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On Putnam's Inequality: Extremal Operators

Mihai Putinar

A basic inequality in the theory of hyponormal operators, due to Putnam /10/, asserts that the norm of the self-commutator of a hyponormal operator is dominated by the area of its spectrum multiplied by a universal constant. From the very beginning it was realized that this inequality is sharp in the case of the unilateral shift operator.

The purpose of this note is to characterize some of the extremal operators with respect to Putnam's inequality, that is, those hyponormal operators for which it becomes an equality. As a byproduct of this characterization and of the recent work of Axler and Shapiro /3/, we describe all subnormal operators with one-dimensional self-commutator. More specifically, we prove that such an irreducible subnormal operator is an affine transformation of the unilateral shift. Also, it turns out from the same considerations, that the unilateral shift is essentially the unique subnormal operator for which Putnam's inequality is sharp.

Our approach is based on the theory of the principal function.

§1. Notations and preliminaries

1.1 Let H denote throughout this paper a complex, separable Hilbert space. A linear bounded operator T, acting on the space H, is called *hyponormal* if $T^*T \geq TT^*$, where T^* stands for the adjoint of T. Every subnormal operator fulfills this condition.

Any hyponormal operator can be uniquely expressed as the sum of a normal operator and a *pure hyponormal* operator, that is an operator without a non-trivial normal direct summand. A hyponormal operator which cannot be decomposed into an orthogonal direct sum of non-trivial operators is said to be <u>irreducible</u>. Every irreducible hyponormal operator is necessarily pure.

1.2 One of the most important examples of subnormal operators, and <u>a fortiori</u>, of hyponormal operators, is the unilateral shift $U_+ : H \longrightarrow H$. It can be defined as follows: let $(e_n)_{n \geqslant 0}$ be an orthonormal basis of the Hilbert space H. Then the shift U_+ associated to the basis (e_n) is the operator:

$$U_+(e_n) = e_{n+1} , \quad n \geqslant 0.$$

There is an extensive literature concerning this universal operator. We mention here only a few facts concerning U_+, which are needed in the sequel.

1.3 The spectrum of the operator U_+ is the unit disc $\mathbb{D} = \{ z \in \mathbb{C} \mid |z| \leqslant 1 \}$: $\sigma(U_+) = \mathbb{D}$.

The operator U_+ is, up to a unitary equivalence, the unique isometry with Fredhølm index -1, and acting on the Hilbert space H. More precisely, the properties $U_+^* U_+ = I$ and $\dim(\mathrm{Ker} U_+^*) = 1$ determine the operator U_+ completely, up to a choice of an orthonormal basis.

The self-commutator $[U_+^*, U_+] = \langle \cdot , e_0 \rangle e_0$ is the one-dimensional orthogonal projection onto the linear space generated by e_0. Consequently we have the following equalities:

$$\| [U_+^*, U_+] \| = \mathrm{Tr}\ [U_+^*, U_+] = \pi^{-1} \mu(\sigma(U_+)).$$

Here μ stands for the planar Lebesgue measure and Tr for the trace.

The above equalities are particular and extremal cases of two important inequalities, as follows.

1.4 Let $D = [T^*,T]$ denote throughout this paper the self-commutator of a hyponormal operator acting on the space $\underline{H}$. D is a non-negative operator and it vanishes if and only if T is a normal operator. We point out the identity:

$$\langle Dh,h\rangle = \|Th\|^2 - \|T^*h\|^2 \quad ,h \in \underline{H}. \tag{1.1}$$

Putnam's inequality asserts that, with the above notation,

$$\|D\| \leq \pi^{-1}\mu(\sigma(T)). \tag{1.2}$$

The original proof of this equality was based on Putnam's subtle techniques of cutting-down the spectrum of T, see /10/ and /6/.

1.5 The operator T is said to be <u>rationally cyclic</u> if there exists a vector $\xi \in \underline{H}$, so that the linear space $\{f(T)\xi \ ,f \in Rat(\sigma(T))\}$ is dense in $\underline{H}$. We have denoted by Rat(K) the algebra of all rational functions with poles off the compact set $K \subset \mathbb{C}$.

The second basic inequality related to hyponormal operators was proved by Berger and Shaw /4/ and it states that

$$Tr\, D \leq \pi^{-1}\mu(\sigma(T)), \tag{1.3}$$

whenever T is a rational cyclic hyponormal operator. In particular, this inequality shows that, if T is rationally cyclic, then the self-commutator D is a trace class operator. Putnam's inequality may be easily derived from Berger and Shaw's inequality.

A direct and simple proof of Berger and Shaw's inequality was given by Voiculescu /11/.

§2. <u>Axler and Shapiro's proof of Putnam's inequality</u>

In the case of a subnormal operator, Axler and Shapiro /3/ have recently discovered a nice proof for Putnam's inequality, starting from some estimates in the rational approximation theory. We shall briefly discuss their method.

2.1 Let S denote a subnormal operator defined on the Hilbert space H and let H_1 denote a S-rational cyclic, closed subspace of H. If S_1 stands for the restriction of the operator S to H_1, then S_1 is again a subnormal operator, and, in virtue of (1.1) one gets the inequality

$$\langle [S^*, S] h_1, h_1 \rangle \leq \langle [S_1^*, S_1] h_1, h_1 \rangle \quad , h_1 \in H_1 .$$

On the other hand, it is immediate to verify the inclusion $\sigma(S_1) \subset \sigma(S)$. Therefore the proof of Putnam's inequality (1.2) for the operator S may be reduced to the case of the rational cyclic operator S_1.

2.2 The rational cyclic operator S_1 is canonically represented as the multiplication with z, the complex coordinate, on a function space supported by the spectrum. By using this model, Axler and Shapiro have derived in /3/ the inequality

$$\| [S_1^*, S_1] \| \leq \operatorname{dist}_{C(\sigma)}(\bar{z}, R(\sigma))^2 ,$$

where $C(\sigma)$ is the Banach space of continuous functions on the set $\sigma = \sigma(S_1)$, and $R(\sigma)$ is the closure of the space $\mathrm{Rat}(\sigma)$ in this topology.

A quantitative version of Hartogs and Rosenthal's theorem, due to Alexander /2/, asserts that

$$\operatorname{dist}_{C(\sigma)}(\bar{z}, R(\sigma))^2 \leq \pi^{-1} \mu(\sigma).$$

This completes the proof of Putnam's inequality for subnormal operators.

2.3 A computation based on Cauchy and Pompeiu's formula leads to the inequality:

$$\mathrm{dist}_{C(\sigma)}(\bar{z},R(\sigma)) \leq \pi^{-1}\sup_{z\in\sigma}\left|\int_\sigma (\zeta-z)^{-1}d\mu(\zeta)\right| .$$

The heart of Alexander's theorem is the next estimate:

$$\sup_{z\in\sigma}\left|\int_\sigma (\zeta-z)^{-1}d\mu(\zeta)\right| \leq [\pi\mu(\sigma)]^{1/2}, \tag{2.1}$$

which was proved by an elementary, but clever computation by Ahlfors and Beurling /1/. A direct inspection of their proof implies the next.

2.4 Lemma. The inequality (2.1) becomes an equality if and only if σ is a closed disc.

§3. The principal function

A central object in the study of hyponormal operators with trace class self-commutators is the principal function of Pincus /8/. It was defined originally in terms of perturmation determinants, and, after the work of Helton and Howe /7/, the principal function turned out to be characterized by an invariant formula with traces of commutators of smooth functions of the initial operator and its adjoint.

We summarize below some of the properties of the principal function needed in this paper.

3.1 Let T be a hyponormal operator with trace-class self-commutator $D\in C_1(H)$. The principal function $g=g_T$ of the operator T is the unique, compactly supported, integrable function g on $\mathbb{C}$, which satisfies Helton and Howe's identity:

$$\mathrm{Tr}\left[p(T,T^*),q(T,T^*)\right] = \pi^{-1}\int(\bar{\partial}p\partial q-\partial p\bar{\partial}q)g_T\,d\mu , \tag{3.1}$$

where p and q are polynomials in two variables. Let us remark that, in view of the assumption $D\in C_1$, the order of the factors T and T^* in the monomials of p and q doesn't affect the trace of the commutator $[p,q]$, see /7/ for details. In particular, formula (3.1) shows that

$$\mathrm{Tr}\, D = \pi^{-1}\int g\, d\mu . \qquad (3.2)$$

3.2 The principal function g of the operator T has the following properties (see for instance Clancey's book /6/):

a). $0\leqslant g\leqslant \mathrm{rank}\, D$.

b). $\mathrm{Supp}(g)\subset\sigma(T)$. If the operator T is pure, then $\mathrm{Supp}(g) = \sigma(T)$.
In particular, this property shows that the spectrum of a pure hyponormal operator has positive planar density.

c). For $\lambda\in\mathbb{C}\setminus\sigma_{ess}(T)$ we have $g(\lambda) = -\mathrm{ind}(T-\lambda)$.
Consequently, the principal function of the unilateral shift is $g_{U_+}=\chi_{\mathbb{D}}$, where χ_K denotes the characteristic function of the set K.

d). The principal function of a subnormal operator is integer valued. This is a deep result proved by Carey and Pincus /5/.

e). For every integrable, compactly supported function g on $\mathbb{C}$, with $0\leqslant g\leqslant 1$, there exists a <u>unique</u>, up to a unitary equivalence, pure hyponormal operator T, with $\mathrm{rank}[T^*,T] = 1$ and so that $[g]=[g_T]$ in $L^1(d\mu)$.
This fact is implicitly proved in the paper of Pincus /8/. Another method of realizing the operator T with prescribed principal function g is presented in Putinar /9/.

f). $g_{(aT+b)} = ag_T + b$, where $a,b\in\mathbb{C}$, $a\neq 0$.

3.3 Properties d) and e) above imply that every pure subnormal operator S with one-dimensional self-commutator is completely determined by its principal function $g=\chi_{\sigma(S)}$,and consequently, by its spectrum. However, as we shall see below, not every compact subset of $\mathbb{C}$ is the spectrum of such an operator.

3.4 Example. Let K be a compact subset of $\mathbb{C}$,with positive planar density. By 3.2.e) there exists a pure hyponormal operator T_K with the self-commutator $D_K = [T_K^*, T_K]$ of rank one and so that its principal function is χ_K. Then, by (3.2) we obtain:

$$\| D_K \| = \operatorname{Tr} D_K = \pi^{-1} \mu(K).$$

The operators in the family $(T_K)_{K \subset\subset \mathbb{C}}$ are, as we shall prove below, the only irreducible operators among a wide class of hyponormal operators for which Putnam's inequality is an equality.

§4. Some extremal hyponormal operators

4.1 We are interested in this section in pure hyponormal operators T which satisfy the condition

$$\| D \| = \pi^{-1} \mu(\sigma(T)). \tag{4.1}$$

Assume (4.1) holds and let us consider a hyponormal operator T' with $\sigma(T') \subset \sigma(T)$. Then by Putnam's inequality (1.2) the operator $T \oplus T'$ still satisfies (4.1). It is natural therefore to take into consideration in this section only irreducible operators.

4.2 Theorem. Let T be an irreducible hyponormal operator with compact self-commutator. If Putnam's inequality is actually an equality for T, then $T=T_K$, with $K=\sigma(T)$.

In other terms, the theorem states that rank $D = 1$ and $g_T = \chi_{\sigma(T)}$ whenever D is compact and fulfills condition (4.1).

Proof. Since D is a compact self-adjoint operator, there exists a unit vector $\xi \in H$, such that

$$\|D\| = \langle D\xi, \xi \rangle . \qquad (4.2)$$

Let H_1 be the T-rational cyclic subspace generated by ξ and denote $T_1 = T|H_1$. Then the operator T_1 is also hyponormal and

$$\langle D\xi, \xi \rangle = \|T\xi\|^2 - \|T^*\xi\|^2 \leq \|T\xi\|^2 - \|PT^*\xi\|^2 = \langle D_1\xi, \xi \rangle , \qquad (4.3)$$

where P stands for the orthogonal projection of $\underline{H}$ onto $\underline{H}_1$.

By using (3.2) and Berger and Shaw's inequality (1.3) one gets

$$\langle D_1\xi, \xi \rangle \leq \operatorname{Tr} D_1 = \pi^{-1} \int g_1 \, d\mu , \qquad (4.4)$$

and

$$\operatorname{Tr} D_1 \leq \pi^{-1} \mu(\sigma(T_1)). \qquad (4.5)$$

But $\sigma(T_1) \subset \sigma(T)$ and T fulfills condition (4.1). Hence the inequalities (4.3)-(4.5) are actually equalities.

Furtheron, we infer from from (4.4) that $\langle D_1\xi, \xi \rangle = \operatorname{Tr} D_1$, so that rank $D_1 = 1$. By taking into account the properties 3.2.a) and 3.2.b) of the principal function we obtain from $\int g_1 \, d\mu = \mu(\sigma(T_1))$ that $g_1 = \chi_{\sigma(T_1)}$. Let us notice also the equalities $D|\underline{H}_1 = D_1 = \langle \cdot, \xi \rangle \xi$.

Concluding, we have proved that $T_1 = T_K$ for some compact set $K \subset \mathbb{C}$.

The proof will be finished if we shall prove that $H_1 = \underline{H}$. This fact follows from the irreducibility assumption and the next.

4.3 <u>Lemma.</u> <u>With the above notations, assume that</u> $T^*\xi \in \underline{H}_1$. <u>Then</u> H_1 <u>is a reducing subspace for</u> T.

Proof. We have to prove that $T^*f(T)\xi \in H_1$ for every function $f \in Rat(\sigma(T))$. In fact, by a density argument, it is enough to check that $T^*f(T)\xi \in H_1$ for a function f of the following type

$$f(z) = \sum_{i=1}^{n} p_i(z)/(z-\lambda_i) ,$$

where p_i are polynomials, $\lambda_i \in \mathbb{C} \setminus \sigma(T)$ and n is arbitrary.

Accordingly, we have to prove that $T^*(T-\lambda)^{-1}\xi \in H_1$ and $T^*T^k\xi \in H_1$ for every natural number k. Both assertions are, via an induction on k, consequences of the following commutator identities:

$$T^*(T-\lambda)^{-1}\xi - (T-\lambda)^{-1}T^*\xi = -\langle (T-\lambda)^{-1}\xi, \xi\rangle (T-\lambda)^{-1}\xi \quad ,$$

and

$$T^*T^k\xi - TT^*T^{k-1}\xi = \langle T^{k-1}\xi, \xi\rangle \xi \quad .$$

4.4 Corollary. _An irreducible subnormal operator with compact self-commutator, for which Putnam's inequality is an equality, is of the form_ $aU_+ + b$, _where_ $a, b \in \mathbb{C}$ _and_ $a \neq 0$.

Proof. The subnormal operator S which fulfills the assumptions in the statement is, in virtue of Theorem 4.2, one of the operators T_K, where $K = \sigma(S)$.

Then Axler ans Shapiro's proof of Putnam's inequality, §2, shows that, in this case

$$\mathrm{dist}_{C(K)}(\bar{z}, R(K))^2 = \pi^{-1}\mu(K),$$

so that, by Lemma 2.4, K is a disc.

As we have already remarked in section 3.3 we must have $T_K = aU_+ + b$, where b is the centrum of the disc K and a is its radius.

4.5 Let us notice finally that, if the irreducibility assumption is dropped from the statements of Theorem 4.2 or Corollary 4.4, then a direct summand of the operator is as prescribed there, namely T_K or $aU_+ + b$, but the other summands cannot be fully controlled.

§5. Subnormal operators with one-dimensional self-commutator

5.1 Let S be an irreducible subnormal operator with one-dimensional self-commutator. Because of 3.2.a) and d) the principal function of S is necessarily the characteristic function of its spectrum $g_S = \chi_{\sigma(S)}$. In that case

$$\|[S^*,S]\| = \mathrm{Tr}[S^*,S] = \pi^{-1}\mu(\sigma(T)),$$

hence, by Corollary 4.4 we can state the next.

5.2 *Theorem*. *Every pure subnormal operator with one-dimensional self-commutator is of the form* $aU_+ + b$ *, with* $a, b \in \mathbb{C}$ *and* $a \neq 0$.

Recall that a pure hyponormal operator with one-dimensional self-commutator is necessarily irreducible.

5.3 Let us mention that the geometry of the spectrum of an irreducible subnormal operator may be rather different from the above picture, when the restriction on the rank is relaxed. For instance, it is expected that a relationship between the rank of the self-commutator and the rank of the fundamental group of the spectrum holds, but this problem has not yet been studied.

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