.

Proof. a) Since the mapping s is affine and w^*-continuous, it is obvious that $s(E_o(B))$ is a compact convex subset of $E_o(A)$. Let $f \in s(E_o(B))$ and let $f = tf' + (1-t)f''$, $o < t < 1$, $f', f'' \in E_o(A)$ be a decomposition of f. Then we have $f' \mid \ker \pi_f = f'' \mid \ker \pi_f = o$ and, therefore, there exist $g_o', g_o'' \in \pi(A)^*$, such that $f' = g_o' \circ \pi$, $f'' = g_o'' \circ \pi$. From $\pi(A)^+ = \pi(A^+)$ we infer that $g_o' \geq o$, $g_o'' \geq o$; whereas from $\pi(A)_1^+ = \pi(A_1^+)$ we infer that $\|f'\| = \|g_o'\|$, $\|f''\| = \|g_o''\|$. If we extend g_o' to $g' \in E_o(B)$, and g_o'' to $g'' \in E_o(B)$, we have $s(g') = f' \in s(E_o(B))$ and $s(g'') = f'' \in s(E_o(B))$.
b) If $f \in s(E(B))$, with the same notations as above, since $\|g_o'\| \leq 1$ and $\|g_o''\| \leq 1$, we can find extensions g' of g_o', and g'' of g_o'', such that $\|g'\| = \|g''\| = 1$; then we have $s(g') = f' \in s(E(B))$ and $s(g'') = f'' \in s(E(B))$.
c) If $1 \in B$, then $E(B)$ is a compact subset of $E_o(B)$. It follows that $s(E(B))$ is a compact subset of $E_o(A)$, and a face of $E_o(A)$, by b).
d) Under the assumptions, we have $s(E(B)) \subset E(A)$.

The Theorem is proved.

THEOREM 16. a) For any orthogonal measure $\mu \in \mathcal{M}_+^1(E_o(A))$, such that $b(\mu) \in s(E_o(B)) \cap E(A)$, there exists an orthogonal measure $\nu \in \mathcal{M}_+^1(E_o(B))$, such that $s_*(\nu) = \mu$, and ν be a stable lifting of μ.
b) If μ is maximal orthogonal, then ν can be chosen to be, moreover, maximal orthogonal.

Proof. a) Since, by Theorem 15, $s(E_o(B))$ is a compact face of $E_o(A)$, we have $\mu(s(E_o(B))) = 1$ (see [28], Proposition 1.5). We then infer that $s_*^{-1}(\{\mu\})$ is a non-empty compact convex subset of $\mathcal{M}_+^1(E_o(B))$. Let

$\nu \in \mathrm{ex}\ s_*^{-1}(\{\mu\})$. By Theorem 12, we infer that ν is a stable lifting μ. We shall prove that ν is also orthogonal.

a') If $[\varphi] \in L^\infty(E_o(A), \mathcal{B}(E_o(A)), \mu)$ is a projection, then $q_\infty([\varphi]) \in L^\infty(E_o(B), \mathcal{B}(E_o(B)), \nu)$ is also a projection, and any projection in $L^\infty(E_o(B), \mathcal{B}(E_o(B)), \nu)$ is of this form, on account of the fact that ν is a stable lifting of μ.

Since μ is orthogonal, $P = K_\mu([\varphi])$ is a projection in $\pi_{b(\mu)}(A)'$ (see [28], Theorem 3.1).

From the fact that $s_*(\nu) = \mu$, we infer that $s(b_B(\nu)) = b_A(\mu)$, and this implies that $\| b_B(\nu) \| = 1$, since $1 = \| b_A(\mu) \| \leq \| b_B(\nu) \| \leq 1$.

By ([12], 2.4.9.) we infer that $\pi_{b(\mu)}$ can be identified with $\pi_{b(\nu)} \circ \pi : A \to \mathcal{L}(e_o H_{b(\nu)})$, where e_o is the projection onto the subspace $\overline{\pi_{b(\nu)}(\pi(A))\xi^o_{b(\nu)}}$; whereas $\xi^o_{b(\mu)}$ can be identified with $\xi^o_{b(\nu)}$. By ([28], Lemma 3.1), $Q = K_\nu(q_\infty([\varphi]))$ is an operator in $\pi_{b(\nu)}(\pi(A))'$ such that $0 \leq Q \leq 1$.

a")Let us now prove that

$$(1) \qquad K_\mu([\varphi]) = e_o K_\nu(q_\infty([\varphi]))e_o,$$

for any $[\varphi] \in L^\infty(E_o(A), \mathcal{B}(E_o(A)), \mu)$. Indeed, we have

$$(K_\mu([\varphi])\pi_{b(\mu)}(a)\xi^o_{b(\mu)} | \xi^o_{b(\mu)}) = \int_{E_o(A)} \varphi \lambda_A(a)\, d\mu =$$
$$= \int_{E_o(B)} (\varphi \circ s)(\lambda_A(a) \circ s)\, d\nu = \int_{E_o(b)} (\varphi \circ s)\lambda_B(\pi(a))\, d\nu =$$
$$= (K_\nu([\varphi \circ s])\pi_{b(\nu)}(\pi(a))\xi^o_{b(\nu)} | \xi^o_{b(\nu)}), \qquad a \in A,$$

and this shows that

$$K_\mu([\varphi]) = e_o K_\nu([\varphi \circ s])e_o,$$

as requested.

a"')From formula (1) we infer that $P = e_o Q e_o$ and, therefore, we have

$$e_o Q e_o Q e_o = e_o Q e_o, \quad Q e_o Q \leq Q^2 \leq Q.$$

We infer that

$$e_o(Q - Q e_o Q)e_o = 0,$$

and, therefore, we have that

$$(Q - Qe_oQ)e_o = 0,$$

whence we get that

$$Qe_o = Qe_oQe_o \ , \quad e_oQ = e_oQe_oQ \ .$$

From $Q^2 \leq Q$ we infer that $e_oQ^2e_o \leq e_oQe_o$ and, if we denote

$$R = (Qe_o - e_oQ)^*(Qe_o - e_oQ),$$

we have that

$$0 \leq R = e_oQ^2e_o - e_oQ - Qe_o + Qe_oQ \ .$$

We infer that

$$0 \leq e_oRe_o = e_oQ^2e_o - e_oQe_o - e_oQe_o + e_oQe_oQe_o =$$

$$= e_oQ^2e_o - e_oQe_o \leq 0$$

and this shows that $e_oRe_o = 0$. We infer that $Re_o = 0$ and, therefore,

$$(Qe_o - e_oQ)e_o = 0,$$

whence we get that

$$Qe_o = e_oQe_o \ ;$$

this implies that

(2) $$Qe_o = e_oQ \ .$$

From (2) we infer that $Q^2e_o = Qe_o$ and, since $e_o\xi^o_{b(\nu)} = \xi^o_{b(\nu)}$, we get that

(3) $$Q^2\xi^o_{b(\nu)} = Q\xi^o_{b(\nu)}.$$

Since $Q \in \pi_{b(\nu)}(B)'$, and since $\xi^o_{b(\nu)}$ is cyclic for $\pi_{b(\nu)}(B)$, from (3) we infer that $Q^2 = Q$;i.e.,Q is a projection.

From ([28],Theorem 3.1) we now infer that ν is orthogonal (We have also to take into consideration the fact that ν is a stable lifting of μ).

b)Let us now assume,moreover,that μ is maximal orthogonal on $E_o(A)$ Since the set $b_B(s_*^{-1}(\{\mu\})) \subset E_o(B)$ is compact,convex and non-empty, we can find a $g_o \in \text{ex } b_B(s_*^{-1}(\{\mu\}))$.Let us define

$$M(g_o) = s_*^{-1}(\{\mu\}) \cap b_B^{-1}(\{g_o\}).$$

It is obvious that $M(g_o)$ is a non-empty compact face of $s_*^{-1}(\{\mu\})$,and, therefore,we have

$$\emptyset \neq \text{ex } M(g_o) = M(g_o) \cap (\text{ex } s_*^{-1}(\{\mu\})) .$$

Let us choose $\nu_1 \in \text{ex } M(g_o)$.By part a) of the Theorem, ν_1 is an orthogonal measure on $E_o(B)$.
If $\nu \in \mathcal{M}_+^1(E_o(B))$ is a maximal orthogonal probability measure on $E_o(B)$,such that

$$(4) \qquad \nu_1 \prec \nu \quad ,$$

then $\nu \sim \nu_1$ and,therefore, $b_B(\nu) = b_B(\nu_1) = g_o$ .It follows that $\nu \in$ $\in b_B^{-1}(\{g_o\})$.On the other hand,from (4) we infer that

$$(5) \qquad \mu = s_*(\nu_1) \prec s_*(\nu) \quad ,$$

and,since μ is maximal orthogonal on $E_o(A)$,from (5) it follows that $\mu = s_*(\nu)$;i.e., $\nu \in s_*^{-1}(\{\mu\})$.We infer that $\nu \in M(g_o)$.Since ν is orthogonal,from ([28],Lemma 3.3 and Corollary 1 to Theorem 3.1),we infer that ν is simplicial;i.e.,

$$\nu \in \text{ex } \mathcal{M}_+^1(E_o(B) ;g_o).$$

We then infer that $\nu \in \text{ex } M(g_o)$,and this implies that

$$\nu \in \text{ex } (s_*^{-1}(\{\mu\})) ;$$

hence, ν is a stable extension of μ .The Theorem is proved.

<u>Remark</u>.If we consider the setting of the proof of the preceding

Theorem, since finite linear combinations of projections in the C^*-algebra $L^\infty(E_o(A), \mathcal{B}(E_o(A)), \mu)$ are uniformly dense in this space, from formulas (1) and (2) we infer that $e_o \in \mathcal{C}_\nu'$ and $\mathcal{C}_\mu = \mathcal{C}_\nu e_o$. Moreover, the mapping $\mathcal{C}_\nu \ni c \mapsto ce_o \in \mathcal{C}_\mu$ is an isomorphism between the abelian von Neumann algebras $\mathcal{C}_\nu$ and $\mathcal{C}_\mu$. We also remark that Theorem 16 partly contains a result of Anderson and Bunce (see [2], Theorem 5).

II. We want to mention here two instances of the situation described in the preceding Theorem.

a) Consider an arbitrary C^*-algebra A and let $f_o \in E(A)$ be given. We can consider the representation $\pi_{f_o} : A \to \mathcal{L}(H_{f_o})$. Let $\mathcal{C} \subset \pi_{f_o}(A)'$ be any maximal abelian von Neumann subalgebra, and define B_o to be the C^*-algebra $C^*(\pi_{f_o}(A), \mathcal{C})$, generated by $\pi_{f_o}(A)$ and $\mathcal{C}$. We have then the C^*-homomorphism $\pi : A \to B_o$, obtained by corestricting π_{f_o} to B_o. We then have that

$$B_o' = \pi_{f_o}(A)' \cap \mathcal{C}' = \mathcal{C},$$

and we can consider the mapping $s : E_o(B) \to E_o(A)$, defined as in section 5.1. If we denote $g_o = \omega_{\xi_{f_o}} | B$, then $s(g_o) = f_o$. (For any $\xi \in H$ we denote by ω_ξ the positive linear functional on $\mathcal{L}(H)$ given by $\omega_\xi(x) = (x\xi | \xi)$, $x \in \mathcal{L}(H)$).

Let ν be the central measure on $E_o(B)$, corresponding to g_o: it is the <u>greatest</u> orthogonal measure on $E_o(B)$, whose barycenter is g_o. Of course, $E(B)$ is compact, and $\nu(E(B)) = 1$.

On the other hand, $\mu = s_*(\nu)$ is the <u>maximal</u> orthogonal measure on $E_o(A)$, whose barycenter is f_o, and which corresponds to $\mathcal{C}$ (see [28], Lemma 3.8). By the Corollary to this Lemma, ν is a stable lifting of μ.

b) Consider now an arbitrary C^* algebra A, let $f_o \in E(A)$ be given and define B_{f_o} to be the C^*-algebra $C^*(\pi_{f_o}(A), \pi_{f_o}(A)')$, generated by $\pi_{f_o}(A)$ and $\pi_{f_o}(A)'$. We have

$$B_{f_o}' = \pi_{f_o}(A)' \cap \pi_{f_o}(A)'' \subset B_{f_o};$$

hence, B_{f_o}' is abelian, and it is the center of B_{f_o}.

If $g_o = \omega_{\xi_{f_o}} | B_{f_o}$, then there exists the <u>greatest</u> orthogonal probability measure ν_{g_o} on $E_o(B_{f_o})$, whose barycenter is at g_o, and which corresponds to the <u>greatest</u> abelian von Neumann subalgebra B_{f_o}' of B_{f_o}'.

We can also consider the C^*-homomorphism $\pi : A \to B_{f_o}$, obtained by corestricting π_{f_o}, and the corresponding mapping $s : E_o(B_{f_o}) \to E_o(A)$, given by $s(g) = g\circ\pi$, $g\in E_0(B_{f_0})$. Of course, $E(B_{f_0})$ is compact and $\nu_{g_0}(E(B_{f_0})) = 1$.

The following Theorem extends to the general case of C^*-algebras A, possibly not possessing the unit element, a well-known Theorem (see [24], Proof of Theorem 3.1.8).

<u>THEOREM 17</u>. $s_*(\nu_{g_0}) = \mu_{f_0}$; i.e., $s_*(\nu_{g_0})$ <u>is the central measure on</u> $E_0(A)$, <u>whose barycenter is at</u> f_0.

<u>Proof</u>. a) We shall identify π_{g_0} with the identical representation of B_f and we can assume that $\xi^0_{g_0} = \xi^0_{f_0}$. It is clear that $b(s_*(\nu_{g_0})) = s(b(\nu_{g_0})) = s(g_0) = f_0$.

b) From ([28], Lemma 3.3) we infer that

$$(1)\qquad K_{s_*(\nu_{g_0})}(\|\cdot\|) = 1.$$

c) For any $\varphi\in\mathcal{L}^\infty(E_0(A), \mathcal{B}(E_0(A)), s_*(\nu_{g_0}))$ we have that

$$(K_{s_*(\nu_{g_0})}([\varphi])\,\pi_{f_0}(a)\xi^0_{f_0}|\xi^0_{f_0}) = \int_{E_0(A)}\varphi\,\lambda_A(a)\; ds_*(\nu_{g_0}) =$$

$$= \int_{E_0(B_{f_0})}(\varphi\circ s)(\lambda_A(a)\circ s)\; d\nu_{g_0} = \int_{E_0(B_{f_0})}(\varphi\circ s)\lambda_B(\pi_{f_0}(a))\; d\nu_{g_0} =$$

$$= (K_{\nu_{g_0}}([\varphi\circ s])\,\pi_{f_0}(a)\xi^0_{f_0}|\xi^0_{f_0}), \qquad a\in A,$$

and since $\xi^0_{f_0}$ is cyclic for $\pi_{f_0}(A)$, we infer that

$$(2)\qquad K_{s_*(\nu_{g_0})}([\varphi]) = K_{\nu_{g_0}}(q_\infty([\varphi])),$$

for any $\varphi\in\mathcal{L}^\infty(E_0(A), \mathcal{B}(E_0(A)), s_*(\nu_{g_0}))$. From formula (2) we immediately infer that the measure $s_*(\nu_{g_0})\in\mathcal{M}^1_+(E_0(A);f_0)$ is orthogonal.

Let e be the orthogonal projection on $B_{f_0}'\xi^0_{f_0}\subset H_{f_0}$. From formula (1) and from ([28], Lemma 3.6), we infer that

$$\int_{E_0(A)}\lambda_A(a_1)\lambda_A(a_2)\ldots\lambda_A(a_n)\; ds_*(\nu_{g_0}) =$$

$$= \int_{E_0(B_{f_0})}\lambda_{B_{f_0}}(\pi_{f_0}(a_1))\ldots\lambda_{B_{f_0}}(\pi_{f_0}(a_n))\; d\nu_{g_0} =$$

$$= (K_{\nu_{g_0}}(\lambda_B(\pi_{f_0}(a_1)))\ldots K_{\nu_{g_0}}(\lambda_B(\pi_{f_0}(a_n)))\xi^0_{f_0}|\xi^0_{f_0}) =$$

$$= (e\,\pi_{f_0}(a_1)e\,\pi_{f_0}(a_2)e\ldots e\,\pi_{f_0}(a_n)e\xi^0_{f_0}|\xi^0_{f_0}) =$$

$$= (K_{\mu_f}(\lambda_A(a_1)\lambda_A(a_2)\ldots\lambda_A(a_n))\xi^0_{f_0}|\xi^0_{f_0}) =$$

$$= \int_{E_o(A)} \lambda_A(a_1)\lambda_A(a_2)\ldots\lambda_A(a_n)\, d\mu_{f_o}, \qquad a_1,a_2,\ldots,a_n \in A.$$

From the Stone-Weierstrass Theorem, by taking into account the fact that $s_*(\nu_{g_o})$ and μ_{f_o} are both supported by $E(A)$, we infer that $s_*(\nu_{g_o}) = \mu_{f_o}$. The Theorem is proved.

THEOREM 18. ν_{g_o} is a stable orthogonal lifting of μ_{f_o}.

Proof. From the preceding Theorem and from formula (2) in its proof we infer that

$$(3) \qquad K_{\mu_{f_o}}([\varphi]) = K_{\nu_{g_o}}(q_\infty([\varphi])),$$

for any $\varphi \in \mathcal{L}^\infty(E_o(A), \mathcal{B}(E_o(A)), \mu_{f_o})$. Let $\psi \in \mathcal{L}^\infty(E_o(B_{f_o}), \mathcal{B}(E_o(B_{f_o})), \nu_{g_o})$; then $K_{\nu_{g_o}}([\psi]) \in B'_{f_o}$ and, therefore, there exists a function $\varphi \in \mathcal{L}^\infty(E_o(A), \mathcal{B}(E_o(A)), \mu_{f_o})$, such that $K_{\mu_{f_o}}([\varphi]) = K_{\nu_{g_o}}([\psi])$. Formula (3) now implies that $K_{\nu_{g_o}}([\psi]) = K_{\nu_{g_o}}(q_\infty([\varphi]))$ and, since $K_{\nu_{g_o}}$ is injective, we infer that $q_\infty([\varphi]) = [\psi]$. Q.E.D.

III. We return to the setting of Theorem 16; i.e., $\pi: A \to B$ is any homomorphism of C^*-algebras. Let $g \in E(B)$ and let $\nu \in \mathcal{M}^1_+(E_o(B))$ be any orthogonal measure, such that $b(\nu) = g$. We have $\overline{\pi_g(\pi(A))\xi^o_g} = e_o H_g$, where $e_o \in \pi_g(\pi(A))'$ is a projection.

THEOREM 19. Let $\mu = s_*(\nu)$ and assume that $s(g) \in E(A)$. Then the following statements are equivalent

a) $e_o \in \mathcal{C}'_\nu$;

b) the measure μ is orthogonal and ν is a stable lifting of μ.

Proof. Let $f = s(g)$; then the representation $\pi_f: A \to \mathcal{L}(H_f)$ can be identified with

$$A \ni a \mapsto \pi_g(\pi(a))e_o \in \mathcal{L}(e_o H_g),$$

whereas ξ^o_f can be identified with ξ^o_g, since $\|f\| = 1$ implies that $e_o \xi^o_g = \xi^o_g$.

By formula (1) in the proof of Theorem 16 we have

$$(1) \qquad K_\mu([\varphi]) = e_o K_\nu(q_\infty([\varphi]))e_o \,; \quad [\varphi] \in L^\infty(E_o(A), \mathcal{B}(E_o(A)), \mu).$$

a) $\Rightarrow$ b). Indeed, if $e_o \in \mathcal{C}'$, then from formula (1) we immediately infer that the mapping K_μ is multiplicative; hence, μ is an orthogonal measure on $E_o(A)$.

From the fact that $f \in E(A)$, we infer that

$$K_\mu(\|\cdot\|) = 1.$$

From ([28], Lemma 3.6) we infer that

$$\int_{E_o(A)} \lambda_A(a_1)\lambda_A(a_2)\ldots\lambda_A(a_n)\, d\mu =$$

$$= \int_{E_o(B)} \lambda_B(\pi(a_1))\lambda_B(\pi(a_2))\ldots\lambda_B(\pi(a_n))\, d\nu =$$

$$= (K_\nu([\lambda_B(\pi(a_1))])K_\nu([\lambda_B(\pi(a_2))])\ldots K_\nu([\lambda_B(\pi(a_n))])\xi_g^o|\xi_g^o) =$$

$$= (e_\nu\pi_g(\pi(a_1))e_\nu\pi_g(\pi(a_2))e_\nu\ldots e_\nu\pi_g(\pi(a_n))e_\nu\xi_g^o|\xi_g^o), \quad a_1,a_2,\ldots,a_n \in A,$$

where e_ν is the projection onto $\overline{\mathcal{C}_\nu\xi_g^o} \subset H_g$.

On the other hand, from $\mathcal{C}_\nu \subset \pi_g(B)'$ and from $\pi_g(\pi(A)) \subset \pi_g(B)$, we infer that $\pi_g(B)' \subset \pi_g(\pi(A))'$ and, therefore, we have

$$\mathcal{C}_\mu \subset \mathcal{C}_\nu e_o \subset e_o\pi_g(\pi(A))'e_o .$$

Since $\mathcal{C}_\nu e_o$ is an abelian von Neumann subalgebra in $e_o\pi_g(\pi(A))'e_o =$ $= \pi_f(A)' \subset \mathcal{L}(e_oH_g)$, from ([28], Theorem 3.3) we infer that there exists an orthogonal measure $\mu_o \in \mathcal{M}_+^1(E_o(A))$, such that $\mu < \mu_o$ and $\mathcal{C}_{\mu_o} = \mathcal{C}_\nu\, e_o$. From $e_o \in \mathcal{C}_\nu'$ we infer that $e_\nu \leq e_o$, whereas from $e_o\xi_g^o = \xi_g^o$ we infer that

$$\overline{\mathcal{C}_{\mu_o}\xi_g^o} = \overline{\mathcal{C}_\nu e_o\xi_g^o} = \overline{\mathcal{C}_\nu\xi_g^o} = e_\nu H_g ;$$

hence, $e_{\mu_o} = e_\nu$. Since $\|b(\mu_o)\| = 1$, from ([28], Lemma 3.6) we infer that

$$\int_{E_o(A)} \lambda_A(a_1)\lambda_A(a_2)\ldots\lambda_A(a_n)\, d\mu_o =$$

(3) $$= (e_{\mu_o}\pi_f(a_1)e_{\mu_o}\ldots e_{\mu_o}\pi_f(a_n)e_{\mu_o}\xi_f^o|\xi_f^o) =$$

$$= (e_\nu\pi_g(\pi(a_1))e_\nu\ldots e_\nu\pi_g(\pi(a_n))e_\nu\xi_g^o|\xi_g^o), \quad a_1,a_2,\ldots,a_n \in A.$$

From (2) and (3) we infer that

(4) $\int_{E_o(A)} \lambda_A(a_1)\lambda_A(a_2)\ldots\lambda_A(a_n)\, d\mu =$

$$= \int_{E_o(A)} \lambda_A(a_1)\lambda_A(a_2)\ldots\lambda_A(a_n)\, d\mu_o ,$$

for any $a_1, a_2, \ldots, a_n \in A$. Since $\mu(E(A)) = \mu_o(E(A)) = 1$, from (4), with the help of the Stone-Weierstrass Theorem, we infer that $\mu = \mu_o$; hence, we have that $\mathcal{C}_\mu = \mathcal{C}_\nu e_o$.

Let us now consider the commutative diagram

$$\begin{array}{ccc} L^\infty(E_o(B), \mathcal{B}(E_o(B)), \nu) & \xrightarrow{K_\nu} & \mathcal{C}_\nu \\ q_\infty \uparrow & & \downarrow q \\ L^\infty(E_o(A), \mathcal{B}(E_o(A)), \mu) & \xrightarrow[K_\mu]{} & \mathcal{C}_\mu \end{array}$$

where q is the mapping

$$q : \mathcal{C}_\nu \ni c \mapsto ce_o \in \mathcal{C}_\mu ,$$

which is correctly defined and surjective, by virtue of the equality just obtained. Since ξ_g^o is cyclic for $\pi_g(B)$, it is separating for $\pi_g(B)'$; hence, ξ_g^o is separating for $\mathcal{C}_\nu$. From the fact that $e_o\xi_g^o = \xi_g^o$, we infer that q is injective; hence, it is an isomorphism. We now easily infer that q_∞ is a surjective isomorphism. It follows that ν is a stable lifting of μ.

b) $\Rightarrow$ a). Indeed, assume that $[\varphi] \in L^\infty(E_o(A), \mathcal{B}(E_o(A)), \mu)$ is a projection. Since μ is assumed to be orthogonal, it follows that $K_\mu([\varphi])$ is a projection. Since ν is assumed to be a stable lifting of μ, from (1) we infer that e_oee_o is a projection, for any projection $e \in \mathcal{C}_\nu$; i.e., $e_oee_oee_o = e_oee_o$. From

$$(e_oee_o - ee_o)^*(e_oee_o - ee_o) = (e_oee_o - e_oe)(e_oee_o - ee_o) =$$

$$= e_oee_oee_o - e_oee_oee_o - e_oee_oee_o + e_oee_o = 0,$$

we infer that $e_oee_o = ee_o$; hence, we have that

$$ee_o = e_oe ,$$

for any projection $e \in \mathcal{C}_\nu$. Since linear combinations of projections are uniformly dense in $\mathcal{C}_\nu$, we infer that

$$ce_o = e_o c \qquad c \in \mathcal{C}_\nu;$$

hence, $e_o \in \mathcal{C}_\nu'$, and the Theorem is proved.

IV. We shall say that a positive linear functional $f \in A^*_+$ is simple if $\pi_f(A)'$ is commutative. We shall denote by $S_o(A)$ the set of all simple quasi-states of A, whereas S(A) will stand for the set of all simple states of A.

THEOREM 20. For any C^*-algebra A we have

a) $P(A) \subset S(A)$;

b) $S(A) \wedge F(A) = P(A)$;

c) $S_o(A)$ and $S(A)$ are extremal subsets of $E_o(A)$;

d) $S_o(A) = E_o(A)$ if, and only if, A is commutative.

Proof. a). For any $p \in P(A)$ we have $\pi_p(A)' = \mathbb{C}\,1_{H_p}$.

b) If $f \in S(A) \wedge F(A)$, then $\pi_f(A)'$ is a commutative factor; hence, $\pi_f(A)' = \mathbb{C}\,1_{H_f}$. It follows that $f \in P(A)$.

c) Let $f_o \in S_o(A)$ and assume that

$$(1) \qquad f_o = tf_1 + (1-t)f_2\,,$$

where $o < t < 1$ and $f_1, f_2 \in E_o(A)$.

From (1) we infer that $f_1 \leq (1/t)f_o$ and, therefore, there exists a $T \in \pi_{f_o}(A)'$, $0 \leq T \leq (1/t)1$, such that

$$f_1(a) = (\pi_{f_o}(a)T\xi^o_{f_o} | \xi^o_{f_o}) = (\pi_{f_o}(a)T^{1/2}\xi^o_{f_o} | T^{1/2}\xi^o_{f_o}),$$

for any $a \in A$. Let us denote $\xi'_{f_o} = T^{1/2}\xi^o_{f_o}$ and denote by e_1 the projection onto $\overline{\pi_{f_o}(A)\xi'_{f_o}} \subset H_{f_o}$. Then $e_1 \in \pi_{f_o}(A)'$ and π_{f_1} can be identified with the subrepresentation $A \ni a \mapsto \pi_{f_o}(a)e_1 \in \mathcal{L}(e_1 H_{f_o})$, whereas $\xi^o_{f_1}$ can be identified with $\xi'_{f_o} \in e_1 H_{f_o}$.

We then have

$$\pi_{f_1}(A)' = e_1\pi_{f_o}(A)'e_1 = \pi_{f_o}(A)'e_1,$$

whence we infer that $\pi_{f_1}(A)'$ is commutative; hence, $f_1 \in S_o(A)$. Similarly, $f_2 \in S_o(A)$. It follows that $S_o(A)$ is an extremal subset of $E_o(A)$. Since $E(A)$ is a face of $E_o(A)$, it immediately follows that $S(A) = S_o(A) \wedge E(A)$ is an extremal subset of $E_o(A)$ (and of $E(A)$).

d) This follows immediately from ([25], Lemma 15). The Theorem is proved.

§6. UNIVERSAL MAXIMAL ORTHOGONAL LIFTINGS OF THE CENTRAL MEASURES

In this section we shall prove that the central measures on $E_o(A)$ induce regular Borel measures on $F(A)$, with respect to the central topology.

I. According to Sakai's Theorem (see [24], Theorem 3.1.8.; and also Theorems 17 and 18 above, for the case of an arbitrary C^*-algebra), any central Radon probability measure on $E_o(A)$, whose barycenter is a state f_o, is the stable projection of a maximal orthogonal Radon probability measure on the state space $E(B_{f_o})$ of a suitably chosen C^*-algebra B_f, with a unit element.

In order to ensure a better control of the properties of the central measures, it seems that a universal construction is better adapted to this aim.

Namely, we shall consider an arbitrary C^*-algebra A and its <u>universal representation</u> $\pi_u : A \to \mathcal{L}(H_u)$, where $H_u = \bigoplus_{f\in E(A)} H_u$, and $\pi_u = \bigoplus_{f\in E(A)} f$. Then $\pi_u(A)''$ can be canonically identified with the second dual A^{**} of A, as a Banach space, endowed with the Arens multiplication (see [12], §12; [24], §1.17, for details).

We shall denote by B the C^*-algebra $C^*(\pi_u(A), \pi_u(A)')$, generated by $\pi_u(A)$ and $\pi_u(A)'$ in $\mathcal{L}(H_u)$. Then

$$B' = \pi_u(A)' \cap \pi_u(A)''$$

is the center of $\pi_u(A)'$, of $\pi_u(A)''$ and of B itself.

Let $\pi : A \to B$ be the corestriction of π_u to B and denote by $s : E(B) \to E_o(A)$ the affine continuous mapping

$$s : E(B) \ni g \longmapsto g \circ \pi \in E_o(A).$$

Let us denote by $V(B)$ the set of all vector states of the C^*-algebra $B \subset \mathcal{L}(H_u)$.

<u>LEMMA 8</u>. a) $V(B) \subset S(B)$; b) $V(B)$ <u>is an extremal subset of</u> $E(B)$.

<u>Proof</u>. a) Let $v \in V(B)$. Then there exists a $\xi \in H_u$, $\|\xi\| = 1$, such that $v = \omega_\xi | B$. Let us now remark that the representation $\pi_v : B \to \mathcal{L}(H_v)$ can be identified with the subrepresentation

$$B \ni b \longmapsto b e_\xi \in \mathcal{L}(e_\xi H_u),$$

where $e_\xi \in \mathcal{L}(H_u)$ is the projection onto $\overline{B\xi} \subset H_u$. Of course, we have that $e_\xi \in B'$; i.e., it is a central projection. It follows that $\pi_v(B)'$ can be identified with $B'e_\xi \subset \mathcal{L}(e H_u)$, which is an abelian von Neumann algebra; hence, $v \in S(B)$.

b) Assume now that $v \in V(B)$ decomposes as

$$v = tv' + (1-t)v'', \quad 0 < t < 1, \; v', v'' \in E(B).$$

Then $v' \leq t^{-1}v$; hence, it exists an $x \in B'e_\xi$, such that $0 \leq x \leq t^{-1}1$, and

$$v'(b) = (bx\xi | \xi) = (bx^{1/2}\xi | x^{1/2}\xi), \; b \in B.$$

We infer that $v' = \omega_{x^{1/2}\xi}|B$; hence, $v' \in V(B)$. Similarly, $v'' \in V(B)$, and the Lemma is proved.

Let us denote $F_o(A) = \{\lambda f \; ; \lambda \in [0,1] \; , \; f \in F(A)\}$.

<u>LEMMA 9</u>. a) <u>For any</u> $p \in P(B)$ <u>we have that</u> $s(p) \in F_o(A)$;
b) <u>For any</u> $f \in F(A)$ <u>we have that</u> $\omega_{\xi^o_f}|B \in P(B) \cap V(B)$ <u>and</u> $s(\omega_{\xi^o_f}|B) = f$.

<u>Proof</u>. a) We obviously have that $\pi_p(\pi_u(A)') \subset \pi_p(\pi_u(A))'$, and, therefore, $\pi_p(\pi_u(A)')' \supset \pi_p(\pi_u(A))''$. From the equality

$$\pi_p(B) = C^*(\pi_p(\pi_u(A)), \pi_p(\pi_u(A)'))$$

and from the fact that $p \in P(B)$, we infer that

$$\mathbb{C}1_{H_p} = \pi_p(B)' = \pi_p(\pi_u(A))' \cap \pi_p(\pi_u(A)')' \supset$$

$$\supset \pi_p(\pi_u(A))' \cap \pi_p(\pi_u(A))'';$$

hence, $\pi_p(\pi_u(A))''$ is a factor in $\mathcal{L}(H_p)$. Let us now remark that $\pi_{s(p)}$ can be identified with the subrepresentation

$$A \ni a \longmapsto \pi_p(\pi_u(a))e_p \in \mathcal{L}(e_p H_p),$$

where $e_p \in \mathcal{L}(H_p)$ is the projection onto $\overline{\pi_p(\pi_u(A))\xi^o_p}$; hence, $e_p \in \pi_p(\pi_u(A))'$. It follows that $\pi_{s(p)}(A)''$ is a factor and, therefore, we have that $s(p) \in F_o(A)$.

<u>Remark</u>. We have $\| s(p) \| = \| e_p \xi^o_p \|^2$.

b) Let us denote $p = \omega_{\xi^o_f}|B$. Then π_p can be identified with the representation

$$B \ni b \longmapsto be_f \in \mathcal{L}(e_f H_u),$$

where $e_f \in B'$ is the projection onto $\overline{B\xi_f^o} \subset H_u$. Since $f \in F(A)$, e_f is a minimal projection of B'; hence, $B''e_f$ is a factor. From $(B''e_f)' = B'e_f = \mathbb{C}e_f$, we infer that $B''e_f = \mathcal{L}(e_f H_u)$; hence, $p \in P(B)$. It is obvious that we have $p \in V(B)$, and also

$$s(p)(a) = (\pi_u(a)\xi_f^o|\xi_f^o) = f(a), \qquad a \in A.$$

The Lemma is proved.

II. For any $g \in E(B)$ we have

$$\pi_g(B) = C^*(\pi_g(\pi_u(A)), \pi_g(\pi_u(A)'))$$

and, therefore,

$$\pi_g(B)' = \pi_g(\pi_u(A))' \cap \pi_g(\pi_u(A)')'.$$

From the fact that $\pi_g(\pi_u(A)') \subset \pi_g(\pi_u(A))'$, we infer that

$$\pi_g(B)' \supset \pi_g(\pi_u(A))' \cap \pi_g(\pi_u(A))''.$$

It is obvious that the von Neumann algebra

$$\mathcal{D}_g = \pi_g(\pi_u(A))' \cap \pi_g(\pi_u(A))''$$

is contained in the center of $\pi_g(B)'$; the corresponding orthogonal measure ν_g on $E(B)$ is, therefore, subcentral. If we denote by $e_g \in \pi_g(\pi_u(A))'$ the projection onto $\overline{\pi_g(\pi_u(A))\xi_g^o} \subset H_g$, then the representation $\pi_{s(g)}: A \to \mathcal{L}(H_{s(g)})$ can be identified with the subrepresentation $A \ni a \longmapsto \pi_g(\pi_u(a))e_g$ of A on $\mathcal{L}(e_g H_g)$. Since we have that $e_g \in \mathcal{D}_g'$, we can apply Theorem 19. From the equality

$$\mathcal{D}_g e_g = (e_g \pi_g(\pi_u(A))' e_g) \cap (\pi_g(\pi_u(A))'' e_g),$$

we infer that $s_*(\nu_g) = \mu_{s(g)}$ <u>is the central measure on</u> $E_o(A)$, <u>corresponding to</u> $s(g)$, <u>if</u> $s(g) \in E(A)$. Of course, ν_g is a stable lifting of $\mu_{s(g)}$.

Let us now consider the function $n: E(B) \to [0,1]$, given by $n(g) = \|s(g)\|$, $g \in E(B)$; it is obviously affine and lower semi-continuous. It follows that the set

$$n^{-1}(\{1\}) = E_1(B) = \{ g \in E(B) \ ; \ s(g) \in E(A) \}$$

is a <u>measure extremal face and a G_δ-subset of</u> $E(B)$; hence, for any $\nu \in \mathcal{M}^1_+(E(B))$, such that $b(\nu) \in E_1(B)$, we have that $\nu(E_1(B)) = 1$.

We shall consider now the set $\Omega_1(E(B)) \subset \Omega(E(B))$ of all orthogonal Radon probability measures ν_o on $E(B)$, such that

a) $g_o = b(\nu_o) \in E_1(B)$, and

b) $\mathcal{C}_{\nu_o} \subset \mathcal{D}_{b(\nu_o)}$.

From a) we infer that $b(s_*(\nu_o)) \in E(A)$, whereas from b) we infer that $e_{b(\nu_o)} \in \mathcal{C}'_{\nu_o}$, for any $\nu_o \in \Omega_1(E(B))$. For any $g_o \in E_1(B)$ let $\Omega_1(E(B); g_o) \subset \Omega_1^o(E(B))$ be the set of all $\nu_o \in \Omega_1(E(B))$, such that $b(\nu_o) = g_o$.

<u>LEMMA 1o</u>. <u>For any</u> $g \in E_1(B)$ <u>the restriction of</u> s_* <u>to</u> $\Omega_1(E(B); g)$ <u>is a bijection between</u> $\Omega_1(E(B); g)$ <u>and the set</u> $\mathcal{Z}^1_+(E_o(A); s(g))$.

<u>Proof</u>. By Theorem 19, since $e_g \in \mathcal{C}'_\nu$, for any $\nu \in \Omega_1(E(B); g)$, the measure $s_*(\nu)$ is orthogonal and $\nu < \nu_g$ implies that $s_*(\nu) \prec s(\nu_g) = \mu_{s(g)}$. Let us now remark that the mapping

$$s_g : \mathcal{D}_g \ni c \longmapsto c e_g \in \mathcal{D}_g e_g$$

is a $*$-isomorphism of abelian von Neumann algebras; hence, it induces a bijection between the set of all von Neumann subalgebras of $\mathcal{D}_g$ and the set of all von Neumann subalgebras of $\mathcal{D}_g e_g = \mathcal{C}_{\mu_{s(g)}}$. Since the diagram

$$\begin{array}{ccccccc}
\Omega_1(E(B); g) & \ni & \nu & \longmapsto & \mathcal{C}_\nu & \subset & \mathcal{D}_g \\
s_* \downarrow & & \updownarrow & & \updownarrow & & \downarrow \ s_g \\
\mathcal{Z}^1_+(E_o(A); s(g)) & \ni & s_*(\nu) & \longmapsto & \mathcal{C}_{s_*(\nu)} & \subset & \mathcal{C}_{\mu_{s(g)}}
\end{array}$$

is commutative, the Lemma now immediately follows from ([28], Theorem 3.4).

We shall now consider the Z_1-extremal subsets $F \subset E(B)$, defined as follows: F is said to be <u>Z_1-extremal</u> if the following conditions are satisfied

a) F is a compact subset of $E(B)$, and

b) $g \in F \cap E_1(B) \Rightarrow \nu(F) = 1$, for any $\nu \in \Omega_1(E(B); g)$.

It is obvious that the set $Z_1(E(B))$ of all (compact) Z_1-extremal sub sets of $E(B)$ is the set of all closed subsets of a topology on $E(B)$, which we shall call the Z_1-topology. We shall also consider the topology $\hat{Z}_1$ induced on $s^{-1}(F(A))$ by Z_1 and we shall denote

$$\hat{Z}_1(s^{-1}(F(A))) = \{ F \cap s^{-1}(F(A));\ F \in Z_1(E(B)) \}.$$

Let $\hat{s} : s^{-1}(F(A)) \to F(A)$ be the restriction of s to $s^{-1}(F(A))$, followe by the corresponding corestriction.

LEMMA 11. *If $F_o \subset E_o(A)$ is a compact Z-extremal subset, then $s^{-1}(F_o)$ is a compact Z_1-extremal subset of $E(B)$.*

Proof. Let $g \in s^{-1}(F_o) \cap E_1(B)$; then $s(g) \in F_o \cap E(A)$ and, therefore, we have that $\mu(F_o) = 1$, for any subcentral measure $\mu \in \mathcal{M}^1_+(E_o(A))$, such that $b(\mu) = s(g)$. We infer that $\nu(s^{-1}(F_o)) = s_*(\nu)(F_o) = 1$, for any $\nu \in \Omega_1(E(B), g)$, because $s_*(\nu)$ is a subcentral measure and $b(s_*(\nu)) = s(g)$. The Lemma is proved.

LEMMA 12. *The mapping $\hat{s}$ is a continuous surjection, if $s^{-1}(F(A))$ is endowed with the $\hat{Z}_1$-topology, whereas $F(A)$ is endowed with the central topology.*

Proof. Any centrally closed subset $\hat{F}_o \subset F(A)$ is of the form $\hat{F}_o = F_o \cap F(A)$, where $F_o \subset E_o(A)$ is a compact Z-extremal subset. Since

$$\hat{s}^{-1}(\hat{F}_o) = s^{-1}(F_o) \cap s^{-1}(F(A)),$$

it will be sufficient to prove that $s^{-1}(F_o)$ is Z_1-extremal. It is clearly compact, whereas from

$$g \in s^{-1}(F_o) \cap E_1(B)$$

we infer that $s_*(\nu) = \mu$ is a subcentral measure corresponding to $s(g)$ for any $\nu \in \Omega_1(E(B);g)$. Since $s(g) \in F_o \cap E(A)$, we infer that $\mu(F_o) = 1$ and, therefore,

$$1 = \mu(F_o) = s_*(\nu)(F_o) = \nu(s^{-1}(F_o)),$$

for any $\nu \in \Omega_1(E(B);g)$, and the Lemma is proved.

LEMMA 13. *Let $F \subset E(B)$ be any compact Z_1-extremal subset of $E(B)$. Then $(\mathrm{ex}\ \overline{\mathrm{co}}(F)) \cap E_1(B) \subseteq F \cap s^{-1}(F(A))$.*

Proof. By Milman's Converse Theorem, we have that $\mathrm{ex}\ \overline{\mathrm{co}}(F) \subset F$. Let now $g \in (\mathrm{ex}\ \overline{\mathrm{co}}(F)) \cap E_1(B)$. Then we have $g \in F \cap E_1(B)$ and, therefore,

$\nu_g(F) = 1$.We infer that $\nu_g(\overline{co}(F)) = 1$.Since $b(\nu_g) = g \in ex\,\overline{co}(F)$,by Bauer's Theorem (see[23],Proposition 1.4) we infer that $\nu_g = \varepsilon_g$,the Dirac measure at g.We infer that $\mathcal{D}_g = \mathfrak{C}1_{H_g}$ and,therefore, $\pi_g(\pi_u(A))''$ is a factor;hence, $\pi_g(\pi_u(A))''e_g$ is a factor,and this implies that $\pi_{s(g)}(A)''$ is a factor.From $g \in E_1(B)$ we infer that $s(g) \in F(A)$.The Lemma is proved.

LEMMA 14.Let $\nu \in \mathcal{M}_+^1(E(B))$ be any maximal measure,such that $\nu(E_1(B)) = 1$,and let $F \subset E(B)$ be any compact Z_1-extremal subset.Then

$$F \cap s^{-1}(F(A)) = \emptyset \implies \nu(F) = o.$$

Proof.By way of contradiction,let us assume that $\nu(F) > o$.Define $\nu_F = \nu(F)^{-1}\chi_F\nu$ .Then ν_F is a maximal Radon probability measure on E(B),such that $\nu_F(F) = 1$ and

$$\nu_F(E_1(B)) = \nu(F)^{-1}\nu(F \cap E_1(B)) = 1.$$

We infer that $\nu_F(\overline{co}(F)) = 1$;hence, $\nu_F | \overline{co}(F)$ is a maximal Radon probability measure on $K = \overline{co}(F)$.If we denote $\psi = n|K$,then ψ is a lower semi-continuous affine function on K,such that $\psi(K) \subset [o,1]$ and $\nu_f(\psi) = 1$.

For any $D \in \mathcal{B}_o(E(B))$ we have $D \cap K \in \mathcal{B}_o(K)$ and also

$$(D \cap K) \cap (ex\,\psi^{-1}(\{1\})) = \emptyset,$$

by Lemma 13.From ([28],Proposition 1.8) we infer that $\nu_F(D) = o$;i.e $\nu_F = o$,a contradiction.The Lemma is proved.

LEMMA 15.Let $F_o \subset E_o(A)$ be any compact Z-extremal subset of $E_o(A)$, and let $\mu \in \mathcal{M}_+^1(E_o(A))$ be any central measure,such that $b(\mu) \in E(A)$ and $\mu(F_o) = 1$.Then we have $\mu(D) = o$,for any $D \in \mathcal{B}_o(E_o(A))$,such that $D \cap F_o \cap F(A) = \emptyset$.

Proof.Let $f = b(\mu)$ and $g = \omega_{\xi_f}|B$.Then $\pi_g(B)' = \mathcal{D}_g$,and ν_g is a maximal orthogonal measure on E(B),such that $s_*(\nu_g) = \mu$ .If $D \in \mathcal{B}_o(E_o(A))$ and $D \cap F_o \cap F(A) = \emptyset$,then $s^{-1}(D) \in \mathcal{B}_o(E(B))$,and

$$s^{-1}(D) \cap s^{-1}(F_o) \cap s^{-1}(F(A)) = \emptyset.$$

Since we have

we infer that $\nu_g(\overline{co}(s^{-1}(F_o))) = 1$.

By Lemmas 11 and 13 we have

$$(\text{ex}\ \overline{co}(s^{-1}(F_o))) \cap E_1(B) \subset s^{-1}(F_o) \cap s^{-1}(F(A))$$

and, therefore,

$$(\text{ex}\ \overline{co}(s^{-1}(F_o))) \cap E_1(B) \cap s^{-1}(D) = \emptyset.$$

If we denote $K = \overline{co}(s^{-1}(F_o))$, $\varphi = n|K$, $\nu_o = \nu_g|\overline{co}(s^{-1}(F_o))$, we can apply ([28], Proposition 1.8) in order to infer that $\nu_o(s^{-1}(D) \cap K) = o$, since $s^{-1}(D) \cap K \in \mathcal{B}_o(K)$, by taking into account also the fact that $\nu_g(E_1(B)) = 1$. It follows that $\mu(D) = \nu_g(s^{-1}(D)) = o$, and the Lemma is proved.

As an immediate consequence of Lemma 14 we obtain the following Theorem, which extends Sakai's Theorem (see [24], Theorem 3.1.8.) to the case of an arbitrary C^*-algebra A, possibly not containing the unit element.

<u>THEOREM 21</u>. <u>Let</u> $\mu \in \mathcal{M}_+^1(E_o(A))$ <u>be any central measure, such that</u> $b(\mu) \in E(A)$. <u>Then</u> $\mu(D) = o$, <u>for any</u> $D \in \mathcal{B}_o(E_o(A))$, <u>such that</u> $D \cap F(A) = \emptyset$.

<u>Proof</u>. In the preceding Lemma take $F_o = E_o(A)$.

III. We shall now consider the σ-algebra $\hat{\mathcal{B}}_o(F(A))$ defined by

$$\hat{\mathcal{B}}_o(F(A)) = \{ D \cap F(A) \ ; \ D \in \mathcal{B}_o(E_o(A)) \}.$$

Let now $\mu \in \mathcal{M}_+^1(E_o(A))$ be any central measure, such that $b(\mu) \in E(A)$. By Theorem 21 we can define correctly a probability measure

$$\hat{\mu}_o : \hat{\mathcal{B}}_o(F(A)) \longrightarrow [o,1],$$

by the formula

$$\hat{\mu}_o(D \cap F(A)) = \mu(D), \qquad D \in \mathcal{B}_o(E_o(A)).$$

From $\hat{\mu}_o$ we can derive the outer measure $\hat{\mu}_o^*$, as usually.

<u>THEOREM 22</u>. <u>For any compact</u> Z-<u>extremal subset</u> $F_o \subset E_o(A)$ <u>we have</u>

$$\hat{\mu}_o^*(F_o \cap F(A)) = \mu(F_o).$$

Proof. Assume first that $\mu(F_o) = o$. Then there exists a set $D_o \in \mathcal{B}_o(E_o(A))$, such that $D_o \supset F_o$ and $\mu(D_o) = o$. We obviously have that $D_o \wedge F(A) \in \hat{\mathcal{B}}_o(F(A))$ and $D_o \wedge F(A) \supset F_o \wedge F(A)$.

From $\hat{\mu}_o(D_o \wedge F(A)) = \mu(D_o) = o$ we infer that $\hat{\mu}_o^*(F_o \wedge F(A)) = o$, and the equality is proved in this case.

If $\mu(F_o) > o$, let us consider the central measure $\mu_{F_o} = \mu(F_o)^{-1}\chi_{F_o}\mu$. We have

$$\|b(\mu_{F_o})\| = \int_{E_o(A)} \|f\| \, d\mu_{F_o} = \mu(F_o)^{-1} \int_{F_o} \|f\| \, d\mu =$$

$$= \mu(F_o)^{-1} \int_{F_o \cap E(A)} \|f\| \, d\mu = \mu(F_o)^{-1} \mu(F_o \wedge E(A)) = 1,$$

since $\|\cdot\|$ is affine and lower semi-continuous, and $\mu(E(A)) = 1$. It follows that $b(\mu_{F_o}) \in E(A)$.

Since $\mu_{F_o}(F_o^o) = 1$, by Lemma 15 we have that $\mu_{F_o}(D) = o$, for any $D \in \mathcal{B}_o(E_o(A))$, such that $D \wedge F_o \wedge F(A) = \emptyset$.

Let then $D_o \in \mathcal{B}_o(E_o(A))$ be such that

$$D_o \cap F(A) \supset F_o \wedge F(A).$$

With $D_1 = \complement D_o$ we have $D_1 \in \mathcal{B}_o(E_o(A))$ and $D_1 \wedge F_o \wedge F(A) = \emptyset$. From Lemma 14 we infer that $\mu_{F_o}(D_1) = o$, and this implies that $\mu_{F_o}(D_o) = 1$. It follows that

$$\mu(F_o) = \mu(F_o \wedge D_o) \leq \mu(D_o) = \hat{\mu}_o(D_o \wedge F(A)),$$

and this implies that

$$\mu(F_o) \leq \hat{\mu}_o^*(F_o \wedge F(A)).$$

On the other hand, there exists a $D \in \mathcal{B}_o(E_o(A))$, such that $F_o \subset D$ and $\mu(F_o) = \mu(D)$. From $F_o \wedge F(A) \subset D \wedge F(A)$, and from

$$\hat{\mu}_o^*(F_o \wedge F(A)) \leq \hat{\mu}_o(D \wedge F(A)) = \mu(D) = \mu(F_o),$$

we infer that $\mu(F_o) = \hat{\mu}_o^*(F_o \wedge F(A))$, and the Theorem is proved.

<u>THEOREM 23</u>. <u>Any Baire-measurable subset</u> $M \subset F(A)$, <u>with respect to the central topology, is</u> $\hat{\mu}_o$<u>-measurable, for any central measure</u> $\mu \in \mathcal{M}_+^1(E_o(A))$, <u>such that</u> $b(\mu) \in E(A)$.

Proof. We recall that the Baire measurable subsets of a topological space are defined to be those belonging to the smallest σ-algebra of subsets of the space, which contains all the closed G_δ-subsets.

It will be sufficient to prove that any centrally closed, centrally G_δ-subset $F_o \cap F(A)$ of $F(A)$, where F_o is a compact Z-extremal subset of $E_o(A)$, is $\hat{\mu}_o$-measurable. Indeed, if $(F_n)_{n\geq 1}$ is an increasing sequence of compact Z-extremal subsets of $E_o(A)$, such that

$$\bigcup_{n=1}^{\infty}(F_n \cap F(A)) = F(A) \setminus F_o,$$

then we have

$$\hat{\mu}_o^*(F(A) \setminus F_o) = \sup \hat{\mu}_o^*(F_n \cap F(A)).$$

From the fact that $F_n \cap F_o \cap F(A) = \emptyset$, for $n \geq 1$, and from Theorem lo we infer that $F_n \cap F_o \cap E(A) = \emptyset$ and, therefore, $\mu(F_n \cap F_o) = o$. It follows that

$$\hat{\mu}_o^*(F(A) \setminus F_o) \leq 1 - \hat{\mu}_o^*(F_o \cap F(A)),$$

and this shows that $F_o \cap F(A)$ is $\hat{\mu}_o$-measurable. The Theorem is proved

Remark. With an obvious notation, we can write the preceding results as follows

$$\mathcal{B}_o(F(A);\hat{Z}(F(A))) \subset \hat{\mathcal{B}}_o(F(A))(\hat{\mu}_o).$$

We shall prove below that $\hat{\mu}_o$ can be extended as a regular probability measure on the σ-algebra $\mathcal{B}(F(A);\hat{Z}(F(A)))$ of all Borel measurable subsets of $F(A)$, with respect to the central topology.

We shall denote by $\hat{\mathcal{B}}_2(F(A))$ the σ-algebra of subsets of $F(A)$, generated by $\hat{\mathcal{B}}_o(F(A))$ and by $\mathcal{B}(F(A);\hat{Z}(F(A)))$. By $\mathcal{B}_2(E_o(A))$ we shall denote the σ-algebra of subsets of $E_o(A)$, generated by $Z(E_o(A))$ and $\mathcal{B}_o(E_o(A))$. It is obvious that

$$\hat{\mathcal{B}}_2(F(A)) = \{ M \cap F(A) \ ; \ M \in \mathcal{B}_2(E_o(A))\}.$$

Also, we shall denote by $\mathcal{B}_1(E_o(A))$ the σ-algebra of subsets of $E_o(A)$ generated by the set $\mathcal{F}(E_o(A))$ of the compact extremal subsets of $E_o(A)$ and by $\mathcal{B}_o(E_o(A))$. From the inclusion $\mathcal{F}(E_o(A)) \subset Z(E_o(A))$ we immediately infer that $\mathcal{B}_1(E_o(A)) \subset \mathcal{B}_2(E_2(A))$.

LEMMA 16. *Let $\nu \in \mathcal{M}_+^1(E(B))$ be a maximal measure, such that $\nu(E_1(B)) = 1$. Then $\nu(D) = 0$ for any $D \in \mathcal{B}_0(E(B))$, such that $D \cap s^{-1}(F(A)) = \emptyset$.*

Proof. From ([31], Theorem 2) we infer that

$$\tilde{\nu}(P(B) \cap E_1(B)) = 1.$$

(Here $\tilde{\nu}$ is the C-Bórel probability measure induced on P(B) by ν; see [31]).

The inclusion $P(B) \cap E_1(B) \subset s^{-1}(F(A))$ now implies that

$$\nu(D) = \tilde{\nu}(D \cap P(B)) = 0,$$

for any $D \in \mathcal{B}_0(E(B))$, such that $D \cap s^{-1}(F(A)) = \emptyset$. The Lemma is proved.

We can now consider the σ-algebra $\hat{\mathcal{B}}_0(s^{-1}(F(A)))$, defined by

$$\hat{\mathcal{B}}_0(s^{-1}(F(A))) = \{ D \cap s^{-1}(F(A)) \ ; \ D \in \mathcal{B}_0(E(B)) \}.$$

The preceding Lemma shows that by the formula

$$\hat{\nu}_0(D \cap s^{-1}(F(A))) = \nu(D), \qquad D \in \mathcal{B}_0(E(B)),$$

one defines correctly a probability measure

$$\hat{\nu}_0 : \hat{\mathcal{B}}_0(s^{-1}(F(A))) \longrightarrow [0,1],$$

from which one can derive the corresponding outer measure ν_0^*.

LEMMA 17. *Let $F \subset E(B)$ be any compact Z_1-extremal subset of E(B), and let $\nu \in \mathcal{M}_+^1(E(B))$ be any maximal measure, such that $\nu(E_1(B)) = 1$ and $\nu(F) = 1$. Then we have $\nu(D) = 0$, for any $D \in \mathcal{B}_0(E(B))$, such that $D \cap F \cap s^{-1}(F(A)) = \emptyset$.*

Proof. Let $K = \overline{co}(F)$; then $\nu(K) = 1$ and $\nu|K$ is a maximal Radon probability measure on K. If $D \in \mathcal{B}_0(E(B))$, then $D \cap K \in \mathcal{B}_0(K)$. If we denote $\psi = n|K$, then ψ is a lower semi-continuous affine function $\psi : K \to [0,1]$, such that $\nu(\psi) = 1$. Since $K \cap E_1(B) = \psi^{-1}(\{1\})$, from the fact that $ex\, \psi^{-1}(\{1\}) = E_1(B) \cap (ex\, K)$, from Lemma 13 and from ([28], Proposition 1.8) the present Lemma now immediately follows.

THEOREM 24. *For any compact Z_1-extremal subset $F \subset E(B)$ we have*

$$\hat{\nu}_0^*(F \cap s^{-1}(F(A))) = \nu(F).$$

(Here $\nu \in \mathcal{M}_+^1(E(B))$ is any maximal measure, such that $\nu(E_1(B)) = 1$).

Proof. If $\nu(F) = 0$, then there exists a $D \in \mathcal{B}_0(E(B))$, such that $F \subset D$ and $\nu(D) = 0$. We then have

$$F \wedge s^{-1}(F(A)) \subset D \wedge s^{-1}(F(A))$$

and, therefore,

$$\hat{\nu}_0^*(F \wedge s^{-1}(F(A))) \leq \hat{\nu}_0(D \wedge s^{-1}(F(A))) = \nu(D) = 0.$$

Let us now assume that $\nu(F) > 0$. We shall then define the maximal measure $\nu_F = \nu(F)^{-1}\chi_F \nu$, for which we also have $\nu_F(E_1(B)) = 1$. By Lemma 17 we have that $\nu_F(D) = 0$, for any $D \in \mathcal{B}_0(E(B))$, such that $D \wedge F \wedge s^{-1}(F(A)) = \emptyset$, because $\nu_F(F) = 1$.

Let then $D_0 \in \mathcal{B}_0(E(B))$ be such that

$$D_0 \wedge s^{-1}(F(A)) \supset F \wedge s^{-1}(F(A)).$$

With $D_1 = \complement D_0$, we have that $D_1 \in \mathcal{B}_0(E(B))$ and $D_1 \wedge F \wedge s^{-1}(F(A)) = \emptyset$. From Lemma 17 we now infer that $\nu_F(D_1) = 0$, and this implies that $\nu_F(D_0) = 1$. It follows that

$$\nu(F) = \nu(F \wedge D_0) \leq \nu(D_0) = \hat{\nu}_0(D_0 \wedge s^{-1}(F(A))),$$

and this implies that

$$\nu(F) \leq \hat{\nu}_0^*(F \wedge s^{-1}(F(A))).$$

On the other hand, there exists a $D \in \mathcal{B}_0(E(B))$, such that $F \subset D$ and $\nu(F) = \nu(D)$. From $F \wedge s^{-1}(F(A)) \subset D \wedge s^{-1}(F(A))$, and from

$$\hat{\nu}_0^*(F \wedge s^{-1}(F(A))) \leq \hat{\nu}_0(D \wedge s^{-1}(F(A))) = \nu(D) = \nu(F),$$

we infer that $\nu(F) = \hat{\nu}_0^*(F \quad F(A))$, and the Theorem is proved.

IV. We shall now use the preceding results in order to prove that any central measure $\mu \in \mathcal{M}_+^1(E_0(A))$, such that $b(\mu) \in E(A)$, induces a regular Borel measure on $F(A)$, with respect to the central topology on $F(A)$. To this end, we shall adapt to our case the method of proof given by Batty to a similar problem (see [5], Theorem 7; [6], Theorem 3.2; [30], Theorem 13).

Let $\mathcal{B}_2(E(B))$ be the σ-algebra of subsets of $E(B)$, generated by $\mathcal{B}_o(E(B))$, and by the set of all (compact) Z_1-extremal subsets of $E(B)$. Of course, any compact extremal subset of $E(B)$ belongs to $\mathcal{B}_2(E(B))$. We shall also consider the σ-algebra $\hat{\mathcal{B}}_2(s^{-1}(F(A)))$ of subsets of $s^{-1}(F(A))$, given by

$$\hat{\mathcal{B}}_2(s^{-1}(F(A))) = \{ M \cap s^{-1}(F(A));\ M \in \mathcal{B}_2(E(B)) \} .$$

It is clear that we have the inclusions

$$\mathcal{B}_o(E(B)) \subset \mathcal{B}_1(E(B)) \subset \mathcal{B}_2(E(B)) \subset \mathcal{B}(E(B)),$$

where by $\mathcal{B}_1(E(B))$ we have denoted the σ-algebra of subsets of $E(B)$ generated by $\mathcal{B}_o(E(B))$ and by the set $\mathcal{F}(E(B))$ of all compact extremal subsets of $E(B)$. Moreover, it is easy to see that $\hat{\mathcal{B}}_2(s^{-1}(F(A)))$ is the σ-algebra of subsets of $s^{-1}(F(A))$, generated by $\hat{\mathcal{B}}_o(s^{-1}(F(A)))$ and $\mathcal{B}(s^{-1}(F(A)); \hat{Z}_1)$.

Let $\mu \in \mathcal{M}^1_+(E_o(A))$ be any central measure, such that $f = b(\mu) \in E(A)$, let $g = \omega_{\xi_f^o}|B$ and consider the corresponding maximal orthogonal measure $\nu = \nu_g$.

For any $M \in \mathcal{B}_2(E(B))$ we shall define

$$\nu'(M) = \sup \{ \nu(F) \ ;\ F \in Z_1(E(B)), F \cap s^{-1}(F(A)) \subset M \}$$

and

$$\nu''(M) = \sup \{ \nu(F) \ ;\ F \in Z_1(E(B)),\ F \subset M \}.$$

We have the following properties

a) $\nu''(M) \leq \nu'(M)$, for any $M \in \mathcal{B}_2(E(B))$; obvious.

b) $\nu''(M) \leq \nu(M)$, for any $M \in \mathcal{B}_2(E(B))$; obvious.

c) $\nu''(D) = \nu(D)$, for any $D \in \mathcal{B}_o(E(B))$. Indeed, it is clear that $\nu''(D) \leq \nu(D)$, by assertion b). Since ν is maximal, and since $Z_1(E(B))$ includes all compact extremal subsets of $E(B)$, the equality follows from ([29], Corollary to Theorem 1).

d) $\nu''(E(B) \setminus F) = \nu(E(B) \setminus F)$, for any $F \in Z_1(E(B))$. Indeed, since $E(B) \setminus F$ is open in $E(B)$, and since ν is maximal, by ([29], Theorem 2), for any $\varepsilon > 0$, there exists a compact extremal subset $F_1 \subset E(B) \setminus F$, such that

$$\nu(E(B) \setminus F) - \varepsilon < \nu(F_1)$$

and, therefore, we have

$$\nu(E(B)\setminus F)\leq \nu''(E(B)\setminus F).$$

Assertion b) provides the reversed inequality.

e) $\nu''(F) = \nu(F)$, for any $F\in Z_1(E(B))$; obvious.

f) $\nu''(M) = \nu(M)$, for any $M\in \mathcal{B}_2(E(B))$. Indeed, as in ([5], proof of Proposition 5) we shall consider the set

$$\mathcal{B}'' = \{M\in \mathcal{B}_2(E(B));\ \nu''(M) = \nu(M),\ \nu''(\complement M) = \nu(\complement M)\}.$$

It is easy to prove that $\mathcal{B}''$ is a σ-algebra, such that

$$\mathcal{B}_o(E(B))\subset \mathcal{B}'' \quad\text{and}\quad Z_1(E(B))\subset \mathcal{B}'',$$

by virtue of assertions c), d) and e). It follows that $\mathcal{B}'' = \mathcal{B}_2(E(B))$

g) We have

$$\nu'(M_1) + \nu'(M_2)\leq \nu'(M_1\cup M_2)$$

and

$$\nu''(M_1) + \nu''(M_2)\leq \nu''(M_1\cup M_2),$$

for any $M_1, M_2\in \mathcal{B}_2(E(B))$, such that $M_1\cap M_2 = \emptyset$. Indeed, the second inequality is an immediate consequence of the definition of ν''; for the first, given $\varepsilon>0$, there exist $F_1, F_2\in Z_1(E(B))$, such that

$$F_i\cap s^{-1}(F(A))\subset M_i \quad\text{and}\quad \nu'(M_i) - \varepsilon<\nu(F_i),\quad i = 1,2.$$

We infer that

$$\nu'(M_1) + \nu'(M_2) - 2\varepsilon<\nu(F_1) + \nu(F_2),$$

whereas from $F_1\cap F_2\cap s^{-1}(F(A)) = \emptyset$ and from Lemma 14 we infer that $\nu(F_1) + \nu(F_2) = \nu(F_1\cup F_2)$. The assertion now immediately follows.

h) $\nu'(M) = \nu(M)$, for any $M\in \mathcal{B}_2(E(B))$. Indeed, by a) and f) we have $\nu(M)\leq \nu'(M)$, for any $M\in \mathcal{B}_2(E(B))$. By g) we have

$$1 = \nu(M) + \nu(\complement M)\leq \nu'(M) + \nu'(\complement M)\leq 1;$$

hence, $\nu'(M) = \nu(M)$, for any $M\in \mathcal{B}_2(E(B))$.

i) For any $M \in \mathcal{B}_2(E(B))$ we have that

$$M \cap s^{-1}(F(A)) = \emptyset \Rightarrow \nu(M) = 0.$$

Indeed, this is an immediate consequence of assertion h).

We infer that by the formula

$$\hat{\nu}(M \cap s^{-1}(F(A))) = \nu(M), \qquad M \in \mathcal{B}_2(E(B)),$$

we define correctly a probability measure

$$\hat{\nu} : \hat{\mathcal{B}}_2(s^{-1}(F(A))) \Rightarrow [0,1],$$

such that $\hat{\nu} | \hat{\mathcal{B}}_0(s^{-1}(F(A))) = \hat{\nu}_0$.

By summarizing the preceding results, we get the following

THEOREM 25. There exists a probability measure

$$\hat{\nu} : \hat{\mathcal{B}}_2(s^{-1}(F(A))) \rightarrow [0,1],$$

such that

a) $\hat{\nu} | \hat{\mathcal{B}}_0(s^{-1}(F(A))) = \hat{\nu}_0$;

and

b) $\hat{\nu}(M \cap s^{-1}(F(A))) = \nu(M), \qquad M \in \mathcal{B}_2(E(B))$;

and possessing the following regularity properties

a) $\hat{\nu}(\hat{M}) = \sup\{\hat{\nu}(\hat{F}); \hat{F} \subset \hat{M}, \hat{F} \in \hat{Z}_1(s^{-1}(F(A)))\}, \quad \hat{M} \in \hat{\mathcal{B}}_2(s^{-1}(F(A)))$

and

b) $\hat{\nu}(\hat{F}) = \inf\{\hat{\nu}(\hat{D}); \hat{D} \supset \hat{F}, \hat{D} \in \hat{\mathcal{B}}_0(s^{-1}(F(A)))\}, \quad \hat{F} \in \hat{Z}_1(s^{-1}(F(A)))$.

Proof. The existence of $\hat{\nu}$ was established just before the statement of the Theorem; assertion a) follows from the equality $\nu' = \nu$, whereas assertion b) follows from Theorem 24.

Let us now recall that, by Lemma 12, the mapping

$$\hat{s} : s^{-1}(F(A)) \rightarrow F(A)$$

is continuous, if $s^{-1}(F(A))$ is equipped with the $\hat{Z}_1$-topology, whereas $F(A)$ is equipped with the $\hat{Z}$-topology (i.e., the central topology). Also, since $s : E(B) \rightarrow E_0(A)$ is continuous, we have that

$$D_o \in \mathcal{B}_o(E_o(A)) \Rightarrow s^{-1}(D_o) \in \mathcal{B}_o(E(B))$$

and, therefore, we have that

$$\hat{s}_*(\hat{\mathcal{B}}_o(s^{-1}(F(A)))) \supset \hat{\mathcal{B}}_o(F(A))$$

and

$$\hat{s}_*(\mathcal{B}(s^{-1}(F(A))); \hat{Z}_1) \supset \mathcal{B}(F(A); Z),$$

where, in the left hand members of these relations the full direct images of the corresponding σ-algebras are denoted. We infer that

$$\hat{s}_*(\hat{\mathcal{B}}_2(s^{-1}(F(A)))) \supset \hat{\mathcal{B}}_2(F(A)),$$

and, therefore, the full direct image of the measure $\hat{\nu}$, i.e.,

$$\hat{s}_*(\nu) : \hat{s}_*(\hat{\mathcal{B}}_2(s^{-1}(F(A)))) \longrightarrow [0,1],$$

is defined on $\hat{\mathcal{B}}_2(F(A))$. It is easy to see that

$$\hat{s}_*(\nu) \mid \hat{\mathcal{B}}_o(F(A)) = \hat{\mu}_o.$$

We shall denote by $\hat{\mu}$ the restriction of $\hat{s}_*(\nu)$ to $\hat{\mathcal{B}}(F(A))$.

LEMMA 18. *The mapping* $\hat{s} : s^{-1}(F(A)) \longrightarrow F(A)$ *is closed*.

Proof. We must prove that

$$\hat{F} \in \hat{Z}_1(s^{-1}(F(A))) \Rightarrow \hat{s}(\hat{F}) \in \hat{Z}(F(A)).$$

Indeed, let $\hat{F} = F \cap s^{-1}(F(A))$, where $F \subset E(B)$ is a compact Z_1-extremal subset. Since we have

$$\hat{s}(\hat{F}) = s(F) \cap F(A),$$

it will be sufficient to prove that $s(F) \subset E_o(A)$ is Z-extremal. Indeed for any $f_o \in s(F) \cap E(A)$ let $\mu_o \in \Omega(E_o(A); f_o)$ be any subcentral measur such that $b(\mu_o) = f_o$. Let $g_o \in F$ be such that $f_o = s(g_o)$. Then we have that $g_o \in E_1(B)$ and, by Lemma 10, there exists a measure $\nu_o \in \Omega_1(E(B); g_o$ such that $s_*(\nu_o) = \mu_o$. From $b(\nu_o) = g_o \in F$ we infer that $\nu_o(F) = 1$ and therefore, we have that

$$\mu_o(s(F)) = \nu_o(s^{-1}(s(F))) \geq \nu_o(F) = 1,$$

and the Lemma is proved.

Remark. The proof of the Lemma shows that we have the implication

$$F \in Z_1(E(B)) \Rightarrow s(F) \in Z(E_o(A)),$$

which will be used below.

We can now prove

THEOREM 26. For any central measure $\mu \in \Omega(E_o(A))$, such that $b(\mu) \in E(A)$, the corresponding measure $\hat{\mu}$ has the following properties

a) $\hat{\mu} \backslash \hat{\mathcal{B}}_o(F(A)) = \hat{\mu}_o$;

and

b) $\hat{\mu}(M_o \wedge F(A)) = \mu(M_o)$, for any $M_o \in \mathcal{B}_2(E_o(A))$;

as well as the following regularity properties

c) $\hat{\mu}(\hat{M}_o) = \sup\{\hat{\mu}(\hat{F}_o); \hat{F}_o \subset \hat{M}_o, \hat{F}_o \in \hat{Z}(F(A))\}$, for any $\hat{M}_o \in \hat{\mathcal{B}}_2(F(A))$;

and

d) $\hat{\mu}(\hat{F}_o) = \inf\{\hat{\mu}(D_o); \hat{D}_o \supset \hat{F}_o, \hat{D}_o \in \hat{\mathcal{B}}_o(F(A))\}$, for any $\hat{F}_o \in \hat{Z}(F(A))$.

Proof. Assertions a) and b) have been established just before the statement of the Theorem.

c) Let $\hat{M}_o \in \hat{\mathcal{B}}(E_o(A))$ and $\varepsilon > o$ be given. Then we have that $\hat{s}^{-1}(\hat{M}_o) \in \hat{\mathcal{B}}_2(s^{-1}(F(A)))$. By Theorem 25,a), there exists a $\hat{F} \in \hat{Z}_1(s^{-1}(F(A)))$, such that $\hat{F} \subset \hat{s}^{-1}(\hat{M}_o)$ and

$$\hat{\mu}(\hat{M}_o) - \varepsilon = \hat{\nu}(\hat{s}^{-1}(M_o)) - \varepsilon < \hat{\nu}(\hat{F}) \leq \hat{\mu}(\hat{s}(\hat{F})).$$

Of course, there exists a compact Z_1-extremal subset $F \subset E(B)$, such that $\hat{F} = F \wedge s^{-1}(F(A))$. Since by Lemma 18 we have that $\hat{s}(\hat{F}) \in \hat{Z}(F(A))$, the assertion now immediately follows.

d) This is an immediate consequence of Theorem 22. The Theorem is proved.

The regularity property c) implies that the "Borel restriction" of $\hat{\mu}$, i.e., the measure $\hat{\mu} \backslash \mathcal{B}(F(A); \hat{Z}(F(A)))$, determines $\hat{\mu}$ on $\hat{\mathcal{B}}_2(F(A))$ as the following Theorem shows.

THEOREM 27. a) $\hat{\mathcal{B}}_o(F(A)) \subset \mathcal{B}(F(A); \hat{Z}(F(A)))(\hat{\mu})$,

and

b) $\hat{\mathcal{B}}_2(F(A)) \subset \mathcal{B}(F(A); \hat{Z}(F(A)))(\hat{\mu})$.

Proof. The two statements are immediate consequences of property c) in Theorem 26.

V. Property c) in the statement of Theorem 26 could be called "the regularity by closed subsets". A more careful analysis of the situation will show that we have, in fact, the stronger property of "the regularity by closed quasi-compact subsets", which we shall present below. In fact, let us return to the proof of statement c) from Theorem 26. Let us first remark that $s^{-1}(F(A)) \subset E_1(B)$. On the other hand, since $b(\nu) \in E_1(B)$, we have that $\nu(E_1(B)) = 1$. Since $E_1(B)$ is a G_δ-subset of $E(B)$, whereas ν is a maximal (orthogonal) measure on $E(B)$, there exists a compact extremal subset $F_1 \subset E_1(B)$, such that $\nu(F_1) > 1 - \varepsilon$. Then F_1 is also Z_1-extremal and, therefore, $s(F \wedge F_1)$ is compact and Z-extremal in $E_o(A)$. Since $s(F \wedge F_1) \subset E(A)$, it is easy to show, with the help of Theorem 10, that the set $s(F \wedge F_1) \wedge F(A)$ is $\hat{Z}(F(A))$-quasi-compact. Moreover, we have that

$$\hat{\mu}(\hat{M}_o) - 2\varepsilon < \hat{\mu}(\hat{s}(\hat{F} \wedge \hat{F}_1)).$$

Thus, we have obtained the following

<u>THEOREM 28</u>. <u>For any central measure</u> $\mu \in \Omega(E_o(A))$, <u>such that</u> $b(\mu) \in E(A)$, <u>any</u> $\hat{M}_o \in \hat{\mathcal{B}}_2(F(A))$ <u>and any</u> $\varepsilon > 0$, <u>there exists a</u> $\hat{Z}$-<u>closed</u>, $\hat{Z}$-<u>quasi-compact set</u> $\hat{F}_o \subset \hat{M}_o$, <u>such that</u> $\hat{\mu}(\hat{M}_o) - \varepsilon < \hat{\mu}(\hat{F}_o)$.

Of course, this Theorem extends the well-known fact that on (Hausdorff) locally compact spaces bounded Radon measures are regular by compact subsets.

The following Theorem will be used below.

<u>THEOREM 29</u>. a) $s_*(\mathcal{B}_2(E(B))(\nu)) = \mathcal{B}_2(E_o(A))(\mu)$;

<u>and</u>

b) $\hat{s}_*(\hat{\mathcal{B}}_2(s^{-1}(F(A)))(\hat{\nu})) = \hat{\mathcal{B}}_2(F(A))(\hat{\mu})$.

<u>Proof</u>. a) Since $s : E(B) \to E_o(A)$ is $(\mathcal{B}_2(E(B)); \mathcal{B}_2(E_o(A)))$-measurable, it immediately follows that

$$\mathcal{B}_2(E_o(A))(\mu) \subset s_*(\mathcal{B}_2(E(B))(\nu)).$$

Let now $M \in s_*(\mathcal{B}_2(E(B))(\nu))$. Then, by the definition of the full direc image, we have that $s^{-1}(M) \in \mathcal{B}_2(E(B))(\nu)$. By Theorem 25 and the proof preceding it (since $\nu'' = \nu$), there exist increasing sequences $(F_n)_{n\geq}$ and $(F'_n)_{n\geq o}$, such that

$$F_n \in Z_1(E(B)),\ \ F_n \subset F_{n+1} \subset s^{-1}(M), \text{for any } n \geq o, \text{and } \nu(F_n) \uparrow \nu(s^{-1}(M))$$

$$F'_n \in Z_1(E(B)),\ F'_n \subset F'_{n+1} \subset \complement s^{-1}(M), \text{for any } n \geq o, \text{and } \nu(F'_n) \uparrow \nu(\complement s^{-1}(M)).$$

By the Remark following the proof of Lemma 18, we have that $s(F_n)$, $s(F'_n) \in Z(E_o(A))$, $n \geq o$, and we have that

$$s^{-1}(s(F_n)) \subset s^{-1}(s(F_{n+1})) \subset s^{-1}(M),\quad n \geq o,$$

$$s^{-1}(s(F'_n)) \subset s^{-1}(s(F'_{n+1})) \subset \complement s^{-1}(M),\ n \geq o,$$

and $\nu(s^{-1}(s(F_n)) \uparrow \nu(s^{-1}(M))$, $\nu(s^{-1}(s(F'_n))) \uparrow \nu(\complement s^{-1}(M))$. We infer that we have

$$s(F_n) \subset s(F_{n+1}) \subset M,\ \ s(F'_n) \subset s(F'_{n+1}) \subset \complement M,\quad n \geq o,$$

and $\mu(s(F_n)) \uparrow \mu(M)$, $\mu(s(F'_n)) \uparrow \mu(\complement M)$; it follows that $M \in \mathcal{B}_2(E_o(A))(\mu)$.

b) Similar proof to that of a), with the help of the fact that the mapping $\hat{s} = s \mid s^{-1}(F(A))$ is $(\hat{\mathcal{B}}_2(s^{-1}(F(A)));\hat{\mathcal{B}}_2(F(A))$-measurable, by taking into account also Lemma 18 and Theorem 26.

From the regularity property we can easily obtain the following extension of Lusin's Theorem, in which we keep the preceding notation.

<u>THEOREM 3o</u>. <u>For any $\hat{\mu}$ -measurable function $\varphi: F(A) \to \mathbb{C}$, and any $\varepsilon > o$, there exists a $\hat{Z}$-closed $\hat{Z}$-quasi-compact subset $\hat{F} \subset F(A)$, such that $\hat{\mu}(\hat{F}) > 1-\varepsilon$ and $\varphi \mid \hat{F}$ be continuous.</u>

<u>Proof</u>. Similar to that of ([33], Theorem 1.5).

Another important property of the measures $\hat{\mu}$ is exhibited by the following Theorem, in which $\hat{\mu}$ is, as above, the measure induced on $\mathcal{B}(F(A);\hat{Z}(F(A)))$ by any central measure $\mu \in \mathcal{M}^1_+(E_o(A))$, such that $b(\mu) \in E(A)$.

<u>THEOREM 31</u>. <u>Any measure $\hat{\mu}$ is perfect.</u>

<u>Proof</u>. Similar to that of ([33], Theorem 1.6).

§7. CENTRAL REDUCTION.

In this section we shall develop a spatial central disintegration (reduction) theory for an arbitrary cyclic representation $\pi : A \to \mathcal{L}(H)$. We shall then show that the disintegration can be extended to the Borel enveloping C^*-algebra $\mathcal{B}(A)$ of A.

I. Let $f_o \in E(A)$ and let $\mu = \mu_{f_o}$ be the corresponding central measure in $\Omega(E_o(A))$. Then, by Theorem 21, we have an induced probability measure $\hat{\mu}_o : \hat{\mathcal{B}}_o(F(A)) \to [o,1]$, whereas by Theorem 27, we have a Borel extension $\hat{\mu} : \mathcal{B}(F(A);\hat{Z}(F(A)))(\hat{\mu}) \to [o,1]$, with good regularity properties, such that $\hat{\mu}_o = \hat{\mu} | \hat{\mathcal{B}}_o(F(A))$.
We shall also denote the measure $\hat{\mu}$ by $\hat{\mu}_{f_o}$, since it is uniquely determined by $f_o \in E(A)$.

In order to carry out the central disintegration of $\pi_{f_o} : A \to \mathcal{L}(H_{f_o})$ the measure $\hat{\mu}_o$ would be sufficient, but if we want to disintegrate over F(A) the restriction of π''_{f_o} to the Borel enveloping C^*-algebra $\mathcal{B}(A)$ of A (see [36], for the definition of $\mathcal{B}(A)$; $\pi''_{f_o} : A^{**} \to \mathcal{L}(H_{f_o})$ is the canonical extension of π_{f_o} to A^{**}), then the measure $\hat{\mu}_o$ does not suffice any more, whereas $\hat{\mu}$ will do the job, as we shall see below.

<u>Remark</u>. Of course, it would be senseless to try to disintegrate over F(A) the representation π''_{f_o}, since the algebra A^{**} is too big, as shown by the commutative case. In other cases, however, as, for instance, that of the elementary C^*-algebras $A = \mathcal{K}(H)$, in which $F(A) = E(A)$, π''_{f_o} can be disintegrated over F(A), but these are the exceptions.

<u>LEMMA 19</u>. a) <u>Let $\varphi : E_o(A) \to \mathbb{C}$ be any Baire measurable complex function. Then $\varphi | F(A)$ is $\hat{\mathcal{B}}_o(F(A))$-measurable.</u>

b) <u>If $\varphi : E_o(A) \to \mathbb{C}$ is $\mathcal{B}_2(E_o(A))$-measurable, then $\varphi | F(A)$ is $\hat{\mathcal{B}}_2(F(A))$-measurable.</u>

<u>In either case, if φ is μ-integrable, then $\varphi | F(A)$ is $\hat{\mu}$-integrable and</u>

$$\int_{E_o(A)} \varphi \, d\mu = \int_{F(A)} \varphi \, d\hat{\mu}.$$

<u>Proof</u>. Similar to that of ([28], Lemma 1.1), by approximating by elementary functions.

II. As in ([28], §4), we shall consider the field of Hilbert spaces

$E_o(A) \ni f \mapsto H_f$, which we shall denote by $(H_f)_{f \in E_o(A)}$. We have also the associated field of triples $E_o(A) \ni f \mapsto (\pi_f, H_f, \xi_f^o)$, corresponding to the GNS-construction. If we define $\theta_f : A \to H_f$ by $\theta_f(a) = \pi_f(a)\xi_f^o$, $a \in A$, $f \in E_o(A)$, then we have a linear mapping $\theta : A \to \prod_{f \in E_o(A)} H_f$, given by $\theta(a) = (\theta_f(a))_{f \in E_o(a)}$, $a \in A$. Let $\Gamma_o(A) = \operatorname{im} \theta$; then $\Gamma_o(A)$ is a vector subspace of $\prod_{f \in E_o(A)} H_f$.

As in ([28], §4), we can consider the L^2-completion $\Gamma^2(A;\mu)$ of $\Gamma_o(A)$. Of course, $\Gamma^2(A;\mu)$ is a vector subspace of $\prod_{f \in E_o(A)} H_f$.

Since the measure μ is orthogonal, the system

$$\text{(o)} \qquad ((H_f)_{f \in E_o(A)},\ \Gamma^2(A;\mu), \mu)$$

is an integrable field of Hilbert spaces, in the sense of W.Wils (see [28], Theorem 4.1). Of course, the scalar product in $\Gamma_o(A)$ is given by the formula

$$((\theta_f(a))_{f \in E_o(A)} \mid (\theta_f(b))_{f \in E_o(A)}) = \int_{E_o(A)} f(b^* a)\, d\mu(f),$$

for any $a, b \in A$, whereas for two vector fields $(\xi_f)_{f \in E_o(A)}, (\eta_f)_{f \in E_o(A)} \in \Gamma^2(A;\mu)$, it is given by the formula

$$((\xi_f)_{f \in E_o(A)} \mid (\eta_f)_{f \in E_o(A)}) = \int_{E_o(A)} (\xi_f \mid \eta_f)_f\, d\mu(f).$$

Endowed with the corresponding semi-norm, $\Gamma^2(A;\mu)$ is a possibly non-Hausdorff, but complete, pre-Hilbert space. Moreover, since μ is orthogonal, for any bounded Borel measurable function $\varphi : E_o(A) \to \mathbb{C}$ we have the implication

$$(\xi_f)_{f \in E_o(A)} \in \Gamma^2(A;\mu) \Rightarrow (\varphi(f)\,\xi_f)_{f \in E_o(A)} \in \Gamma^2(A;\mu).$$

We shall denote by $\tilde{\Gamma}^2(A;\mu)$ the associated (Hausdorff, complete) Hilbert space, and by $\rho_\mu : \Gamma^2(A;\ ) \to \tilde{\Gamma}^2(A;\mu)$ the corresponding canonical surjection.

We shall also consider the "restricted" field of Hilbert spaces $F(A) \ni f \mapsto H_f$, which we shall denote by $(H_f)_{f \in F(A)}$.

We can now define the linear mapping $\hat{\theta} : A \to \prod_{f \in F(A)} H_f$, given by $\hat{\theta}(a) = (\theta_f(a))_{f \in F(A)}$, $a \in A$. Let us define $\hat{\Gamma}_o(A) = \operatorname{im} \hat{\theta}$. Then $\hat{\Gamma}_o(A)$ is a vector subspace of $\prod_{f \in F(A)} H_f$ and we have

$$(1)\qquad f_o(a_2^* a_1) = \int_{E_o(A)} f(a_2^* a_1)\, d\mu(f) = \int_{F(A)} f(a_2^* a_1)\, d\hat{\mu}(f)$$

for any $a_1, a_2 \in A$, by virtue of Lemma 19. We can, therefore, define a scalar product on $\hat{\Gamma}_o(A)$ by the formula

$$(2)\qquad \begin{aligned} (\hat{\theta}(a_1) \mid \hat{\theta}(a_2)) &= \int_{F(A)} f(a_2^* a_1)\, d\hat{\mu}(f) = \\ &= \int_{F(A)} (\theta_f(a_1) \mid \theta_f(a_2))\, d\hat{\mu}(f), \qquad a_1, a_2 \in A. \end{aligned}$$

We can consider the "restriction mapping" $\rho : \Gamma_o(A) \to \hat{\Gamma}_o(A)$, given by $\rho(\theta(a)) = \hat{\theta}(a)$, $a \in A$, which is a unitary linear surjection, by formulas (1) and (2).

Let $\hat{\Gamma}^2(A;\hat{\mu})$ be the L^2-completion of $\hat{\Gamma}_o(A)$ with respect to the measure $\hat{\mu}$, constructed as in ([28], §4). Then $\hat{\Gamma}^2(A;\hat{\mu})$ is a vector subspace of $\prod_{f \in F(A)} H_f$, on which the scalar product

$$(2')\qquad ((\xi_f)_{f \in F(A)} \mid (\eta_f)_{f \in F(A)}) = \int_{F(A)} (\xi_f \mid \eta_f)_f\, d\hat{\mu}(f),$$

for $(\xi_f)_{f \in F(A)}, (\eta_f)_{f \in F(A)} \in \hat{\Gamma}^2(A;\hat{\mu})$, is defined, along with the associated semi-norm $\|\cdot\|$, given by

$$(2'')\qquad \|(\xi_f)_{f \in F(A)}\|^2 = \int_{F(A)} \|\xi_f\|_f^2\, d\hat{\mu}(f),$$

for any $(\xi_f)_{f \in F(A)} \in \Gamma^2(A;\hat{\mu})$.

Proposition 4.2 from [28] implies that $\hat{\Gamma}^2(A;\hat{\mu})$ is a complete, possibly non-Hausdorff, pre-Hilbert space.

The restriction mapping ρ extends to the restriction mapping $\rho_\mu : \Gamma^2(A;\mu) \to \hat{\Gamma}^2(A;\hat{\mu})$, given by $\rho_\mu((\xi_f)_{f \in E_o(A)}) = (\xi_f)_{f \in F(A)}$, for $(\xi_f)_{f \in E_o(A)} \in \Gamma^2(A;\mu)$. It is obvious that we have

$$(3)\qquad \int_{E_o(A)} (\xi_f \mid \eta_f)_f\, d\mu(f) = \int_{F(A)} (\xi_f \mid \eta_f)_f\, d\hat{\mu}(f),$$

for any $(\xi_f)_{f \in E_o(A)}, (\eta_f)_{f \in E_o(A)} \in \Gamma^2(A;\mu)$. It immediately follows that we have the implication

$$(4)\qquad (\xi_f)_{f \in F(A)} \in \hat{\Gamma}^2(A;\hat{\mu}) \Longrightarrow (\varphi(f)\xi_f)_{f \in F(A)} \in \hat{\Gamma}^2(A;\hat{\mu}),$$

for any bounded $\hat{\mu}$-measurable function $\varphi : F(A) \to \mathbb{C}$ (by stability properties, any $\hat{\mu}$-measurable function on $F(A)$ coincides $\hat{\mu}$-a.e. with

the restriction to $F(A)$ of a Baire measurable function on $E_o(A)$).
From (4) we immediately infer that the system

$$((H_f)_{f\in F(A)}, \hat{\Gamma}^2(A;\hat{\mu}), \hat{\mu})$$

is an integrable field of Hilbert spaces, in the sense of W.Wils.

Let $q:\Gamma^2(A;\mu)\to\tilde{\Gamma}^2(A;\mu)$, respectively $\hat{q}:\hat{\Gamma}^2(A;\hat{\mu})\to\tilde{\hat{\Gamma}}^2(A;\hat{\mu})$, be the canonical unitary linear mappings onto the corresponding Hilbert direct integrals of the fields $\Gamma^2(A;\mu)$, resp., $\hat{\Gamma}^2(A;\hat{\mu})$; here $\tilde{\Gamma}^2(A;\mu)$, resp., $\tilde{\hat{\Gamma}}^2(A;\hat{\mu})$, are the associated (Hausdorff, complete) Hilbert spaces and they are, sometimes, less properly denoted as

$$\int_{E_o(A)}^{\oplus} H_f\, d\mu(f) \ , \quad \text{resp.,} \quad \int_{F(A)}^{\oplus} H_f\, d\hat{\mu}(f) \ .$$

It is easy to see that the mappings

$$u:\Gamma_o(A)\ni\vartheta(a)\mapsto\pi_{f_o}(a)\xi^o_{f_o}\in H_{f_o}\ , \qquad a\in A,$$

and

$$\hat{u}:\hat{\Gamma}_o(A)\ni\hat{\vartheta}(a)\mapsto\pi_{f_o}(a)\xi^o_{f_o}\in H_{f_o}\ , \qquad a\in A,$$

are correctly defined unitary linear mappimgs. They extend in a unique manner to unitary linear mappings

$$U_\mu:\Gamma^2(A;\mu)\longrightarrow H_{f_o} \quad \text{and} \quad \hat{U}_{\hat{\mu}}:\hat{\Gamma}^2(A;\hat{\mu})\longrightarrow H_{f_o}\ ,$$

which clearly factorize through q, respectively $\hat{q}$

$$U_\mu=\tilde{U}_\mu\circ q\ , \quad \text{resp.,} \quad \hat{U}_{\hat{\mu}}=\tilde{\hat{U}}_{\hat{\mu}}\circ\hat{q},$$

where $\tilde{U}_\mu:\tilde{\Gamma}^2(A;\mu)\longrightarrow H_{f_o}$, resp., $\tilde{\hat{U}}_{\hat{\mu}}:\tilde{\hat{\Gamma}}^2(A;\hat{\mu})\longrightarrow H_{f_o}$, are uniquely determined unitary isomorphisms of Hilbert spaces.

The preceding construction yields the <u>central reduction</u> of the representation π_f: namely, we can consider, for any $a\in A$, the field of operators $(\pi_f(a^\rho))_{f\in F(A)}$, which is $\hat{\mu}$-<u>integrable</u> in the sense that

$$(\xi_f)_{f\in F(A)}\in\hat{\Gamma}^2(A;\hat{\mu}) \implies (\pi_f(a)\xi_f)_{f\in F(A)}\in\hat{\Gamma}^2(A;\hat{\mu})\ ,$$

for any $a\in A$. It is easy to see that we have

(5) $$\hat{U}[(\pi_f(a)\xi_f)_{f\in F(A)}] = \pi_{f_o}(a)\,\hat{U}[(\xi_f)_{f\in F(A)}]\ ,$$

for any $a \in A$ and any $(\xi_f)_{f\in F(A)} \in \hat{\Gamma}^2(A;\hat{\mu})$.

By Theorem 6, with the help of the considerations made in §4.5, we infer that there exists a surjective homomorphism of von Neumann algebras $\hat{\Phi}: Z(A^{**}) \to L^\infty(\hat{\mu})$, such that

$$(6) \qquad \hat{U}[(\pi_f(a)\hat{\Phi}(z)(f)\xi_f)_{f\in F(A)}] = \pi''_{f_0}(z)\,\pi_{f_0}(a)\,\hat{U}[(\xi_f)_{f\in F(A)}],$$

for any $a\in A, z\in Z(A^{**})$, and $(\xi_f)_{f\in F(A)} \in \hat{\Gamma}^2(A;\hat{\mu})$.

(Here the function $F(A) \ni f \mapsto \hat{\Phi}(z)(f)$ is, for any $z\in Z(A^{**})$, an arbitrarily chosen $\hat{\mu}$-measurable representative of $\hat{\Phi}(z)$).

Formula (5) yields the central reduction (disintegration) of the representation π_{f_0}. Formula (6) shows that, by this disintegration, the von Neumann algebra of the diagonalizable operators corresponds to the center of $\pi_{f_0}(A)''$.

Given a field of operators $\hat{a} = (a_f)_{f\in F(A)}$, $a_f \in \mathcal{L}(H_f), f\in F(A)$, we shall say that it is $\hat{\mu}$-integrable (here $\hat{\mu}$ is the measure on $F(A)$ corresponding to a central measure $\mu\in \mathcal{M}^1_+(E_0(A))$, such that $f_0 = b(\mu) \in$ $\in E(A)$), if the following conditions hold

a) $\hat{\mu}$-vrai sup$\{\|a_f\|;\ f\in F(A)\} < +\infty$;

and

b) $(\xi_f)_{f\in F(A)} \in \hat{\Gamma}^2(A;\hat{\mu}) \Rightarrow (a_f\xi_f)_f \in \hat{\Gamma}^2(A;\hat{\mu})$.

It is obvious that to any such field of operators there corresponds an operator $\tilde{\hat{a}}_\mu \in \mathcal{L}(\tilde{\hat{\Gamma}}^2(A;\hat{\mu}))$. It is easy to see that $\tilde{\hat{U}}\,\tilde{\hat{a}}_\mu\,\tilde{\hat{U}}^{-1} \in$ $\in (\pi_{f_0}(A)' \cap \pi_{f_0}(A)'')'$.

The field of operators $(a_f)_{f\in F(A)}$ will be said to be universally centrally integrable, if it is $\hat{\mu}$-integrable, for any central measure $\mu\in\mathcal{M}^1_+(E_0(A))$, such that $b(\mu)\in E(A)$.

Remark. In the preceding argumentation we had to consider also the integrable field of Hilbert spaces (o), in order to derive, with the help of Theorem 4.1 from [28], the implication (4), which means that $\hat{\Gamma}^2(A;\hat{\mu})$ is an $\mathcal{L}^\infty(\hat{\mu})$-module. This property, together with the completeness with respect to the semi-norm (2"), is an essential ingredient in Wils' definition of the integrable fields of Hilbert spaces (see [39]; and, also, [28], §4).

III. As one can easily see, the central reduction of a representation π_{f_0} of A can be carried out only with the help of the measure $\hat{\mu}_0$: $\hat{\mathcal{B}}_0(F(A)) \to [0,1]$. The need for an extension of $\hat{\mu}_0$ appears when one wants to disintegrate extensions of π_{f_0} to larger C^*-algebras. We shall prove now that, with the help of the completion of the measure

$\hat{\mu}:\mathcal{B}(F(A));\hat{Z}(F(A)))\to[0,1]$,one can carry out the central disintegration of the representation $\pi''_{f_0}:\mathcal{B}(A)\to\mathcal{L}(H_{f_0})$;i.e.,of the restriction of the representation π''_{f_0} to the Borel enveloping C^*-algebra $\mathcal{B}(A)$ of A.We refer to [36] for the results on $\mathcal{B}(A)$ we shall use below.

LEMMA 20.Let K be any compact convex subset of a Hausdorff locally convex topological real vector space,let $\mu\in\mathcal{M}_+^1(K)$ be any (Choquet-Meyer) maximal Radon probability measure and let $h_o:K\to\mathbb{R}$ be any semi-continuous affine real function.Then h_o is measurable with respect to the completion $\mathcal{B}_1(K)(\mu)$ of $\mathcal{B}_1(K)$,with respect to $\mu|\mathcal{B}_1(K)$.

Proof.We recall that $\mathcal{B}_1(K)$ is the σ-algebra of subsets of K,generated by $\mathcal{B}_o(K)$ and by the set $\mathcal{F}(K)$ of all compact extremal subsets of K.For the proof,we shall use the notations and results from ([31],p.11-14).Indeed,let $\mu_1=\mu|\mathcal{B}_1(K)$.Then,by ([31],Theorem 2), we have that $(\mu_1)_*(F_a)=\mu(F_a)$, $a\in\mathbb{R}$.

In order to prove that $\mu_1^*(F_a)\leq\mu(F_a)$,it will be sufficient (equivalent) to prove that $\mu(E_a)\leq(\mu_1)_*(E_a)$.To this end,we shall consider the set $G_a'=G_a\cap\Gamma(h_o)$.It is obvious that $E_a=p(G_a')$.Since G_a is a compact Baire measurable subset of K_o,by ([29],Corollary to Theorem 1),for any $\varepsilon>0$,there exists a compact extremal subset $D_1'\subset G_a$,such that $\nu(D_1)>\nu(G_a)-\varepsilon$.Then,for $D_o'=D\cap D_1'$,we have that D_o' is a compact extremal subset of $G_a\cap\Gamma(h_o)$ and $\nu(D_o')>\nu(G_a)-2\varepsilon$.We infer that $p(D_o')$ is a compact extremal subset of E_a,and $\mu(p(D_o'))>\mu(E_a)-2\varepsilon$. We immediately infer that $(\mu_1)_*(E_a)\geq\mu(E_a)$,and the Lemma is proved.

Remark 1.As proved in ([19],Satz 2.1),any semi-continuous affine real function $h_o:K\to\mathbb{R}$ is bounded.This is an immediate consequence of the barycentric calculus,which holds for such functions.

Remark 2.The preceding Lemma was contained in an answer given to a question put by G.Păltineanu in a private conversation.

We recall that by A_{sa}^m one denotes the subset of all lower semicontinuous elements in A_{sa}^{**} ,over A;by $\mathcal{U}(A)$ one denotes the real vector subspace of A_{sa}^{**} ,consisting of all the (strongly) universally measurable elements over A (see [22],p.104;[36],p.7).

LEMMA 21.a)For any $a\in\mathcal{U}(A)$,the function $\lambda_A(a)$ is $(\mu|\mathcal{B}_2(E_o(A)))$-measurable,for any central measure $\mu\in\mathcal{M}_+^1(E_o(A))$,such that $f=b(\mu)\in E(A)$.

b)For any $a\in\mathcal{U}(A)$,the function $\lambda_A(a)|F(A)$ is $\hat{\mu}$-measurable,for any central measure $\mu\in\mathcal{M}_+^1(E_o(A))$,such that $f=b(\mu)\in E(A)$.

c)We have that

$$(*)\qquad f(a) = \int_{E_o(A)} \lambda_A(a)\, d\mu = \int_{F(A)} \lambda_A(a)\, d\hat{\mu} ,$$

for any $a \in \mathcal{U}(A)$ and any central measure $\mu \in \mathcal{M}_+^1(E_o(A))$, such that $f = b(\mu) \in E(A)$.

Proof. a) Let $a \in A_{sa}^m$; then there exists a (bounded) increasing net $(a_\alpha)_\alpha$ in A_{sa}, such that

$$\lambda_A(a_\alpha) \uparrow \lambda_A(a)$$

on $E_o(A)$. Let $b_\alpha = \pi_u(a_\alpha) \in B$; then $(b_\alpha)_\alpha$ is a (bounded) increasing net in B and, therefore, we have $b_\alpha \Uparrow b$ in B_{sa}^{**}, where $b \in B_{sa}^m$. It is easy to see that

$$(1)\qquad \lambda_A(a) \circ \ddot{s} = \lambda_B(b).$$

Let $\nu \in \mathcal{M}_+^1(E(B))$ be the maximal orthogonal measure on E(B), corresponding to μ, as in §6.4. Then we have $s_*(\nu) = \mu$ and, by Lemma 20, the function $\lambda_B(b)$ is $(\nu | \mathcal{B}_1(E(B)))$-measurable; since we have that

$$\mathcal{B}_1(E(B)) \subset \mathcal{B}_2(E(B)),$$

it follows that $\lambda_B(b)$ is $(\nu | \mathcal{B}_2(E(B)))$-measurable. By formula (1) and Theorem 29, we infer that $\lambda_A(a)$ is $(\mu | \mathcal{B}_2(E_o(A)))$-measurable.

b) This follows immediately from statement a) and from Theorem 26, b).

c) This follows imm ediately from the fact that the barycentric calculus holds (with respect to any measure in $\mathcal{M}_+^1(E_o(A))$) for any semi-continuous affine real function on $E_o(A)$, and also from Lemma 19.

The Lemma is thus proved for semi-continuous elements in A_{sa}^{**} (over A).

For the case of an arbitrary $a \in \mathcal{U}(A)$, one can use the method of proof given in ([31], proof of Theorem 3). The Lemma is proved.

LEMMA 22. For any $a \in \mathcal{U}(A)$ and any $b, c \in A$, the function

$$F(A) \ni f \longmapsto f(c^* ab) \in \mathbb{C}$$

is $\hat{\mu}$-measurable, for any central measure $\mu \in \mathcal{M}_+^1(E_o(A))$, such that $f_o = b(\mu) \in E(A)$, and we have the equality

$$f_o(c^*ab) = \int_{F(A)} f(c^*ab)\, d\hat{\mu}(f).$$

Proof. By ([36], Lemma 7), we have that

$$b^*\mathcal{U}(A)b \subset \mathcal{U}(A),$$

for any $b \in A$. From the polarization formula

$$f(c^*ab) = \frac{1}{4}[f((b+c)^*a(b+c))-f((b-c)^*a(b-c))+$$

$$+if((b+ic)^*a(b+ic))-if((b-ic)^*a(b-ic))],$$

and from Lemma 21 the assertion now immediately follows.

By analogy with the definition given in ([36], p.24), we shall say that an element $a \in A^{**}$ is universally centrally disintegrable if the following conditions are satisfied

a) The field of operators $(\pi_f''(a))_{f\in F(A)}$ is universally centrally integrable;

and, if we denote by $\pi''_{\hat{\mu}_{f_o}}(a)$ the operator in $\mathcal{L}(\tilde{\hat{\Gamma}}^2(A;\hat{\mu}))$, determined by the field $(\pi_f''(a))_{f\in F(A)}$, we should also have

b) $\tilde{\hat{U}}_{\hat{\mu}_{f_o}} \pi''_{\hat{\mu}_{f_o}}(a) = \pi''_{f_o}(a)\tilde{\hat{U}}_{\hat{\mu}_{f_o}}$,

where $f_o = b(\mu_{f_o})$ is the barycenter of the central measure $\mu_{f_o} \in \mathcal{M}_+^1(E(A))$ to which $\hat{\mu}_{f_o}$ corresponds, for any $f_o \in E(A)$.

With a similar proof as for ([36], Lemma 5), we can state

LEMMA 23. i) *If $a, b \in A^{**}$ are universally centrally disintegrable, then $a+b$ and ab are universally centrally disintegrable, and αa is universally centrally disintegrable, for any $\alpha \in \mathbb{C}$.*

ii) *The norm limit of a sequence of universally centrally disintegrable elements of A^{**} is universally centrally disintegrable.*

iii) *If $(a_n)_{n\in\mathbb{N}}$ is a bounded monotone sequence of universally centrally disintegrable elements in A^{**}_{sa}, then its limit is a universally centrally disintegrable element.*

We can also prove

LEMMA 24. *For any universally centrally disintegrable element $a \in A^{**}$, the formula*

$$\int_{F(A)} f(c^*ab)\, d\hat{\mu}_{f_o}(f) = f_o(c^*ab)$$

holds for any $b, c \in A$. In particular, the "central barycentric calculus" holds for a; i.e.,

$$f_o(a) = \int_{F(A)} f(a)\, d\hat{\mu}_{f_o}(f).$$

Proof. Since $(\theta_f(b))_{f\in F(A)} \in \hat{\Gamma}^2(A;\hat{\mu}_{f_o})$, we also have $(\pi''_f(a)\theta_f(b))_f \in \hat{\Gamma}^2(A;\hat{\mu}_{f_o})$, and for any $b, c \in A$ we shall have

$$\int_{F(A)} f(c^*ab)\, d\hat{\mu}(f) = \int_{F(A)} (\pi''_f(a)\theta_f(b)|\theta_f(b))_f\, d\hat{\mu}(f) =$$

$$= ((\pi''_f(a)\theta_f(b))_f\,|\,(\theta_f(c))_f) = (\hat{U}_{\hat{\mu}}(\pi''_f(a)\theta_f(b))_f\,|\,(\theta_f(c))_f) =$$

$$= (\pi''_{f_o}(a)\theta_{f_o}(b)|\theta_{f_o}(c)) = f_o(c^*ab).$$

On the other hand, since we have that $(\xi^o_f)_{f\in F(A)} \in \hat{\Gamma}^2(A;\hat{\mu})$, (see [28], Proposition 4.5 and Theorem 4.3), we immediately infer that

$$\int_{F(A)} f(a)\, d\hat{\mu}_{f_o}(f) = f_o(a).$$

The Lemma is proved.

Let $\mathcal{C}(A) \subset A^{**}$ be the set of all universally centrally disintegrable elements in A^{**}. We define

$$\mathcal{C}_o(A) = \mathcal{C}(A) \cap \mathcal{C}(A)^* ,$$

where $\mathcal{C}(A)^* = \{ a^* \in A^{**} ;\ a \in \mathcal{C}(A) \}$.

THEOREM 32. $\mathcal{C}_o(A)$ is a C^*-algebra whose self-adjoint part is sequentially monotone closed.

Proof. Immediate consequence of Lemma 23.

As in ([36], p.7) we denote by $\mathcal{B}^o_{sa}(A)$ the smallest real vector subspace of A^{**}_{sa}, which contains $(A_{sa})^m$ and is closed with respect to the sequential bounded monotone convergence in A^{**}_{sa}; convergence which is to be understood either with respect to the order relation in A^{**}_{sa} or, equivalently, strongly on the space H_u.

LEMMA 25. Any $a \in \mathcal{B}^o_{sa}(A)$ is universally centrally disintegrable.

Proof. By ([36], §4) we have that

$$\mathcal{B}^{o}_{sa}(A) \subset \mathcal{U}(A)$$

and, by ([36], Proposition 1) we have that

$$a \in \mathcal{B}^{o}_{sa}(A) \Longrightarrow a^2 \in \mathcal{B}^{o}_{sa}(A).$$

We shall now consider the vector space $\hat{\Gamma}_a(A) \subset \prod_{f\in F(A)} H_f$, defined by

$$\hat{\Gamma}_a(A) = \{(\pi''_f(a)\theta_f(b) + \theta_f(c))_{f\in F(A)}\ ;\ b,c \in A\}.$$

It is obvious that

$$\hat{\Gamma}_o(A) \subset \hat{\Gamma}_a(A) \subset \prod_{f\in F(A)} H_f .$$

Since we have that

$$f((ab_2+c_2)^*(ab_1+c_1)) = f(b_2^*a^2b_1)+f(b_2^*ac_1)+f(c_2^*ab_1)+f(c_2^*c_1),$$

for any $f \in E_o(A)$ and any $b_1, b_2, c_1, c_2 \in A$, we infer that the function

$$F(A) \ni f \longmapsto (\xi_f | \eta_f)_f \in \mathbb{C}$$

is $\hat{\mu}$-integrable, for any $(\xi_f)_{f\in F(A)}$, $(\eta_f)_{f\in F(A)} \in \hat{\Gamma}_a(A)$. We can, therefore, consider the L^2-completion $\hat{\Gamma}^2_a(A;\hat{\mu})$ of $\hat{\Gamma}_a(A)$, with respect to $\hat{\mu}$ (see [28], §4). We can define correctly a linear mapping

$$V_a : \hat{\Gamma}_a(A) \longrightarrow H_{f_o} ,$$

by the formula

$$V_a [(\pi''_f(a)\theta_f(b) + \theta_f(c))_{f\in F(A)}] = \pi''_{f_o}(a)\theta_{f_o}(b) + \theta_{f_o}(c),$$

for any $b, c \in A$. The correctness of the definition follows from the fact that if $f((ab+c)^*(ab+c)) = o$, for any $f \in F(A)$, then, by Lemma 24, we have

$$f_o((ab+c)^*(ab+c)) = \int_{F(A)} f((ab+c)^*(ab+c))\, d\hat{\mu}_{f_o}(f) = o,$$

and this implies that $\pi''_{f_o}(a)\theta_{f_o}(b)+\theta_{f_o}(c)=o.$

It is easy to see that V_a is an isometric linear mapping;therefore,it extends uniquely to an isometric linear surjective mapping

$$W_a:\hat{\Gamma}^2_a(A;\hat{\mu})\longrightarrow H_{f_o}.$$

We infer that

$$\hat{\Gamma}^2(A;\hat{\mu})=\hat{\Gamma}^2_a(A;\hat{\mu}),$$

and,therefore,for any $b\in A$,there exists a strongly integrable vector field $(\xi_f)_{f\in F(A)}\in\hat{\Gamma}^2(A;\hat{\mu})$,such that $\pi''_f(a)\theta_f(b)=\xi_f$, $\hat{\mu}$-a.e.We infer that

$$(\xi_f)_{f\in F(A)}\in\hat{\Gamma}^2(A;\hat{\mu})\Longrightarrow(\pi''_f(a)\xi_f)_{f\in F(A)}\in\hat{\Gamma}^2(A;\hat{\mu}).$$

On the other hand,from the formula

$$\int_{F(A)}f(b^*a^2b)\,d\hat{\mu}_{f_o}(f)=f_o(b^*a^2b)=\|\pi''_{f_o}(a)\theta_{f_o}(b)\|^2,$$

which holds for any $b\in A$,we infer that

$$\tilde{\hat{U}}_{\hat{\mu}_{f_o}}\pi''_{\hat{\mu}_{f_o}}(a)=\pi''_{f_o}(a)\tilde{\hat{U}}_{\hat{\mu}_{f_o}},$$

and the Lemma is proved.

We can now prove the main result of the paper:

THEOREM 33.Any $a\in\mathcal{B}(A)$ is universally centrally disintegrable.

Proof.Let us define

$$\mathcal{A}(A)=\left\{\sum_{k=1}^{n_1}a'_{k1}\cdots a'_{km_1}+i\sum_{k=1}^{n_2}a''_{k1}\cdots a''_{km_2}\ ;\ a'_{kl},a''_{kl}\in\mathcal{B}^o_{sa}(A)\right\};$$

then $\mathcal{A}(A)$ is a $*$-subalgebra of $\mathcal{B}(A)$,and its norm closure $\mathcal{B}_1(A)$ is a C^*-subalgebra of $\mathcal{B}(A)$.From Lemma 25 and from part i) of Lemma 23 we infer that any element $a\in\mathcal{A}(A)$ is universally centrally disintegrable.From part ii) of Lemma 25 we infer that any element in $\mathcal{B}_1(A)$ is universally centrally disintegrable.If we denote

$$\mathcal{M}=\{a\in\mathcal{B}(A)_{sa}\ ;\ a\text{ is univ. centr. disitegr.}\},$$

we infer that

$$\mathcal{B}_1(A)_{sa} \subset \mathcal{M} \subset \mathcal{B}(A)_{sa} .$$

By part iii) of Lemma 25 we infer that $\mathcal{M}$ is a sequentially monotone closed subset of A^{**}_{sa}. Lemma 4 from [36] implies that $\mathcal{M} = \mathcal{B}(A)_{sa}$, whereas part i) of Lemma 25 now ends the proof.

Remark. The preceding Theorem is a generalization to the general, possibly non-separable, case of a Theorem of Sakai (see [24], Theorem 3.5.2). It is clear that even for the commutative (non-separable) case the use of the measures $\hat{\mu}$, instead of the measures $\hat{\mu}_o$, is essential for the obtention of this generalization.

IV. In this section we continue the study of the behaviour of the "Borel enveloping C^*-algebra" functor $A \mapsto \mathcal{B}(A)$ with respect to C^*-algebra morphisms $\pi : A \longrightarrow B$, which we began in [36].

We recall that by virtue of ([36], Theorem 2), for any surjective C^*-algebra morphism $\pi : A \longrightarrow B$, if we denote by π^{**} the second transpose mapping $\pi^{**} : A^{**} \longrightarrow B^{**}$, we have $\pi^{**}(\mathcal{B}(A)) = \mathcal{B}(B)$.

Let A and B be two C^*-algebras and let $\pi : A \longrightarrow B$ be an injective C^*-homomorphism, which can be, in particular, an inclusion $A \subset B$. Then it is easy to prove that π^{**} is injective, too.

THEOREM 34. $\pi^{**}(\mathcal{B}(A)) \subset \mathcal{B}(B)$.

Proof. a) Let $\alpha \mapsto a_\alpha \in A_{sa}$ be a bounded increasing net, such that $a_\alpha \uparrow a \in A^m_{sa}$; then $\alpha \mapsto \pi(a_\alpha)$ is an increasing net in B and $\pi(a_\alpha) \uparrow \pi^{**}(a)$. It follows that $\pi^{**}(a) \in B^m_{sa}$. We infer that $\pi^{**}(A^m_{sa}) \subset B^m_{sa}$, and, therefore, we have that $\pi^{**}(A^m_{sa} - A^m_{sa}) \subset B^m_{sa} - B^m_{sa} \subset \mathcal{B}^o_{sa}(B)$ (here, as in [36], we denote by $\mathcal{B}^o_{sa}(A)$, respectively $\mathcal{B}^o_{sa}(B)$, the smallest real vector subspace of A^{**}_{sa}, respectively of B^{**}_{sa}, which contains the cone A^m_{sa}, respectively B^m_{sa}, and is closed in A^{**}_{sa}, respectively in B^{**}_{sa}, with respect to the bounded sequential monotone convergence). If we denote

$$\mathcal{V} = \{ a \in A^{**}_{sa} ; \pi^{**}(a) \in \mathcal{B}^o_{sa}(B) \} ,$$

then it is easy to see that $\mathcal{V}$ is a real vector subspace of A^{**}_{sa}, which contains A^m_{sa} and it is closed with respect to the bounded sequential monotone convergence. It follows that $\mathcal{B}^o_{sa}(A) \subset \mathcal{V}$ and, therefore, we have that

$$(1) \qquad \pi^{**}(\mathcal{B}^o_{sa}(A)) \subset \mathcal{B}^o_{sa}(B).$$

b) Let us now denote by $\mathcal{A}(A)$, resp., $\mathcal{A}(B)$, the $*$-algebras defined as in the proof of Theorem 33, for the C^*-algebras A, resp., B. Then, from (1), we infer that

$$\pi^{**}(\mathcal{A}(A)) \subset \mathcal{A}(B). \tag{2}$$

Since the self-adjoint part $\mathcal{B}(A)_{sa}$ of $\mathcal{B}(A)$ is the smallest real vector subspace of A^{**}_{sa}, which contains the self-adjoint part $\mathcal{A}(A)_{sa}$ of $\mathcal{A}(A)$ and is closed with respect to bounded sequential monotone convergence (see [36], Lemma 4, and take into account the fact that from (2) it immediately follows that

$$\pi^{**}(\mathcal{B}_1(A)) \subset \mathcal{B}_1(B),$$

where $\mathcal{B}_1(A)$, resp., $\mathcal{B}_1(B)$, is the norm closure of $\mathcal{A}(A)$, resp., $\mathcal{A}(B)$) from (2) we immediately infer that

$$\pi^{**}(\mathcal{B}(A)) \subset \mathcal{B}(B),$$

and the Theorem is proved.

We can now prove the following

THEOREM 35. *For any morphism $\pi: A \longrightarrow B$ of C^*-algebras we have that*

$$\pi^{**}(\mathcal{B}(A)) \subset \mathcal{B}(B).$$

Proof. Let $q: A \longrightarrow A/_{\ker\pi}$ be the canonical surjective C^*-morphism and denote by $j: A/_{\ker\pi} \longrightarrow B$ the canonical injective C^*-morphism, such that $j \circ q = \pi$. By ([36], Theorem 2) we have that

$$q^{**}(\mathcal{B}(A)) = \mathcal{B}(A/_{\ker\pi}), \tag{1}$$

whereas by Theorem 34 above, we have that

$$j^{**}(\mathcal{B}(A/_{\ker\pi})) \subset \mathcal{B}(B). \tag{2}$$

Since we have that $\pi^{**} = j^{**} \circ q^{**}$, from (1) and (2) we infer that

$$\pi^{**}(\mathcal{B}(A)) = j^{**}(q^{**}(\mathcal{B}(A))) = j^{**}(\mathcal{B}(A/_{\ker\pi})) \subset \mathcal{B}(B),$$

and the Theorem is proved.

V. For the elements belonging to the center $Z(\mathcal{B}(A))$ of the C^*-algebra $\mathcal{B}(A)$ we can obtain an explicit expression for the element $\Phi_{f_o}(z) \in L^{\infty}(\hat{\mu}_{f_o})$, given by Sakai's mapping Φ_{f_o}, $f_o \in E(A)$ (see §2.4, above).

Indeed, let us first remark that for any $z \in Z(A^{**})$ and any $a \in A^{**}$, we have that

$$f(za) = f(z)f(a), \qquad f \in F(A). \tag{1}$$

On the other hand we have the equality

$$Z(\mathcal{B}(A)) = \mathcal{B}(A) \cap Z(A^{**}). \tag{2}$$

Let now $f_o \in E(A)$ and $z \in Z(\mathcal{B}(A))$ be given. Then, by Lemma 24 and Theorem 33 we have that

$$f_o(za) = \int_{E_o(A)} \lambda_A(za)\, d\mu_{f_o} = \int_{F(A)} \lambda_A(za)\, d\hat{\mu}_{f_o} =$$

$$= \int_{F(A)} \lambda_A(z)\lambda_A(a)\, d\hat{\mu}_{f_o} = (K_{\hat{\mu}_{f_o}}([\lambda_A(z)])\pi_{f_o}(a)\xi^o_{f_o} | \xi^o_{f_o}) =$$

$$= (\pi''_{f_o}(z)\,\pi_{f_o}(a)\xi^o_{f_o} | \xi^o_{f_o}),$$

for any $a \in A$, and this implies that

$$\Phi_{f_o}(z) = [\lambda_A(z)] \pmod{\hat{\mu}_{f_o}}.$$

We have thus proved the following

THEOREM 36. *For any state* $f_o \in E(A)$ *and any* $z \in Z(\mathcal{B}(A))$ *we have that*

$$\Phi_{f_o}(z) = [\lambda_A(z)] \pmod{\hat{\mu}_{f_o}}$$

on $F(A)$.

Remark. It would be interesting to see whether the equality

$$\Phi_{f_o}(Z(\mathcal{B}(A))) = L^{\infty}(\hat{\mu}_{f_o})$$

holds for any $f_o \in E(A)$. An affirmative answer would give an impr-
version to the description of Sakai's mapping Φ_{f_o}.

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