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Laurenţiu PAUNESCU

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Laurenţiu PAUNESCU*)

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*) Faculty of Mathematics, University of Bucharest, str.Academiei nr.14, Bucharest 70109, ROMANIA.

EQUIVARIANT MAPS FOR PROPERLY

DISCONTINUOUS ACTIONS

by LAURENTIU PAUNESCU

In this paper we study the fundamental group of an orbit space, and we give necessary and sufficient conditions for the existence of φ-equivariant maps for properly discontinuous actions.

As a consequence, we give a generalization of a classical theorem of Borsuk-Ulam type, regarding the nonexistence of equivariant maps between certain spaces.

From now on, G and H will be two topological groups and X a G-space and Y an H-space.

Let $\underline{G}$ ($\underline{H}$) be the normal subgroup of G (H) generated by the path components of those elements of G (H) which have fixed points.

We have the following proposition, whose proof is obvious.

PROPOSITION 1.

Let N be a normal subgroup of G. Then G/N acts on X/N and we have a canonical homeomorphism $(X/N)/(G/N) \approx X/G$.

Let $p: X \longrightarrow X/G$, $q: Y \longrightarrow Y/H$ be the canonical projections. By $C\widetilde{\pi}(X,x_0)$ we shall denote the consequence in $\widetilde{\pi}(X/G, p(x_0))$ of $p_{\#}(\widetilde{\pi}(X,x_0))$, and whenever there is no danger of confusion we shall omit the base points.

Let X be a path connected G-space and $g \in G$. We consider $\sigma: I \longrightarrow X$ a path such that $\sigma(0)=x_0$, $\sigma(1)=gx_0$ and we take the right coset of $p\circ\sigma$ in $\widetilde{\pi}(X/G)/p_{\#}\widetilde{\pi}(X)$. This correspondence defines a group morphism

$$\begin{tikzcd} \psi_G : G \arrow[r, "\psi'_G"] \arrow[dr] & \widetilde{\pi}(X/G)/p_{\#}\widetilde{\pi}(X) \arrow[r, two heads, "c"] & \widetilde{\pi}(X/G)/C\widetilde{\pi}(X) \\ & Np_{\#}\widetilde{\pi}(X)/p_{\#}\widetilde{\pi}(X) \arrow[u] & \end{tikzcd}$$

Since it is known that $\underline{G} \subseteq \ker\left\{G \longrightarrow Np_{\#}\widetilde{\pi}(X)/p_{\#}\widetilde{\pi}(X)\right\}$ (see [A]) we can factorise ψ_G' in the following way :

$\psi_G' : G \twoheadrightarrow G/\underline{G} \xrightarrow{\overline{\psi}_G'} \tilde{\pi}(X/G)/p_{\#}\tilde{\pi}(X)$. By $\overline{\psi}_G$ we shall denote the composite $G/\underline{G} \xrightarrow{\overline{\psi}_G'} \tilde{\pi}(X/G)/p_{\#}\tilde{\pi}(X) \twoheadrightarrow \tilde{\pi}(X/G)/C\,\tilde{\pi}(X)$.

If we consider a continuous group morphism $\varphi : G \longrightarrow H$ and a φ-equivariant map $f : X \longrightarrow Y$ (i.e. $f(gx) = \varphi(g)f(x)$ for any $x \in X$, $g \in G$, we see that $\varphi(\underline{G}) \subset \underline{H}$ and we have the following commutative diagrams (with obviously induced arrows).

(1)
$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\
X/\underline{G} \arrow[r, "\bar{\bar{f}}"] \arrow[d, "p'"] & Y/\underline{H} \arrow[d, "q'"] \\
X/G \arrow[r, "\bar{f}"] & Y/H
\end{tikzcd}$$

(2)
$$\begin{tikzcd}
G \arrow[r, two heads] \arrow[d, "\varphi"] & G/\underline{G} \arrow[r, "\overline{\psi}_G'"] \arrow[d, "\bar{\varphi}"] & \tilde{\pi}(X/G)/p_{\#}\tilde{\pi}(X) \arrow[r, two heads] \arrow[d, "\tilde{f}"] & \tilde{\pi}(X/G)/C\,\tilde{\pi}(X) \arrow[d, "\tilde{\tilde{f}}"] \\
H \arrow[r, two heads] & H/\underline{H} \arrow[r, "\overline{\psi}_H'"] & \tilde{\pi}(Y/H)/q_{\#}\tilde{\pi}(Y) \arrow[r, two heads] & \tilde{\pi}(Y/H)/C\,\tilde{\pi}(Y)
\end{tikzcd}$$

Since $\bar{\bar{f}} : X/\underline{G} \longrightarrow Y/\underline{H}$ is a $\bar{\varphi}$-equivariant map ,a necessary condition for the existence of a φ-equivariant map is the existence of a $\bar{\varphi}$-equivariant map between certain orbit spaces, on which the associated actions turn out to be,in many interesting cases,properly discontinuous,i.e. much easier to handle .

Definition. 1.

Let N be a normal subgroup of G.We say that G acts N-discontinuously on X if G/N acts properly discontinuously on X/N (see [A]).

We have the following characterization of this property.

PROPOSITION 2.

G acts N-discontinuously on X if and only if for each $x \in X$ there is a neighborhood V of x,such that $gV \cap V = \emptyset$ for each $g \notin N$.

Proof. Trivial ∎

PROPOSITION 3.

If G acts $\underline{G}$-discontinuously on X,and X is a path connected space, we have:

a) $G \neq \underline{G}$ implies $\tilde{\pi}(X/G) \neq C\tilde{\pi}(X)$.

b) If ψ_G is onto then $\tilde{\pi}(X/G)/C\tilde{\pi}(X) \approx G/\underline{G}$ and $C\tilde{\pi}(X) = p'_{\#}\tilde{\pi}(X/\underline{G})$.

Proof. – Suppose more generally, that G acts N-discontinuously on X.

We remark that we have a regular covering $X/N \xrightarrow{p'} X/G$, $p'_{\#}\tilde{\pi}(X/N)$ is a normal subgroup of $\tilde{\pi}(X/G)$, and $\psi_{G/N}$ becomes an isomorphism (see [S]).

Consider the following commutative diagram

$$(3)\qquad \begin{tikzcd}
\psi_G : G \arrow[r] \arrow[d] & \tilde{\pi}(X/G)/p_{\#}\tilde{\pi}(X) \arrow[r, two heads] \arrow[d] & \tilde{\pi}(X/G)/C\tilde{\pi}(X) \arrow[d] \\
\psi_{G/N} : G/N \arrow[r, tail, two heads, "\sim"] & \tilde{\pi}(X/G)/p'_{\#}\tilde{\pi}(X/N) \arrow[r, tail, two heads, "\sim"] & \tilde{\pi}(X/G)/C\tilde{\pi}(X/N)
\end{tikzcd}$$

and we deduce the inclusion $\operatorname{Ker}\psi_G \subseteq N$. Since we have already remarked that the other inclusion holds for $N=\underline{G}$, a simple inspection of diagram (3) establishes our assertion a).

If ψ_G is supposed to be onto the diagram (3) becames

$$\begin{tikzcd}
G \arrow[r, two heads, "\psi_G"] \arrow[d] & \tilde{\pi}(X/G)/C\tilde{\pi}(X) \arrow[d, two heads] \\
G/\underline{G} \arrow[r, "\psi_{G/\underline{G}}", "\sim"'] \arrow[ur, dashed, "\bar{\Psi}_G \simeq"] & \tilde{\pi}(X/G)/p'_{\#}\tilde{\pi}(X/\underline{G})
\end{tikzcd}$$

and our claims follow ∎

Remarks. – In [A] one may find several exemples in which ψ'_G is onto (and thus ψ_G is epimorphic), and also an analogue of b) of proposition 3, in the case when X is 1-connected.

We list here the examples :

a) G acts simplicially on a triangulation of X

b) The action of G on X is discontinuous (in the sense of [A]), and the stabiliser of any point is finite

c) G is a compact Lie group

d) G is a locally compact Lie group and acts properly on X

e) X/G is semilocally simply connected

On the other hand if X is a G-space, G a finite group, G acts $\underline{G}$-discontinuously on X, [illegible]

COROLLARY 1.

Let X be a path connected,Hausdorff space and G a finite group which acts on X.

We have the following facts :

a) $\tilde{\pi}(X/G)/C\tilde{\pi}(X) \approx G/\underline{G}$

b) If X is simply connected then $\tilde{\pi}(X/G) \approx G/\underline{G}$ and thus $\tilde{\pi}(X/G)=0$ if and only if $G=\underline{G}$.

c) If $G=Z_p$, where p is a prime integer then $\tilde{\pi}(X/G)/C\tilde{\pi}(X)\neq 0$ if and only if the action is free ∎

COROLLARY 2.

Let G be a finite group which acts on a path connected,Hausdorff, paracompact space X.There exists a map $\alpha_X : X/G \longrightarrow K(\tilde{\pi}(X/G)/C\tilde{\pi}(X),1)$ such that $(\alpha_X)_{\#} : \tilde{\pi}(X/G) \longrightarrow \tilde{\pi}(X/G)/C\tilde{\pi}(X)$ is the canonical projection.

Proof. - $X/\underline{G} \xrightarrow{p'} X/G$ is a regular covering with structure group $G/\underline{G} \approx \tilde{\pi}(X/G)/C\tilde{\pi}(X)$, see proposition 3. Since X/G is paracompact the results of Dold [D] give the map α_X. The assertion about $(\alpha_X)_{\#}$ follows easily ∎

From now on we are going to study the connections between the morphism φ and the applications $\tilde{f},\tilde{\tilde{f}}$ induced by $f_{\#}$, where f is a φ-equivariant map.

PROPOSITION 4.

Let $f:(X,x_0) \longrightarrow (Y,y_0)$ be a φ-equivariant map, where $\varphi : G \longrightarrow H$ is a group morphism.

The followings hold :

a) $\tilde{f}$ nontrivial implies φ nontrivial

b) ψ'_H surjective and $\bar{\varphi}$ epimorphic imply $\tilde{f}$ surjective

c) Let N be the kernel of φ and assume that G acts N-discontinuously on X, and H acts $\underline{H}$-discontinuously on Y. If $\bar{\varphi}$ is nontrivial then both $\tilde{f}$ and $\tilde{\tilde{f}}$ are nontrivial

d) Suppose that ψ'_H, ψ'_G are surjective and that H acts $\underline{H}$-discontinuously . Then $\bar{\varphi}$ is epimorphic if and only if $\tilde{f}$ is surjective, if and only if $\tilde{\tilde{f}}$ is epimorphic, and if $\bar{\varphi}$ is monomorphism then $\tilde{f}$ and $\tilde{\tilde{f}}$ are injective . If in addition, G acts $\underline{G}$-discontinuously, then $\bar{\varphi}$ is a monomorphism if and only if $\tilde{f}$ is injective, if and only if $\tilde{\tilde{f}}$ is a monomorphism.

<u>Proof.</u> - a) If φ is trivial, then f factorises as $f_1 \circ p$, where $f_1 : X/G \longrightarrow Y$. Moreover $q \circ f_1 = \tilde{f}$ and thus $\tilde{f}$ and $\tilde{\tilde{f}}$ are trivial .

b) With our assumptions the diagram (2) becames

$$\begin{tikzcd}
G \arrow[r, two heads] \arrow[d, "\varphi"'] & G/\underline{G} \arrow[r] \arrow[d, two heads, "\bar{\varphi}"'] & \widetilde{\pi}(X/G)/p_{\#}\widetilde{\pi}(X) \arrow[r, two heads] \arrow[d, "\tilde{f}"'] & \widetilde{\pi}(X/G)/C\,\widetilde{\pi}(X) \arrow[d, "\tilde{\tilde{f}}"] \\
H \arrow[r, two heads] & H/\underline{H} \arrow[r, two heads] & \widetilde{\pi}(Y/H)/q_{\#}\widetilde{\pi}(Y) \arrow[r, two heads] & \widetilde{\pi}(Y/H)/C\,\widetilde{\pi}(Y)
\end{tikzcd}$$

and b) follows.

c) Remember that $\underline{G} \subseteq \ker \Psi_G \subseteq N$ (see the proof of proposition 3).

We have therefore the following commutative diagram

$$(4)\quad \begin{tikzcd}
G/\underline{G} \arrow[r, "\bar{\Psi}'_G"] \arrow[d, two heads] \arrow[dd, bend right, "\bar{\varphi}"'] & \widetilde{\pi}(X/G)/p_{\#}\widetilde{\pi}(X) \arrow[r, two heads] \arrow[d, two heads] \arrow[dd, bend right, "\tilde{f}"'] & \widetilde{\pi}(X/G)/C\,\widetilde{\pi}(X) \arrow[d, two heads] \arrow[dd, bend right, "\tilde{\tilde{f}}"'] \\
G/N \arrow[r, "\Psi_{G/H}", "\sim"'] \arrow[d, "\bar{\varphi}'"] & \widetilde{\pi}(X/G)/p'_{\#}\widetilde{\pi}(X/N) \arrow[r, equal] \arrow[d, "\tilde{f}'"] & \widetilde{\pi}(X/G)/p'_{\#}\widetilde{\pi}(X/N) \arrow[d, "\tilde{\tilde{f}}'"] \\
H/\underline{H} \arrow[r, tail, "\bar{\Psi}'_H"] \arrow[rr, bend right, tail, two heads, "\bar{\Psi}_H"'] & \widetilde{\pi}(Y/H)/q_{\#}\widetilde{\pi}(Y) \arrow[r, two heads] & \widetilde{\pi}(Y/H)/C\,\widetilde{\pi}(Y)
\end{tikzcd}$$

which gives all our statements .

d) Inspect the commutative diagram:

$$\begin{tikzcd}
G \arrow[r, two heads] \arrow[d, "\varphi"'] & G/\underline{G} \arrow[r, two heads] \arrow[d, "\bar{\varphi}"] & \widetilde{\pi}(X/G)/p_{\#}\widetilde{\pi}(X) \arrow[r, two heads] \arrow[d, "\tilde{f}"'] & \widetilde{\pi}(X/G)/C\,\widetilde{\pi}(X) \arrow[d, "\tilde{\tilde{f}}"'] \\
H \arrow[r, two heads] & H/\underline{H} \arrow[r, "\sim"'] & \widetilde{\pi}(Y/H)/q_{\#}\widetilde{\pi}(Y) \arrow[r, equal] & \widetilde{\pi}(Y/H)/C\,\widetilde{\pi}(Y)
\end{tikzcd}$$

<u>Remark</u> : If for example, H acts freely , notice that $\bar{\varphi}$ is epimorphic if and only if φ is epimorphic, and if φ is monomorphic, then $\bar{\varphi} = \varphi$.

We go on giving some theorems about the existence of φ-equivariant maps.

THEOREM 1.

Let X be a G-space (X connected and locally path connected). Let Y be an H-space and suppose that H acts properly discontinuously on Y. If we suppose that there is an application $\bar{f}:(X/G,p(x_0))\longrightarrow(Y/H,q(y_0))$ such that $\bar{f}_{\#}(p_{\#}\pi(X))\subseteq q_{\#}\pi(Y)$, then there is a group morphism $\varphi:G\longrightarrow H$ and a φ-equivariant map $f:X\longrightarrow Y$ such that :

a) If $\tilde{f}$ is nontrivial then φ is nontrivial

b) If Y is a connected space and ψ'_G is epimorphic then $\tilde{f}$ surjective implies φ epimorphic. If moreover G acts G-discontinuously then $\tilde{f}$ injective implies $\bar{\varphi}$ monomorphic.

Proof. - Denote by f the unique lift $f:(X,x_0)\longrightarrow(Y,y_0)$ of $\bar{f}\circ p:(X,x_0)\longrightarrow(Y/H,q(y_0))$ (see [S,p.76]). Pick any $g\in G$. Since $q\circ f(gx_0)=\bar{f}\circ p(gx_0)=\bar{f}\circ p(x_0)=q(y_0)$, we know that there is an unique element $\varphi(g)\in H$ such that $f(gx_0)=\varphi(g)f(x_0)=\varphi(g)y_0$. We consider the following applications : $f_1=f\circ\mu_g$, $f_2=\mu_{\varphi(g)}\circ f$, where $\mu_g(x)=gx$, and we observe that $q\circ f_1(x)=q\circ f(gx)=\bar{f}\circ p(gx)=\bar{f}\circ p(x)$, $q\circ f_2(x)=$ $=q(\varphi(g)f(x))=q\circ f(x)=\bar{f}\circ p(x)$, any x. Moreover $f_1(x_0)=f_2(x_0)$, hence $f_1=f_2$ and thus $f(gx)=\varphi(g)f(x)$ for any $x\in X, g\in G$. The fact that H acts freely on Y implies that φ is a group morphism. The rest of the theorem follows from proposition 4 ∎

COROLLARY 3.

Let X be a G-space (X connected, locally path connected) where G is a finite group with square free order. If H acts properly discontinuously on Y and there exists $\bar{f}:X/G\longrightarrow Y/H$ such that $\bar{f}_{\#}\circ p_{*}\pi(X)\subseteq q_{\pi}\pi(Y)$ and such that $\tilde{f}$ is nontrivial, then there exists a nontrivial subgroup $L\subseteq G$ and a ψ-equivariant map $f:X\longrightarrow Y$ for some monomorphism $\psi:L\longrightarrow H$.

Proof. -Theorem 1 provides a φ-equivariant map $f:X\longrightarrow Y$, for some nontrivial morphism $\varphi:G\longrightarrow H$. Let L be a subgroup of G, such that $L\cap\ker\varphi=\{1\}$ (remember that the order of G is squarefree) We may then take $\psi=\varphi/L$ ∎

If the groups G,H are both finite we have the following theorem:

THEOREM 2.

Let X be a connected, locally path connected, paracompact G-space, and let Y be a connected, paracompact H-space, with H acting properly discontinuously on Y. Suppose that G and H are finite and let $\varphi : G \longrightarrow H$ be a group morphism.

The following are equivalent :

a) There exists a φ-equivariant map $f : X \longrightarrow Y$

b) There exists a lift up to homotopy in the following diagram

$$\begin{tikzcd} & & Y/H \arrow[d, "\alpha_Y"] \\ X/G \arrow[r, "\alpha_X"] & K(\tilde{\pi}(X/G)/C\tilde{\pi}(X),1) \arrow[r, "\tilde{\varphi}"] & K(\tilde{\pi}(Y/H)/C\tilde{\pi}(Y),1) \end{tikzcd}$$

for some $\tilde{\varphi}$ making the diagram below commutative

$$\begin{tikzcd} \tilde{\pi}(X/G)/C\tilde{\pi}(X) \arrow[rr, "\tilde{\varphi}_{\#}"] & & \tilde{\pi}(Y/H)/C\tilde{\pi}(Y) \\ G/\underline{G} \arrow[u, "\overline{\Psi}_G"', "\wr"] & & H \arrow[u, "\wr"', "\overline{\Psi}_H"] \\ G \arrow[u, two heads] \arrow[rr, "\varphi"] & & H \arrow[u, equal] \end{tikzcd}$$

, where α_X and α_Y are given by corollary 2.

Proof.- Given the φ-equivariant map f , use $\overset{\approx}{f}$ in order to construct $\tilde{\varphi}$. The desired lift is then $\tilde{f}$, which may be easily checked using corollary 2 and the classification theorem of [D].

The second implication follows from theorem 1, which gives a ψ-equivariant map $f : X \longrightarrow Y$ for some group morphism $\psi : G \longrightarrow H$

Knowing that f lifts $\tilde{\varphi}\circ\alpha_X$, we deduce that $\tilde{\varphi}_{\#} = \overset{\approx}{f}$. On the other hand, the ψ-equivariance of f gives the commutative diagram :

$$\begin{tikzcd} \tilde{\pi}(X/G)/C\tilde{\pi}(X) \arrow[rr, "\overset{\approx}{f}"] & & \tilde{\pi}(Y/H)/C\tilde{\pi}(Y) \\ G \arrow[u, "\psi_G"] \arrow[rr, "\psi"] & & H \arrow[u, "\wr"', "\psi_H"] \end{tikzcd}$$

which still commutes, with φ in place of ψ, by our assumption.

We thus see that actually $\psi = \varphi$, and we are done.

If we no more insist on fixing the morphism φ we obtain the following form of theorem 2.

THEOREM 2'.

Let X be a connected, locally path connected G-space, and Y a connected H-space, with H acting properly discontinuously on Y. The following are equivalent

a) There exist a nontrivial (respectively surjective, respectively such that $\bar\varphi$ is injective) group morphism $\varphi : G \longrightarrow H$ and a φ-equivariant map $f : X \longrightarrow Y$.

b) There exists an application $\bar f : X/G \longrightarrow Y/H$ such that $\bar f_\# \circ p_\# \pi(X) \subseteq q_\# \pi(Y)$ and such that $\tilde f : \pi(X/G)/p_\# \pi(X) \longrightarrow \pi(Y/H)/q_\# \pi(Y)$ is nontrivial (respectively surjective, respectively injective).

If in addition X and Y are paracompact these are also equivalent with :

c) There exists an application $\tilde\varphi : K(G/\underline{G},1) \longrightarrow K(H,1)$ such that $\tilde\varphi_\#$ is nontrivial (respectively surjective, respectively injective), and there exists a lift in the diagram

$$\begin{tikzcd} & & Y/H \arrow[d, "\alpha_Y"] \\ X/G \arrow[r, "\alpha_X"] & K(G/\underline{G},1) \arrow[r, "\tilde\varphi"] & K(H,1) \end{tikzcd}$$

Proof. - Analogous to the proof of theorem 2 ∎

Now we give some corollaries .

COROLLARY 4.

Let G,H be finite groups and let X be a simply connected, locally path connected G-space and let Y be a simply connected H-space, on which H acts freely.

There exists a nontrivial (respectively surjective, respectively such that $\bar\varphi$ is injective) group morphism $\varphi : G \longrightarrow H$ and a φ-equivariant map $f : X \longrightarrow Y$ if and only if there exists $\bar f : X/G \longrightarrow Y/H$ with $\bar f_\#$ nontrivial (respectively surjective, respectively injective)

COROLLARY 5.

Let G,H be finite groups and let X be a connected, locally path connected G-space and let Y be a simply connected H-space, with H acting freely on Y.

There exists a nontrivial (epimorphism) $\varphi : G \longrightarrow H$, and a φ-equivariant map $f: X \longrightarrow Y$ if and only if there exists $\bar{f}: X/G \longrightarrow Y/H$ such that $(\bar{f}\circ p)_{\#}$ is trivial and $\bar{f}_{\#}$ is nontrivial (epimorphism).

Proof.- Follows from theorem 2'. ∎

COROLLARY 6.

Let G be a finite group acting freely on a connected space Y, and let X be a connected, path connected G-space and suppose that there exists $\bar{f}: X/G \longrightarrow Y/H$ such that $(\bar{f}\circ p)_{\#}(\widetilde{\Pi}(X)) \subseteq q_{\#}\widetilde{\Pi}(Y)$ and $\bar{f}_{\#}$ is epimorphic. Then there exists an automorphism of G and a φ-equivariant map $f: X \longrightarrow Y$ (and consequently G also acts freely on X).

Proof.- $\bar{f}_{\#}$ being epimorphic, theorem 1 implies that there is an epimorphism $\varphi : G \longrightarrow G$ and a φ-equivariant map $f: X \longrightarrow Y$. Since G is finite φ must be an automorphism ∎

COROLLARY 7.

Let p be a prime integer and let G be equal to Z_p or Z. Let X be a connected , locally path connected G-space and suppose that G acts properly discontinuously on a connected space Y. Assume also that there exists $\bar{f}: X/G \longrightarrow Y/G$ such that $(\bar{f}\circ p)_{\#}(\widetilde{\Pi}(X)) \subseteq q_{\#}\widetilde{\Pi}(Y)$.

Then there exists an application $f: X \longrightarrow Y$ and a positive integer r such that $f(gx) = g^r f(x)$ for each $x \in X, g \in G$. If we have $\mathrm{im}\bar{f}_{\#} \not\subseteq \mathrm{im}.q_{\#}$ then $(r,p)=1$ (when $G=Z_p$), and $r \neq o$ (when $G=Z$). In both cases it follows that the action on X must be free.

Proof.- We make use of the description of the group endomorphisms of G ∎

COROLLARY 8.

Let r be a prime integer and let X be a connected, locally path connected space. We suppose that Z_r acts freely on X in two ways.

Let p,q be the canonical projections corresponding to the given actions, $X \xrightarrow{p} X/Z_r = X_1$, $X \xrightarrow{q} X/Z_r = X_2$. If there exists $\bar{f}: X_1 \longrightarrow X_2$ such that $\mathrm{im.}(\bar{f}\circ p)_\# \subseteq \mathrm{im.}q_\#$ and $\mathrm{im.}\bar{f}_\# \not\subseteq \mathrm{im.}q_\#$ then there exists an equivariant map $f: X \longrightarrow X$.

Proof.- From the previous corollary, there exists $s \in N$ such that $(r,s)=1$ and there exists $f_1 : X \longrightarrow X$ such that $f_1(gx)=g^s f_1(x)$ for each $x \in X$, $g \in Z_r$. We take then $f = f_1^{(r-1)}$.

Finally we shall give a generalization of a classical theorem (Borsuk-Ulam's theorem [G-p.168]) regarding the nonexistence of equivariant maps between spheres of certain dimensions.

The universal Z_p-space and the m-universal Z_p-space will play an important role; therefore we recall some of their properties, which are of interest for us (see [S-E-p.67-68]).

Let $Z_p \hookrightarrow EZ_p \longrightarrow BZ_p = EZ_p/Z_p$ be the universal principal bundle (see [H-p.53]). Let $E^m Z_p$ be the m-skeleton of the equivariant cellular structure of EZ_p (see [S-E p.67-68]), and consider the m-universal principal bundle $Z_p \hookrightarrow E^m Z_p \longrightarrow B^m Z_p = E^m Z_p/Z_p$.

The m-universal Z_p -space $E^m Z_p$ is m-dimensional and (m-1) connected. The cohomology algebra $H^*(B^m Z_p; Z_p)$ is generated by the homogenous components $H^1(B^m Z_p; Z_p)$ and $H^2(B^m Z_p; Z_p)$ in dimensions $\leq m-1$.

Moreover ,denoting by $A_m \subseteq H^*(B^m Z_p; Z_p)$ the subalgebra generated by H^1 and H^2 one knows that A_m is nonzero in degree m (see [S-E,p.68]).

Lemma 1.

Let $m > n \geqslant 1$ be positive integers and let φ be an automorphism of Z_p . There are no φ-equivariant maps $f: E^m Z_p \longrightarrow E^n Z_p$.

Proof.- Supposing the contrary we must have the following commutative diagram

$$\begin{tikzcd}
Z_p \arrow[r, "\varphi"] \arrow[d, hook] & Z_p \arrow[d, hook] \\
E^m Z_p \arrow[r, "f"] \arrow[d] & E^n Z_p \arrow[d] \\
B^m Z_p \arrow[r, "\bar{f}"] & B^n Z_p
\end{tikzcd}$$

We may safely suppose from now on that $m > 2$ (if m=2 then necessarily n=1 and it follows easily that there are no equivariant maps $f:E^2Z_p \longrightarrow E^1Z_p$, by looking at $f_{*}j$.

Since $i:E^nZ_p \hookrightarrow E^mZ_p$ is an equivariant map, the composite $i\circ f$ passes to orbit spaces $B^mZ_p \xrightarrow{\bar{f}} B^nZ_p \xrightarrow{\bar{i}} B^mZ_p$, inducing a 2-equivalence for $m > 2$.

It follows that $\bar{f}$ induces epimorphisms between H^1 and H^2 (Z_p-coefficients), and consequently $A_m \subseteq \mathrm{im}.\bar{f}^*$.

But this is a contradiction in degree m, since $H^m(B^nZ_p;Z_p)=0$, for dimensional reasons ∎

THEOREM 3.

Let X be a CW-complex which is r-connected ($r \leq \infty$), and let Y be a CW-complex which is n-dimensional ($n < \infty$). Assume that the group $G=Z_p$, p a prime integer, acts freely on X and on Y. If $r \geq n$ then there are no φ-equivariant maps $f:X \longrightarrow Y$, with $\varphi:G \longrightarrow G$ non-trivial.

Proof.- Suppose that there exists such a map $f:X \longrightarrow Y$. We consider the following diagrams

$$\begin{tikzcd}
 & E^nZ_p \arrow[d] \\
Y \arrow[ur, dashed, "h"] \arrow[d] & E^nZ_p/Z_p \arrow[d, "\alpha_n"] \\
Y/Z_p \arrow[ur, dashed, "\bar{h}"] \arrow[r, "\alpha_Y"] & K(Z_p,1)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & X \arrow[d] \\
E^{r+1}Z_p \arrow[ur, dashed, "g"] \arrow[d] & X/Z_p \arrow[d, "\alpha_X"] \\
E^{r+1}Z_p/Z_p \arrow[ur, dashed, "\bar{g}"] \arrow[r, "\alpha_{r+1}"'] & K(Z_p,1)
\end{tikzcd}$$

where the maps labelled by α are classifying maps (see corollary 2), and we deduce, via obstruction theory on Y ([S-ch.8]) and the globalization theorem 2.7 from [D] , the existence of a lift $\bar{h}$ (and similarly for $\bar{g}$). Use then corollary 5 to obtain a ψ-equivariant map h, and a ϕ-equivariant map g, for some automorphisms ψ and ϕ of G.

It follows that $h\circ f\circ g$ is a $\psi\circ\varphi\circ\phi$ -equivariant map and this contradicts the previous lemma ∎

COROLLARY 9.

Let X and Y be as in the previous theorem. Assume that the finite group G acts freely on X and that the group H acts freely on Y.

If the order of G is square free, then there are no φ-equivariant maps $f:X \longrightarrow Y$ with φ a nontrivial morphism.

Proof.- Restrict to the action of some $Z_p \subseteq G$, p a prime(compare with corollary 3) ∎

COROLLARY 10.

Let X be a CW-complex, connected and finite dimensional. Assume that the group G has nontrivial torsion and acts freely on X. Denote by n_X the connectivity of X (i.e. $\tilde{\pi}_i(X)=0, i \leq n_X$). If $n_X \geq \frac{\dim.X}{2}$ then X can not equivariantly split as a product of two free finite dimensional G-spaces which are CW-complexes.

Proof.- If $X = X_1 \times X_2$ as G-spaces then one of the canonical projections will contadict theorem 3 ∎

Remarks

We can proove theorem 3 by means of the same idea, using the criterion of the existence of equivariant maps (see [H]-pg. 46).

After I had written this paper I learned about Professor Dold's article [D_1] ,whose aim is to prove this theorem. Thus, I thank him for his kindness of sending me his article.

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Faculty of Mathematics,
University of Bucharest,
str. Academiei nr.14,
Bucharest,Romania.