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MULTIPLICATIVE PROPERTIES IN THE STRUCTURE OF CONTRACTIONS

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ABSTRACT. The canonical form of the product of two-by-two matrix contractions is given (Section 4). As preparation, two ways of composing elementary rotations are studied (Sections 1 and 2). Interpretations using multiports are sketched.

1. For two (complex) Hilbert spaces H and H', let $L(H, H')$ denote the set of all (linear, bounded) operators from H into H'. For a contraction $T \in L(H, H')$ (i.e. $\|T\| \leq 1$; we will write $T \in L_1(H, H')$), consider its defect operator $D_T = (I - T^*T)^{\frac{1}{2}}$ and its defect space $\mathcal{D}_T = D_T(H)^-$ (the upper bar denotes the norm closure).

One important point in dilation theory is the study of contractions using special classes of operators (unitaries, isometries or coisometries) on larger spaces (see [18]). In this respect, the use of the following unitary operator is always fundamental. For a contraction $T \in L(H, H')$, denote by $J(T)$ the operator:

$$(1.1)\qquad \begin{cases} J(T): H \oplus \mathcal{D}_{T^*} \to H' \oplus \mathcal{D}_T \\ J(T) = \begin{bmatrix} T & D_{T^*} \\ D_T & -T^* \end{bmatrix} \end{cases}$$

We will call the unitary operator $J(T)$ the *elementary rotation* associated to the contraction T. It is the core of the minimal unitary dilation of T (see [18, Section I.5]), and several other terminologies from operator theory or systems theory are used for it in the literature.

A systematic study of dilations of contractions asks for the structure of two-by-two matrix contractions. We refer to [8] for the structure presented below, and for a history of the subject. It is worth to add the reference [16] where the use of so-called Redheffer product in the study of matrix contractions was initiated; this was explicitly done in [14]. The mentioned structure is described by the one-to-one correspondence between the set of contractions $\mathring{A} \in L(H_1 \oplus H_2, K_1 \oplus K_2)$ and the set of 4-tuples $\{A, \Gamma_1, \Gamma_2, \Gamma\}$ with $A \in L_1(H_1, H_2)$, $\Gamma_1 \in L_1(H_2, \mathcal{D}_{A^*})$, $\Gamma_2 \in L_1(\mathcal{D}_A, K_2)$, $\Gamma \in L_1(\mathcal{D}_{\Gamma_1}, \mathcal{D}_{\Gamma_2^*})$, given by the formula

(1.2) $$\mathring{A} = \begin{bmatrix} A & D_{A^*}\Gamma_1 \\ \Gamma_2 D_A & -\Gamma_2 A^* \Gamma_1 + D_{\Gamma_2^*}\Gamma D_{\Gamma_1} \end{bmatrix}.$$

We write in this case:

(1.3) $$\mathring{A} = \mathbb{C}(A, \Gamma_1, \Gamma_2, \Gamma);$$

it is clear that $\mathring{A}^* = \mathbb{C}(A^*, \Gamma_2^*, \Gamma_1^*, \Gamma^*)$. Note that $J(A) = \mathbb{C}(A, I, I, 0)$, with $H_2 = D_{A^*}$ and $K_2 = D_A$. We will call (1.2) the canonical form of the two-by-two matrix contraction $\mathring{A}$.

Moreover, one has a description of the defect space of $\mathring{A}$, namely, the following operator is a unitary one:

(1.4) $$\begin{cases} \alpha(\mathring{A}) = \alpha : D_{\mathring{A}} \to D_{\Gamma_2} \oplus D_{\Gamma} \\ \alpha(\mathring{A}) = \begin{bmatrix} D_{\Gamma_2} D_A & -(D_{\Gamma_2} A^* \Gamma_1 + \Gamma_2^* \Gamma D_{\Gamma_1}) \\ 0 & D_\Gamma D_{\Gamma_1} \end{bmatrix} \in L(H_1 \oplus H_2, D_{\Gamma_2} \oplus D_\Gamma). \end{cases}$$

The defect space $D_{\mathring{A}^*}$ can be identified with $D_{\Gamma_1^*} \oplus D_{\Gamma^*}$ using $\alpha(\mathring{A}^*) = \alpha_*$.

Among the applications of these facts we mention here only the *triangularization of the defect operators* induced by (1.4); this was used in [5] and [6] for computing some determinants connected with angles in Gaussian processes (and with Szegö-type theorems for them).

Particular cases of this description (i.e. for row – or column – operators, or for A = 0; see [9], [3], [4], [17], and so on) are also useful; we list here the case of a row operator, which (together with column case) will be used later. This corresponds to the case where $K_2 = \{0\}$, and thus $\Gamma_2 = 0 \in L(D_A, \{0\})$, $\Gamma = 0 \in L(D_{\Gamma_1}, \{0\})$. Then $A_r \in L_1(H_1 \oplus H_2, K_1)$ iff

$(1.2)_r$ $$A_r = (A, D_{A^*}\Gamma_1),$$

where $A \in L_1(H_1, K_1)$, $\Gamma_1 \in L_1(H_2, D_{A^*})$; moreover, one has the unitary operators:

$(1.4)_r$ $$\begin{cases} \alpha(A_r) : D_{A_r} \to D_A \oplus D_{\Gamma_1} \\ \alpha(A_r) D_{A_r} = \begin{bmatrix} D_A & -A^*\Gamma_1 \\ 0 & D_{\Gamma_1} \end{bmatrix} \in L(H_1 \oplus H_2, D_A \oplus D_{\Gamma_1}), \end{cases}$$

$$(1.4)_r^* \qquad \begin{cases} \alpha(A_r^*) : D_{A_r^*} \to D_{\Gamma_1^*} \\ \alpha(A_r^*) D_{A_r^*} = D_{\Gamma_1^*} D_{A^*} \in L(K_1, D_{\Gamma_1^*}). \end{cases}$$

After this review, let us tackle the question of describing $J(\mathring{A})$ for $\mathring{A} \in L_1(H_1 \oplus H_2, K_1 \oplus K_2)$ in terms of elementary rotations of A, Γ_1, Γ_2, and Γ, where $\mathring{A} = \mathbf{C}(A, \Gamma_1, \Gamma_2, \Gamma)$. We have the following:

PROPOSITION 1.1. *If* $\mathring{A} = \mathbf{C}(A, \Gamma_1, \Gamma_2, \Gamma)$, *then:*

$$(1.5) \qquad [I_{K_1 \oplus K_2} \oplus \alpha] J(\mathring{A}) [I_{H_1 \oplus H_2} \oplus \alpha_*^*] =$$

$$= (I_{K_1} \oplus J(\Gamma_2) \oplus I_{D_\Gamma})(J(A) \oplus J(\Gamma))(I_{H_1} \oplus J(\Gamma_1) \oplus I_{D_{\Gamma^*}})$$

where $\alpha = \alpha(\mathring{A})$ *and* $\alpha_* = \alpha(\mathring{A}^*)$ *are defined in* (1.4).

PROOF. Both sides of (1.5) act between $H_1 \oplus H_2 \oplus D_{\Gamma_1^*} \oplus D_{\Gamma^*}$ and $K_1 \oplus K_2 \oplus D_{\Gamma_2} \oplus D_\Gamma$. The right hand side of (1.5) is, after making the products, the matrix:

$$(1.6) \qquad \begin{bmatrix} A & D_{A^*}\Gamma_1 & D_{A^*}D_{\Gamma_1^*} & 0 \\ \Gamma_2 D_A & -\Gamma_2 A^* \Gamma_1 + D_{\Gamma_2^*} \Gamma D_{\Gamma_1} & -(\Gamma_2 A^* D_{\Gamma_1^*} + D_{\Gamma_2^*} \Gamma \Gamma_1^*) & D_{\Gamma_2^*} D_{\Gamma^*} \\ D_{\Gamma_2} D_A & -(D_{\Gamma_2} A^* \Gamma_1 + \Gamma_2^* \Gamma D_{\Gamma_1}) & \Gamma_2^* \Gamma \Gamma_1^* - D_{\Gamma_2} A^* D_{\Gamma_1^*} & -\Gamma_2^* D_{\Gamma^*} \\ 0 & D_\Gamma D_{\Gamma_1} & -D_\Gamma \Gamma_1^* & -\Gamma^* \end{bmatrix}$$

Taking into consideration (1.4), we have that (1.5) follows from (1.6) and the equality:

$$(1.7) \qquad \alpha \mathring{A}^* \alpha_*^* = \begin{bmatrix} -\Gamma_2^* \Gamma \Gamma_1^* + D_{\Gamma_2} A^* D_{\Gamma_1^*} & \Gamma_2^* D_{\Gamma^*} \\ D_\Gamma \Gamma_1^* & \Gamma^* \end{bmatrix} \in L(D_{\Gamma_1^*} \oplus D_{\Gamma^*}, D_{\Gamma_2} \oplus D_\Gamma).$$

Formula (1.7) is a routine computation. Indeed, it is enough to prove (1.7) for elements $\gamma \in D_{\Gamma_1^*} \oplus D_{\Gamma^*}$ of the form

(1.8)
$$\gamma = \alpha_* D_{\mathring{A}^*}(k_1 \oplus k_2) = \begin{bmatrix} D_{\Gamma_1^*} D_{A^*} k_1 - D_{\Gamma_1^*} A \Gamma_2^* k_2 - \Gamma_1 \Gamma^* D_{\Gamma_2^*} k_2 \\ D_{\Gamma^*} D_{\Gamma_2^*} k_2 \end{bmatrix}$$

where $k_1 \in K_1$ and $k_2 \in K_2$. Now, the action of the left hand side of (1.7) to γ is

$$\alpha \mathring{A}^* \alpha_*^* \gamma = \alpha \mathring{A}^* \alpha_*^* \alpha_* D_{\mathring{A}^*}(k_1 \oplus k_2) = \alpha D_{\mathring{A}} \mathring{A}^*(k_1 \oplus k_2),$$

which can be computed using (1.4) and (1.2). The result proves to be equal with the action of the matrix from the right hand side of (1.7) to γ from (1.8). The computations repeatedly use that for a contraction T one has $TD_T = D_{T^*}T$ and $D_T^2 = I - T^*T$. □

REMARKS 1.2. a) The matrix (1.6) appeared in Proposition 1.1 from [15].

b) In the case of a row operator A_r as in $(1.2)_r$ the formula (1.5) becomes

$(1.5)_r$
$$[I_{K_1} \oplus \alpha(A_r)] J(A_r) [I_{H_1 \oplus H_2} \oplus \alpha(A_r^*)^*] = (J(A) \oplus I_{D_{\Gamma_1}})(I_{H_1} \oplus J(\Gamma_1))$$

which is Lemma 2.2 from [6]. This might give an alternate way of proving Proposition 1.1: write $\mathring{A}$ from (1.2) as a row operator (the two components being some column operators) and apply $(1.5)_r$ and a similar one for columns.

c) The right hand side of (1.5) suggests a sort of simmetry between A and Γ in (1.2). This is also contained in (1.6); indeed, when A, Γ_1, Γ_2 and Γ are pure contractions (T is a pure contraction if $\ker D_T = \{0\}$), then the right hand side of (1.7) is (after intertwining the rows and the columns) exactly $\mathbb{C}(\Gamma^*, \Gamma_1^*, \Gamma_2^*, A^*) \in L(K_2 \oplus K_1, H_2 \oplus H_1)$. It is maybe worth mentioning that if $T \in L_1(H, H')$ is decomposed into the direct sum $T_u \oplus T_p$, where $T_u \in L(\ker D_T, \ker D_{T^*})$ is unitary and $T_p \in L(D_T, D_{T^*})$ is a pure contraction, then

(1.9)
$$J(T) = T_u \oplus J(T_p),$$

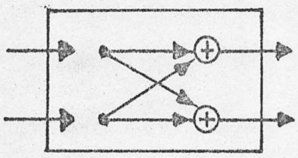
with respect to the orthogonal decompositions $H \oplus D_{T^*} = (\ker D_T) \oplus (D_T \oplus D_{T^*})$ and $H' \oplus D_T = (\ker D_{T^*}) \oplus (D_{T^*} \oplus D_T)$.

d) Formula (1.10)' from [8] shows that if $\mathring{A}$ is as in (1.2), then

(1.10)
$$\mathring{A} = (I \oplus \Gamma_2) J(A)(I \oplus \Gamma_1) + (0 \oplus D_{\Gamma_2^*} \Gamma D_{\Gamma_1});$$

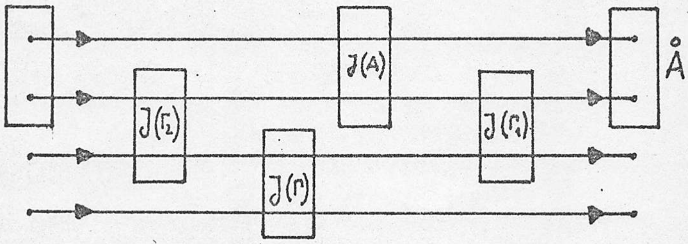
here the "imposed" and the "free" parts of $\mathring{A}$ (when A, Γ_1, and Γ_2 are given) are separated. Formula (1.5) gives the reason why (1.10) is true.

e) The elementary rotation of a contraction T can be interpreted as the elementary lattice section



J(T).

Thus, formula (1.5) shows that the elementary rotation of $\mathring{A}$ from (1.2) is



$J(\mathring{A})$.

f) The results of [8] were generalized in [7] to the case of operators (in Pontrjagin spaces) whose "defect operator" has a finite number of negative squares. The facts of this section can be accordingly generalized to that case.

2. Let $T \in L_1(H, H')$, $S \in L_1(H', H'')$, and $R = ST \in L_1(H, H'')$. The aim of this section is to compute J(R) in terms of J(T) and J(S); this will show a different way (from (1.5)) of composing elementary rotations. A close connection with the notion of regular factorization (see [18, Chapter VII]) will appear.

It is natural to try to embed D_R into the direct sum of D_S and D_T. This can be done using the isometry

$$\hat{\gamma}(ST) = \hat{\gamma} : D_R \to D_S \oplus D_T \tag{2.1}$$

$$\hat{\gamma} D_R = (D_S T,\ D_T)^t ,$$

("t" stands for the matrix transpose). Simple examples (e.g. S = T = 0) show that this operator can be nonunitary. Define

$$F(ST) = (\mathrm{Im}\ \gamma)^- = \{D_S Th \oplus D_T h;\ h \in H\}^- \subset D_S \oplus D_T \tag{2.2}$$

and

$$R(ST) = (D_S \oplus D_T) \ominus F(ST) . \tag{2.3}$$

We have then the unitary operator

$$\gamma(ST) = \gamma : D_R \to F(ST), \quad \gamma d = \hat{\gamma} d, \quad d \in D_R . \tag{2.4}$$

Analogously, $\mathcal{D}_{R^*}$ can be identified with $\mathcal{F}(T^*S^*) \subset \mathcal{D}_{T^*} \oplus \mathcal{D}_{S^*}$ through the unitary operator $\gamma_* = \gamma(T^*S^*)$. We have now the following

PROPOSITION 2.1. *For the factorization* $R = ST$

$$(2.5) \qquad (I_{H''} \oplus \gamma)J(R)(I_H \oplus \gamma_*^*) = (J(S) \oplus I_{\mathcal{D}_T})(I_{H'} \oplus J(0))(J(T) \oplus I_{\mathcal{D}_{S^*}}) \,|\, H \oplus \mathcal{F}(T^*S^*)$$

where the zero operator in right hand side is considered in $L(\mathcal{D}_T, \mathcal{D}_{S^*})$.

PROOF. We proceed as in the proof of Proposition 1.1. The product from the right hand side is the matrix

$$(2.6) \qquad M = \begin{bmatrix} ST & SD_{T^*} & D_{S^*} \\ D_S T & D_S D_T & -S^* \\ D_T & -T^* & 0 \end{bmatrix} \in L(H \oplus \mathcal{D}_{T^*} \oplus \mathcal{D}_{S^*}, H'' \oplus \mathcal{D}_S \oplus \mathcal{D}_T).$$

Let now $x = h \oplus D_{T^*}S^*h'' \oplus D_{S^*}h'' \in H \oplus \mathcal{F}(T^*S^*)$, where $h \in H$ and $h'' \in H''$. Then

$$(2.7) \qquad Mx = (Rh + D_{R^*}^2 h'', D_S T(h - R^*h''), D_T(h - R^*h'')).$$

From (2.7) it follows that $M(H \oplus \mathcal{F}(T^*S^*)) = H'' \oplus \mathcal{F}(ST)$ and that

$$(2.8) \qquad -\gamma R^* \gamma_*^* = \begin{bmatrix} D_S D_{T^*} & -S^* \\ -T^* & 0 \end{bmatrix} \,|\, \mathcal{F}(T^*S^*).$$

Formulas (2.1) and (2.2) finish the proof. □

REMARKS 2.2. a) The operator $J(0)$ is the operator of changing the sumands in a direct sum.

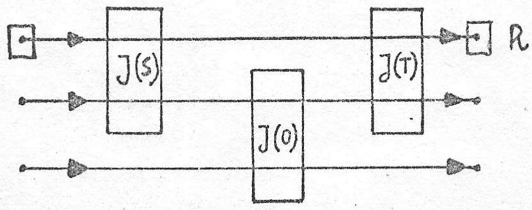
b) Replacing $J(0)$ by $J(Z)$ with $Z \in L_1(\mathcal{D}_T, \mathcal{D}_{S^*})$, the product from the right hand side of (2.5) (replacing $I_{\mathcal{D}_{S^*}}$ by $I_{\mathcal{D}_{Z^*}}$ and $I_{\mathcal{D}_T}$ with $I_{\mathcal{D}_Z}$) is connected with the computation of the elementary rotation of the *contraction* $ST + D_{S^*}ZD_T$. Operators like this last one appeared in the study of positivity for arbitrary block-matrices (see [13]). If Z is factorized through a Hilbert space $\mathcal{X}$ as $Z = Z_2 Z_1$, one obtains the elementary rotation of the product between the row $(S, D_{S^*}Z_1)$ and the column $(T, Z_2 D_S)^t$; see also Section 4.

c) The factorization ST of R is called *regular* if $\mathcal{R}(ST) = \{0\}$. This notion was intensively used in the study of invariant subspaces, namely any factorization of the characteristic function of a completely non-unitary contraction provides an invariant subspace for the direct sum of the considered contraction by a unitary operator, which is zero iff the factorization is regular ([18, p.321]). Regular factorizations appeared later in the uniqueness criterium for contractive intertwining dilations (see [1]), and then in the Schur-type analysis of the set of all contractive dilations ([2], [10], [11], [3], [4]). Now, in the setting of Proposition 2.1, it is clear that the contraction

$$Z(S,T) = Z = \begin{bmatrix} -D_S D_{T^*} & S^* \\ T^* & 0 \end{bmatrix} \in L(\mathcal{D}_{T^*} \oplus \mathcal{D}_{S^*}, \mathcal{D}_S \oplus \mathcal{D}_T) \tag{2.9}$$

has the property that its pure contractive part acts between $F(T^*S^*)$ and $F(ST)$ (and is canonically unitary equivalent with the pure contractive part of R^*), while its unitary part acts between $R(T^*S^*)$ and $R(ST)$. This is a way of proving the symmetry of regularity with respect to the adjoint operation (see [18, Proposition 3.2 of Chapter VII], [3], [9] for more about this). What Proposition 2.1 says is related with the beginning of this remark: the matrix M "is" the elementary rotation of the product plus a unitary operator which is zero when the factorization is regular.

d) The multiport interpretation of the regular case of Proposition 2.1 is



J(R).

3. In this section we indicate the canonical form (as a two-by-two matrix contraction) of the product of a column operator with a row operator. This will help doing this for the case to be considered in Section 4: the canonical form of the product of two-by-two matrix contractions.

Consider the operators:

$$T_r = (T,\ D_{T^*}\Delta_1) \in L_1(H_1 \oplus H_2,\ K) \tag{3.1}$$

where $\Delta_1 \in L(H_2, \mathcal{D}_{T^*})$, and

$$S_c = (S,\ \Delta_2 D_S)^t \in L_1(K,\ G_1 \oplus G_2), \tag{3.2}$$

where $\Delta_2 \in L(\mathcal{D}_S, G_2)$.

Denote

$$R = ST \in L(H_1, G_1), \text{ and} \tag{3.3}$$

$$(3.4) \qquad \overset{o}{R} = S_c T_r \in L(H_1 \oplus H_2, G_1 \oplus G_2).$$

Then

$$(3.5) \qquad \overset{o}{R} = \begin{bmatrix} R & SD_{T^*}\Delta_1 \\ \Delta_2 D_S T & \Delta_2 D_S D_{T^*}\Delta_1 \end{bmatrix}.$$

For writing $\overset{o}{R}$ in the canonical form it is clear that we have to use the analysis of defect spaces of R from Section 2. To this end, consider

$$(3.6) \qquad P(S,T) = P = P_{F(ST)}^{D_S \oplus D_T} \quad \text{and}$$

$$(3.6)_* \qquad P_* = P(T^*, S^*).$$

Then, using the operators γ and γ_* from (2.4), define

$$(3.7) \qquad \Gamma_1 = \gamma_*^* P_* (\Delta_1, \ 0)^t : H_2 \to D_{R^*}$$

$$(3.8) \qquad \Gamma_2 = (\Delta_2, \ 0)\gamma : D_R \to G_2.$$

From (3.7) it follows

$$(3.9) \qquad D_{R^*}\Gamma_1 = SD_{T^*}\Delta_1,$$

which is the canonical form of the (1, 2) entry of $\overset{o}{R}$ from (3.5). Analogously,

$$(3.10) \qquad \Gamma_2 D_R = \Delta_2 D_S T$$

is the canonical form of the (2, 1) entry of $\overset{o}{R}$.

For the (2,2) entry, note first that

$$(3.11) \qquad -\Gamma_2 R^* \Gamma_1 = -(\Delta_2, \ 0)\gamma R^* \gamma_*^* P_* (\Delta_1, \ 0)^t = (\Delta_2, \ 0) Z P_* (\Delta_1, \ 0)^t$$

where we used (2.8) and (2.9). Because

$$(3.12) \qquad (\Delta_2, \ 0) \begin{bmatrix} D_S D_{T^*} & -S^* \\ -T^* & 0 \end{bmatrix} \begin{bmatrix} \Delta_1 \\ 0 \end{bmatrix} = \Delta_2 D_S D_{T^*}\Delta_1,$$

the operator $\Gamma \in L(D_{\Gamma_1}, D_{\Gamma_2^*})$ from the canonical form of the (2,2) entry of $\overset{o}{R}$ must be obtained from

$$(3.13) \qquad (\Delta_2, \ 0) Z (1 - P_*)(\Delta_1, \ 0)^t = D_{\Gamma_2^*} \Gamma D_{\Gamma_1}.$$

We will use now the usual method to "identify" the defect operators D_{Γ_1} and

$D_{\Gamma_2^*}$. For any $h_2 \varepsilon H_2$,

$$(3.14)\quad \|D_{\Gamma_1} h_2\|^2 = \|h_2\|^2 - \|\Gamma_1 h_2\|^2 = \|h_2\|^2 - \|\gamma_*^* P_*(\Delta_1 h_2 \oplus 0)\|^2 =$$

$$= \|h_2\|^2 - \|P_*(\Delta_1 h_2 \oplus 0)\|^2 = \|h_2\|^2 - \|\Delta_1 h_2\|^2 + \|(1 - P_*)(\Delta_1 h \oplus 0)\|^2 =$$

$$= \|D_{\Delta_1} h_2\|^2 + \|(1 - P_*)(\Delta_1, 0)^t h_2\|^2 .$$

The relation (3.14) shows that the operator :

$$(3.15)\quad \begin{cases} \Phi : H_2 \to D_{\Delta_1} \oplus R(T^* S^*) \\ \Phi = (D_{\Delta_1}, \ (1 - P_*)(\Delta_1, \ 0)^t)^t \end{cases}$$

is an isometry; denote $\Phi(H_2) = M$. Then, the operator

$$(3.16)\quad \begin{cases} \phi : D_{\Gamma_1} \to M \\ \phi D_{\Gamma_1} = \Phi \end{cases}$$

is a unitary one.

Similar considerations for $D_{\Gamma_2^*}$ imply that the operator

$$(3.17)\quad \begin{cases} \Phi_* : G_2 \to D_{\Delta_2^*} \oplus R(ST) \\ \Phi_* = (D_{\Delta_2^*}, \ (1 - P)(\Delta_2^*, \ 0)^t)^t \end{cases}$$

is an isometry; denoting $\Phi_*(G_2) = M_*$, the operator

$$(3.18)\quad \begin{cases} \phi_* : D_{\Gamma_2^*} \to M_* \\ \phi_* D_{\Gamma_2^*} = \Phi_* \end{cases}$$

is a unitary one.

Now it is easy to verify that the solution for Γ from (3.13) is:

$$(3.19)\quad \Gamma = \phi_*^* P_{M_*}^{D_{\Delta_2^*} \oplus R(ST)} (0 \oplus Z)\phi \ \varepsilon \ L(D_{\Gamma_1}, D_{\Gamma_2^*}) .$$

Indeed, since Φ_* is an isometry, $P_{M_*} = \Phi_* \Phi_*^*$, and then

$$(3.20)\quad D_{\Gamma_2^*}\Gamma D_{\Gamma_1} = D_{\Gamma_2^*}\phi_*^*\Phi_*\Phi_*^*(0\oplus Z)\phi D_{\Gamma_1} = \Phi_*^*(0\oplus Z)\Phi = (\Delta_2,\ 0)Z(I-P_*)(\Delta_1,\ 0)^t .$$

Thus, we proved:

PROPOSITION 3.1. *With the notation from* (1.3) *we have that* $\overset{\circ}{R} = \mathbb{C}(R,\Gamma_1,\Gamma_2,\Gamma)$, *where* R, Γ_1, Γ_2 *and* Γ *are defined by* (3.3), (3.7), (3.8), *and* (3.19), *respectively.*

The connection with regular factorizations appears also in the following observation:

COROLLARY 3.2. a) *If the factorization* $R = ST$ *from* (3.3) *is regular, then for any choice of* H_2, Δ_1, G_2, Δ_2 *from* (3.1) *and* (3.2) *the operator* Γ *from* (3.19) *is zero.*

b) *If there exist* $H_2 \neq \{0\}$ *and* $G_2 \neq \{0\}$ *such that for any choice of* Δ_1 *and* Δ_2, *the operator* Γ *from* (3.19) *is zero, then the factorization* $R = ST$ *is regular.*

PROOF. a) follows immediately from (3.19), because $\mathcal{R}(ST) = \{0\}$ implies $Z = 0$.

b) follows from the fact that $\overline{(I-P)(\mathcal{D}_S\oplus\{0\})} = \mathcal{R}(S,T)$ (see Lemma 1.1 in [3]). □

REMARK 3.3. The methods from Proposition 3.1 gives the possibility of carrying out the analysis done in Section 1 for the operator $\overset{\circ}{R}$ in (3.4). So, one can obtain descriptions of $\mathcal{D}_{\overset{\circ}{R}}$ and $\mathcal{D}_{\overset{\circ}{R}^*}$, as well as some equivalences between multiports interpretations of $\overset{\circ}{R}$ resulting from (3.4) and (3.21). We omit the details.

4. We shortly indicate now how to apply the results of Section 3 to the general case of two-by-two matrices.

Let $\overset{\circ}{A} \in L_1(H_1\oplus H_2, K_1\oplus K_2)$, $\overset{\circ}{A} = \mathbb{C}(A,\Gamma_1,\Gamma_2,\Gamma)$ and $\overset{\circ}{B} \in L_1(K_1\oplus K_2, G_1\oplus G_2)$, $\overset{\circ}{B} = \mathbb{C}(B,\Delta_1,\Delta_2,\Delta)$; see (1.3) for notation. Denote

$$(4.1)\qquad \overset{\circ}{C} = \overset{\circ}{B}\overset{\circ}{A} \in L_1(H_1\oplus H_2, G_1\oplus G_2).$$

Write $\overset{\circ}{C}$ in the canonical form:

$$(4.2)\qquad \overset{\circ}{C} = \mathbb{C}(C,\Theta_1,\Theta_2,\Theta);$$

we indicate below the operators C, Θ_1, Θ_2, and Θ in terms of the stucture operators of $\overset{\circ}{A}$ and $\overset{\circ}{B}$. Having in mind the analysis done in Section 3 we will write $\overset{\circ}{A}$ as a row operator and $\overset{\circ}{B}$ as a column operator.

To this aim, put $\overset{\circ}{A}$ in the form:

$$\mathring{A} = (A_c,\ D_{A_c^*}\tilde{\Gamma}), \tag{4.3}$$

where

$$A_c = (A,\ \Gamma_2 D_A)^t, \tag{4.4}$$

and

$$\begin{cases} \tilde{\Gamma} \in L_1(H_2, D_{A_c^*}) \\ \tilde{\Gamma} = \alpha^*(A_c^*)(\Gamma_1,\ \Gamma D_{\Gamma_1})^t . \end{cases} \tag{4.5}$$

This is indeed the case, because $(\Gamma_1\ \ \Gamma D_{\Gamma_1})^t$ is a column contraction acting between H_2 and $D_{A^*} \oplus D_{\Gamma^*}$, while $\alpha^*(A_c^*)$ is a unitary operator acting between $D_{A^*} \oplus D_{\Gamma_2^*}$ and $D_{A_c^*}$; moreover - as in $(1.4)_2$ - $D_{A_c^*}\alpha^*(A_c^*) \in L(D_{A^*} \oplus D_{\Gamma_2^*}, K_1 \oplus K_2)$ has the matrix

$$D_{A_c^*}\alpha^*(A_c^*) = \begin{bmatrix} D_{A^*} & 0 \\ -\Gamma_2 A^* & D_{\Gamma_2^*} \end{bmatrix}, \tag{4.6}$$

which implies (4.3).

Similarly

$$\mathring{B} = (B_r,\ \tilde{\Delta} D_{B_r})^t, \tag{4.7}$$

where

$$B_r = (B,\ D_{B^*}\Delta_1), \tag{4.8}$$

and

$$\begin{cases} \tilde{\Delta} \in L_1(D_{B_r}, G_2) \\ \tilde{\Delta} = (\Delta_2,\ D_{\Delta_2^*}\Delta)\alpha(B_r), \end{cases} \tag{4.9}$$

where

$$\alpha(B_r)D_{B_r} = \begin{bmatrix} D_B & -B^*\Delta_1 \\ 0 & D_{\Delta_1} \end{bmatrix}. \tag{4.10}$$

Now, we can apply Proposition 3.1 to $\mathring{C} = (B_r,\ \tilde{\Delta}D_{B_r})^t(A_c,\ D_{A_c^*}\tilde{\Gamma})$. From (3.3) it follows that

$$C = B_r A_c = BA + D_{A^*}\Gamma_1\Delta_2 D_B . \tag{4.11}$$

For describing Θ_1, note first that D_{C^*} can be identified with $F(A_c^* B_r^*) \subset D_{A_c^*} \oplus D_{B_r^*}$ via the unitary operator $\gamma(A_c^* B_r^*)$ from (2.4). Using the analysis from $(1.4)_r$ and $(1.4)_r^*$ one has that $D_{A_c^*} \oplus D_{B_r^*}$ can be identified with $(D_{A^*} \oplus D_{\Gamma_2^*}) \oplus D_{\Delta_1^*}$ using the unitary operator $\alpha(A_c^*) \oplus \alpha(B_r^*)$. Composing these identifications we have that if we denote

$$(4.12)\quad \tilde{F}(A_c^* B_r^*) = \{(D_{A^*} B^* g_1 - A\Gamma_2^* \Delta_1^* D_{B^*} g_1) \oplus D_{\Gamma_2^*} \Delta_1^* D_{B^*} g_1 \oplus D_{\Delta_1^*} D_{B^*} g_1 ;\; g_1 \in G_1\} \subset$$
$$\subset D_{A^*} \oplus D_{\Gamma_2^*} \oplus D_{\Delta_1^*},$$

then the operator

$$(4.13)\quad \begin{cases} \tilde{\gamma}_* : D_{C^*} \to \tilde{F}(A_c^* B_r^*) \\ \tilde{\gamma}_* = [\alpha(A_c^*) \oplus \alpha(B_r^*)]\gamma(A_c^* B_r^*) \end{cases}$$

is a unitary one. Denote by $\tilde{P}_*$ the projection of $D_{A^*} \oplus D_{\Gamma_2^*} \oplus D_{\Delta_1^*}$ onto $\tilde{F}(A_c^* B_r^*)$. Then the relations (3.7), (4.3), (4.5), (4.13) and the definition of $\tilde{P}_*$ imply that

$$(4.14)\quad \Theta_1 = \tilde{\gamma}_*^* \tilde{P}_* (\Gamma_1 ,\; \Gamma D_{\Gamma_1} ,\; 0)^t \in L_1(H_2 , D_{C^*}) .$$

The description of Θ_2 is similar. One has to define

$$(4.15)\quad \tilde{F}(B_r A_c) = \tilde{\gamma} F(B_r A_c) \subset D_B \oplus D_{\Delta_1} \oplus D_{\Gamma_2} ,$$

where

$$(4.16)\quad \tilde{\gamma} = [\alpha(B_r) \oplus \alpha(A_c)]\gamma(B_r A_c) ,$$

and $\tilde{P}$ the projection of $D_B \oplus D_{\Delta_1} \oplus D_{\Gamma_2}$ onto $\tilde{F}(B_r A_c)$. Then, using (3.8), we obtain

$$(4.17)\quad \Theta_2 = (\Delta_2 ,\; D_{\Delta_2^*} \Delta ,\; 0)\tilde{\gamma} \in L_1(D_C , G_2) .$$

For indicating Θ, we have to identify D_{Θ_1} and $D_{\Theta_2^*}$. Using (3.15), (3.16), and the previous analysis of Θ_1, we infer that the operator

$$(4.18)\quad \begin{cases} \tilde{\phi} : D_{\Theta_1} \to \tilde{M} \subset D_\Gamma \oplus D_{A^*} \oplus D_{\Gamma_2^*} \oplus D_{\Delta_1^*} \\ \tilde{\phi} D_{\Theta_1} = (D_\Gamma D_{\Gamma_1} ,\; (I - \tilde{P}_*)(\Gamma_1 ,\; \Gamma D_{\Gamma_1} ,\; 0)^t)^t \end{cases}$$

is a unitary one ($\tilde{M} = \tilde{\phi} D_{\Theta_1}$). Analogous, from (3.17) and (3.18) - and the structure of Θ_2 - it results that the operator

$$(4.19)\qquad \begin{cases} \tilde{\phi}_* : \mathcal{D}_{\Theta_2^*} \to \tilde{M}_* \subset \mathcal{D}_{\Delta^*} \oplus \mathcal{D}_B \oplus \mathcal{D}_{\Delta_1} \oplus \mathcal{D}_{\Gamma_2} \\ \tilde{\phi}_* D_{\Theta_2^*} = (D_{\Delta^*} D_{\Delta_2^*},\ (I - \tilde{P})(\Delta_2^*,\ \Delta^* D_{\Delta_2^*},\ 0)^t)^t \end{cases}$$

is a unitary one ($\tilde{M}_* = \tilde{\phi}^* \mathcal{D}_{\Theta_2^*}$). Consider also the operator:

$$(4.20)\qquad \begin{cases} \tilde{Z} : \mathcal{D}_{A^*} \oplus \mathcal{D}_{\Gamma_2^*} \oplus \mathcal{D}_{\Delta_1^*} \to \mathcal{D}_B \oplus \mathcal{D}_{\Delta_1} \oplus \mathcal{D}_{\Gamma_2} \\ \tilde{Z} = [\alpha(B_r) + \alpha(A_c)] Z(B_r, A_c) \alpha^*(A_c^*) + \alpha^*(B_r^*) \end{cases}$$

(see (2.9)). The explicit matrix of $\tilde{Z}$ is

$$(4.21)\qquad \tilde{Z} = \begin{bmatrix} -(D_B D_{A^*} + B^* \Delta_1 \Gamma_2 A^*) & B^* \Delta_1 D_{\Gamma_2^*} & B^* D_{\Delta_1^*} \\ D_{\Delta_1} \Gamma_2 B^* & -D_{\Delta_1} D_{\Gamma_2^*} & \Delta_1^* \\ D_{\Gamma_2} A^* & \Gamma_2^* & 0 \end{bmatrix}.$$

Using (3.19) we finally have

$$(4.22)\qquad \Theta = \tilde{\phi}_*^* P_{\tilde{M}_*} (0 \oplus \tilde{Z}) \tilde{\phi} \in L_1(\mathcal{D}_{\Theta_1}, \mathcal{D}_{\Theta_2^*}).$$

Summing up, we obtain

THEOREM 4.1. *If* $\mathring{A} = \mathbf{C}(A, \Gamma_1, \Gamma_2, \Gamma)$ *and* $\mathring{B} = \mathbf{C}(B, \Delta_1, \Delta_2, \Delta)$, *then* $\mathring{C} = \mathring{B}\mathring{A}$ *has the canonical form* $\mathring{C} = \mathbf{C}(C, \Theta_1, \Theta_2, \Theta)$, *where* C, Θ_1, Θ_2, *and* Θ *are defined by* (4.11), (4.14), (4.17), *and* (4.22), *respectively*.

REMARKS 4.2. a) Corollary 3.2 and Remark 3.3 can be transcribed for the case of this section.

b) The computations connected with angles in Gaussian processes and with Szegö-type theorems ask for the value of the determinants of the defect operators of some products of contractions. As remarked in Section 1, the knowledge of the canonical form of these products implies a triangularization of defect operators, and thus the possibility of iterative computations of their determinants. This was done directly in [5, Theorem 4.5] and in [6, Theorem 3.4] for cases which can now be included in Theorem 4.1.

c) The so-called Schur analysis of contractivity (see [10], [11], [3], [4], [9], [12]) gives algorithms and formulas for computing the general solution of some operatorial extrapolations problems in terms of Schur-type parameters; for this, various manipulations with rotation operators are used. Some of these computations are included in the results of the present paper.

d) A natural question in the Schur analysis of contractivity is the structure of parameters for a product of two solutions; Theorem 4.1 is a step in this respect.

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