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The present paper deals with unital *-homomorphisms $\Phi : C(X) \otimes M_n \to C(Y) \otimes M_{kn}$, where X and Y are compact. Our interest in such homomorphisms arose in connection with a question of E.G. Effros [3] concerning the structure of inductive limits of C*-algebras of the form $C(X) \otimes M_n$.

If X=Y and k=1 we have the case of automorphisms studied by R.V. Kadison-J. Ringrose [6], J. Phillips-I. Raeburn [9], [10] and K. Thomsen [13]. Also in connection with the question of E.G. Effros certain classes of *-homomorphisms related to a covering $X \to Y$ have been considered by C. Pasnicu in [7]. Our results concern the situation when no connection between the spaces X and Y is imposed a priori.

The paper consists of three sections. In the first section we show that under certain topological restrictions on Y the study of homomorphism Φ can be reduced to the case when n=1. The topological conditions involve that the homotopy type of Y be that of a CW-complex of dimension $\leq 2k$ and the absence of n-torsion in K^0. Section 2 provides a way for describing *-homomorphisms $\Phi : C(X) \to C(Y) \otimes M_k$ (Thm 2.2). Of course if k=1 such homomorphisms correspond to continuous maps $Y \to X$. To give an idea about the additional complications arising in the case when k>1, our general result being somewhat technical, let us consider the homogeneous case. This means that for every $y \in Y$ there is some $f \in C(X)$ such that the matrix

$\Phi(f)(y)$ has k distinct eigenvalues. Then Φ is completely characterized by the following objects:

(1) A k-fold covering space $\pi : Z \longrightarrow Y$.

(2) A continuous map $\varphi : Z \longrightarrow X$ which is injective on each fibre $\pi^{-1}(y)$, $y \in Y$.

(3) One dimensional projections $p(z)$ depending continuously on $z \in Z$ such that

$$\sum_{z \in \pi^{-1}(y)} p(z) = I_k \quad (I_k \text{ denotes the unit of } M_k) \quad ,$$

Then Φ is given by the following formula

(4) $$\Phi(f)(y) = \sum_{z \in \pi^{-1}(y)} f(\varphi(z))p(z), \quad f \in C(X), \; y \in Y.$$

The third section contains results concerning *-homomorphisms compatible with the k-fold covering $\psi : X \longrightarrow Y$ in the sense of C. Pasnicu [7]. This means that

(5) $$\Phi((g \circ \psi) \otimes I_n) = g \otimes I_{kn} \quad \text{for all} \quad g \in C(Y).$$

This property appeared as the essential feature of the homomorphisms in the description of the Bunce-Deddens algebras [2] as inductive limits . ψ-compatible homomorphisms can be also viewed as a kind of "sections" for $\psi : X \longrightarrow Y$. More precisely they make the following diagram commutative

$$\begin{array}{ccc} C(X) \otimes M_n & \xrightarrow{\;\Phi\;} & C(Y) \otimes M_{kn} \\ & {}_{\psi^*}\nwarrow \quad \nearrow_{\alpha} & \\ & C(Y) & \end{array}$$

where $\alpha(g) = g \otimes I_{kn}$ and $\psi^*(g) = (g \circ \psi) \otimes I_n$.

ψ-compatible homomorphisms are homogeneous under natural connectedness assumptions. Assuming $n=1$ a global description of the structure of the ψ-compatible homomorphisms is provided in Proposition 3.2. We also give certain results concerning the existence of ψ-compatible maps and discuss some examples.

For instance if G is a finite abelian group acting freely on X then the natural injection $C(X) \hookrightarrow C(X) \rtimes G$ can be turn into a homomorphism compatible with the covering $X \to X/G$ provided the crossed-product $C(X) \rtimes G$ is isomorphic to some $C(X/G) \otimes M_k$.

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1. UNITAL *-HOMOMORPHISMS

We need in this section some elementary sheaf cohomology. It is known that every short sequence of sheaves of abelian groups induces a long sequence of cohomology groups ([4]). We may also consider sheaves of nonabelian groups and homogeneous spaces but in this case we get only a short sequence of cohomological sets. We shall give some details for the convenience of the reader.

A sequence of pointed sets (or pointed topological spaces) and maps

$$\to (A_i, x_i) \xrightarrow{F_i} (A_{i+1}, x_{i+1}) \xrightarrow{F_{i+1}} (A_{i+2}, x_{i+1}) \to$$

is called exact if the equality

$$F_i(A_i) = F_{i+1}^{-1}(x_{i+2})$$

holds for every i.

For an arbitrary pointed topological space (A,a) we denote by A^c the sheaf of the germs of continuous functions from X to A. For $x \in X$ we denote by $(A^c)_x$ the stalk at x of the sheaf A^c. We distinguish in $(A^c)_x$ the function which is equal to a on a neighborhood of x.

Let G be a Lie group and let H be a closed subgroup of G. We have the following short exact sequence of pointed topological spaces

$$1 \to (H, 1_H) \to (G, 1_G) \to (G/H, \langle H \rangle) \to 1 \tag{6}$$

Recall that G acts on the homogeneous space G/H in the canonical way $g.(xH)=(gx)H$.

The following sequence of sheaves is exact

$$(8) \qquad 1 \to H^c \to G^c \to (G/H)^c \to 1 .$$

We mean by this that the following sequence of pointed sets and maps is exact for every $x \in X$:

$$(9) \qquad 1 \to (H^c)_x \to (G^c)_x \to (G/H)^c_x \to 1$$

This is nothing more that the fibering $H \to G \to G/H$ admits continuous local sections and this is clear since H is a closed subgroup of the Lie group G.

Now we can derive from (8) the following exact sequence of pointed cohomological sets

$$(10) \quad 1 \to H^\circ(X,H^c) \to H^\circ(X,G^c) \to H^\circ(X,(G/H)^c) \xrightarrow{\delta} H^1(X,H^c) \xrightarrow{\gamma} H^1(X,G^c)$$

We have that $H^\circ(X,H^c)$ is equal to $C(X,H)$ the set (group) of all continuous functions from X to H pointed by $f \equiv 1_H$. Similarly $H^\circ(X,G^c) = C(X,G)$ is pointed by $f \equiv 1_G$ and $H^\circ(X,(G/H)^c) = C(X,G/H)$ is pointed by the constant function $f \equiv \langle H \rangle$. The cohomological sets $H^1(X,H^c)$ and $H^1(X,G^c)$ are pointed by the trivial cocycles $(X,1_H)$ and $(X,1_G)$ respectively ([4]). Given $f \in C(X,G/H)$ the cocycle $\delta(f) \in H^1(X,H^c)$ represents the obstruction for lifting f to a function in $C(X,G)$. By the exactness of (10) the function f has a lifting if and only if $\delta(f) = (X,1_H)$ i.e. the cocycle $\delta(f)$ is trivial in $H^1(X,H^c)$. Furthermore the action of G on G/H induces an action of the group $C(X,G)$ on $C(X,G/H)$. If $f_1, f_2 \in C(X,G/H)$ then $\delta(f_1) = \delta(f_2)$ if and only if $f_2 = g \cdot f_1$ for some $g \in C(X,G)$.

Next, we describe the sequence in (10) in the case of the fibering

$$(11) \qquad U(k) \xrightarrow{\gamma} U(kn) \xrightarrow{j} U(kn)/U(k)$$

where the embedding $\gamma: U(k) \to U(kn)$ is given by

(12) $\gamma: U(k) \ni u \mapsto u \otimes I_n \in U(kn) \simeq U(M_k \otimes M_n)$.

First we need some notation.

Let $\mathrm{Vect}_m(Y)$ denote the set of isomorphism classes of complex vector bundles of rank m on Y. In $\mathrm{Vect}_m(Y)$ we have one naturally distinguished element - the class of the trivial bundle of rank m. Let $T_n(\mathrm{Vect}_m(Y))$ be the subset of $\mathrm{Vect}_m(Y)$ consisting of classes of all vector bundles E for which the Whitney sum $nE = E \oplus \ldots \oplus E$ (n-times) is isomorphic to the trivial bundle of rank mn.

If A and B are unital C*-algebras we denote by Hom(A,B) the set of unital *-homomorphisms from A to B. Hom(A,B) is a topological space with the topology of pointwise convergence.

Two homomorphisms $\phi_1, \phi_2 \in \mathrm{Hom}(A,B)$ are said to be inner equivalent if there is a unitary $u \in B$ such that $\phi_2 = u\phi_1 u^*$. Let Hom(A,B)/inn be the set of classes of inner equivalent homomorphisms from A to B.

PROPOSITION 1.1. Assume that Y is a compact space. Then there is an exact sequence of pointed sets

(13) $$1 \to C(Y,U(k)) \to C(Y,U(kn)) \xrightarrow{j} \mathrm{Hom}(M_n, C(Y,M_{kn})) \xrightarrow{\delta'} \mathrm{Vect}_k(Y) \to \xrightarrow{\gamma'} \mathrm{Vect}_{kn}(Y)$$

which induces an isomorphisms

$$\mathrm{Hom}(M_n, C(Y,M_{kn}))/\mathrm{inn} \xrightarrow{\sim} T_n(\mathrm{Vect}_k(Y)) .$$

Proof. We have the following commutative diagram of pointed sets and maps

$$\begin{array}{ccccccccccc} 1 \to C(Y,U(k)) & \to & C(Y,U(kn)) & \xrightarrow{j} & C(Y,U(kn)/U(k)) & \xrightarrow{\delta} & H^1(Y,U(k)^c) & \xrightarrow{\gamma} & H^1(Y,U(kn)^c) \\ \| & & \| & & \downarrow \alpha & & \downarrow \beta & & \downarrow \beta_1 \\ 1 \to C(Y,U(k)) & \to & C(Y,U(kn)) & \xrightarrow{j'} & \mathrm{Hom}(M_n,C(Y,M_{kn})) & \xrightarrow{\delta'} & \mathrm{Vect}_k(Y) & \xrightarrow{\gamma'} & \mathrm{Vect}_{kn}(Y) \end{array}$$

The vertical arrows are bijections. To describe α recall that $\mathrm{Hom}(M_n,M_{kn}) \simeq U(kn)/U(k)$ as topological spaces, the isomorphism being induced by the map $\eta : U(kn) \to \mathrm{Hom}(M_n,M_{kn})$ given by $\eta(v)(a)=v(I_k \otimes a)v^*$, $a\in M_n$, $M_{kn} \simeq M_k \otimes M_n$. Now η induces a bijection $\eta_*: C(Y,U(kn)/U(k)) \to C(Y,\mathrm{Hom}(M_n,M_{kn}))$ and it is clear that $C(Y,\mathrm{Hom}(M_n,M_{kn})) \simeq \mathrm{Hom}(M_n, C(Y,M_{kn}))$ by the map α_1 which takes the continuous function $\psi: Y \to \mathrm{Hom}(M_k,M_{kn})$ to the homomorphism $\alpha_1(\psi)(a)(y)=\psi(y)(a)$, $a\in M_n$, $y\in Y$. By definition we set $\alpha = \alpha_1 \eta_*$. In $\mathrm{Hom}(M_n,C(Y,M_{kn}))$ we distinguish the homomorphism $a \mapsto I_k \otimes a$. Thus α is a morphism of pointed sets. The maps β and β_1 are the natural ones. Namely if (U_i,g_{ij}) is a $U(k)$-cocycle then $\beta(U_i,g_{ij})$ is the isomorphism class of the vector bundle E obtained by clutching the trivial bundles $U_i \times \mathbb{C}^k$ with the transition functions (g_{ij}). β_1 is defined in a similar way. The other maps are defined to make commutative the diagram. We describe them below.

If $u\in C(Y,U(kn))$ then $j'(u): M_n \to C(Y,M_{kn})$ is defined by $j'(u)(a)(y)=u(y)(I_k \otimes a)u(y)^*$, $a\in M_n$, $y\in Y$. The map γ' takes the vector bundle E to the Whitney sum $nE=E \oplus \ldots \oplus E$.

If $\phi_1,\phi_2 \in \mathrm{Hom}(M_n,C(Y,M_{kn}))$ then $\delta'(\phi_1)=\delta'(\phi_2)$ if and only if ϕ_1 and ϕ_2 are inner equivalent. The isomorphism class of the vector bundle $\delta'(\phi_1)$ represents the obstruction for lifting ϕ_1 to an unitary $u\in C(Y,U(kn))$. □

We shall use the following well-known result concerning triviallity of the vector bundles which is contained in [5, Ch.8 Thm. 1.5].

LEMMA 1.2. Let Y be a finite CW-complex of dimension r and let E be a complex vector bundle of rank k over Y. Assume that $r \leq 2k$ and that $E \oplus F$ is trivial for some trivial vector bundle F. Then E is trivial. The main result of this section is the following:

THEOREM 1.3. Let X and Y be compact topological spaces and let ϕ be an unital *-homomorphism

$$(14) \qquad \phi : C(X) \otimes M_n \longrightarrow C(Y) \otimes M_{kn} \simeq C(Y) \otimes M_k \otimes M_n \quad .$$

Assume the following

(1) Y is homotopic equivalent with a finite CW-complex of dimension $\leq 2k$.

(2) $K^{\circ}(Y)$ does not contain nontrivial elements whose order divides n.

Then there is an unitary $u \in C(Y, U(kn))$ and an unital *-homomorphism $\phi' : C(Y) \longrightarrow C(Y) \otimes M_k$ such that

$$(15) \qquad \phi = u(\phi' \otimes id_{M_n})u^*$$

Proof. a) Consider first a particular case. Let us suppose that ϕ acts on matrices as an amplification. That is:

$$(16) \qquad \phi(1_{C(X)} \otimes a) = 1_{C(Y)} \otimes I_k \otimes a \ , \qquad a \in M_n \ .$$

Using (16) we get for every $f \in C(X)$ and $a \in M_n$

$$(17) \quad \phi(f \otimes a) = \phi(f \otimes I_n)\phi(1 \otimes a) = \phi(1 \otimes a)\phi(f \otimes I_n) = \\ = 1 \otimes I_k \otimes a \ . \ \phi(f \otimes I_n) \ .$$

The computation in (17) shows us that the algebra $\phi(C(X) \otimes I_n)$ lies in the relative commutant of $1 \otimes I_k \otimes M_n$ in

$C(Y) \otimes M_k \otimes M_n$, which is equal to $C(Y) \otimes M_k \otimes I_n$. It follows that there is an unique unital *-homomorphism $\phi ' : C(X) \to C(Y) \otimes M_k$ such that

(18) $$\phi(f \otimes I_n) = \phi '(f) \otimes I_n .$$

Now using again (17) we get

(19) $$\phi = \phi ' \otimes id_{M_n}$$

b) Consider now the general case of an arbitrary homomorphism . Under the hypotheses 1) and 2) we shall find an unitary $u \in C(Y,U(kn))$ such that the homomorphism $u^* \phi u$ will verify the additional condition given in (16). Define $\phi_1 \in Hom(M_n, C(Y,M_{kn}))$ by $\phi_1(a) = \phi(1 \otimes a)$, $a \in M_n$. Our problem is to find $u \in C(Y,U(kn))$ such that $\phi_1(a)(y) = u(y)(I_k \otimes a)u(y)^*$, $y \in Y$, $a \in M_n$.

But in virtue of Proposition 1.1 this can be done if and only if the vector bundles $E = \delta '(\phi_1)$ is trivial. Having in mind the map $\gamma ' : Vect_k(Y) \to Vect_{kn}(Y)$ we define a homomorphism of groups $\gamma ' : K^\circ(Y) \to K^\circ(Y)$ given by $\gamma '(x) = nx$. The second hypothesis of the Theorem allows us to conclude that $\gamma '$ is injective. Since $E \in T_n(Vect_k(Y))$ it follows that the class of K-theory of the vector bundle E is zero hence E becomes trivial after summing some trivial vector bundle. Since rank(E)=k and $\dim(Y) \leq 2k$ it follows from Lemma 1.2 that E is trivial. ☐

<u>REMARK 1.4</u>. Assume k=1. Since the line bundles on Y are classified by the second Cech cohomology group $H^2(Y,\mathbb{Z})$ it follows that the conclusion of Theorem 1.3 remains true if we drop both hypotheses (1) and (2) but we assume that $H^2(Y,\mathbb{Z})$ has not n-torsion.

In this way we recover a result of Knus (cf.[9])

2. THE "COVERING" ASSOCIATED WITH A *-HOMOMORPHISM

Let X and Y be compact spaces and let

$$\phi : C(X) \to C(Y, M_k)$$

be an unital *-homomorphism. For every $y \in Y$ the map $\phi_y : f \mapsto \phi(f)(y)$ defines a *-representation of C(X) on $\mathbb{C}^k$. This representation decomposes into a direct sum of characters of the C*-algebra C(X): $\phi_y = m_1 x_1 + \ldots + m_r x_r$ where $x_1, \ldots, x_r$ are distinct points in X and $1 \leq m_1 \leq \ldots \leq m_r$ are the multiplicities with they occur as characters in the decomposition of ϕ_y. Of course we must have $m_1 + \ldots + m_r = k$. Let us denote by F(y) the set $\langle x_1, \ldots, x_r \rangle$. It will be useful to take into consideration the spectral projections.

Let G(k,j) be the space of all j-dimensional selfadjoint projections acting on $\mathbb{C}^k$ and let $G(k) = \bigcup_{j=1}^{k} G(k,j)$. Then there are $p_1 \in G(k, m_1), \ldots, P_r \in G(k, m_r)$ satisfying $p_1 + \ldots + p_r = I_n$ and such that

$$(20) \qquad \phi(f)(y) = \phi_y(f) = \sum_{i=1}^{r} f(x_i) p_i \quad , \ f \in C(X).$$

Of course x_i and p_i depends on $y \in Y$ and they have some continuity properties. We try to store these properties in some construction given below.

For $1 \leq r \leq y$ define Y_r to be the set of all points y in Y such that card $F(y) \leq r$ and define $Y(m_1, \ldots, m_r)$ to be the set of all points y in Y such that F(y) has exactly r elements with the multiplicities $m_1, \ldots, m_r$. By definition we have that

$$(21) \qquad Y = Y_k \supset Y_{k-1} \supset \ldots \supset Y_1 \qquad \text{and}$$

$$(22) \qquad Y_r \setminus Y_{r-1} = \bigcup Y(m_1, \ldots, m_r)$$

LEMMA 2.1. Let $y^\circ \in Y(m_1,\dots,m_r)$ and $F(y^\circ)=\{x_1^\circ,\dots,x_r^\circ\}$. Assume that $V_1,\dots,V_r$ are disjoint neighbourhoods of the points $x_1^\circ,\dots,x_r^\circ$. Then there is an open neighbourhood V of y° such that for every $y\in V$ we have that $F(y)\subset V_1\cup\dots\cup V_r$ and the number of points (counted with multiplicities) in $F(y)\cap V_i$ is equal to m_i, $i=1,\dots,r$.

Proof. a) There is some open $V\ni y^\circ$ such that $F(y)\subset V_1\cup\dots\cup V_r$ whenever $y\in V$. To get a contradiction suppose that there are two nets (y_V) and (x_V) indexed by the neighbourhoods of y° such that y_V converges to y° and $x_V\in F(y_V)\setminus(V_1\cup\dots\cup V_r)$. Now choose $f\in C(X)$ such that $f=0$ on $F(y^\circ)$ and $f=1$ on $X\setminus(V_1\cup\dots\cup V_r)$. Since $y_V\to y^\circ$ it follows that $\phi(f)(y_V)\to\phi(f)(y^\circ)$. But this is imposible since $\phi(f)(y^\circ)=0$ and $\|\phi(f)(y_V)\|\geq|f(x_V)|=1$. (Recall the formula (20)).

b) Let V be the open neighbourhood of y° found at a). Consider open sets $W_1,\dots,W_r$ such that $V_i\subset W_i$ and $\overline{W}_i$ are disjoint. For every $1\leq i\leq r$ choose $f_i\in C(X)$ such that $f_i=1$ on V_i and $f_i=0$ on $X\setminus W_i$. Shrinking V we may suppose that

$$\|\operatorname{tr}\phi(f_i)(y)-\operatorname{tr}\phi(f_i)(y^\circ)\|<1 \tag{23}$$

But the functions f_i were choosen such that $\operatorname{tr}\phi(f_i)(y^\circ)=m_i$ and such that $\operatorname{tr}\phi(f_i)(y)$ equals the number of points (counted with multiplicities) in $F(y)\cap V_i$. With this remark the Lemma follows from (23). ▨

Let $Z=\{(y,x)\in Y\times X : x\in F(y)\}$. It follows from Lemma 2.1 that Z is closed in $Y\times X$, hence Z is a compact space. Consider the canonical projections on factors $\pi: Z\to Y$, $\pi(y,x)=y$ and $\gamma: Z\to X$, $\gamma(y,x)=x$. Define also the projection valued map

$p: Z \to G(k)$ by taking $p(z)$ to be the orthogonal projection on the spectral space that corresponds to the character given by $x = \varphi(z)$ in the decomposition (20).

THEOREM 2.2. 1) The map $\pi : Z \to Y$ is open.

2) $Y = Y_k \supset Y_{k-1} \supset \ldots \supset Y_1$ is a filtration with closed sets.

3) $Y(m_1, \ldots, m_r)$ is open in Y_r.

4) $\pi^{-1}(Y(m_1, \ldots, m_r)) \xrightarrow{\pi} Y(m_1, \ldots, m_r)$ and $\pi^{-1}(Y_r \setminus Y_{r-1}) \xrightarrow{\pi} Y_r \setminus Y_{r-1}$ are covering spaces.

5) The projection valued map $p: Z \to G(k)$ is continuous on every $Y_r \setminus Y_{r-1}$.

Proof. Essentially the theorem is a reformulation of Lemma 2.1.

1) Let $W \subset Y$ and $U \subset X$ be open sets. Then $\pi((W \times U) \cap Z)$ is open in Y. Indeed if $(y^\circ, x^\circ) \in (W \times U) \cap Z$, then by Lemma 2.1 there is an open set $V \subset W$, $y^\circ \in V$ such that $F(y) \cap U$ is nonvoid for all y in V. Hence $V \subset ((W \times U) \cap Z)$.

2) It follows from 3) since $Y_r \setminus Y_{r-1} = \bigcup Y(m_1, \ldots, m_r)$.

3) Let $y^\circ \in Y(m_1, \ldots, m_r)$ and $F(y^\circ) = \{x_1^\circ, \ldots, x_r^\circ\}$. By Lemma 2.1 we can choose disjoint neighbourhoods $V_i \ni x_i^\circ$, $i = 1, \ldots, r$ and $V \ni y^\circ$ open such that $F(y) \subset V_1 \cup \ldots \cup V_r$ and $F(y) \cap V_i$ has exactly m_i points (counted with multiplicities) whenever $y \in V$. We prove that $V \cap Y_r \subset Y(M_1, \ldots, m_r)$. If $y \in V \cap Y_r$ then $F(y)$ has at most r elements. Since each $F(y) \cap V_i$ is nonvoid it follows that $F(y)$ has exactly r elements and the multiplicities are as desired.

4) It suffices to prove the first assertion.

Let V_i be as in Lemma 2.1. By 2) every point $y^\circ \in Y(m_1, \ldots, m_r)$ has an open neighbourhood V such that $V \cap Y_r \subset Y(m_1, \ldots, m_r)$ and

$\pi^{-1}(V \cap Y_r) = W_1 \cup \ldots \cup W_r$. Here $W_i = \pi^{-1}(V \cap Y_r) \cap (Y \times V_i)$ are open sets in $\pi^{-1}(Y(m_1,\ldots,m_r))$ and $(\pi|W_i): W_i \to V \cap Y_r$ is continuous and bijective.

The map $(\pi|W_i)$ is open by a similar argument to that given at 1).

5) Let $z^\circ = (y^\circ, x^\circ) \in \pi^{-1}(Y_r \setminus Y_{r-1})$. Choose open $U \ni x^\circ$ such that $U \cap F(y^\circ) = \{x^\circ\}$. If $f \in C(X)$ is equal to 1 on a neighbourhood of x° and $\mathrm{supp}(f) \subset U$ then $p(z) = \phi(f)(\pi(z))$ in a small enough neighbourhood of z° in $\pi^{-1}(Y_r \setminus Y_{r-1})$. ▨

With the above notation we get the following formula for ϕ:

$$\phi(f)(y) = \sum_{z \in \pi^{-1}(y)} f(\varphi(z))p(z) \qquad (24)$$

Note that φ is injective when restricted to the fibres of π and ϕ is isometric if and only if φ is onto.

<u>REMARK 2.4</u>. The homomorphism ϕ is called homogeneous if for every $y \in Y$, $F(y)$ has k elements.

For homogeneous ϕ it follows from Theorem 2.2 that $\pi : Z \to Y$ is a k-fold covering space and that the map $p(z)$ is continuous on Z. Conversely if $\pi : Z \to Y$ is a covering space, $\varphi : Z \to X$ is a continuous map that is injective on fibres of π and $p: Z \to G(k,1)$ is continuous and verifies $\sum_{z \in \pi^{-1}(y)} p(z) = I_k$ for all $y \in Y$ then the formula (24) defines a homogeneous homomorphism.

<u>REMARK 2.5</u>. Suppose that Y is simply connected, locally pathwise connected and that ϕ is homogeneous. Then it follows from the general theory of the covering spaces [12] that $Z = Y \sqcup \ldots \sqcup Y$ (k-times). Therefore there exist continuous maps

$\varphi_1,\dots,\varphi_k:Y\to X$, $\varphi_i(y)\neq\varphi_j(y)$, $y\in Y$ $(i\neq j)$; $p_1,\dots,p_k:Y\to G(k,1)$, $\sum_{i=1}^{k} p_i(y)=I_k$ such that $\Phi(f)(y)=\sum_{i=1}^{k} f(\varphi_i(y))p_i(y)$, $y\in Y$. If in addition we assume that $H^2(Y,\mathbb{Z})=0$ then it can be shown that there is some unitary $u\in C(Y,U(k))$ such that $p_i(y)=$ $=u(y)p_i(y^\circ)u(y)^*$, $y\in Y$, $i=1,\dots,k$, for some fixed $y^\circ\in Y$. Therefore we get the following formula

$$(25)\qquad \Phi(f)(y)=u(y)\begin{bmatrix} f(\varphi_1(y)) & & 0 \\ & \ddots & \\ 0 & & f(\varphi_k(y)) \end{bmatrix}u(y)^*,\quad f\in C(X),\ y\in Y.$$

Consequently, under the previous assumptions, the homogeneous homomorphisms are classified modulo a inner equivalence by the set $\langle\varphi_1,\dots,\varphi_k\rangle$. To sketch a proof let $T=U(1)\times\dots\times U(1)$ be the maximal torus of $U(k)$. The fibration

$$(26)\qquad T\to U(k)\to U(k)/T$$

induces an exact sequence of pointed cohomological sets

$$C(X,T)\to C(T,U(k))\to C(Y,U(k)/T)\xrightarrow{\ \delta\ } H^1(Y,T^c)$$

Since $H^1(Y,T^c)=\bigoplus_{i=1}^{k} H^1(Y,U(1)^c)=\bigoplus_{i=1}^{k} H^2(Y,\mathbb{Z})=0$ (see [4]) we get that $\delta=0$ and so every continuous map $\psi:Y\to U(k)/T$ can be lifted to a function in $C(Y,U(k))$. Let N be the space of all k-uples $(q_1,\dots,q_k)$ of one dimensional projections acting on $\mathbb{C}^k$, $q_1+\dots+q_k=I_k$. $U(k)$ operates transitively on N by the formula $U(k)\ni u\mapsto(up_1(y^\circ)u^*,\dots,up_k(u^\circ)u^*)$, $y^\circ\in Y$ is fixed. Since the corresponding stabilizer is T it follows that N is homeomorphic to $U(k)/T$. Thus the map $y \mapsto (P_1(y),\dots,p_k(y))$ can be lifted to a continuous map $u\in C(Y,U(k))$.

3. HOMOMORPHISMS COMPATIBLE WITH A COVERING

Let $\psi : X \to Y$ be a k-fold covering space and A, B be unital $\mathbb{C}^*$-algebras. A homomorphism $\phi \in Hom(C(X) \otimes A, C(Y) \otimes B)$ is called compatible with the covering ψ or ψ-compatible if $\phi(g\circ\psi \otimes 1_A) = g \otimes 1_B$ for all $g\in C(Y)$ (see [7]). In this section we consider only the case $A=M_n$, $B=M_{kn}$.

REMARK 3.1. Let $\phi \in Hom(C(X) \otimes M_n, C(Y) \otimes M_{kn})$ be a ψ-compatible homomorphism. Assume that the hypotheses of Theorem 1.3 are fulfilled. Then the homomorphism ϕ' that appears in the decomposition $\phi = u(\phi' \otimes id)u^*$ is ψ-compatible. This is clear from the following computation:

$$u\phi'((g\circ\psi) \otimes I_n)u^* = g \otimes I_{kn} \Rightarrow \phi'(g\circ\psi) \otimes I_n = u^*(g \otimes I_{kn})u = g \otimes I_{kn}$$

PROPOSITION 3.2. Assume that X and Y are connected, locally pathwise connected and that $\phi : C(X) \to C(Y) \otimes M_k$ is compatible with the k-fold covering $\psi : X \to Y$. Then there is a continuous projection valued map $p : X \to G(k,1) = P(\mathbb{C}^k)$ (the projective complex (k-1)space) such that

$$\sum_{x\in\psi^{-1}(y)} p(x) = I_k\ , \quad y\in Y \qquad \text{and} \tag{27}$$

$$\phi(f)(y) = \sum_{x\in\psi^{-1}(y)} f(x)p(x) \qquad f\in C(X),\ y\in Y. \tag{28}$$

Proof. We shall use the notation in section 2. It follows from the condition of ψ-compatibility that $\varphi(\pi^{-1}(y)) \subset \psi^{-1}(y)$, $y \in Y$. Combining with Lemma 2.1 this shows that every subset $Y(m_1,\dots,m_r)$ is open in Y. Since Y is connected and it is a disjoint union of such subsets we infer that Y is equal to some $Y(m_1,\dots,m_r)$. Therefore $\pi: Z \to Y$ is a r-fold covering space. Now, since $\psi\varphi = \pi$ it follows from [12] that $\varphi: Z \to X$ is a covering space. This implies $r=k$ and we conclude that φ is an isomorphism of covering spaces. Thus we may take $Z=X$ and $\pi = \psi$. ◻

REMARK 3.3. It is easily seen that every projection valued continuous map $p: X \to P(\mathbb{C}^k)$ which verifies (2.7) defines a ψ-compatible homomorphism. Under the connectedness assumptions of Proposition 3.2 it follows that ψ-compatible homomorphisms must be isometric.

We give below some criteria for the existence of ψ-compatible homomorphism. Suppose that the cover $\psi: X \to Y$ is regular. If G denote the group of the covering automorphisms, this means that G operates transitively on fibres $\psi^{-1}(y)$ and we can recover Y from X and G as X/G. Of course $|G| = |\psi^{-1}(y)| = k$.

PROPOSITION 3.3. Suppose that G is commutative and that $H^2(Y,\mathbb{Z})$ is torsion free. Then there exists a ψ-compatible homomorphism $\Phi \in \mathrm{Hom}(C(X), C(X) \otimes M_k)$.

Proof. First we prove that for every character $\omega \in \hat{G}$ ($\hat{G}$=the Pontrjagin dual of G) there is a continuous function $f_\omega: X \to U(1)$ such that

$$f_\omega(g(x)) = \omega(g) f_\omega(x) \qquad x \in X,\ g \in G. \tag{2.9}$$

For $\omega \in \hat{G}$ let us denote by $E(\omega)$ the complex line bundle over Y obtained as

$$E(\omega)=X\times\mathbb{C}/(x,\lambda) \quad (g(x),\ \omega(g)\lambda)$$

It follows from Theorem 2.6 Ch.II in [1] that the C(Y)-module of continuous linear sections in $E(\omega)$ is isomorphic to the C(Y)--module of continuous function $f:X\to\mathbb{C}$ which satisfy (29). Our aim is to find such a function which does not vanish. This is equivalent to prove that the line bundle $E(\omega)$ is trivial.
If $\omega_1,\omega_2 \in G$ it is not hard to check that

$$E(\omega_1)\otimes E(\omega_2)\simeq E(\omega_1\omega_2)$$

This isomorphism allows us to define a morphism of groups

$$\hat{G}\ni\omega\longmapsto c_1(E(\omega))\in H^2(Y,\mathbb{Z}) \tag{30}$$

(Here $c_1(E)$ is the first Chern class of the line bundle E).

Since $\hat{G}$ is finite and $H^2(Y,\mathbb{Z})$ is torsion free the above morphism must be zero. This implies that $E(\omega)$ is trivial.

Let $\rho:G\to B(l^2(G))\simeq M_k$ be the right regular representation ([8]). Using the first step of the proof we shall define a continuous map $u:X\to U(l^2(G))\simeq U(k)$ such that

$$u(g(x))=\rho(g)u(x) \qquad x\in X,\quad g\in G. \tag{31}$$

Let $\rho(g)=\sum_{\omega\in\hat{G}}\omega(g)p_\omega$ be the decomposition of ρ as a direct sum of characters. Here p_ω is the projection on the one dimensional spectral subspace corresponding to ω . For every $\omega\in\hat{G}$ we choose a continuous function $f_\omega:X\to U(1)$ which satisfies (29) and we define

$$(32)\qquad u(x)=\sum_{\omega} f_{\omega}(x)p_{\omega}$$

It is easy to check that u verifies (31).

Let us denote by $\langle e_g : g \in G\rangle$ the projections on the subspaces $[\delta_g]$ spanned by the elements of the canonical basis $\langle \delta_g : g \in G\rangle$ of $l^2(G)$. Now we are able to define

$$\phi : C(X) \to C(X/G) \otimes B(l^2(G)) \simeq C(Y) \otimes M_k \quad \text{by}$$

$$(33)\qquad \phi(f)(\Psi(x)) = u(x)^*\Big(\sum_{g\in G} f(g(x))e_g\Big)u(x) \qquad x \in X,\quad f \in C(X).$$

The homomorphism ϕ is well defined since if $\Psi(x)=\Psi(z)$ then $h(x)=z$ for some $h\in G$ and we have

$$u(z)^*\Big(\sum_g f(g(z))e_g\Big)u(z) = u(x)^*\varrho(h)^*\Big(\sum_g f(hg(x))e_g\Big)\varrho(h)u(x) =$$
$$= u(x)^*\Big(\sum_g f(g(x))e_g\Big)u(x)\ .$$

In the above computation we used (31) and $\varrho(h)^* e_g \varrho(h) = e_{gh}$. ▨

REMARK 3.4. The conclusion of Proposition 3.3 remains valid if we drop the assumption on $H^2(Y,Z)$ but we suppose that X is a commutative compact group and G is a finite subgroup of it which acts on X by translations. In this case the line bundle $E(\omega)$ will be trivial since every character of G may be extended to some character of X ([11]).

REMARK 3.5. Assume the hypothesis of Proposition 3.3. The action of G on X induces an action of G on the C*-algebra C(X): $g(f)(x)=f(g^{-1}(x))$. Therefore we can consider the crossed product $C(X) \rtimes G$ and we realize it as the following subalgebra of $C(X) \otimes B(l^2(G))$: $C(X) \rtimes G = \{F \in C(Y) \otimes B(l^2(G)) : F(g(x)) = \rho(g)F(x)\rho(g)^*,\ x \in X,\ g \in G\}$.

The algebra $C(X)$ can be canonically imbedded into $C(X) \rtimes G$ $C(X) \ni f \longmapsto j(f) \in C(X) \rtimes G$, $j(f)(x) = \sum_{g \in G}' f(g(x)) e_g$. Let $u : X \to U(l^2(G))$ be the unitary constructed in the proof of Proposition 3.3. Identify $C(Y, M_k)$ with $\{F \in C(X, M_k) : F(g(x)) = F(x),\ x \in X,\ g \in G\}$ Then the isomorphism $\Psi : C(X) \rtimes G \to C(Y) \otimes B(l^2(G)) \simeq C(Y, M_k)$ $\Psi(F) = u^* F u$ is such that $\Psi \circ j = \phi$ where ϕ is the homomorphism given by Proposition 3.3.

<u>EXAMPLE 3.6</u>. Let $\mathbb{P}^n = S^n / Z_2$ be the real n-dimensional projective space. Since $H^2(\mathbb{P}^n, Z) = \mathbb{Z}_2$ Proposition 3.3 doesn't apply. $(n \geq 2)$ However the following statements are true:

a) The set of all unital *-homomorphisms $\phi : C(S^2) \to C(\mathbb{P}^2, M_2)$ which are compatible with the canonical covering $S^2 \to \mathbb{P}^2$ is in bijection with the set of continuous functions $P' : S^2 \to S^2$ which takes antipodal points to antipodal points, i.e.

$$p'(-x) = -p'(x) \ . \tag{34}$$

b) If $n \geq 3$ do not exist homomorphisms $\phi \in \mathrm{Hom}(C(S^n), C(\mathbb{P}^n, M_2))$ compatible with the canonical covering $S^n \to \mathbb{P}^n$.

<u>Proof</u>. a) In virtue of Proposition 3.2 it is enough to consider continuous maps $p : S^2 \to P(\mathbb{C}^2)$ for which $p(x) + p(-x) = I_2$. But $P(\mathbb{C}^2)$ is homeomorphic to S^2 by a homeomorphism that sends orthogonal projections to antipodal poins. Therefore every p is given by some p' that satisfies (34).

b) By the Theorem of Borsuk -Ulam [12] does not exist continuous maps $f : S^n \to S^2$, $n \geq 3$ such that $f(-x) = -f(x)$. ☐

COROLLARY 3.7. Let $n=2k+1$. The homomorphisms $\phi: C(S^2, M_n) \to C(S^2, M_{2n})$ which are compatible with the covering $S^2 \to \mathbb{P}^2$ are classified modulo a inner equivalence by the set of continuous maps $p: S^2 \to S^2$ which verify $p(-x) = -p(x)$, $x \in S^2$.

Proof. Since $K^\circ(\mathbb{P}^2) = Z_2$ we may apply Theorem 1.3. Now the assertion follows from Example 3.6 (a).

We have to mention that it was proved in [7] that all the homomorphisms $\phi: C(U(1)^2, M_n) \to C(U(1)^2, M_{nrs})$ which are compatible with the covering $U(1)^2 \ni (z_1, z_2) \mapsto (z_1^r, z_2^s) \in U(1)^2$ are inner equivalent. No general results there are known on the problem of classifying ψ -compatible homomorphisms.

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ADDED IN PROOF

After this work was complete, thanks to a preprint of K. Thomsen, I learn about the following reference

14. K. GROVE and G.K. PEDERSEN, Diagonalizing Matrices over C(X), J. Funct.Analysis 59, 65-89 (1984).

The problem of diagonalizing normal elements or abelian *-subalgebras of $C(Y) \otimes M_k$ is somewhat related to the study of homomorphisms $C(X) \to C(Y) \otimes M_k$.

In this context we may infer from the results of section 2 that the various bundles that arised in [14] as obstruction to diagonalization are in fact basic constituents of homomorphisms.