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§0. INTRODUCTION

Let E be a G structure of type one on M and K a group of automorphisms of E. If $\omega\in\Omega^1(E,\mathfrak{g})$ is a K invariant symmetric connexion on E, then for each fixed G-adapted frame $u_0\in E$, K naturally embeds in E by $\phi_{u_0}(k)=L(k)(u_0)$. In this paper we analyse this embedding, when $G=O(p,q)$ and K is transitive on M, relating the geometry of K and E by ϕ_{u_0}.

Thus in the second section we show that the structure equations of K are the pull-back of the structure equations of $E=O_g(M)$ (Theorem 2.2.), so that the Jacobi conditions of K are a consequence of the Bianchi relations on $O_g(M)$ (Lemma 2.3) and we give a simple necessary and sufficient condition for the local equivalence of two $O(p,q)$-homogeneous spaces (Corrolary 2.1). In order to classify germs of homogeneous Lorentz spaces, we find in the first section that there is only one class of conjugated maximal subalgebras of noncompact type in $\mathfrak{so}(n,1)$ (Proposition 1.1) and derive then a first gap in the dimension of automorphism groups of $O(p,q)$-structures (Theorem 1.1).

Section 3 is devoted to the classification of normal forms of homogeneous Lorentz 3-manifolds (Theorem 3.3.) and Sections 4 and 5 to the classification of classes of equivalence of germs of homogeneous Lorentz 4-manifolds.

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§1. FIRST GAP IN DIMENSION OF AUTOMORPHISM GROUPS OF LORENTZ GEOMETRY

It is well known that the maximal compact group of SO(p,q) is SO(p)×SO(q) ([18]). If $n \geq 2$ it is known too that $\mathfrak{so}(n,1)$ has a subalgebra of dimension $\frac{n(n-1)}{2}+1$, that we shall denote by $\mathfrak{m}(n)$; a description of $\mathfrak{m}(n)$ is the following ([12]): let $\mathfrak{so}(n,1)=\mathfrak{so}(n)\oplus\mathfrak{p}$ be a Cartan decomposition of $\mathfrak{so}(n,1)$ and let b be a bilinear symmetric positive definite form on $\mathfrak{p}$, that gives a structure of constant curvature -1 on $SO(n,1)/SO(n)$. If we choose $X\in\mathfrak{p}$, with $b(X,X)=1$ and $\mathfrak{p}^{\perp}$ is the b-orthogonal complement of X in $\mathfrak{p}$, then $\mathfrak{m}(n)=\mathfrak{g}\oplus\langle X\rangle+\mathfrak{m}$ where $\mathfrak{g}=\ker Ad(X)$ and
$\mathfrak{m}=\{Z+Ad\,X(Z),\ Z\in\mathfrak{p}^{\perp}\}$

PROPOSITION 1.1. For $n\neq 4$, any maximal subalgebra of $\mathfrak{so}(n,1)$ is conjugated with $\mathfrak{m}(n)$.

The ideea of our proof is to show that the trace of a maximal subalgebra $\mathfrak{a}$ of $\mathfrak{so}(n,1)$ on $\mathfrak{so}(n)$ is a maximal subalgebra of $\mathfrak{so}(n)$. If not $\mathfrak{a}\cap\mathfrak{so}(n)=\mathfrak{so}(n)$. Thereis an Y in $\mathfrak{a}\setminus\mathfrak{so}(n)$; as $Ad(Y)(\mathfrak{so}(n))\subseteq\mathfrak{a}$ it follows that $\mathfrak{so}(n,1)\subseteq\mathfrak{a}$.

Let $\mathfrak{b}=\mathfrak{a}\cap\mathfrak{so}(n)$. Up to a conjugation $\mathfrak{b}\subseteq\mathfrak{so}(n-1)$ ([10],[12]) so that we can suppose that $\mathfrak{b}=\mathfrak{so}(n-1)$. Let us denote by f^i_j the canonical basis of $\mathfrak{so}(n,1)$ (see § 2). An elementary argument using matrices shows us that $f^{n+1}_n\in\mathfrak{a}$, and for any $k=\overline{1,n-1}$, $f^n_k+f^{n+1}_k\in\mathfrak{a}$ or for any $k=\overline{1,n-1}$, $-f^n_k+f^{n+1}_k\in\mathfrak{a}$. First case leads us to $\mathfrak{m}(n)\subseteq\mathfrak{a}$ and second to $Ad\,T(\mathfrak{m}(n))\subseteq\mathfrak{a}$, where $T=\begin{pmatrix} I_n & 0\\ 0 & -1\end{pmatrix}$; but $\mathfrak{m}(n)$ is maximal.

Using Proposition 1.1. we can prove the analogous of a

Wang Theorem ([17]), namely

THEOREM 1.1. For any $d \in (\frac{n(n+1)}{2}+1, \frac{(n+1)(n+2)}{2})$ there is no Lorentz manifold M^{n+1} having a group of automorphisms of dimension d.

PROPOSITION 1.2. Two points of a Lorentz manifold can be joined by a path γ that is piecewise a spacelike geodesic, whenever they belong to the same component.

If $x,y \in M$ are joined by a path $\alpha:[a,b]\to M$, and $p=\alpha(t)$, we take a Whitehead neighbourhood $\exp_p V_p$ of p. Let $\varepsilon:[a,b]\to(0,\infty)$ be a path such that the disk $D(0,\varepsilon(t)) \subseteq V_{\alpha(t)}$ for any $t\in[a,b]$. If $S_p(\varepsilon)=\{\exp_p tX \mid |t|<\varepsilon, g_p(X,X)>0\}$, as $\lim_{t\to t_0} S_{\alpha(t)}(\varepsilon(t)) = S_{\alpha(t_0)}(\varepsilon(t_0))$ we see that for any $t\in[a,b]$, there is some δ_{t_0} such that $S_{\alpha(t)}(\varepsilon(t)) \cap S_{\alpha(t_0)}(\varepsilon(t_0)) \neq \phi$ whenever $|t-t_0|<\delta_{t_0}$. We cover $[a,b]$ with finitely many intervals $(t_i-\delta_{t_i}, t_i+\delta_{t_i})$ such that $S_{\alpha(t_i)}(\varepsilon(t_i)) \cap S_{\alpha(t_{i+1})}(\varepsilon(t_{i+1})) \neq \phi$; if p_i stay in this intersection, we can join p_i and p_{i+1} by a spacelike geodesic γ_i.

Now Theorem 1.1. follows, since if M is connected and has a group of isometries K of dimension d, K is an open subset of $O_g(M)$, and its isotropy group at o, K_o is an open subgroup of $O(n,1)$, so that for any spacelike vector $X\in T_oM$ there is one $k\in K_o$ with $d_ok(X)=-X$. Then if γ is a spacelike geodesic and $k\in K_{\gamma(\frac{t}{2})}$ with $dk(\dot\gamma(\frac{t}{2}))=-\dot\gamma(\frac{t}{2})$, $k(\gamma(0))=\gamma(t)$. From Proposition 1.2 it follows that given $x,y\in M$, there are $k_1,\ldots k_s\in K$, such that $k_s\circ\cdots\circ k_1(x)=y$, that is M is homogenous. Then $\dim K=\frac{(n+1)(n+2)}{2}$.

§2.HOMOGENEOUS PSEUDORIEMANNIAN MANIFOLDS

The problem of the local classification of multiple transitive homogeneous pseudoRiemannian manifolds is essentially an algebraic one.It reduces ,as Elie Cartan showed,to the next two problems:

A. Find the conjugation classes of subalgebras of $\mathfrak{so}(n)$

B.For a subalgebra $\mathfrak{h}$ of $\mathfrak{so}(n)$ that is the isotropy algebra of a group of isometries K,find all Ad $(\mathfrak{k})$ invariant bilinear symmetric positive definite forms on $\mathfrak{k}/\mathfrak{h}$ ([4],[8]).

We have taken over this view point on homogeneous spaces, trying to explain it in a bundle viewpoint for pseudoriemannian manifolds.We must say that the method described below is available for other G structures then O(p,q) ones.

We shall abreviate with Ψ.R. the word pseudoriemannian. From now on,we shall denote by M a homogeneous Ψ.R. manifold of type (p,q),i.e.a manifold that has an O(p,q) structure,and by K a group of automorphisms of M that is transitive on M.$H=K_o$ will be its isotropy group (at o), $\mathfrak{k}$ and $\mathfrak{h}$ the Lie algebras of K and H.

If $h\in H$ the diagram

$$\begin{tikzcd} T_oM \arrow[r, "d_oh"] \arrow[d, "\exp_o"'] & T_oM \arrow[d, "\exp_o"] \\ M \arrow[r, "h"] & M \end{tikzcd}$$

is commutative,so that the isotropic representation of H in O(p,q) is faithfull and $\mathfrak{h}$ can beembeded as a subalgebra of $\mathfrak{so}(p,q)$.

Let $L(M) \xrightarrow{\pi} M$ be the framebundle of M and $O_g(M)$ its O(p,q) reduction induced by the metric g.If $u_o\in O_g(M)$ is a fixed frame, the differential structure of K is induced by the mapping $\phi_{u_0}: K \longrightarrow O_g(M)$, $\phi_{u_0}(k)=L(k)(u_0)$, where $L(k)$ is the map induced on $L(M)$ by k. From now on we shall suppose that dim $H \geqslant 1$, that is K is a multiply transitive group on M, or shortly M is a multiply transitive homogeneous Ψ.R. manifold.

PROPOSITION 2.1. Let $M=K/H$ be a homogeneous Ψ.R. manifold and $\theta=\theta^i e_i \in \Omega^1(L(M), \mathbb{R}^{p+q})$ be the canonical 1 form on $L(M)$. Then $\theta^i_{u_0}=\phi^*_{u_0}\theta^i$ are left invariant 1-forms on K.

Suppose that $\alpha: K\times N \longrightarrow N$ is a left action on N and $\alpha_k: N\to N$, $\alpha_p: K\longrightarrow N$ are induced by α, then we have:

LEMMA 2.1. If $\theta\in\Omega^1(N)$ is K invariant on N, then for any $p\in N$, $\theta_p=\alpha_p^*\theta$ is left invariant on K.

In fact if L_k is the left translation by k on K, we have $\alpha_p \circ L_k=\alpha_k\circ\alpha_p$, so that

$$L_k^*\theta_p = L_k^*\alpha_p^*\theta=(\alpha_p\circ L_k)^*\theta=(\alpha_k\circ\alpha_p)^*\theta=\alpha_p^*\alpha_k^*\theta=\alpha_p^*\theta=\theta_p$$

Now Proposition 2.1. follows if we take $N=O_g(M)$ and α the action induced on $L(M)$ by the action of K on M.

Let $\omega\in\Omega^1(L(M), gl(p+q,\mathbb{R}))$ be the Levi Civita connection form of M. In $gl(n)=\mathcal{M}_n(\mathbb{R})$ $n=p+q$ we consider the canonical basis $(e^i_j)_{i,j=\overline{1,n}}$, $e^i_j: \mathbb{R}^n\to\mathbb{R}^n$, $e^i_j(e_k)=\delta^i_k e_j$; we have $\omega=\omega^j_i e^i_j$, than

PROPOSITION 2.2. The forms $\omega^i_{j,u_0}=\phi^*_{u_0}\omega^i_j$ are left invariant forms on K. $(\omega^i_{j,u_0}, \theta^i_{u_0})$, $i,j=\overline{1,n}$ are generators of the Lie algebra of left invariant forms on K, $\theta^i_{u_0}$, $i=\overline{1,n}$ being linearly independent.

Proof. As ω^i_j are uniquely determined by the conditions $\omega^i_j e^j_i \in SO(p,q)$, $d\theta^i=-\omega^i_j\wedge\theta^j$ and θ^i are K invariant, it follows that ω^i_j are K invariant and we can use Lemma 2.1. Because $(\omega^i_j(e), i,j=\overline{1,n}, \theta^i(e), i=\overline{1,n})$ form a basis of $T^*_e L(M)$,

$(\omega^i_{j,u_0}, \theta^i_{u_0}, i,j=\overline{1,n})$ generate $\mathfrak{k}$. Since K is transitive on M, given $w \in T_0M$, there is an one parameter subgroup $K_w(t)$ such that $\frac{d}{dt}K_w(\cdot)(p)\big|_{t=0} = w$. Then for each $j=\overline{1,n}$, we find one parameter subgroups of K, K_j, with $\frac{d}{dt}K_j(\cdot)(p)\big|_{t=0} = u_{0,j}$. Let $\xi_u \in T_u O_g(M)$, then $\pi_* \xi_{u_0} = \theta^j(\xi_{u_0})u_{0,j}$ and $\theta^i_{u_0}(\dot{K}_j(0)) = \delta^i_j$ which shows that $\theta^i_{u_0}$ are l.i.(linearly independent).

We shall denote by $(f^i_j)_{i>j}$ the basis of $\mathfrak{SO}(p,q)$ given by

$$f^i_j = \begin{cases} e^i_j - e^j_i & , j < i \leq p \text{ or } p+1 \leq j < i \\ e^i_j + e^j_i & , i > p \geq j \end{cases}$$

and by $\xi^j_i, j<i$ the componehts of $\xi \in \mathfrak{SO}(p,q)$ relatively to this basis.

THEOREM 2.1. Let K be a transitive Lie group of automorphisms of the Ψ.R. manifold M=K/H, such that $\dim \mathfrak{h} = d$ and $\mathfrak{h}$ is given by

$$\mathfrak{h} = \{\xi \in \mathfrak{SO}(p,q) \mid \sum_{j<i} A^i_{rj}\xi^j_i = 0, \ r = \overline{1, \tfrac{n(n-1)}{2} - d}\}$$

Then there are constants $c_{i,j}$ $i=\overline{1,\frac{n(n-1)}{2}-d}$, $j=\overline{1,n}$, $n=p+q$ such that

$$\sum_{j<i} A^i_{rj}\omega^j_{i,u_0} = c_{r,s}\theta^s_{u_0}, \quad r = \overline{1, \tfrac{n(n-1)}{2} - d} \qquad (1)$$

Proof. Let f be a parametrization of M and L(f) the induced parametrization of L(M). The representatives of the forms θ^i, ω^i_j relatively to f are given by

$\theta^i_f(x,v) = y^i_k dx^k$ where $y=v^{-1}$ and

$\omega^i_{j,f}(x,v) = y^i_k(dv^k_j + \Gamma^k_{r,s}v^s_j dx^r)$ (see [8])

Since $dx^i = v^i_j\theta^j_f$ if we put ${}_0\omega^i_{j,f} = y^i_k dv^k_j$

$\sigma^i_{j,k} = \Gamma^p_{r,s}v^s_j y^i_p v^r_k$ the equality $\omega^i_{j,k} = {}_0\omega^i_{j,f} + \sigma^i_{j,k}\theta^k_f$

does not change when it is pulled back on $f^{-1}(\Phi_{u_0}(K))$. Since the manifold structure of K is induced by the map Φ_{u_0}, it follows that relatively to a parametrization φ of K, the local forms ${}_0\omega^i_{j,\varphi}$ and the local reprezentatives $\omega^i_{j,u_0;\varphi}, \theta^i_{u_0;\varphi}$

verify $\omega^i_{j,u_0;\varphi} = {}_0\omega^i_{j,\varphi} + \sigma^i_{j,k}\,\theta^k_{u_0;\varphi}$

But $\phi_{u_0}(H) = \phi_{u_0}(K) \cap \pi^{-1}(\theta)$, so that $\omega_0|H = \phi^*_{u_0}\omega|\pi^{-1}(\theta)$

If $X \in \mathfrak{h}$, $\omega_{u_0}(X) = \phi^*_{u_0}(\omega(\sigma(X)))$ where $\sigma(X)$ is the vertical vector field generated by $\exp(tX)(u_0)$. Since $\omega(\sigma(X)) = \lambda(X)$ where λ is the isotropic reprezentation of $\mathfrak{h}$ in $\mathfrak{so}(p,q)$ and $\theta^i_{u_0}|H = 0$, it follows that ${}_0\omega_\varphi = {}_0\omega^i_{j,\varphi}\, e^j_i$ is a $\lambda(\mathfrak{h})$ valauated form so that $\sum_{j<i} A^i_{r,j}\,{}_0\omega^j_{i,\varphi} = 0$ and we have (1). The forms ω^j_{i,u_0} and $\theta^i_{u_0}$ being left invariant, they must be constant on $\operatorname{im}\varphi$. Because $\theta^i_{u_0}$ are l.i. by Proposition 2.2., $c^{\varphi}_{i,j} = c^{\varphi'}_{i,j}$.

PROPOSITION 2.3. Any linear dependence relation between ω^i_{j,u_0}, $i<j$, and $\theta^i_{u_0}$ is a linear combination of relations (1).

Let $\sum A^i_j \omega^j_{i,u_0} = c_r\theta^r_{u_0}$. If $X_{\mathfrak{h}}$ is the projection of X on $\mathfrak{h}$ in a decomposition $\mathfrak{k} = \mathfrak{m} \oplus \mathfrak{h}$, , if $(A^i_j)_{j<i}$ would be l.i. of $(A^i_{r,j})_{j<i}$ $r = \overline{1, \frac{n(n-1)}{2} - d}$, then $\dim \mathfrak{h} \leq d-1$. From Theorem 2.1. it follows that $A^i_j = \lambda^r A^i_{rj}$ $i>j$, i.e. $(\lambda^r c_{r,i} - c_i)\theta^i_{u_0} = 0$.

From Theorem 2.1. and Propositions 2.1-2.3. it results that if $\mathfrak{h}$ projects isomorphically on the space generated by $\{(f^{i_\alpha}_{j_\alpha}), \alpha = \overline{1,d}, i_\alpha > j_\alpha\}$, then $\theta^i_{u_0}$, $i = \overline{1,n}$ together with $(\omega^{j_\alpha}_{i_\alpha}, j_\alpha < i_\alpha)_{\alpha=\overline{1,d}}$ is a basis of left invariant l-forms on K.

In this situation, if $I_d = \{(i,j)\ i<j, (i,j) \neq (j_\alpha, i_\alpha)\ \alpha = \overline{1,d}\}$.

$$\xi^i_j = \sum_{\alpha=1}^{d} A^{i\,j_\alpha}_{j\,i_\alpha}\,\xi^{i_\alpha}_{j_\alpha}, \quad (i,j) \in I_d \tag{2}$$

are the equations of $\mathfrak{h}$, there are $c^i_{j,k} \in \mathbb{R}$, $k = \overline{1,p+q}$, $i<j$, $(i,j) \in I_d$, such that if $(i,j) \in I_d$

$$\omega^i_{j,u_0} = \sum_{\alpha=1}^{d} A^{i,j_\alpha}_{j,i_\alpha}\,\omega^{j_\alpha}_{i_\alpha} + c^i_{j,k}\,\theta^k_{u_0} \tag{3}$$

LEMMA 2.2. The structure equations of K are a consequence of the Levi Civitae connexion of the $O(p,q)$ structure $O_g(M)$.

The proof of Lemma 2.2 stands on the next simple remarks:

- ω^i_{j,u_0}, $\theta^i_{u_0}$ are the pull backs by ϕ_{u_0} of ω^i_j, θ^i and pull-back, d and $\wedge$ are commuting operations on forms,

- ω^j_{i,u_0} $(j,i) \in I_d$ are linear combinations of $(\omega^{j_\alpha}_{i_\alpha,u_0})_{\alpha=\overline{1,d}}, (\theta^i_{u_0}), i=\overline{1,n}$

If we remark that the Bianchi relations on $O_g(M)$ and the Jacobi conditions on the structure equations of K are the integrability conditions of the stucture equations, we can say

LEMMA 2.3. The Jacobi conditions on $\mathfrak{k}$ are a consequence of the Bianchi relations on $O_g(M)$.

If K^i_{jrs} are the components of the curvature forms Ω^i_j, Lemmas 2.2, 2.3. state that the existence of K is assured by the existence of a solution of a system of equations of degree at most 3, named S, with the unknowns $(c^j_{i,k})_{(j,i)\in I_d}, k=\overline{1,n}$ and $K^{j_\alpha}_{i_\alpha rs}$ $\alpha=\overline{1,d}$, $r<s\leq n$

The system S ever has solutions by the next simple remark

PROPOSITION 2.4. Any Lie subalgebra of $\mathfrak{so}$ (p,q) is the isotropy algebra of a flat space M.

This is trivial, since if H is a subgroup of $GL(n,\mathbb{R})$, then $K=H\times\mathbb{R}^n$ acts transitively on $\mathbb{R}^n$ by $(h,a)x=hx+a$, and $K_0=H$.

THEOREM 2.2. The structure equations of M are determined by the inclusion $\mathfrak{h}\subset\mathfrak{so}$ (p,q) and the structure equations of K.

This is a converse of Lemma 2.2.; let $\Omega\in\Omega^1(O_g(M),\mathfrak{so}(p,q))$ be the curvature form of M and $\Omega_{u_0}=\phi^*_{u_0}\Omega$. If $\Omega^i_j=\sum_{r<s}K^i_{jrs}\theta^r\wedge\theta^s$, then $\Omega^i_{j,u_0}=\sum_{r<s}K^i_{jrs}\theta^r_{u_0}\wedge\theta^s_{u_0}$ but the forms $(\theta^r_{u_0})_{r=\overline{1,n}}$ being l.i., $(\theta^r_{u_0}\wedge\theta^s_{u_0})_{r<s}$ are l.i. too, so that K^i_{jrs} are uniquely determined by the structure equations of $\mathfrak{k}$ and the constants $A^{i,i_\alpha}_{j,j_\alpha}$ in relations (2). The structure equations of $\mathfrak{k}$ are a consequence of the structure equations of $\mathfrak{h}\subset\mathfrak{so}(p,q)$, and of the choice of the solution of the system S.

COROLLARY 2.1. For $a=1,2$, let K_a be a transitive group on the Ψ.R. manifold M_a, and ${}_aK^i_{jkl}$ the components of its curvature forms $\phi^*_{u_0}\Omega^i_j$. Then M_1 and M_2 are locally equivalent iff there is $c = (c^\alpha_i) \in O(p,q)$ such that ${}_1K^i_{jkp} = (c^{-1})^i_\alpha c^\beta_j c^\lambda_k c^\rho_p {}_2K^\alpha_{\beta\lambda\rho}$ (e)

Proof. The necessity of (e) is obvious. If (e) hold, we can suppose without loss of generality that $c=id$. Then the structure equations, pulled back on M_1 and M_2 are idendical. By changing K_2 with the automorphism group of M_2, we can suppose that $\mathfrak{h}_1 \subset \mathfrak{h}_2$. Then the equations of K_a are ${}_a\omega^i_{j,u_0} = 0$, $(i,j) \in I_{d_a}$; $I_{d_1} \supset I_{d_2}$ and we see that ${}_2\omega^i_{j,u_0} = 0$, $(i,j) \in I_{d_1}$ are the equations of a subgroup K' of K_2 which has same constants of structure as K_1. We find then a local morphism into from K_1 to K_2, f, such that $f(H_1) \subset H_2$ and $f^*\,{}_2\theta^i_{u_0} = {}_1\theta^i_{u_0}$ $i=\overline{1,n}$. From Theorem 2.3. we find that M_1 and M_2 are locally equivalent.

If (M,g) is a Ψ.R. manifold of signature (p,q), then $\pi^*g = \theta^1\otimes\theta^1 + \dots + \theta^p\otimes\theta^p - \theta^{p+1}\otimes\theta^{p+1} - \dots - \theta^{p+q}\otimes\theta^{p+q}$, if M is homogeneous and $\pi_{u_0} = \pi\circ\phi_{u_0}$, then $\pi^*_{u_0}g = \sum_{i=1}^{p}\theta^i_{u_0}\otimes\theta^i_{u_0} - \sum_{i=p+1}^{p+q}\theta^i_{u_0}\otimes\theta^i_{u_0} = \Psi$ (4), Conversely, given $\mathfrak{h} \subset \mathfrak{so}(p,q)$ and $\mathfrak{k}$ by equations (3), the left invariant tensor field Ψ given in (4) induces a nondegenerate quadratic form B of signature (p,q) on $\mathfrak{k}/\mathfrak{h}$, which is Ad ($\mathfrak{k}$) invariant. If we carry B by left translations on every tangent space $T_x(K/H)$, K/H becomes a homogeneous Ψ.R. manifold, that admits K as a group of isometries with isotropy group H ([8]); this Ψ.R. stucture is named $\tilde{g}$.

An alternate description of $\tilde{g}$ in a neighbourhood of H in K/H is the next one: let I be the ideal generated by $(\theta^i_{u_0}, i=\overline{1,n})$ then kH is the maximal integral manifold of I, that passes through k. If we take a coordinate system $u = x\times y: V \longrightarrow \mathbb{R}^n \times \mathbb{R}^d$, such that $V_a = \{p \mid x(p)=a\}$ are integral manifolds of I, and if we put

$V^b = \{p \mid y(p)=b\}$.

THEOREM 2.3. 1. $\varphi|V_a$ is constant

2. The natural projection $\pi_{u_0}:(V^b,\varphi|V^b)\to(K/H,\bar{g})$ is a local isometry.

2. is obvious from the definitions, because if we choose a frame $u_0 \in O_{\bar{g}}(K/H)$, $\pi_{u_0}^*\bar{g}=\varphi$.

1. Let $\omega^\alpha_{\beta,u_0}=n^\alpha_{\beta,\gamma}dx^\gamma+m^\alpha_{\beta,r}dy^r$, $\theta^\alpha_{u_0}=l^\alpha_\beta dx^\beta$. The structure equations imply that for any r, $(m^\alpha_{\beta,r})\in \mathfrak{so}(p,q)$ and $\frac{\partial l^\alpha_\beta}{\partial y^r}=-m^\alpha_{\gamma,r}l^\gamma_\beta$. Since $g_{\beta\lambda}=\sum_{\alpha=1}^{p} l^\alpha_\beta l^\alpha_\lambda-\sum_{p+1}^{p+q} l^\alpha_\beta l^\alpha_\gamma$, from the structure equations we find that $\frac{\partial g_{\beta\lambda}}{\partial y^r}=0$.

PROPOSITION 2.5. Let $\mathfrak{h},\mathfrak{h}'\subset\mathfrak{so}(p,q)$ be two conjugated subalgebras. If $\mathfrak{h}$ is the isotropy algebra of a homogeneous $\Psi.R.$ manifold M, there is an homogeneous $\Psi.R.$ manifold M' isomorphic to M, whose isotropy algebra is $\mathfrak{h}'$.

It is enough to see that $\mathfrak{h},\mathfrak{h}'$ are conjugated iff they have same equations relatively to two $\Psi.R.$-orthogonal basis, so that a solution of the system S implies the exixtence of two homogeneous manifolds M,M' with isotropy algebras $\mathfrak{h},\mathfrak{h}'$. Once we choose the isomorphic pairs of algebras $(\mathfrak{k},\mathfrak{h})$, $(\mathfrak{k}',\mathfrak{h}')$ of isomorphic pairs of groups (K,H), (K',H'), it follows that $(K/H,\bar{g})$, $(K'/H',\bar{g}')$ are isomorphic.

If we put $\omega^i=\theta^i_{u_0}$, $i=\overline{1,p+q}$, $\omega^{p+q+\alpha}=\omega^{j_\alpha}_{i_\alpha,u_0}$, $\alpha=\overline{1,d}$ and $-C^i_{jk}$ are the constants of structure of $\mathfrak{k}$ relatively to the dual basis of $(\omega^i, i=\overline{1,p+q+d})$, then

PROPOSITON 2.6. 1. M is reductive iff $C^i_{jk}=0$, for $j\leq p+q<k$, $i>p+q$

2. M is locally symmetric iff $C^i_{jk}=0$ for $(j\leq p+q<k, i>p+q)$ or $(i<p+q, j<k\leq p+q)$.

If $(X_i)_{i=\overline{1,p+q+d}}$ is the dual of $(\omega^i, i=\overline{1,n+d})$, $\mathfrak{m}$ is the subspace of $\mathfrak{k}$ generated by $X_1,\dots,X_{p+q}$, the conditions given in 1. respectively 2. of Proposition 2.6. are equivalent to $[\mathfrak{h},\mathfrak{m}]\subset\mathfrak{m}$ respectively to $[\mathfrak{h},\mathfrak{m}]\subset\mathfrak{m}$ and $[\mathfrak{m},\mathfrak{m}]\subseteq\mathfrak{h}$ conditions which are equivalent to the local reductivity respectively the local symmetry of($K/H,\bar{\bar{g}}$) ([8],[11]).

&3. NORMAL FORMS OF 3-DIMENSIONAL HOMOGENEOUS ψ.R. MANIFOLDS

We recall that a ψ.R. manifold M is extendible if it embeds as a proper open submanyfold of another ψ.R. manifold,M'([1]) We shall say that a ψ.R. manifold(M,g) is a normal form if it is connected,1-connected and inextendible.Normal homogeneous forms of Riemannian geometry were settled in small dimensions by E.Cartan([4]),and Ishihara([6]).Roughly speaking,in this case, for a given H, the topological type of M is determined by the one of K. The key of this fact is the completeness of homogeneous Riemannian manifolds. In the ψ.R.case, the geodesic completeness is a consequence of one of the following hypothesis:

H.1. -M is compact ([9])

H.2. -M is a naturally reductive([8],[11])

The question of the natural reductivity of a given ψ.R.manifold of dimension $\geqslant 4$ is more difficult, as we shall see in &5.

THEOREM3.1. Any homogeneous ψ.R. manyfold M of dimension 3, is reductive.M is naturally reductive iff it is symmetric.

In the Riemannian case this is a consequence of the compacity of H([8]),and of some calculations,not presented in this paper.In the Lorentz case,if K is simply transitive,we have nothing to prove. For multiply transitive K, we use the method described in §2. The algebraic support of Theorem 3.1. is

THEOREM 3.2. There are 3 conjugacy classes of 1-dimensional subspaces of $\mathfrak{so}(2,1)$, denoted by c_+, c_0 and c_-; c_+ is the conjugacy class of $\mathfrak{so}(2)$ and corresponds to a spacelike isotropy subgroup, c_0 is the conjugacy class of $\langle X_0\rangle$, where $X_0=\begin{pmatrix}0&1&0\\-1&0&1\\0&1&0\end{pmatrix}$ and corresponds to a null isotropy subgroup, and c_- is the conjugacy class of $\mathfrak{so}(1)\times\mathfrak{so}(1,1)$ and corresponds to a timelike isotropy subgroup.

We shall not prove this theorem. Using Proposition 1.1., we see that there is one conjugacy class of 2 dimensional subalgebras of $\mathfrak{so}(2,1)$, the class of $\mathfrak{m}(2)$. If M has $\mathfrak{m}(2)$ as isotropy algebra, M is a space of nonpositive constant curvature.

Case c_+. $\mathfrak{so}(2)$ has the equations $\xi^1_3=\xi^2_3=0$, that is $\omega^1_{3,u_0}=a\,\omega^1_{u_0}+b\,\omega^2_{u_0}+c\,\omega^3_{u_0}$, $\omega^2_{3,u_0}=a'\omega^1_{u_0}+b'\omega^2_{u_0}+c'\omega^3_{u_0}$ (5). From now on we shall omit the subscript u_0; we shall also omit the symbol $\wedge$ for exterior multiplication of forms. To understand how is applied the machinery of §2., in this case we shall give all the details. The forms ω^i, ω^i_j are subject to the structure equations:

$$d\omega^1=-\omega^1_2\omega^2-\omega^1_3\omega^2 \qquad d\omega^1_2=-\omega^1_3\omega^2_3+\Omega^1_2$$
$$d\omega^2=\omega^1_2\omega^2-\omega^2_3\omega^3 \qquad d\omega^1_3=-\omega^1_2\omega^2_3+\Omega^1_3$$
$$d\omega^3=-\omega^1_3\omega^1-\omega^2_3\omega^2 \qquad d\omega^2_3=\omega^1_2\omega^1_3+\Omega^2_3$$

If we idwntify the terms in $\omega^i\omega^1_2$, when exterior differentiateing relations (5), we find that

$$-\omega^1_2(a'\omega^1+b'\omega^2+c'\omega^3)=-a\,\omega^1_2\omega^2+b\,\omega^1_2\omega^1 \quad \text{and}$$
$$\omega^1_2(a\omega^1+b\omega^2+c\omega^3)=-a'\omega^1_2\omega^2+b'\omega^1_2\omega^1 \quad \text{that is}$$

$a=b'$, $b=-a'$, $c=c'=0$. The structure equations of K are

(6) $d\omega^1=-a\omega^1\omega^3-b\omega^2\omega^3+\omega^2\omega^1_2$, $d\omega^2=b\omega^1\omega^3-a\omega^2\omega^3-\omega^1\omega^1_2$
$d\omega^3=2b\,\omega^1\omega^2$, $d\omega^1_2=-(a^2+b^2)\omega^1\omega^2+K_{12}\omega^1\omega^2+K^1_{213}\omega^1\omega^3+K^1_{223}\omega^2\omega^3$

We see that the conditions in 1. of Prop.2.6. are verified, i.e. M is reductive. The Jacobi conditions imposed to (6) give us $K^1_{213} = K^1_{223} = 0$, $ab=0$, $a(K^1_{212}-a^2-b^2)=0$. We put $c=K^1_{212}-a^2-b^2$; if $a\neq 0$ then $b=c=0$ and $\Omega^1_2=a^2\omega^1\omega^2$. Now $\Omega^1_3=d\omega^1_3+\omega^1_2\omega^2_3=-a^2\omega^1\omega^3$ and $\Omega^2_3=$ $=-a^2\omega^2\omega^3$, i.e. M is a space of constant nonnegative curvature a^2, and then M is reductive. If $a=0$, then $[X_1,X_2]_m=2bX_3$ $[X_1,X_3]_m=-bX_2$, and then $B([X_1,X_2]_m,X_3)+B(X_2,[X_1,X_3]_m)=b$

Then M is naturally reductive iff $b=0$, in which case M is symmetric.

Case c_0. Similar computations, show us that in this case, $\omega^1_2-\omega^1_3=\omega^2_3=0$ i.e. M is locally symmetric.

Case c_-. If $\omega^4=\omega^2_3$, the structure equations pulled back on K, give us

$$\begin{aligned} d\omega^1 &= 2c\,\omega^2\omega^3 \\ d\omega^2 &= -b\,\omega^1\omega^2 - c\,\omega^1\omega^3+\omega^3\omega^4 \\ d\omega^3 &= -c\,\omega^1\omega^2 - b\,\omega^1\omega^3+\omega^2\omega^4 \\ d\omega^4 &= -a\,\omega^2\omega^3, \quad bc=ab=0 \end{aligned}$$

For $b\neq 0$, M is a space of constant curvature $-b^2$; for $b=0$, it may be shown that M is reductive too. M is naturally reductive iff $c=0$, and in this case M is locally symmetric too.

THEOREM 3.3. A normal homogeneous 4.R. form of dimension 3, is diffeomorphic to $\mathbb{R}^3$, S^3 or $S^2\times\mathbb{R}$.

Proof. If K is simply transitive, M = K, and a 3-dimensional 1-connected Lie group is diffeomorphic to S^3 or $\mathbb{R}^3$. If K is multiply transitive, we have to check only the cases with dim H=1

Case c_+. If $a\neq 0$,

If $c\neq 0=b$, from Corrolary 2.1. we see that M is the product of a complete Riemannian surface of constant curvature and $(\mathbb{R},g_0)$.

If $c\neq 4b^2$, $b\neq 0$, we put $\omega=\omega^4-b\omega^3$, then $d\omega^1=-\omega\omega^2$, $d\omega^2=\omega\omega^1$, $d\omega^3=2b\omega^1\omega^2$, $d\omega=l\omega^1\omega^2$ where $l = 2b^2+c$.

If $l=0, \omega=0$ gives a subgroup of K,T with structure equations $d\omega^1=d\omega^2=0$ $d\omega^3=2b\omega^1\omega^2$ .Since $\mathfrak{t}$ the Lie algebra of T cuts $\mathfrak{h}$ in 0,T is transitive on M;then T = M,but T = $\mathbb{R}^3$. If $l\neq 0$,the subgroup G given by $2b\omega+l\omega^3=0$ is transitive on M,then G=M,but G is topologically S^3 or $\mathbb{R}^3$.

Case c_0 Because $\mathfrak{k}$ is solvable , $K\simeq\mathbb{R}^4$, $H\simeq\mathbb{R}$, $M\simeq\mathbb{R}^3$

Case c_- Symilar arguments as in the case c_+ show us that M is topological $\mathbb{R}^3$.

REMARK 3.1.We nottice that in the case c_0 ,M may be one of the next spaces:-Minkowski space

-(M,g_α) with

$$g_\alpha=(dx^1)^2-2dx^1dx^2-\frac{1}{4\alpha}e^{-2\sqrt{-\alpha}x^1}(dx^3)^2,\text{when } \alpha<0$$

topological $\mathbb{R}^3$ when $\alpha>0$,where

$$g_\alpha=(1+(x^2)^2\operatorname{ctg}^2x^1\sqrt{\alpha}+2x^3\frac{\cos(x^1\sqrt{\alpha})}{\sin^2x^1\sqrt{\alpha}})(dx^1)-\frac{2x^2}{\sqrt{\alpha}}\operatorname{ctg}(x^1\sqrt{\alpha})dx^1dx^2-\frac{2}{\sqrt{\alpha}\sin(x^1\sqrt{\alpha})}dx^1dx^3.$$

We shall say that M is a space $M_0^{2,1}$

In the case c_- ,apart of Minkowski space,M has one of the metrics g given below:

$$g=c^2(dx^1+x^2dx^3-x^3dx^2)^2+(dx^2)^2-(dx^3)^2\quad ,\text{if } a=2c^2$$

$$(a-2c^2)^2g=((dx^1+2c\operatorname{sh}x^2dx^3)^2+(a-2c^2)((dx^2)^2-\operatorname{ch}^2x^2(dx^3)^2),$$

if $a-2c^2\neq 0$.We shall say that M is a $M_-^{2,1}$.

In the case c_+,apart of Minkowski space,M has one of the metrics given below:$g=(dx^1)^2+(dx^2)^2-b^2(2x^1dx^2+dx^3)^2$, if $l=0$

$$l^2g=l((dx^1)^2+\cos^2(x^1)(dx^2)^2)-b^2(2\sin x^1dx^2+dx^3)^2\quad ,\text{if } l>0$$

$$l^2g=-l((dx^1)^2+\operatorname{ch}^2x^1(dx^2)^2)-b^2(2\operatorname{sh}x^1dx^2+dx^3)^2,\text{if } l<0.$$ We shall say that M is a $M_+^{2,1}$.

§4. REAL SUBALGEBRAS OF $\mathfrak{sl}(2,\mathbb{C})$

In order to find the subalgebras of $\mathfrak{so}(p,q)$, we were lead to the next problem in linear algebra: find the ''canonical form'' of a (p,q)-skew symmetric matrice, that is a matrice $X \in \mathcal{M}_{p+q}(\mathbb{R})$, with $X + Q\,{}^tX\,Q = 0$, where $Q = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$. For our purposes we shall solve this problem only for $p=3, q=1$. Let us denote by $\langle X \rangle$ the linear subspace generated by X.

LEMMA 4.1. If $X \in \mathfrak{so}(p,q)$ and $P_X(t) = t^{p+q} + \sum_{k\geq 1} a_k(X)t^{p+q-k}$ is its characteristic polynomial, then $a_{2r+1}(X)=0$ and the following numbers are invariants of the conjugacy class of $\langle X \rangle$:

1) $\mathrm{sgn}(a_{2r}(X))$, 2) $(a_{2s}(X))^r(a_{2r}(X))^{-s}$, if $a_{2r}(X) \neq 0$.

$a_{2r+1}(X)=0$ since $P_X(t) = (-1)^{p+q} P_X(-t)$. The numbers in 1) and 2) are invariants because P_X is invariant under Ad.

We shall now specialize in $\mathfrak{so}(3,1)$. If $X = (x^i_j) \in \mathfrak{so}(3,1)$,

$a_2(X) = (x^1_2)^2 + (x^1_3)^2 + (x^2_3)^2 - (x^1_4)^2 - (x^2_4)^2 - (x^3_4)^2$ and

$a_4(X) = -(x^1_2x^3_4 + x^1_4x^2_3 - x^1_3x^2_4)^2$. Looking at the eigenvalues of X we get to

LEMMA 4.2. If $a_4(X) \neq 0$, X is conjugated with a matrice of the form

$$X_{ab} = \begin{pmatrix} \begin{matrix} 0 & a \\ -a & 0 \end{matrix} & O_2 \\ O_2 & \begin{matrix} 0 & b \\ b & 0 \end{matrix} \end{pmatrix}, \text{with } a,b > 0.$$

LEMMA 4.3. If $a_4(X) = 0, a_2(X) \neq 0$, X is conjugated to X_{0b} with $b>0$, or to X_{a0} with $a>0$.

LEMMA 4.4. If $a_2(X) = a_4(X) = 0$, X is conjugated to Y_a, $a \geq 0$,

$$\text{where } Y_a = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & -a & 0 & a \\ 0 & 0 & a & 0 \end{pmatrix}$$

Another simplification of the problem of determination of the conjugancy classes of subalgebras of $\mathfrak{so}(3,1)$ is furnished by the remarkable coincidence $\mathfrak{so}(3,1) \simeq \mathfrak{sl}(2,\mathbb{C})$ ([5],[15]) If we exhibit an explicit isomorphism I $:\mathfrak{so}(3,1)\to\mathfrak{sl}(2,\mathbb{C})^{\mathbb{R}}$, then Lemmas 4.2-4.4.can be red in the next form:

PROPOSITION 4.1.Any $X\in\mathfrak{sl}(2,\mathbb{C})$ is conjugated over $\mathbb{R}$ with a matrice of one of the following forms: $\begin{pmatrix} z & 0\\ 0 & -z\end{pmatrix}$ or $i\begin{pmatrix} a & a\\ a & -a\end{pmatrix}$, $a\in\mathbb{R}$.

Because conjugated subalgebras are isomorphic and the isomorphism classes of 2 and 3 dimensional Lie algebras are known ([3]),we can find all the real subalgebras of $\mathfrak{sl}(2,\mathbb{C})$ in the next manner:we take $X\in\mathfrak{sl}(2,\mathbb{C})$ of the form given in Proposition 4.1.,and we complete it to a basis of a Lie subalgebra of dimension 2 or 3 ,relatively to that basis,the subalgebra having the canonical structural equation ([15]).So,after solving the problem of classification of real conjugacy classes of subalgebras of $\mathfrak{sl}(2,\mathbb{C})$,we've got back on $\mathfrak{so}(3,1)$ and obtained the following

THEOREM 4.1.The conjugacy classes of subalgebras of $\mathfrak{so}(3,1)$ are:-in dimension 5 -none

-in dimension 4 -class of $\mathfrak{m}(3)$,with equations: $\xi_3^1-\xi_4^1=\xi_3^2-\xi_4^2=0$

-in dimension 3 -classes of

$\mathfrak{a}:\ \xi_3^1-\xi_4^1=\xi_3^2-\xi_4^2=\xi_4^3=0$

$\mathfrak{so}(3):\ \xi_4^1=\xi_4^2=\xi_4^3=0$

$\mathfrak{so}(2,1):\ \xi_2^1=\xi_3^1=\xi_4^1=0$

$\mathfrak{a}_k:\ \xi_2^1+k\,\xi_4^3=\xi_3^1-\xi_4^1=\xi_3^2-\xi_4^2=0\quad k>0$

-in dimension 2 -classes of $\mathfrak{so}(2)\times\mathfrak{so}(1,1):\ \xi_3^1=\xi_4^1=\xi_3^2=\xi_4^2=0$

$\mathfrak{m}(2):\ \xi_2^1=\xi_3^1=\xi_4^1=\xi_3^2-\xi_4^2=0$

$\mathfrak{b}\ :\ \xi_2^1-\xi_4^2=\xi_3^1-\xi_4^3=\xi_4^1-\xi_3^2=0$

-in dimension 1 -classes of

$\langle X_0\rangle:\ \xi_2^1=\xi_3^1=\xi_4^1=\xi_3^2-\xi_4^3=\xi_4^2=0$

$\mathfrak{so}(2):\ \xi_3^1=\xi_4^1=\xi_3^2=\xi_4^2=\xi_4^3=0$

$\mathfrak{so}(1,1):\ \xi_2^1=\xi_3^1=\xi_4^1=\xi_3^2=\xi_4^2=0$

$\mathfrak{c}_m\ :\ \xi_3^1=\xi_4^1=\xi_3^2=\xi_4^2=\xi_4^3-m\,\xi_2^1=0$

§5.THE LOCAL CLASSIFICATION OF 4 DIMENSIONAL HOMOGENEOUS LORENTZ MANIFOLDS

In this paragraph we shall give the list of all germs of 4-dimensional multiply transitive homogeneous Lorentz manifolds. This list was obtained after some lenghty and rather irrelevant calculations, that we shall skip without exception, as a direct application of the method presented in §2, and of Theorem 4.1. There are few examples that do not verify H.1. or H.2. in §3. This fact does not allow us to describe completely the normal homogeneous 4 dimensional Lorentz forms. We shall point them out, but we shall examine them elsewhere. We shall enumerate the spaces after their isotropy algebra, which will be underlined at the begining of the row.

$\underline{\mathfrak{m}(3)}$ M is locally Minkovski

$\underline{\mathfrak{a}_k}$ M is locally Minkovski

$\underline{\mathfrak{a}}$ $\Omega^1_2=\Omega^3_4=0 \quad \Omega^1_3=\Omega^1_4=b\omega^1(\omega^3+\omega^4)$

$\Omega^2_3=\Omega^2_4=b\omega^2(\omega^3+\omega^4)$ M is locally symmetric,

$g=(dx^1)^2+(dx^2)^2+(1-b((x^1)^2+(x^2)^2))(dx^3)^2-dx^3dx^4$

$\underline{\mathfrak{so}(3)}$ 1.M has constant curvature ; 2.M is locally the Lorentz product of a 3-dimensional Riemannian space of constant curvature and the Euclidean line.

$\underline{\mathfrak{so}(2,1)}$ 1. M has constant curvature ;2.M is locally the Lorentz product of an Euclidean line and a 3-Lorentz manifold of constant curvature

$\underline{\mathfrak{so}(2)\times\mathfrak{so}(1,1)}$.M is locally the Lorentz product of two surfaces of constant curvature, one Riemannian and one Lorentz.

$\underline{\mathfrak{m}(2)}$ M is locally the Lorentz product of an Euclidean line and a Lorentz manifold of constant nonpositive curvature

$\mathfrak{b}$ 1. $\Omega^1_2=\Omega^3_4=0 \quad -\Omega^1_3=\Omega^1_4=(\alpha\omega^1+\beta\omega^2)(\omega^3-\omega^4)$

$$-\Omega^2_3=\Omega^2_4=(\beta\omega^1+\gamma\omega^2)(\omega^3-\omega^4)$$

If we consider the basis of left invariant vector fields dual to $-\omega^1, -\omega^2, -\omega^3, \omega^4-\omega^3, -\omega^5, -\omega^6$ we find out the structure equations of a G_6 on a homogeneous space considered in ([13]). This is not a reductive, neither a compact space, $\mathfrak{b}$ being a solvable algebra.

2. $\Omega^1_2=-(a^2+b^2)\omega^1\omega^2 \quad \Omega^1_3=-(a^2+b^2+(\lambda+1)(a^2+4b^2))\omega^1\omega^3$
$+(\lambda+1)(a^2+4b^2)\omega^1\omega^4-3ab(\lambda+1)\omega^2\omega^4$, $\Omega^2_3=3(\lambda+1)ab\omega^1\omega^3-$
$-(a^2+b^2+(\lambda+1)(4a^2+b^2))\omega^2\omega^3-3(\lambda+1)ab\omega^1\omega^4+(\lambda+1)(4a^2+b^2)\omega^2\omega^4$

$\Omega^1_4=(\lambda+1)(a^2+4b^2)\omega^1\omega^3-3(\lambda+1)ab\omega^2\omega^3+3(\lambda+1)ab\omega^2\omega^4-$
$-(\lambda(a^2+b^2)+3(\lambda+1)b^2)\omega^1\omega^4$

$\Omega^2_4=-3(\lambda+1)ab\omega^1\omega^3+(\lambda+1)(4a^2+b^2)\omega^2\omega^3+3(\lambda+1)ab\omega^1\omega^4-$
$-(\lambda(a^2+b^2)+3(\lambda+1)a^2)\omega^2\omega^4$

$\Omega^3_4=(a^2+b^2)\omega^3\omega^4$

This is not a reductive space

$\langle X_0\rangle$ 1. M is locally the product of an Euclidean line and $M^{2,1}_0$

2. $\Omega^1_2=-2c^2\omega^1\omega^2-c^2\omega^2\omega^4 \quad \Omega^1_3=(5c+2b)c\omega^1\omega^3+2c(b+c)\omega^3\omega^4$

$\Omega^1_4=c^2\omega^1\omega^4 \quad \Omega^2_3=-c^2\omega^2\omega^3 \quad \Omega^2_4=-c^2\omega^1\omega^2 \quad \Omega^3_4=2c(b+c)\omega^1\omega^3+c(2b+c)\omega^3\omega^4$

$$g=(dx^1)^2+(2+\tfrac{b}{c})e^{2(b-c)x^1}(dx^2)^2+e^{-2cx^1}(dx^3)^2+2e^{-2cx^1}dx^2dx^4$$

3. $\alpha\neq 0$. $\Omega^1_2=(\alpha-2c^2)\omega^1\omega^2+(\alpha-c^2)\omega^2\omega^4 \quad \Omega^1_3=-c^2\omega^1\omega^3$

$\Omega^1_4=c^2\omega^1\omega^4 \quad \Omega^2_3=-c^2\omega^2\omega^3 \quad \Omega^2_4=(\alpha-c^2)\omega^1\omega^2+\alpha\omega^2\omega^4 \quad \Omega^3_4=+c^2\omega^3\omega^4$

If $\alpha>0$, $g=((dx^1)^2-(dx^2)^2)+e^{-2cx^1}\Big((x^3)^2\operatorname{tg}^2(x^2\sqrt{\alpha})+\dfrac{2x^4}{\cos(x^2\sqrt{\alpha})}\operatorname{tg}(x^2\sqrt{\alpha})\Big)\cdot$
$\cdot(dx^2)^2+(dx^3)^2+\dfrac{2x^3}{\cos(x^2\sqrt{\alpha})}dx^2dx^3+\dfrac{2}{\cos(x^2\sqrt{\alpha})}dx^2dx^4$

If $\alpha<0$, $g=(dx^1)^2-(dx^2)^2+e^{-2cx^1}\Big(2dx^1dx^4-\dfrac{e^{2x^2\sqrt{-\alpha}}}{4\alpha}(dx^3)^2\Big)$

$\underline{SO(2)}$ 1. $\Omega^1_2=(\alpha^2-a^2)\omega^1\omega^2$ $\Omega^1_3=(\alpha r-a^2)\omega^1\omega^3+(ar-\alpha a)\omega^1\omega^4$

$\Omega^1_4=(ar-\alpha a)\omega^1\omega^3+(as-\alpha^2)\omega^1\omega^4$ $\Omega^2_3=(\alpha r-a^2)\omega^2\omega^3+(ar-\alpha a)\omega^2\omega^4$

$\Omega^2_4=(\alpha s-a\alpha)\omega^1\omega^3+(as-\alpha^2)\omega^2\omega^4$ $\Omega^3_4=(s^2-r^2)\omega^3\omega^4$, $ar-\alpha s=0$

$$g=e^{2x^3}((dx^1)^2+(dx^2)^2)+\frac{1}{a^2}((dx^3)-\frac{2\alpha}{a}e^{sx^3}dx^3dx^4+\frac{\alpha^2-a^2}{a^2}(dx^4)^2)$$

2. M is locally the Lorentz product of a Riemannian surface of constant curvature and a Lorentz surface of constant curvature.

3. $\Omega^1_2=\alpha^2\omega^1\omega^2$ $\Omega^1_3=\alpha r\omega^1\omega^3$ $\Omega^2_3=\alpha r\omega^2\omega^3$

$\Omega^1_4=-\alpha^2\omega^1\omega^4$ $\Omega^2_4=-\alpha^2\omega^2\omega^4$ $\Omega^3_4=-r^2\omega^3\omega^4$

$$g=e^{2\alpha x^4}((dx^1)^2+(dx^2)^2)+e^{2rx^4}(dx^3)^2-(dx^4)^2$$

4. $\Omega^1_2=c\omega^1\omega^2$ $\Omega^1_3=b\omega^1(b\omega^3+\beta\omega^4)$ $\Omega^1_4=\beta\omega^1(b\omega^3+\beta\omega^4)$

$\Omega^2_3=\beta\omega^2(b\omega^3+\beta\omega^4)$ $\Omega^2_4=b\omega^2(b\omega^3+\beta\omega^4)$ $\Omega^3_4=0$

If $l=c-2b^2+2\beta^2$, and g is the expression of the metric of $M^{2,1}_+$ with the above l, then M has the metric g_+ given by

$g_+=g+\beta^2(2x^1dx^2+dx^4)^2$, if $l=0$, $l^2g=l^2g_++\omega^2$, if $l\neq 0$, where

$$\omega=\begin{cases}\beta(2\sin(x^1)dx^2+dx^4), & \text{for } l>0\\ \beta(2\operatorname{sh}(x^1)dx^2+dx^4), & \text{for } l<0\end{cases}$$

. For $\beta=0$ M is locally the Lorentz product of $M^{2,1}_+$ and the Euclidean line. For $b^2=\beta^2$, $c=0$, M is locally isomorphic to the space with isotropy algebra $\mathfrak{a}$, given on p.16.

5. $ab=\alpha\beta$ $a\beta=\alpha b$ $\Omega^1_2=(\alpha^2+\beta^2-a^2-b^2)\omega^1\omega^2$

$\Omega^1_3=(2\alpha^2+b^2-a^2)\omega^1\omega^3+(a\alpha+b\beta)\omega^1\omega^4$

$\Omega^1_4=(a\alpha+b\beta)\omega^1\omega^3+(2a^2+\beta^2-\alpha^2)\omega^1\omega^4$

$\Omega^2_3=(2\alpha^2+\beta^2-a^2)\omega^1\omega^3+(a\alpha+b\beta)\omega^2\omega^4$

$\Omega^2_4=(a\alpha+b\beta)\omega^2\omega^3+(2a^2+\beta^2-\alpha^2)\omega^2\omega^4$

$\Omega^3_4=4(a^2-\alpha^2)\omega^3\omega^4$

If $a=0$, $g=(dx^1)^2+(dx^2)^2+(dx^3)^2-(dx^4)^2+4(b^2-\beta^2)(x^1)^2(dx^2)^2+$

$+4x^1dx^2(b\,dx^3-\beta dx^4)$

If $a \neq 0$, $g=(dx^1)^2+(dx^2)^2-(dx^3)^2+2e^{2ax^3}(2bx^1dx^2dx^3-\frac{\alpha}{a}dx^3dx^4)+$

$+e^{4ax^3}(4(\beta^2-b^2)(x^1)^2(dx^2)^2+\frac{a^2-\alpha^2}{a^2}(dx^4)^2+\frac{4}{a}(a+\alpha)x^1dx^2dx^4))$.

$SO(1,1)$ 1. M is the Lorentz product of a Riemannian surface of constant nonpositive curvature and a Lorentz surface of constant curvature.

2. $\Omega^1_2=-(a^2+b^2)\omega^1\omega^2$, $\Omega^1_3=(ad-c^2)\omega^1\omega^3-(cd+ac)\omega^2\omega^3$
$\Omega^1_4=(c^2-ad)\omega^1\omega^4+(cd+ac)\omega^2\omega^4$; $\Omega^2_3=(bd-dc)\omega^1\omega^3-$
$-(d^2+bc)\omega^2\omega^3$, $\Omega^2_4=(cd-bd)\omega^1\omega^4+(d^2+bc)\omega^2\omega^4$,
$\Omega^3_4=(c^2+d^2)\omega^3\omega^4$.

$g=(c^2+d^2)e^{2ax^2}(dx^1)^2-2ce^{ax^2}dx^1dx^2+(dx^2)^2-2e^{2dx^2}dx^3dx^4$

3. M is locally the Lorentz product of the Euclidean line and $M^{2,1}$

4. $\Omega^1_2=0$ $\Omega^1_3=-e(e\omega^1+f\omega^2)\omega^3$ $\Omega^1_4=e(e\omega^1+f\omega^2)\omega^4$
$\Omega^2_3=-f(e\omega^1+f\omega^2)\omega^3$, $\Omega^2_4=f(e\omega^1+f\omega^2)\omega^4$ $\Omega^3_4=\lambda\omega^3\omega^4$

If $L=\lambda+3(f^2+e^2)=0$,

$g=4(e^2+f^2)(x^3)(dx^4)^2+(dx^1)^2+(dx^2)^2+4x^3dx^4(edx^1+fdx^2)+$
$+(dx^3)^2-(dx^4)^2$,

If $L \neq 0$, $L^2g=4(e^2+f^2)sh^2x^3(dx^4)^2-4Lshx^3dx^4(edx^1+fdx^2)-$
$-L((dx^3)^2-ch^2x^3(dx^4)^2)$.

The above list gives us the local classification of 4-dimensional multiply transitive homogeneous Lorentz manifolds. Simply transitive Lorentz manifolds (Lie groups endowed with left invariant Lorentz metrics) were enumerated in ([14]), without distinction of isometric manifolds. Our Corrolary 2.1. shows that two homogeneous Lorentz manifolds are locally isometric iff their linear curvature tensors (K^i_{jkr}) are Ad(O(3,1)) equivalent.Using quadratic forms in curvature(see [2] in the Riemannian case) and the above list of linear curvature tensors,we can now answer the problem of equivalence of two manifolds in our list;the details will be given elsewhere.

In the years '50-'60,the Soviet mathematicians Egorov,Kruckovich,A.Z.Petrov and others,obtained many results concerning groups of automorphisms of Ψ.R. manifolds;for some of them we send to [13] ,where is given a list of Ψ.R.4-manifolds and their groups of motions in Chap. .Our classification shows that this list is not complete:for instance our b .2.,which is a family of spaces,that contains for each (a,b) a one parameter family of homogeneous Lorentz manifolds,passing through de Sitter space of constant curvature $-(a^2+b^2)$,does not appear there.

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