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by

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INTRODUCTION

This work brings ont some explanations and corrections concerning our first preprint on the subject [Pă].

In its first part, we give a simple algebraic criterion for the local equivalence of two pseudoriemannian homogeneous manifolds (T.1.1.).

As a corrolary, we give the local classification of 3 dimensional pseudoriemannian manifolds with a multiple transitive group of automorphisms.

In the second part, we settle in detail the complete list of germs of Lorentz 4-manifolds which admit a multiple transitive group of automorphisms, and we point out some of the omissions in the list of A.Z. Petrov [Pe].

In this paper we shall frequently use some notations and abridgements, namely

$\Psi.R_{n,q}$ manifold	-	pseudoriemannian manifold of dimension n, furnished with a metric tensor of index q
$\Psi.R_{n,q}$ m-space	-	$\Psi.R_{n,q}$ manifold with a multiple transitive group of automorphisms. We shall point out that the group is of a specified dimension d, by changing m with d
M	-	current notation for a $\Psi.R_{n,q}$ m-space
g	-	tensor metric of M
K	-	transitive group of automorphisms of (M,g)
$\mathfrak{k}$	-	Lie algebra of K
Og(M)	-	total space of the bundle of g-ortonormal frames of M
Aut(M,g)	-	the group of automorphisms of (M,g)

$\mathcal{O}_q(n)$ - Lie algebra of $O(n-q,q)$. If $q = 0$, we shall omit it

$\Omega^p(N,V)$ - the space of the differential forms of degree p on M, that are V - valuated (V real vector space)

e_i - canonical basis of $\mathbb{R}^n$

e_i^j - canonical basis of $\mathfrak{gl}(n, \mathbb{R}) \simeq L(\mathbb{R}^n, \mathbb{R}^n)$, $(e_i^j(e_k) = \delta_k^j \, e_i)$

$${}_q\delta_{ij} = \begin{cases} \delta_{ij}, & i \quad n-q \\ -\delta_{ij}, & i \quad n-q \end{cases}$$

$Sp(x_1, \ldots, x_p)$ - vector subspace generated by $x_1, \ldots, x_p$

- both exterior and symmetric product of and

- exterior product

$\ell.i$ - linearly independent

must thank dr. M.Martin for waisting his time to read part of the manuscript. I must thank my wife Adina too, for correcting part of my bad English.

1. Local equivalence of two homogeneous $\Psi.R_{n,q}$ - manifolds

Let (M,g) be a $\Psi.R_{n,q}$ - manifold and u $O_g(M)$. If $u = (x,u_1,\dots,u_n)$, we associate to u, an imbedding ϕ_u of $\mathrm{Aut}(M,g)$ in $O_g(M)$, given by

$$\phi_u(k) = (k(x),\ d_xk(u_1),\dots,\ d_xk(u_n))\ [\mathrm{Ko}]$$

If V is a real vector space and $\lambda \in \Omega^r(O_g(M), V)$ we shall note $\phi_u^*\lambda$ by ${}_u\lambda$. Let $\theta = \theta^i e_i \in \Omega^1(O_g(M), \mathbb{R}^n)$ be the dual form, $\omega = \omega_i^j\ e_j^i \in \Omega^1(O_g(M), \mathfrak{gl}(n, R))$ be the connexion form of the Levi Civita connexion of g, and $\Omega \in \Omega^2(O_g(M), \mathfrak{o}_q(n))$ its curvature form.

We shall note by $(\Omega_j^i)_{i\ j}$ the components of Ω relatively to the basis $(f_i^j)_{i<j}$ of $\mathfrak{O}_q(n)$ given by

$$f_i^j = \begin{cases} e_i^j + e_j^i, & i < j \leq n - q \vee n - q \leq i < j \\ e_i^j - e_j^i, & i \leq n - q \leq j \end{cases}$$

and by K^i_{jrs} the components of the curvature forms relatively to the dual forms: $\Omega^i_j = \frac{1}{2} K^i_{jrs}\ \theta^r \wedge \theta^s$.

If $K \subset \mathrm{Aut}(M,g)$ is transitive on M, ${}_uK^i_{jrs}$ are constant on K and ${}_uR_{ijrt} = {}_q\delta_{is}\cdot{}_uK^s_{jrt}$ are the components of a curvature tensor in the sense of [B.G.M.], ${}_uR$, named the <u>punctual curvature tensor</u> associated to the pair (K,u).

Let $\mathcal{C}_n$ be the space of curvature tensors on $\mathbb{R}^n$ ([B.G.M.] - p.68); then $O_q(n)$ acts on $\mathcal{C}_n$ by

$$A_q(R,c)(x_1,x_2,x_3,x_4) = R(cx_1,cx_2,cx_3,cx_4)$$

<u>Theorem 1.1</u>. For each index $\alpha = 1,2$, we take one $\Psi.R_{n,q}$ manifold (M_α, g_α), one transitive group of automorphisms

K of (M, g) and one frame u $O_g(M)$.

The following are equivalent :

1) (M_1, g_1) and (M_2, g_2) are locally equivalent

2) u_1 and u_2 belong to the same orbit of A_q.

Proof. It is enough to show that 2) 1), 1) 2) being evident.

If $u, u' \in O_g(M)$ and $c \in O_q(n)$ are such that $u' = u.c$, then ${}_{u}R = A_q({}_{u'}R, c)$, so that if we change in a suitable way the frame u_1, we a suppose that ${}_{u_1}R = {}_{u_2}R = R$.

We know (T.2.3 in [Pă]) that if we choose in a neighbourhood V_α of the identity of K_α, a coordinate system $(x^1_\alpha, \ldots, x^n_\alpha, y^1_\alpha, \ldots, y^d_\alpha) : V_\alpha \subset R^n \times R^d$, such that $x^1_\alpha, \ldots, x^n_\alpha$ are functionally independent prime integrals, of the Pfaff system ${}_{u_\alpha}\theta_\alpha = 0$, then $(V^{b_\alpha}_\alpha, {}_q\delta_{rs}\, {}_{u_\alpha}\theta^r_\alpha \otimes {}_{u_\alpha}\theta^s_\alpha \mid V^{b_\alpha}_\alpha)$ is a $\Psi.R_{n,q}$ manifold, locally equivalent to (M_α, g_α) (here $V^{b_\alpha}_\alpha = \{k \in K_\alpha \mid y_\alpha(k) = b_\alpha\}$).

If $j_\alpha : V^{b_\alpha}_\alpha \hookrightarrow K_\alpha$ and $f_\alpha = \Phi_{u_\alpha} \circ j_\alpha : V^{b_\alpha}_\alpha \longrightarrow O_{g_\alpha}(M_\alpha)$, the forms $\overline{\theta}^j_\alpha = f^*_\alpha(\theta^j_\alpha)$ $\overline{\omega}^i_{j,\alpha} = f^*_\alpha({}_{u_\alpha}\omega^i_{j,\alpha})$ verify the structure equations (E) on $V^{b_\alpha}_\alpha$ with the same constant coefficients K^i_{jrs}, for $\alpha = \overline{1,2}$:

$$d\,\overline{\theta}^i_\alpha = -\overline{\omega}^i_{j,\alpha} \wedge \overline{\theta}^j_\alpha$$

$$\text{(E)} \qquad d\,\overline{\omega}^i_{k,\alpha} + \overline{\omega}^i_{j,\alpha} \wedge \overline{\omega}^j_{k,\alpha} = \frac{1}{2} K^i_{krs}\, \overline{\theta}^r_\alpha \wedge \overline{\theta}^s_\alpha$$

$${}_q\delta_{ir}\, \overline{\omega}^r_{k,\alpha} + \overline{\omega}^r_{i,\alpha}\, {}_q\delta_{rk} = 0 \qquad i,k = \overline{1,n}$$

Lemma 1.2. For $= 1,2$, let $({}_\alpha\omega^i)_{i=\overline{1,n}}$ be fields of orthonormal coframes on the $R_{n,q}$-manifold (M_α, g_α).

If, for $\alpha = 1,2$, the structure equations relatively to $({}_\alpha\omega^i)_{i=\overline{1,n}}$ are the same, then (M_1, g_1) and (M_2, g_2) are locally

Proof. Let $({}_{\alpha}\omega^{i}_{j})$ be the connexion forms associated to the fiéld $({}_{\alpha}\omega^{i})_{i=\overline{1,n}}$ ([Sp]). Let us consider on $M_1 \times M_2 \times O_q(n)$ the distribution $\Delta_{1,2}$ that annihilates the ideal generated by the Pfaff forms $(\theta^{i}, \theta^{i}_{j})_{i,j=\overline{1,n}}$ defined as follows.

If π_1, π_2, π_3 are the projections on the factors of the product $M_1 \times M_2 \times O_q(n)$ and ω is a differential form on M_{α}, ${}^{\alpha}\omega = \pi^{*}_{\alpha}(\omega)$, then :

$$\theta^{i}(x^1,x^2,a) = {}^{2}\omega^{i} - a^{i}_{j}\,{}^{1}\omega^{j} \qquad , \; i,j = \overline{1,n}$$

$$\theta^{i}_{j}(x^1,x^2,a) = {}^{2}\omega^{i}_{j} - a^{i}_{r}\,{}^{1}\omega^{r}_{s}(a^{-1})^{s}_{j} - a^{i}_{r}\,{}^{3}d(a^{-1})^{r}_{s}$$

Let $O_q(n)_R$ be the isotropy subgroup of R, relatively to the action A_q. Let us proof that $\Delta_{1,2} \,|\, M_1 \times M_2 \times O_q(n)_R$ is an integrable distribution, i.e. that the ideal J generated by the restrictions of $\theta^{i}, \theta^{i}_{j}$ is d - closed.

We have successively :

$$d\,\theta^{i} = d({}^{2}\omega^{i}) - {}^{3}d\,a^{i}_{j} \wedge {}^{1}\omega^{j} - a^{i}_{j}\,d({}^{1}\omega^{j}) =$$

$$= - {}^{2}\omega^{i}_{k} \wedge {}^{2}\omega^{k} - {}^{3}da^{i}_{j} \wedge {}^{1}\omega^{j} + a^{i}_{j}\,{}^{1}\omega^{j}_{k} \wedge {}^{1}\omega^{k} =$$

$$= - {}^{2}\omega^{i}_{k} \wedge (\theta^{k} + a^{k}_{r}\,{}^{1}\omega^{r}) + a^{r}_{j}\,{}^{3}d(a^{-1})^{k}_{r}\,a^{i}_{k} \wedge {}^{1}\omega^{j} +$$

$$+ a^{i}_{r}\,{}^{1}\omega^{r}_{j} \wedge {}^{1}\omega^{j} = - {}^{2}\omega^{i}_{k} \wedge \theta^{k} - {}^{2}\omega^{i}_{r} \wedge a^{r}_{j}\,{}^{1}\omega^{j} + a^{i}_{k}\,{}^{3}d(a^{-1})^{k}_{r} \wedge$$

$$\wedge\, a^{r}_{j}\,{}^{1}\omega^{j} + a^{i}_{k}\,{}^{1}\omega^{k}_{s}(a^{-1})^{s}_{r} \wedge a^{r}_{j}\,{}^{1}\omega^{j} = - {}^{2}\omega^{i}_{k} \wedge \theta^{k} +$$

$$+ a^{r}_{j}\,{}^{1}\omega^{j} \wedge ({}^{2}\omega^{i}_{r} - a^{i}_{k}\,{}^{3}d(a^{-1})^{k}_{r} - a^{i}_{k}\,{}^{1}\omega^{k}_{s}(a^{-1})^{s}_{r}) =$$

$$= - {}^{2}\omega^{i}_{k} \wedge \theta^{k} + a^{r}_{j}\,{}^{1}\omega^{j} \wedge \theta^{i}_{r}$$

$$d\,\theta^{i}_{j} = d({}^{2}\omega^{i}_{j}) - {}^{3}da^{i}_{r} \wedge {}^{1}\omega^{r}_{s}\,(a^{-1})^{s}_{j} - a^{i}_{r}\,d\,{}^{1}\omega^{r}_{s}(a^{-1})^{s}_{j} +$$

$$+ a^{i}_{r}\,{}^{1}\omega^{r}_{s} \wedge {}^{3}d(a^{-1})^{r}_{j} - {}^{3}d\,a^{i}_{r} \wedge {}^{3}d(a^{-1})^{r}_{j} = - {}^{2}\omega^{i}_{k} \wedge {}^{2}\omega^{k}_{j} +$$

$$+ \frac{1}{2} K^{i}_{jkr}\,{}^{2}\omega^{k} \wedge {}^{2}\omega^{r} - {}^{3}d\,a^{i}_{k} \wedge {}^{1}\omega^{k}_{r}(a^{-1})^{r}_{j} -$$

$$- a^i_k \left(- {}^1\omega^k_p \wedge {}^1\omega^p_s + \frac{1}{2} K^k_{spq}\, {}^1\omega^p \wedge {}^1\omega^q\right)(a^{-1})^s_j +$$

$$+ a^i_k\, {}^1\omega^k_r \wedge {}^3(da^{-1})^r_j - {}^3da^i_k \wedge {}^3d(a^{-1})^k_j =$$

$$= -(\theta^i_k + a^i_r\, {}^1\omega^r_q\, (a^{-1})^q_k + a^i_r\, {}^3d(a^{-1})^r_k) \wedge (\theta^k_j + a^k_s\, {}^1\omega^s_p (a^{-1})^p_j +$$

$$+ a^k_s\, {}^3d(a^{-1})^s_j) + \frac{1}{2} K^i_{jkp}(\theta^k + a^k_r\, {}^1\omega^r) \wedge (\theta^p + a^p_s\, {}^1\omega^s) -$$

$$- {}^3da^i_k \wedge {}^1\omega^k_p (a^{-1})^p_j - a^i_k(-{}^1\omega^k_p \wedge {}^1\omega^p_s + \frac{1}{2} K^k_{spq}\, {}^1\omega^p \wedge {}^1\omega^q)(a^{-1})^s_j +$$

$$+ a^i_k\, {}^1\omega^k_p\, {}^3d(a^{-1})^p_j - {}^3da^i_k \wedge {}^3d(a^{-1})^k_j =$$

$$= \sigma(\theta) + \frac{1}{2} (K^i_{jkp}\, a^k_r\, a^p_s - a^i_p\, K^p_{qrs} (a^{-1})^q_j)\, {}^1\omega^r \wedge {}^1\omega^s = \sigma(\theta) \in J$$

If the dimension of $O_q(n)_R$ is d, J is generated by n + d l.i. Pfaff forms. Because $\pi_{\alpha *(x_1,x_2,a)} (\Delta_{1,2}) = T_{x_\alpha} M_\alpha$, $\alpha = 1,2$, we may consider that an integral manifold of dimension n of J is locally the graph of a differential map $(f,a) : D_1 \longrightarrow D_2 \times O_q(n)_R$, where D_α are open sets in M_α and f is a local diffeomorphism, with $f^* \omega^i_2 = a^i_j\, \omega^j_1$.

<u>Corrolary 1.3</u>. For each $\alpha = 1,2$, let (M_α, g_α) be a naturally reductive, connected 1-connected, $\Psi.R_{n,q}$ homogeneous manifold. The following are equivalent :

1) (M_1, g_1) and (M_2, g_2) are isomorphic
2) u_1^R is A_q - equivalent to u_2^R.

<u>Proof</u>. Natural reductive $\Psi.R_{n,q}$-manifolds are geodesically complete, analytic $\Psi.R_{n,q}$-manifolds. From C.6.2.in ([K.N.I.]) it follows that a local equivalence of M_1 with M_2 extends to an analytic isomorphism F of analytic manifolds furnished with analytic connexions. Since F is a local

isomorphism of analytic $\Psi.R_{n,q}$-manifolds, it follows that F is an isomorphism.

Theorem 1.1 shows that the invariants of the local equivalence class of an homogeneous $\Psi.R_{n,q}$-manifold are precisely the invariants of the A_q orbit of its punctual curvature tensor. Among these invariants we distinguish:

a) - the invariants of the orbit of the associated q - Ricci tensor $_q\rho_R$ (relatively to the action induced by A_q on bilinear symmetric forms on $\mathbb{R}^n$) :

$$_q\rho_R(u,v) = \sum_r K^r_{jrk}\ u^k\ v^j = {}_q\delta^{ps}\ R_{sjpk}\ u^k\ v^j$$

b) - the q - norm of R, given by :

$$\| R\|^2_q = \sum_{i,j,k} {}_q\delta_{rs}\ K^r_{kij}\ K^s_{kij}\,.$$

The punctual curvature tensor is uniquely determined by the matrix $\widetilde{\Omega}\in\mathcal{M}_{\frac{n(n-1)}{2}}(\mathbb{R})$, with indices pairs (i,j) $i<j$ lexicographycally ordered :

$$\widetilde{\Omega} = (K^i_{jkr})_{((i,j),(k,r))}$$

Because $\widetilde{\Omega}$ is symmetric, we shall ommit in its writing the terms under the diagonal.

Now on, we shall study only $\Psi.R_{n,q}$ m-spaces.

<u>Corrolary 1.4</u>. The local equivalence class of a $\Psi.R_{3,q}$ m-space M is uniquely determined by the eigenvalues of the q-Ricci form associated to its punctual curvature tensor.

<u>Proof</u>. We may suppose that $2q \leqslant 3$.

If q = 0, one finds the next matrices $\widetilde{\Omega}$ (Car)

$$\Omega_{-1}(a,0) = \begin{pmatrix} -a^2 & 0 & 0 \\ & -a^2 & 0 \\ & & -a^2 \end{pmatrix}, \quad \Omega_1(a,\alpha) = \begin{pmatrix} \alpha & 0 & 0 \\ & a^2 & 0 \\ & & a^2 \end{pmatrix}$$

If $q = 1$, one finds the next matrices $\widetilde{\Omega}$ (Pă).

$$\Omega_1(a,\alpha) = \begin{pmatrix} 3a^2-\alpha & 0 & 0 \\ & a^2 & 0 \\ & & a^2 \end{pmatrix}, \quad \Omega_2(a,0) = \begin{pmatrix} a^2 & 0 & 0 \\ & -a^2 & 0 \\ & & -a^2 \end{pmatrix}$$

$$\Omega_{-1}(a,\alpha) = \begin{pmatrix} -a^2 & 0 & 0 \\ & a^2 & 0 \\ & & -a^2-\alpha \end{pmatrix}, \quad \Omega_0(a,\alpha) = \begin{pmatrix} \alpha & \alpha+a^2 & 0 \\ & \alpha+2a^2 & 0 \\ & & a^2 \end{pmatrix}.$$

If we determine the associated q-Ricci form in every case, we see that its eigenvalues determine the orbit of the punctual curvature tensor.

A $R_{3,q}$ - m-space with the matrix $\widetilde{\Omega} = \Omega_{\varepsilon}(a,\alpha)$ will be named a space of <u>type $M^3_{q,\varepsilon}(a,\alpha)$</u>.

Since two m-spaces of the same type are locally equivalent, we obtain the following.

<u>Classification 1.5</u>. Any $\Psi_{3,q}$ m-space ($q = 0,1$) is of one and only one of the following types :

- $M^3_{o,-1}(a,0)$, $a > 0$
- $M^3_{o,1}(a,\alpha)$, $a \geq 0$, $\alpha \in \mathbb{R}$
- $M^3_{1,1}(a,\alpha)$, $\alpha \neq 4a^2$, $\alpha \in \mathbb{R}$, $a \geq 0$
- $M^3_{1,o}(a,\alpha)$, $a \geq 0$, $\alpha \in \mathbb{R} \setminus \{-a^2\}$
- $M^3_{1,2}(a,0)$, $a > 0$
- $M^3_{1,-1}(a,\ )$, $\alpha \in \mathbb{R}$, $a > 0$.

2. The list of germs of $\Upsilon.R_{4,1}$ m-spaces

We showed in [Pă] how one may regain the list of $\Upsilon.R_{n,q}$ m-spaces as soon as one knons the conjugacy classes of subalgebras of $\mathcal{O}_q(n)$. We gave, using that method a list o metrics of $\Upsilon.R_{4,1}$ m-spaces. As this list had some omissions, in order to give a complete one, we have resumed more times the calculations. To be more precise, the only point when one can loose some cases is that one of imposing the integrability conditions (the correct settling of the system $S_{\mathfrak{h}}$ in the terminology of Pă). In order to determine the basis of left invariant 1-forms on K from the structure equations, we applied the next elementary facts :

Lemma 2.1. If the local l.i. 1-forms γ^{α}, $\alpha = \overline{1,3}$ verify the equations $d\gamma^1 = -\varepsilon\gamma^2\wedge\gamma^3$, $d\gamma^2 = \gamma^3\wedge\gamma^1$, $d\gamma^3 = \gamma^1\wedge\gamma^2$, then, up to a change of variables, we have :

$$\begin{pmatrix}\gamma^1\\ \gamma^2\\ \gamma^3\end{pmatrix} = \begin{cases}\begin{pmatrix}\mathrm{sh}x\,dy + \mathrm{ch}x\,\mathrm{ch}y\,dz\\ \mathrm{ch}x\,dy + \mathrm{sh}x\,\mathrm{ch}y\,dz\\ dx - \mathrm{sh}y\,dz\end{pmatrix}, & \text{if } \varepsilon = -1\\[2ex] \begin{pmatrix}\cos x\,dy + \sin x\cos y\,dz\\ -\sin x\,dy + \cos x\cos y\,dz\\ -dx + \sin y\,dz\end{pmatrix}, & \text{if } \varepsilon = 1\end{cases}$$

Lemma 2.2. If $A = (a^i_j)\in\mathcal{M}_n(\mathbb{R})$, and the local 1-forms $(\omega^1,\dots,\omega^n)$ verify the system :

$$d\omega^i = a^i_j\,dx\wedge\omega^j\ ,\quad i = \overline{1,n}\ ,$$

for each eigenvector v of ${}^tA^{\mathbb{C}}$ corresponding to the eigenvalue λ, the $\mathbb{C}$-valuated 1-form $e^{-\lambda x}\sum_{i=1}^{n} v^i\omega^i$ is exact.

We remind the reader the list of subalgebras of $\mathcal{O}_1(4)$.

Proposition 2.3. Any nonnul subalgebra of $\mathfrak{o}_1(4)$ is conjugated to one and only one of the following :

$\mathfrak{m}(3)$: $\xi^1_3 - \xi^1_4 = \xi^2_3 - \xi^2_4 = 0$

$\mathfrak{a}$: $\xi^1_3 - \xi^1_4 = \xi^2_3 - \xi^2_4 = \xi^3_4 = 0$

$\mathfrak{o}(3)$: $\xi^1_4 = \xi^2_4 = \xi^3_4 = 0$

$\mathfrak{o}_1(3)$: $\xi^1_2 = \xi^1_3 = \xi^1_4 = 0$

$\mathfrak{a}_k$, $k>0$: $\xi^1_3 - \xi^1_4 = \xi^2_3 - \xi^2_4 = \xi^1_2 + k\xi^3_4 = 0$

$\mathfrak{o}(2)\times\mathfrak{o}_1(2)$: $\xi^1_3 = \xi^1_4 = \xi^2_3 = \xi^2_4 = 0$

$\mathfrak{m}(2)$: $\xi^1_2 = \xi^1_3 = \xi^1_4 = \xi^2_3 - \xi^2_4 = 0$

$\mathfrak{b}$: $\xi^1_2 = \xi^3_4 = \xi^1_3 + \xi^1_4 = \xi^2_3 + \xi^2_4 = 0$

$Sp(X_o)$: $\xi^1_3 = \xi^2_3 = \xi^2_4 - \xi^1_2 = \xi^1_4 = \xi^3_4 = 0$

$\mathfrak{o}(2)$: $\xi^1_3 = \xi^1_4 = \xi^2_3 = \xi^2_4 = \xi^3_4 = 0$

$\mathfrak{o}_1(2)$: $\xi^1_2 = \xi^1_3 = \xi^1_4 = \xi^2_3 = \xi^2_4 = 0$

$\mathfrak{c}_m$, $m>0$: $\xi^1_3 = \xi^1_4 = \xi^2_3 = \xi^2_4 = \xi^3_4 - m\xi^1_2 = 0$

From now on, using the method of É. Cartan presented in [Pă], we shall locally determine the $\Psi.R_{4,1}$ m-spaces. In each nontrivial case, we start from the equations of the isotropy subalbegra $\mathfrak{h}$ of $\mathfrak{k}$, as a subalgebra of $\mathfrak{o}_1(4)$, and we find the structure equations of K, the punctual curvature matrix $\tilde{\Omega}$, we integrate the structure equations and we find the K-invariant metric g of M = K/H.

$\mathfrak{h} = \mathfrak{m}(3)$, M is locally Minkowski

$\mathfrak{h} = \mathfrak{o}(3)$. We distinguish the cases :

a) M has constant positive curvature

b) M is locally the Lorentz product of one Riemann manifold of one Riemann manifold of constant curvature with the Euclidean line.

$\mathfrak{h} = \mathfrak{O}_1(3)$ a) M is locally the product of the Euclidean line with a Lorentz 3-space of constant curvature.

$\mathfrak{h} = \mathfrak{a}_b$ M is locally Minkowski.

$\mathfrak{h} = \mathfrak{a}$ Then $(\omega^1, \omega^2, \omega^3, \omega^4, \omega^1_2, \omega^1_3, \omega^2_3)$ is a basis of left invariant 1-forms on K. We shall note by $\sim$ the equivalence relation modulo the subspace generated by $(\omega^i \wedge \omega^j)_{1 \leq i < j \leq 4}$. From T.2.1 [Pă] and the equation of $\mathfrak{a}$, we obtain :

$$(\mathfrak{a}.1)\begin{cases} \omega^i_4 = \omega^i_3 + \theta^i, \ i = 1,2 \\ \omega^3_4 = \theta^3, \text{ where } \theta^i = a^i_j \omega^j, \ i = \overline{1,3} \end{cases}$$

If we replace $(\mathfrak{a}.1)$ into the structure equations of K, obtained pulling-back the structure equations of $O_g(M)$, we find :

$$(\mathfrak{a}.2)\quad \begin{aligned} & d\omega^1 \sim -\omega^1_2\omega^2 - \omega^1_3(\omega^3+\omega^4), && d\omega^1_2 \sim -\omega^1_3\theta^2 + \omega^2_3\theta^1 \\ & d\omega^2 \sim \omega^1_2\omega^1 - \omega^2_3(\omega^3+\omega^4), && d\omega^1_3 \sim -\omega^1_2\omega^2_3 - \omega^1_3\theta^3 \\ & d\omega^3 \sim \omega^1_3\omega^1 + \omega^2_3\omega^2, && d\omega^1_4 \sim -\omega^1_2\omega^2_3 - \omega^1_2\theta^2 - \omega^1_3\theta^3 \\ & d\omega^4 \sim -\omega^1_3\omega^1 - \omega^2_3\omega^2, && d\omega^2_3 \sim \omega^1_2\omega^1_3 - \omega^2_3\theta^3 \\ & d\omega^2_4 \sim \omega^1_2\omega^1_3 + \omega^1_2\theta^1 - \omega^2_3\theta^3, && d\omega^3_4 \sim \omega^1_3\theta^1 + \omega^2_3\theta^2 \end{aligned}$$

If we differentiate $(\mathfrak{a}.1)$ and we use $(\mathfrak{a}.2)$ we find $\theta^i = 0$, $i = \overline{1,3}$. The equations of structure of K become

$$(\mathfrak{a}.3)\quad \begin{aligned} & d\omega^1 = \omega^2\omega^1_2 + (\omega^3 + \omega^4)\omega^1_3 \\ & d\omega^2 = -\omega^1\omega^1_2 + (\omega^3 + \omega^4)\omega^2_3 \\ & d(\omega^3 + \omega^4) = 0 \\ & d\omega^4 = \omega^1\omega^1_3 + \omega^2\omega^2_3 \end{aligned}$$

$$d\omega_2^1 = \Omega_2^1 \quad d\omega_3^1 = -\omega_2^1\omega_3^2 + \Omega_3^1$$

$$d\omega_3^2 = \omega_2^1\omega_3^1 + \Omega_3^2$$

We impose the integrability conditions to (a.3), and find:

(a.4)
$$\Omega_2^1 = 0,\ \Omega_3^1 = b\omega^1(\omega^3 + \omega^4) = \Omega_4^1$$
$$\Omega_4^3 = 0,\ \Omega_3^2 = b\omega^2(\omega^3 + \omega^4) = \Omega_4^2$$

The local solution of (a.4) is :

(a.5)
$$\begin{cases} \omega^1 + i\omega^2 = e^{ix}\,(du - \bar{v}dy) \\ \omega_3^1 + i\omega_3^2 = e^{ix}\,(dv + budy) \\ \omega^3 + \omega^4 = dy \quad \omega_2^1 = dx \\ 2\omega^4 = dt + (bu\bar{u} - v\bar{v})dy - (\bar{v}du + vd\bar{u}), \end{cases}$$
where $u = u^1 + iu^2$, $v = v^1 + iv^2$. The Pfaff system $\omega^i = 0$, $i = \overline{1,4}$ has the prime integrals $x^1 + ix^2 = u$, $y = x^3$, $t = x^4$, and we find:

(1) $$g = (dx^1)^2 + (dx^2)^2 + (1 - b((x^1)^2 + (x^2)^2))(dx^3)^2 - dx^3dx^4.$$

$\underline{\mathfrak{h} = O(2) \times O_1(2)}$ M is locally the product of two surfaces of constant curvature.

$\underline{\mathfrak{h} = \mathfrak{b}}$. We have

(b.1)
$$\omega_2^1 = \widetilde{\theta^1} = \sum_{i=1}^{4} a_i\omega^i\ ,\ \omega_4^3 = \widetilde{\theta^2} = \sum_{i=1}^{4} b_i\omega^i$$
$$\omega_3^i + \omega_4^i = \widetilde{\theta^{2+i}} = \sum_{j=1}^{4} a_j^i\omega^j,\quad i = 1,2$$

From the structure equations written modulo the subspace generated by $\{\omega^i.\omega^j\}\ i<j\leq 4$, it follows that :

$$\tilde{\theta}^1 = a\omega^1 \pm b\omega^2 + c\omega'^4, \ \tilde{\theta}^3 = b\omega'^4$$

$$\tilde{\theta}^2 = -b\omega^1 + a\omega^2 + d\omega'^4, \ \tilde{\theta}^4 = -a\omega'^4, \text{ where } \omega'^4 = \omega^3 - \omega^4.$$

If we impose the integrability conditions, we obtain the following cases.

b.1. $\tilde{\theta}^1 = c\omega'^4$, $\tilde{\theta}^2 = d\omega'^4$, $\tilde{\theta}^3 = \tilde{\theta}^4 = 0$. The punctual curvature tensor is given by :

$$\text{(b.2)} \qquad \Omega^1_4 = -\Omega^1_3 = (\alpha\omega^1 + \beta\omega^2)\omega'^4, \Omega^1_2 = 0$$

$$\Omega^2_4 = -\Omega^2_3 = (\beta\omega^1 + \gamma\omega^2)\omega'^4, \Omega^3_4 = 0$$

The structure equations of K become ($\omega^5 = \omega^1_4, \omega^6 = \omega^2_4$)

$$\text{(b.3)} \quad \begin{aligned} d\omega^1 &= (c\omega^2 + \omega^5)\omega'^4 \qquad d\omega^3 = \omega^1\omega^5 + \omega^2\omega^6 + d\omega^3\omega'^4 \\ d\omega^2 &= (-c\omega^2 + \omega^6)\omega'^4 \qquad d\omega'^4 = 0 \\ d\omega^5 &= (\alpha\omega^1 + \beta\omega^2 + c\omega^6 + d\omega^5)\omega'^4 \\ d\omega^6 &= (\beta\omega^1 + \gamma\omega^2 + d\omega^6 - c\omega^5)\omega'^4 \end{aligned}$$

Proposition 2.4. The family of m-spaces with transitive groups K given by (b.3) coincide with the family of spaces (33.54) in the list of Petrov ([Pe]).

Proof. Let $\xi_1, \dots, \xi_6$ be the left invariant vector fields on K dual to $\omega^1, \omega^2, \omega^3, \omega'^4, \omega^5, \omega^6$. If we change the notation as follows: $d = b$, $c = c$, $\alpha = -d$, $\beta = -e$, $\gamma = f$, $\xi_1 = -X_2$, $\xi_2 = X_1$, $\xi_3 = X_1$, $\xi_4 = -X_6$, $\xi_5 = X_4$, $\xi_6 = X_3$ we obtain the structure equations (33.53) in [Pe].

In [Pe], the author did not succed to integrate the structure equations of K. If we use T.1.1 and (b.2), we see that c and d are unessential parameters, so that we may take $c = d = 0$.

We may write $\omega'^4 = dx$, then the $\mathbb{R}^4$-valuated 1 - form $\omega = \omega^1 e_1 + \omega^2 e_2 + \omega^5 e_3 + \omega^6 e_4$, verifies the system $d\omega = dx.A\omega$, the characteristic equation of A being $(\lambda^2 - \alpha)(\lambda^2 - \gamma) - \beta^2 = 0$.

If $\alpha - \gamma = \beta = 0$, we see, from T.1.1 that (M,g) is locally equivalent to (M',g'), with g' given by (1).

We may suppose that $(\alpha - \gamma)^2 + \beta^2 \neq 0$, and we integrate (b.3) with the help of L.2.1., L.2.2, depending on the signs of the roots μ_1, μ_2 of the equation $(\mu - \alpha)(\mu - \gamma) - \beta^2 = 0$.

If $\beta \neq 0, \mu_1, \mu_2 > 0$, we consider the forms

(b.4) $$\theta^i = \beta\omega^1 + (\mu_i - \alpha)\omega^2, \quad \theta^{i+2} = \beta\omega^5 + (\mu_i - \alpha)\omega^6, \quad i = 1,2$$

From L.2.2. it follows that

(b.5) $$\sqrt{\mu_i}\,\theta^i + \theta^{i+2} = 2\,e^{-\sqrt{\mu_i}x}\,du^{2i-1}$$
$$-\sqrt{\mu_i}\,\theta^i + \theta^{i+2} = 2\,e^{\sqrt{\mu_i}x}\,du^{2i}, \quad i = 1,2$$

From (b.3) and (b.5),

$$\beta^2(\mu_1 - \mu_2)^2\,d\omega^3 = ((\mu_2 - \alpha)^2 + \beta^2)\,\theta^1\theta^3 + ((\mu_1 - \alpha)^2 + \beta^2)\,\theta^2\theta^4 =$$
$$= 2(\mu_1^{-\frac{1}{2}}((\mu_2 - \alpha)^2 + \beta^2)\,du^1\,du^2 + \mu_2^{-\frac{1}{2}}((\mu_1 - \alpha)^2 + \beta^2)\,du^3\,du^4)$$
$$\beta^2(\mu_1 - \mu_2)^2\omega^3 = du^0 - 2\mu_1^{-\frac{1}{2}}((\mu_2 - \alpha)^2\ \ \beta^2)\,u^2\,du^1 -$$
$$- 2\mu_2^{-\frac{1}{2}}((\mu_1 - \alpha)^2 + \beta^2)\,u^4\,du^3$$

The system $\omega^i = 0$, $i = \overline{1,4}$, equivalent to $\theta^1 = \theta^2 = = \omega^3 = \omega'^4 = 0$ has the prime integrals x^i, $i = \overline{1,4}$ given by :

(b.6) $$x^4 = x,\quad u^1 = e^{2\sqrt{\mu_1}x}\,u^2 + \mu_1^{\frac{1}{2}}\,x^1,\quad u^3 = e^{2\sqrt{\mu_1}x}\,u^4 + \mu_2^{\frac{1}{2}}\,x^2$$
$$x^3 = u^0 - \mu_1^{-\frac{1}{2}}((\mu_2 - \alpha)^2 + \beta^2)e^{2\sqrt{\mu_1}x}\,(u^2)^2 -$$

$$-\mu_2^{-\frac{1}{2}}\,((\mu_1-\alpha)^2+\beta^2)\; e^{2\sqrt{\mu_2}x}\,(u^1)^2$$

We find

(2) $$\beta^2(\mu_1-\mu_2)^2 g = ((\mu_2-\alpha)^2+\beta^2)\; e^{-2\sqrt{\mu_1}x^4}(dx^1)^2 +$$

$$+((\mu_1-\alpha)^2+\beta^2)\; e^{-2\sqrt{\mu_2}x^4}(dx^2)^2 + 2dx^3dx^4 - \beta^2(\mu_1-\mu_2)^2(dx^4)^2$$

If $\underline{\beta \neq 0,\ \mu_1 < 0 < \mu_2}$, from L.2.2. we have

$$i\,\sqrt{-\mu_1}\,\theta^1 + \theta^3 = e^{-i\sqrt{-\mu_1}x}\,(du^1 - idu^2)$$

(b.7) $$-i\,\sqrt{-\mu_1}\,\theta^1 + \theta^3 = e^{i\sqrt{-\mu_1}x}\,(du^1 + idu^2)$$

$$\mu_2\theta^2 + \theta^4 = 2\,e^{-\sqrt{\mu_2}x}\,du^3$$

$$-\mu_2\theta^2 + \theta^4 = 2\,e^{\sqrt{\mu_2}x}\,du^4$$

From (b.3) it follows

$$\beta^2(\mu_1-\mu_2)^2\,d\omega^3 = ((\mu_2-\alpha)^2+\beta^2)(-\mu_1)^{-\frac{1}{2}}\,du^1du^2 +$$

$$+\,2((\mu_1-\alpha)^2+\beta^2)\mu_2^{-\frac{1}{2}}\,du^3du^4;\ \text{then :}$$

$$\omega'^4 = dx,\ \sqrt{\mu_2}\,\theta^2 = e^{-\sqrt{\mu_2}x}\,du^3 - e^{\sqrt{\mu_2}x}\,du^4$$

(b.8) $$\sqrt{-\mu_1}\,\theta^1 = -\sin(x\sqrt{-\mu_1})\,du^1 - \cos(x\sqrt{-\mu_1})\,du^2$$

$$\beta^2(\mu_1-\mu_2)^2\omega^3 = du^0 - ((\mu_2-\alpha)^2+\beta^2)(-\mu_1)^{\frac{1}{2}}\,u^2du^1 -$$

$$-\,2\mu_2^{-\frac{1}{2}}\,((\mu_1-\alpha)^2+\beta^2)\,u^4du^3.$$

The Pfaff system $\theta^1 = \theta^2 = \omega^3 = \omega'^4 = 0$ has the prime integrals x^i, $i = \overline{1,4}$, given by :

$$x^4 = x\ , -u^1 = \mathrm{ctg}(\sqrt{-\mu_1}x)\, u^2 + x^1\sqrt{-\mu_1}\ \mathrm{cosec}\,(\sqrt{-\mu_1}x)$$

(b.9) $$u^3 = e^{2\sqrt{\mu_2}x} u^4 + \mu_2^{\frac{1}{2}}\ x^2.$$

$$2u^0 + ((\mu_2-\alpha)^2+\beta^2)(-\mu_1)^{-\frac{1}{2}}\ \mathrm{ctg}(\sqrt{-\mu_1}x)(u^2)^2 -$$

$$-2\mu_2^{-\frac{1}{2}}((\mu_1-\alpha)^2+\beta^2)\ e^{2\sqrt{\mu_2}x}(u^4)^2 - 2x^3 = 0\ .$$

We find out :

(3) $$\beta^2(\mu_1-\mu_2)^2 g = ((\mu_2-\alpha)^2+\beta^2)(dx^1 - x^1\sqrt{-\mu_1}\ \mathrm{ctg}(x^4\sqrt{-\mu_1})dx^4)^2 +$$

$$+ ((\mu_1-\alpha)^2+\beta^2)\ e^{-\sqrt{\mu_2}x^4}(dx^2) + 2dx^3dx^4 - \beta^2(\mu_1-\mu_2)^2(dx^4)^2$$

If $\underline{\beta \neq 0 > \mu_1, \mu_2}$, from L.2.2, it follows :

(b.10) $$i\sqrt{-\mu_\alpha}\,\theta^\alpha + \theta^{\alpha+2} = e^{-i\sqrt{-\mu_\alpha}x}(du^{2\alpha-1} - i du^{2\alpha})\quad \alpha = 1,2$$

where θ^α, $\alpha = 1,2$ are given by (b.4). From (b.3)

$$\beta^2(\mu_1-\mu_2)^2\ d\omega^3 = ((\mu_2-\alpha)^2+\beta^2)\,\theta^1\theta^3 + ((\mu_1-\alpha)^2+\beta^2)\,\theta^2\theta^4 =$$

$$= ((\mu_2-\alpha)^2+\beta^2)(-\mu_1)^{-\frac{1}{2}}\ du^1du^2 + ((\mu_1-\alpha)^2+\beta^2)(-\mu_2)^{-\frac{1}{2}}du^3du^4,$$

$$\beta^2(\mu_1-\mu_2)^2\omega^3 = du^0 - ((\mu_2-\alpha)^2+\beta^2)(-\mu_1)^{-\frac{1}{2}}\ u^2\ du^1 -$$

$$-((\mu_1-\alpha)^2+\beta^2)\ (-\mu_2)^{-\frac{1}{2}}\ u^4\ du^3.$$

The Pfaff system $\theta^1 = \theta^2 = \omega'^4 = \omega^3 = 0$ has the prime integrals x^i, $i = \overline{1,4}$ given by :

$$x^4 = x,$$

(b.11) $$-u^1 = \mathrm{ctg}\,(x\sqrt{-\mu_1})\ u^2 + x^1\sqrt{-\mu_1}\ \ \mathrm{cosec}\ (x\sqrt{-\mu_1})$$

$$(b.11)\quad \begin{aligned} -u^2 &= \operatorname{ctg}(x\sqrt{-\mu_2}) + x^2\sqrt{-\mu_2}\operatorname{cosec}(x\sqrt{-\mu_2}) \\ 2u^0 &= -((\mu_2-\alpha)^2+\beta^2)(-\mu_1)^{-\frac{1}{2}}\operatorname{ctg}(\sqrt{-\mu_1}x)(u^2)^2 - \\ &\quad -((\mu_1-\alpha)^2+\beta^2)(-\mu_2)^{-\frac{1}{2}}\operatorname{ctg}(\sqrt{-\mu_2}x)(u^4)^2 + 2x^3 \end{aligned}$$

We obtain

$$(4):\ \beta^2(\mu_1-\mu_2)^2 g = ((\mu_1-\alpha)^2+\beta^2)(dx^2+x^2\sqrt{-\mu_2}\operatorname{ctg}(x^4\sqrt{-\mu_2})dx^4)^2 + \\ + ((\mu_2-\alpha)^2+\beta^2)(dx^1+x^1\sqrt{-\mu_1}\operatorname{ctg}(x^4\sqrt{-\mu_1})dx^4)^2 + 2dx^3dx^4 - \\ - \beta^2(\mu_1-\mu_2)^2(dx^4)^2$$

If $\underline{\beta \neq 0 = \mu_2 < \mu_1 = \mu}$, let θ^α, $\alpha = \overline{1,4}$ be given by (b.4). From L.2.2. it follows that :

$$(b.12)\quad \begin{aligned} \sqrt{\mu}\theta^1 + \theta^3 &= 2e^{-\sqrt{\mu}x}du^1 \\ -\sqrt{\mu}\theta^1 + \theta^3 &= 2e^{\sqrt{\mu}x}du^2 \\ \theta^4 &= du^3 \end{aligned}$$

Because $\mu = \alpha + \gamma$, from (b.12) we have

$$\mu\beta\omega^5 = (e^{-\sqrt{\mu}x}du^1 + e^{\sqrt{\mu}x}du^2) + \gamma du^3$$

$$\omega^6 = e^{-\sqrt{\mu}x}du^1 + e^{\sqrt{\mu}x}du^2 - du^3$$

and from (b.3) :

$$(b.13)\quad \begin{aligned} \mu\omega^2 &= \psi^2 + \mu^{-\frac{1}{2}}\psi^1 &&, \psi^1 = e^{-\sqrt{\mu}x}du^1 - e^{\sqrt{\mu}x}du^2 \\ \mu\beta\omega^1 &= \alpha\mu^{-\frac{1}{2}}\psi^1 - \gamma\psi^2 &&, \psi^2 = du^4 - xdu^3; \\ \mu\beta^2\omega^3 &= du^0 + 2\alpha\mu^{-\frac{1}{2}}u^1du^2 + \gamma u^3du^4 = \psi^3 \end{aligned}$$

The Pfaff system $\omega^i = 0$, $i = \overline{1,4}$ has the prime integrals x^i, $i = \overline{1,4}$ given by :

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$$(b.14)\quad \begin{aligned} & x^4 = x,\ x^1 = u^2 - e^{-2\sqrt{\mu}x}u^1,\ \ x^2 = u^4 + xu^3 \\ & 2x^3 = 2u^0 + 2\alpha\mu^{-\frac{1}{2}}e^{-2\sqrt{\mu}x}(u^1)^2 - \gamma x(u^3)^2 \end{aligned}$$

(b.13) and (b.14) give :

$$(5)\quad \alpha\gamma(\alpha+\gamma)^2 g = e^{2\sqrt{\alpha+\gamma}\,x^4}(dx^1)^2 + (dx^2)^2 +$$

$$+ 2(\alpha+\gamma)\,dx^3\,dx^4 - (\alpha+\gamma)^2\alpha\gamma(dx^4)^2 .$$

If $\underline{\beta \neq 0 = \mu_2 > \mu_1} = -f = -(a+c), \alpha+a = \gamma+c = 0,$

from L.2.2. it follows :

$$(b.15)\quad \begin{aligned} & i\sqrt{f}\,(b\omega^1 + c\omega^2) + b\omega^5 + c\omega^6 = e^{-i\sqrt{f}x}(du^1 - idu^2) \\ & b\omega^5 - a\omega^6 = du^3 \end{aligned}$$

If we solve (b.15) and introduce ω^5, ω^6 into (b.3) we obtain :

$$(b.16)\quad f\omega^2 = du^4 + xdu^3 - f^{-\frac{1}{2}}(\sin(x\sqrt{f})du^1 + \cos(x\sqrt{f})du^2),\ \text{then}$$

$$(b.17)\quad \begin{cases} bf\omega^1 = -af^{-\frac{1}{2}}\theta^1 - c\,\theta^2 & \theta^1 = \sin(x\sqrt{f})du^1 + \cos(x\sqrt{f})du^2 \\ f\omega^2 = \theta^2 - f^{-\frac{1}{2}}\theta^1 , & \theta^2 = du^4 + xdu^3 \end{cases}$$

We finnaly find :

$$b^2 f\omega^3 = af^{-\frac{1}{2}}u^1du^2 + cu^3du^4 + du^0 = \theta^3$$

The Pfaff system $\omega^i = 0$, $i = \overline{1,4}$, equivalent to $\omega'^4 = \theta^i = 0$, $i = \overline{1,3}$ has the prime integrals :

$$(\ .18)\quad \begin{aligned} & x^4 = x,\ x^2 = u^4 + xu^3,\ \ x^1 = u^1\sin(x\sqrt{f}) + u^2\cos(x\sqrt{f}) \\ & 2x^3 = 2u^0 - af^{-\frac{1}{2}}\,tg(x\sqrt{f})(u^1)^2 - cx(u^3)^2.\ \text{We find :} \end{aligned}$$

$$(6)\quad (\alpha+\gamma)^2\alpha\gamma g = -(dx^1 + \sqrt{-\alpha-\gamma}\,x^1\,tg(x^4\sqrt{-\alpha-\gamma})dx^4)^2 + (\alpha+\gamma)(dx^2)^2 -$$

$$- 2(\alpha+\gamma)dx^3\,dx^4 + (\alpha+\gamma)^2\alpha\gamma(dx^4)^2.$$

If $\underline{\beta = 0, \mu_1 = \alpha \quad \mu_2 = \gamma > 0}$, from L.2.2. we obtain

(b.19) $$\sqrt{\alpha}\,\omega^1 + \omega^5 = 2\, e^{-\sqrt{\alpha}\,x}\, du^1 \qquad \sqrt{\gamma}\,\omega^2 + \omega^6 = 2\, e^{-\sqrt{\gamma}\,x}\, du^3$$
$$-\sqrt{\alpha}\,\omega^1 + \omega^5 = 2\, e^{\sqrt{\alpha}\,x}\, du^2 \qquad -\sqrt{\gamma}\,\omega^2 + \omega^6 = 2\, e^{\sqrt{\gamma}\,x}\, du^4,$$

and after that we find :

$$\omega^3 = -2\,\alpha^{-\frac{1}{2}}\, u^2\, du^1 - 2\,\gamma^{-\frac{1}{2}}\, u^4\, du^3 + du^o$$

The prime integrals of $\omega^i = 0$, $i = \overline{1,4}$ are x^i, $i = \overline{1,4}$, given by :

(b.20) $$x^4 = x,\ u^1 = e^{2\sqrt{\alpha}\,x}\, u^2 + x^1\sqrt{\alpha},\ u^3 = e^{2\sqrt{\gamma}\,x}\, u^4 + x^2\sqrt{\gamma}$$
$$x^3 = u^o - \alpha^{-\frac{1}{2}}\, e^{2\sqrt{\alpha}\,x}(u^2)^2 - \gamma^{-\frac{1}{2}}\, e^{2\sqrt{\gamma}\,x}\,(u^4)^2$$

We find :

(7) $$g = e^{-2\sqrt{\alpha}\,x^4}(dx^1)^2 + e^{-2\sqrt{\gamma}\,x^4}\,(dx^2)^2 + 2dx^3dx^4 - (dx^4)^2$$

If $\underline{\beta = 0, \mu_1, \mu_2 < 0}$, $\alpha + a = \gamma + c = 0$, from L.2.2.

(b.21) $$i\,\sqrt{a}\,\omega^1 + \omega^5 = e^{-i\sqrt{a}\,x}\,(du^1 - i du^2)$$
$$i\,\sqrt{c}\,\omega^2 + \omega^6 = e^{-i\sqrt{c}\,x}\,(du^3 - i du^4)$$
$$\omega^3 = -a^{-\frac{1}{2}}\, u^2 du^1 - c^{-\frac{1}{2}}\, u^4 du^3 + du^o.$$

The prime integrals x^i, $i = \overline{1,4}$ of the system $\omega^i = 0$, $i = \overline{1,4}$ are given by :

(b.22) $$x^4 = x,\quad -u^1 = \mathrm{ctg}(x\sqrt{a})u^2 + x^1\sqrt{a}\,\mathrm{cosec}(x\sqrt{a})$$
$$-u^3 = \mathrm{ctg}(x\sqrt{c})u^4 + x^2\sqrt{c}\,\mathrm{cosec}(x\sqrt{c})$$
$$2x^3 = 2u^o + a^{-\frac{1}{2}}\,\mathrm{ctg}(x\, a)(u^2)^2 + c^{-\frac{1}{2}}\,\mathrm{ctg}(x\, c)(u^4)^2$$

Then :

(8) $$g = (dx^1 - x^1\sqrt{a}\,\mathrm{ctg}(x^4\sqrt{a})dx^4)^2 + (dx^2 - x^2\sqrt{c}\,\mathrm{ctg}(x^4\sqrt{c})dx^4)^2 + 2dx^3\, dx^4 - (dx^4)^2$$

If $\underline{\beta = 0, \mu_1 = \alpha > 0 > \mu_2} = \gamma = -c$, from L.2.2. it follows :

(b.23)
$$\sqrt{\alpha}\,\omega^1 + \omega^5 = 2\, e^{-\sqrt{\alpha}x}\, du^1$$
$$-\sqrt{\alpha}\,\omega^1 + \omega^5 = 2\, e^{\sqrt{\alpha}x}\, du^2$$
$$i\sqrt{c}\,\omega^2 + \omega^6 = e^{-i\sqrt{c}\,x}\,(du^3 - i du^4)$$

$$\omega^3 = du^0 - 2\alpha^{-\frac{1}{2}} u^2 du^1 - c^{-\frac{1}{2}} u^4 du^3 ,$$

The prime integrals x^i, $i = \overline{1,4}$ of $\omega^i = 0$, $i = \overline{1,4}$ are given by

(b.24)
$$x^4 = x, \quad u^1 = e^{2\sqrt{\alpha}x} u^2 + x^1\sqrt{\alpha}$$
$$-u^1 = ctg(x\sqrt{c}) + x^2\sqrt{c}\ cosec\ (x\sqrt{c})$$
$$2x^3 = 2u^0 - 2\alpha^{-\frac{1}{2}} e^{2\sqrt{\alpha}x}(u^2)^2 + c^{-\frac{1}{2}} (u^4)^2\ ctg\ (x\sqrt{c})$$

Then :

(9)
$$g = e^{-2\sqrt{\alpha}\,x^4}(dx^1)^2 + (dx^2 - x^2\sqrt{c}\ ctg(x^4\sqrt{c})\ dx^4)^2 +$$
$$+ 2dx^3\, dx^4 - (dx^4)^2$$

If $\underline{\beta = 0,\ \gamma = \mu_2 = 0 < \mu_1 = \alpha}$, from L.2.2. if follows

(b.25)
$$\sqrt{\alpha}\,\omega^1 + \omega^5 = 2\sqrt{\alpha}\, e^{-\sqrt{\alpha}\,x}\, du^1$$
$$-\sqrt{\alpha}\,\omega^1 + \omega^5 = 2\sqrt{\alpha}\, e^{\sqrt{\alpha}x}\, du^2, \quad \omega^6 = du^3$$

Returning to (b.3), we obtain

(b.26)
$$\omega^2 = -x\, du^3 + du^4$$
$$\omega^3 = du^0 - 2\sqrt{\alpha}\, u^2\, du^1 - u^3\, du^4$$

The prime integrals of the system $\omega^i = 0$, $i = \overline{1,4}$ are

(b.27)
$$x^4 = x, \quad x^1 = u^1 - e^{2\sqrt{\alpha}\,x}\, u^2, \ x^2 = u^4 - xu^3$$
$$x^3 = u^0 - \tfrac{1}{2}\, x^4\, (u^3)^2 - \sqrt{\alpha}\, e^{2\sqrt{\alpha}x}\, (u^2)^2.$$

Then :

(10)
$$g = e^{-2\sqrt{\alpha}\,x^4}\,(dx^1)^2 + (dx^2)^2 + 2dx^3\, dx^4 - (dx^4)^2$$

If $\beta = 0 = \gamma = \mu_1 > \mu_2 = \alpha = -a$, from L.2.2. it follows

$$
(b.28) \quad \begin{aligned} i\sqrt{a}\,\omega^1 + \omega^5 &= e^{-i\sqrt{a}\,x}\,(du^1 - i du^2) \\ \omega^6 &= du^3 \end{aligned}
$$

It results $\omega^2 = -x du^3 + du^4$

$$
\omega^3 = a^{\frac{1}{2}}\, u^1\, du^2 - u^3\, du^4 + du^0
$$

The prime integrals of the system $\omega^i = 0$, $i = \overline{1,4}$ are

$$
(b.29) \quad \begin{aligned} & x^4 = x, \quad x^1 = u^1 \sin(x\sqrt{a}) + u^2 \cos(x\sqrt{a}) \\ & x^2 = u^4 - x u^3 \\ & 2x^3 = 2u^0 - (u^3)^2 x - a^{-\frac{1}{2}}\, tg(x\sqrt{a})(u^1)^2. \end{aligned}
$$

Then

$$
(11) \quad g = (x^1\, tg(x^4\sqrt{-\alpha})dx^4 + (-\alpha)^{-\frac{1}{2}}\, dx^1)^2 + (dx^2)^2 + \\ + 2dx^3\, dx^4 - (dx^4)^2
$$

b.2. $(a,b) \neq 0$. We note $\theta^i = \tilde{\theta}^i$ $i = 1,2$, $a^2 + b^2 = r$, $\theta^3 = r\omega^3$, $\theta^4 = \omega'^4$, $\theta^5 = a\omega^5 + b\omega^6$, $\theta^6 = -b\omega^5 + a\omega^6$.

The system of the structure equations becomes

$$
(b.30) \quad \begin{aligned} d\theta^1 &= -\theta^1\theta^2 + (c\theta^2 + \theta^5)\theta^4 \\ d\theta^2 &= -(c\theta^1 + \theta^3 - \theta^6)\theta^4 \\ d\theta^3 &= \theta^1\theta^5 + \theta^2\theta^6 + 2r\,\theta^2\theta^4 - \theta^2\theta^3 + d\,\theta^3\theta^4 \\ d\theta^4 &= 2\theta^2\theta^4 \\ d\theta^5 &= \Omega - (\theta^1 + c\theta^4)\ {}^6 - (\theta^2 + d\theta^4)\theta^5, \quad \Omega = a\Omega^1_4 + b\Omega^2_4 \\ d\theta^6 &= \Lambda + (\theta^1 + c\theta^4)\theta^5 - (\theta^2 + d\theta^4)\ {}^6 - r\theta^2\theta^4, \\ \Lambda &= -b\Omega^1_4 + a\Omega^2_4 \end{aligned}
$$

If we impose the integrability conditions, we find that $c = d = 0$

$$(\Omega - \theta^1\theta^3)\theta^4 = (\Lambda - \theta^2\theta^3)\theta^4 = \theta^1\Omega + \theta^2\Lambda = 0$$

$$\Lambda_{24} = 3r + \Omega_{14}, \; \Omega_{24} = 0, \quad \text{where} \tag{b.31}$$

$$\Omega = \sum_{i<j\leq 4} \Omega_{ij}\,\theta^i\theta^j, \quad \Lambda = \sum_{i<j\leq 4} \Lambda_{ij}\,\theta^i\theta^j$$

From (b.31) it results that $\Omega = \theta^1\theta^3 + k\,\theta^1\theta^4$

$\Lambda = \theta^2\theta^3 + (3r + 4k)\,\theta^2\theta^4$, where $k = \Omega_{14} = \lambda r$. Then :

$$\widetilde{\Omega} = \begin{pmatrix} -r & 0 & 0 & 0 & 0 & 0 \\ & -r-(\lambda+1)(r+3b^2) & (\lambda+1)(r+3b^2) & 0 & -3ab(\lambda+1) & 0 \\ & & r-(\lambda+1)(r+3b^2) & -3ab(\lambda+1) & 3ab(\lambda+1) & 0 \\ & & & -r-(\lambda+1)(r+3a^2) & (\lambda+1)(r+3a^2) & 0 \\ & & & & r-(\lambda+1)(r+3a^2) & 0 \\ & & & & & r \end{pmatrix} \tag{b.32}$$

In order to integrate (b.30) we distinguish the cases $k \neq 0$, $k = 0$.

If $\underline{k \neq 0}$, we note $\beta^1 = \theta^6 - \theta^3$, $\beta^2 = 2\theta^2$, $\beta^3 = 2k\,\theta^4 + \theta^3 - \theta^6$, then $d\beta^1 = \beta^2\beta^3$, $d\beta^2 = \frac{1}{k}\beta^1\beta^3$, $d\beta^3 = -\beta^1\beta^2$.

If $\underline{k > 0}$, $\beta^1 = \sqrt{k}\,\gamma^1$, $\beta^2 = \gamma^3$, $\beta^3 = \sqrt{k}\,\gamma^2$, $\Omega^1 = \sqrt{k}\,\theta^1$, $\Omega^2 = -\theta^5$, the system (b.30) becomes :

$$\begin{aligned} d\gamma^1 &= \gamma^3\gamma^2 \\ d\gamma^2 &= \gamma^3\gamma^1 \\ d\gamma^3 &= \gamma^1\gamma^2 \\ d\Omega^1 &= \tfrac{1}{2}\gamma^3\Omega^1 + \tfrac{1}{2}(\gamma^1 + \gamma^2)\Omega^2 \\ d\Omega^2 &= \tfrac{1}{2}(\gamma^2 - \gamma^1)\Omega^1 - \tfrac{1}{2}\gamma^3\Omega^2 \\ d\theta^3 &= -\frac{1}{\sqrt{k}}\Omega^1\Omega^2 + \frac{\sqrt{k}}{2}\,d(\gamma^2) + \frac{r}{2\sqrt{k}}\,d(\gamma^1 + \gamma^2) \end{aligned} \tag{b.33}$$

Then γ^i, $i = \overline{1,3}$ are given in L.2.1 ($\varepsilon = -1$). After the successive transformations $\Omega^1 = e^{\frac{x}{2}} A^1$, $\Omega^2 = e^{-\frac{x}{2}} A^2$, $A^1 + A^2 = e^{\frac{y}{2}} C^1$, $A^1 - A^2 = e^{-\frac{y}{2}} C^2$, $C^1 + iC^2 = e^{-\frac{iz}{2}} D$, one finds $dD = 0$; then $D = d(u + iv)$.

The last equation of (.33) becomes

$$d\theta^3 = d\left(\frac{1}{2\sqrt{k}} v\, du + \frac{\sqrt{k}}{2}\gamma^2 + \frac{r}{2\,k}(\gamma^1 + \gamma^2)\right).$$ One finds

$$(b.34)\quad \begin{cases} \theta^1 = \dfrac{e^{\frac{x}{2}}}{2\sqrt{k}}\Big(e^{\frac{y}{2}} (\cos \frac{z}{2} du + \sin \frac{z}{2} dv) - \\ \qquad\qquad - e^{-\frac{y}{2}} (\sin \frac{z}{2} du - \cos \frac{z}{2} dv)\Big) \\ \theta^2 = \frac{1}{2} (dx - \mathrm{sh}y dz) \\ \theta^4 = \dfrac{1}{2\sqrt{k}} e^x (dy + \mathrm{ch}y dz) \\ 2\sqrt{k}\,\theta^3 = dt - udv + k(\mathrm{ch}x dy - \mathrm{sh}x dy) \end{cases}$$

The system $\omega^i = 0, i = \overline{1,4}$ equivalent to $\theta^i = 0$, $i = \overline{1,4}$, has the prime integrals :

$$(b.35)\quad \begin{aligned} x^1 &= z + 2 \,\mathrm{arctg}\, (e^y) \\ x_2 &= x + \ln(\mathrm{ch}y) \\ x^3 &= v - u \,\mathrm{tg}\, \frac{x^1}{2} \end{aligned}$$

$x^4 = t - \frac{u^2}{2} \mathrm{tg} \frac{x^1}{2} - ke^{-x_2}\, \mathrm{sh}y$. If we renote $x_2 = x^2$, then :

$$(12)\quad g = \frac{e^{x^2}}{2rk} \left(\cos^2 \frac{x^1}{2} (dx^3)^2 + dx^1 dx^4\right) +$$

$$\frac{1}{4r} \left((dx^2)^2 - (dx^1)^2 + e^{2x^2}(dx^1)^2\right) + \frac{e^{2x^2}}{4k} (dx^1)^2 = g_{r,k}$$

One sees that g depends only on r and k.

If $\underline{k < 0}$, we note $\beta^1 = \sqrt{-k}\,\gamma^1$, $\beta^3 = \sqrt{-k}\,\gamma^1$, $\beta^2 = \gamma^2$, $\sqrt{-k}\,\theta^1 = \Omega^1$, $\theta^5 = \Omega^2$. Then γ^i, Ω^j, $i = \overline{1,3}$, $j = 1,2$ verify the first 5 equations of (b.33). If we choose the same solutions as before, we find :

$$d\theta^3 = -\frac{1}{2\sqrt{|k|}}(k\gamma^1 + r(\gamma^1+\gamma^2) - udv), \text{ then } \theta^1, \theta^2, \theta^4 \text{ are}$$

given in (b.34) and

$$-2\sqrt{-k}\,\theta^3 = dt - udv + k\gamma^1 + r(\gamma^1+\gamma^2).$$

The Pfaff system $\theta^i = 0$, $i = \overline{1,4}$ has the prime integrals (b.35); we finnally find $g = g_{r,-k}$

If $\underline{k = 0}$, we consider the forms

(b.36) $\gamma^1 = \theta^3 - \theta^4 - \theta^6$, $\gamma^2 = \theta^6 - \theta^3 - \theta^4$, $\gamma^3 = 2\theta^2$,

$\Omega^1 = \theta^1$, $\Omega^2 = \theta^5$; which verify the first 5 equations of (b.33). Then

$d\theta^3 = \frac{1}{4}d(\gamma^1 - \gamma^2) - \frac{r}{2}d(\gamma^1+\gamma^2) + \frac{1}{2}du\,dv$; and we find :

$$\text{(b.37)}\quad \begin{cases} \theta^1 = \frac{1}{2}e^{\frac{x}{2}}\left(e^{\frac{y}{2}}\left(\cos\frac{z}{2}du + \sin\frac{z}{2}dv\right) + e\right. \\ \qquad\qquad \left. + e^{-\frac{y}{2}}\left(\sin\frac{z}{2}du - \cos\frac{z}{2}dv\right)\right) \\ \theta^2 = \frac{1}{2}(dx - \mathrm{sh}y\,dz) \\ \theta^3 = \frac{1}{2}\left(udv + dt + \frac{1}{2}e^{-x}(\mathrm{ch}y\,dz - dy)\right) - re^x(dy + \mathrm{ch}y\,dz) \\ \theta^4 = -\frac{e^x}{2}(dy + \mathrm{ch}y\,dz) \end{cases}$$

The Pfaff system $\theta^i = 0$, $i = 1,2,4$ has the prime integrals x^i, $i = \overline{1,3}$ given in (b.35). If $\theta^3 = 0$, we find the prime integral $x^4 = t + \frac{u^2}{2}\,\mathrm{tg}\,\frac{x^1}{2} - e^{-x_2}\,\mathrm{sh}y$.

Renoting $x_2 = x^2$, we find :

$$(13)\quad 4rg = (dx^2)^2 - 2e^{x^2}dx^1dx^4 + 2e^{x^2}\cos^2\frac{x^1}{2}(dx^3)^2 - (1 + re^{2x^2})(dx^1)^2$$

Proposition 2.5. There are [illegible] $R_{4,1}$ 6-spaces [illegible] [illegible], which are not on the l st of A.Z. Petrov.

Proof. In the classification of $R_{4,1}$ - 6-spaces [illegible], given by A.Z. Petrov, beside the spaces with the groups ($\mathfrak{h}$.3) there appear, spaces with constant curvature, products of spaces of constant curvature and two spaces admitting the Killing vector fields $(X_i)_{i=\overline{1,6}}$ given by (33.44), (33.45) in [Pe]. One may show that the transformation group generated by $(X_i)_{i=\overline{1,6}}$ is not transitive on M, in both cases (33.44) and (33.45).

The Ricci form of the punctual curvature tensor of a space from the family ($\mathfrak{h}$.2) is $\varrho = -2\beta u^2u^4 + (\alpha+\gamma)(u^3-u^4)^2$, that is, ϱ is degenerated, while the Ricci form of one space from the family ($\mathfrak{h}$.32) is

$$\varrho = -r(3\lambda(u^1)^2 + (\lambda+4)(u^2)^2 + (5\lambda+8)(u^3)^2 + 6(\lambda+1)\, u^3u^4 + $$
$$+ (4\lambda+1)(u^4)^2 \quad ,$$
form which is gennerally non singular.

$\mathfrak{h} = \mathfrak{m}(2)$. Starting rrom the equations of $\mathfrak{m}(2)$, we distinguish the cases :

$\mathfrak{m}(2).1$ $\omega^1_2 = \omega^1_3 = \omega^1_4 = 0,\ \omega^2_3 - \omega^2_4 = b(\omega^3 + \omega^4)$,

$$(\mathfrak{m}(2).1)\quad \Omega^1_j = 0,\ j = \overline{2,4},\ \Omega^k_r = -{}_1\delta_{kr}\, b^2\omega^k\omega^r,\ 2 \leq k < r \leq 4.$$

M is the local product of the Euclidean line with a Lorentz space of constant negative curvature

$\mathfrak{m}(2).2$ $\omega^1_j = p\,\omega^j,\quad j = \overline{2,4},\ \omega^2_3 = \omega^2_4,$ M has constant nonpositive curvature.

$\mathfrak{h} = Sp(X_o)$

$(X_o.1)\quad \omega^i_j - \delta^i_1\delta^2_j\omega^1_2 = \theta^i_j = a^i_j\omega^1 + b^i_j\omega^2 + c^i_j\omega^3 + d^i_j\omega^4,$

$$\forall (i,j),\ i < j,\ j \neq 2$$

Using the structure equations one finds :

$$d\omega^1 \sim -\omega^1_2\omega^2,\ d\omega^2 \sim \omega^1_2(\omega^1 - \omega^4),\ d\omega^3 \sim 0,\ d\omega^4 \sim -\omega^1_2\omega^2$$

$(X_o.2)$ $d(A\omega^1 + B\omega^2 + C\omega^3 + D\omega^4) \sim ((A+D)\omega^2 - B\omega'^1)\omega^1_2$, where $\omega'^1 = \omega^1 - \omega^4$.

We have also :

$d\theta^1_3 \sim \theta^2_3\omega^1_2,\ d\theta^1_4 \sim \theta^2_4\omega^1_2,\ d\theta^3_4 \sim \theta^2_3\omega^1_2,\ d\theta^2_3 \sim (\theta^3_4 - \theta^1_3)\omega^1_2$, $d\theta^2_4 \sim 0$; from $(X_o.1)$, $(X_o.2)$, we obtain the system

$$(S)\begin{cases} \theta^2_3 = (a^1_3 + d^1_3)\omega^2 - b^1_3\omega'^1 = (a^3_4 + d^3_4)\omega^2 - b^3_4\omega'^1,\ 0 = (a^2_4 + d^2_4)\omega^2 - b^2_4\omega'^1 \\ \theta^2_4 = (a^1_4 + d^1_4)\omega^2 - b^1_4\omega'^1,\ \theta^3_4 - \theta^1_3 = (a^2_3 + d^2_3)\omega^2 - b^2_3\omega'^1 \end{cases}$$

Solving (S) we find :

$$(X_o.3)\quad \begin{aligned} \theta^1_3 &= (p-c)\omega'^1 + b\omega^2 + h\omega^3 + p\omega^4 \\ \theta^3_4 &= -c\omega'^1 + b\omega^2 + h\omega^3 + p\omega^4 \\ \theta^2_3 &= -b\omega'^1 + p\omega^2 \\ \theta^1_4 &= m\omega'^1 + f\omega^2 + q\omega^3 \\ \theta^2_4 &= -f\omega'^1. \end{aligned}$$

The structure equations of K are :

$$(X_o 4)\quad \begin{aligned} d\omega'^1 &= (2f\omega^2 + (p+q)\omega^3)\omega'^1 \\ d\omega^3 &= -h\omega'^1\omega^3 \\ d\omega^2 &= \omega'^1(-\omega^1_2 + b\omega^3 + f\omega^4) - p\omega^2\omega^4 \end{aligned}$$

$$d\omega^4 = \omega^2\omega_2^1 + \omega'^1(2f\omega^2 + (q+c)\omega^3 - m\omega^4) +$$
$$+\omega^3(b\omega^2 + (p-q)\omega^4) - f\omega^2\omega^4$$
$$d\omega_2^1 = \omega_2^1(m\omega'^1 + f\omega^2 + q\omega^3) + \Omega_2^1 - p\omega^2(h\omega^3 + p\omega^4) +$$
$$+\omega'^1((p^2-pc+b^2-f^2)\omega^2 + (bd-fq)\omega^3 + bp\omega^4)$$

The conditions $d^2\omega'^1 = d^2\omega^3 = 0$, imply

$$(X_0.5)\qquad pf = hf = 0$$

1. <u>Case f = 0.</u> $d^2\omega^2 = 0 \Leftrightarrow \omega'^1(\Omega_2^1 - p^2\omega^2\omega^4) = 0$,

$$d^2\omega^4 = \omega^2(-\Omega_2^1 + b(m-2h)\omega'^1\omega^3) + (m(p+q)-h(p-q))\omega'^1\omega^3\omega^4 =$$
$= 0$, then:

$$(X_0.6)\begin{cases} \Omega_2^1 = \alpha\omega'^1\omega^2 + b(m-2h)\omega'^1\omega^3 + p^2\omega^2\omega^4 \\ (p+q)m = h(p-q) \end{cases}$$

We find the next form for $\tilde{\Omega}$:

$$\tilde{\Omega} = \begin{pmatrix} \alpha & b(m-2h) & 0 & 0 & \alpha+p^2 & b(m-2h) \\ bm & 2c(p+q)+b^2+h(m-h)-p^2 & 0 & -b(2p+q) & bm & 2c(p+q)+b^2+h(m-h) \\ 0 & 0 & p^2 & 0 & 0 & 0 \\ 0 & bp & 0 & -p^2 & 0 & bp \\ p^2+\alpha & b(m-2h) & 0 & 0 & \alpha+2p^2 & b(m-2h) \\ bm & 2c(p+q)+b^2+h(m-h) & 0 & -b(2p+q) & bm & 2c(p+q)+b^2+h(m-h)+p^2 \end{pmatrix}$$
$$(X_0.8)$$

As $\tilde{\Omega}$ is symmetric, we find $bh = b(3p+q) = 0$; then we analyse the resulting cases $b = 0$, and $b \neq 0$.

<u>X .1.1.</u> $f = b = 0$. $d^2\omega_2^1 = 0 \Longleftrightarrow$

$\Leftrightarrow$ $(X_o.9)$ $\quad 2(p+q)(p^2-pc+\alpha) + ph\,(h-m) = 0.$

If we use the terminology in [Pă], in the case $(X_o.1.1.)$ the system S_h is equivalent with $(X_o.7) + (X.9)$. In order to integrate $(X_o.4)$ we find all the solutions of S_h.

$X_o.1.1.1.$ $\quad p,\ h,\ m+h,\ h-m \neq 0,\ m = \lambda(p-q),\ h = \lambda(p+q),$

$$\alpha = pc - p^2 - \lambda pq$$

Replacing these values in $(X_o.8)$ we find that the punctual curvature tensor depends only of $p, q, c-\lambda q$. We solve $(X_o.4)$ and we find

$$(X_o.10)\qquad \begin{aligned} \omega'^1 &= e^{(p+q)x}\,dy \\ \omega^2 &= e^{px}\,(zdy + du) \\ \omega^3 &= \lambda e^{(p+q)x}\,dy + dx \\ \omega^4 &= e^{(p-q)x}\,(dt - zdu - \frac{z^2}{2}dy - \frac{q+c}{2q}\,e^{2qx}\,dy \\ \omega^1_2 &= e^{-qx}\,dz + \lambda p\omega^2 \end{aligned}$$

$x^1 = x,\ x^2 = y,\ x^3 = u,\ x^4 = t$ are prime integrals for $\omega^i = 0,\ i = \overline{1,4}$

$$(14)\quad g = -\frac{c}{q}\,e^{2(p+q)x^1}\,(dx^2)^2 + (dx^1 + \lambda e^{(p+q)x^1}\,dx^2)^2 + $$
$$+\, e^{2px^1}\,(dx^3)^2 + 2\,e^{2px^1}\,dx^2\,dx^4$$

$X_o.1.1.2.$ $\quad p \neq 0 \neq h = m \qquad q = p^2 - pc + \alpha = 0.$

Replacing the determined parameters in $(X_o.8)$, one finds that the punctual curvature tensor depends only on p and c. We may take $h = 1$. The solution of $(X_o.4)$ is

$$\omega'^1 = e^x dy, \ \omega^2 = e^x zdy + du, \ p\omega^3 = dx + e^x dy$$

$$(X_0.11) \quad \omega^4 = e^x \left(-\frac{z^2}{2} dy - zdu - \frac{c}{p} xdy + dt\right)$$

$$\omega^1_2 = dz + e^x zdy + du$$

The prime integrals of the system $\omega^i = 0$, $i = \overline{1,4}$ are

$x^1 = x$, $x^2 = y$, $x^3 = u$, $x^4 = t$

$$(15) \quad g = e^{2x^1}(dx^2)^2 + 2\, e^{2x^1} dx^2(dx^4 - \frac{c}{p} x^1 dx^2) + (dx^3)^2 + $$
$$+ \frac{1}{p^2} (dx^1 + e^{x^1} dx^2)^2$$

<u>$X_0.1.1.3.$</u> $m = 0$, $h \neq 0 \neq p = q$, $h^2 + 4(p^2 - pc + \alpha) = 0$

If we replace the given parameters in $(X_0.8)$, we find that Ω depends only on p, $4pc-h^2$. We may take $h = 2$. The solution of $(X_0.4)$ is

$$(X_0.12) \quad \begin{cases} \omega'^1 = e^{2x} dy \\ \omega^2 = e^x(zdy + du) \\ \omega^3 = p^{-1}(dx + dye^{2x}) \\ \omega^4 = dt - \frac{z^2}{2} dy - zdu - \frac{p+c}{2p} e^{2x} dy \\ \omega^1_2 = \omega^2 + e^{-x} dz \end{cases}$$

$x^1 = x$, $x^2 = y$, $x^3 = u$, $x^4 = t$ are prime integrals of $\omega^i = 0$, $i = \overline{1,4}$.

$$(16) \quad g = 2\, e^{2x^1} dx^2 dx^4 + e^{2x^1}(dx^3)^2 - \frac{c}{p} e^{4x^1}(dx^2)^2 + $$
$$+ \frac{1}{p^2} (dx^1 + e^{2x^1} dx^2)^2$$

<u>$X_0.1.1.4.$</u> $q = -p \neq 0 = h \neq m$

As Ω does not depend on m and c, we may take $m = 1$,

$c = 0$. $(X_o.4)$ is equivalent to :

$$\omega'^1 = dx \qquad d\omega^2 = -dx\,\omega^1_2 + p\,dy\,\omega^2$$

$$(X_o.13) \qquad \omega^3 = dy \qquad d\omega^1_2 = (p\,dy - dx)\omega^1_2 + (\alpha - p^2)dx\,\omega^2$$

$$d\omega^4 = -\omega^1_2\omega^2 + (2p\omega^3 - \omega'^1)\omega^4 - p\omega'^1\omega^3$$

If $\omega^1_2 = e^{py}\theta^1$, $\omega^2 = e^{py}\theta^2$, then :

$$(X_o.14) \qquad \begin{aligned} d\theta^1 &= dx(-\theta^1 + (\alpha - p^2)\theta^2) \\ d\theta^2 &= -dx\,\theta^1 \end{aligned}$$

In order to apply L.2.2., we note $\Delta = 1 - 4(\alpha - p^2)$.

<u>$\Delta > 0$.</u> $\lambda_1 \neq \lambda_2$ are the roots of $\lambda^2 + \lambda + \alpha - p^2 = 0$. Using L.2.2. we find :

$$\theta^1 - (1 + \lambda_i)\theta^2 = e^{\lambda_i x}dz^i, \; i = 1,2$$

$$(X_o.15) \qquad \omega^4 = e^{2py-x}$$

$$\theta = \frac{1}{1^- 2}z^1dz^2 - \frac{1}{2}e^{x-2py}dx + dt$$

If we consider the prime integrals of the system $\omega^i = 0$, $i = \overline{1,4}$

$$(X_o.16) \qquad \begin{aligned} &x^1 = x, \; x^2 = y, \; x^3 = e^{\lambda_2 x}z^2 - e^{\lambda_1 x}z^1 \\ &x^4 = t + 2(\lambda_2 - \lambda_1)^{-1}z^1e^{(\lambda_1 - \lambda_2)x}, \end{aligned}$$ we find :

$$(17) \qquad g = 2\,e^{2px^2 - x^1}dx^1dx^4 + \Delta^{-1}e^{2px^2}(dx^3 - \lambda_2x^3dx^1)^2 + (dx^2)^2$$

If <u>$\Delta < 0$.</u> From L.2.2. it follows that :

$$\theta^1 - (\tfrac{1}{2} + i\rho)\theta^2 = e^{(-\frac{1}{2} + i\rho)x}(du^1 - i\,du^2)$$

We take $\omega^4 = e^{2py-x}\theta$, and $d\theta = -\frac{1}{\rho}du^1du^2 - pe^{x-2py}dxdy$, then $\theta = dt - \frac{u^1}{\rho}du^2 - \frac{1}{2}e^{x-2py}dx$.

The system $\omega^i = 0$, $i = \overline{1,4}$ has the prime integrels

$$(X_o.17)\quad \begin{aligned} &x^1 = x,\ x^2 = y,\ x^4 - 2t - \rho^{-1}(u^1)^2\, tg(\rho x) \\ &x^3 = \rho^{-1}(u^2\cos(\rho x) - u^1 \sin(\rho x)). \text{ Then} \end{aligned}$$

$$(18)\quad g = (dx^2)^2 + e^{2px^2 - x^1}((dx^3 + \frac{\sqrt{-\Delta}}{2}\, x^3 tg(\frac{\sqrt{|\Delta|}}{2}x^1)dx^1)^2 + 2dx^1dx^4)$$

If $\underline{\Delta = 0}$. Using L.2.2., we find :

$$\theta^1 - r\,\theta^2 = e^{-\frac{x}{2}}\, dz$$

$$(X_o.18)\quad \theta^2 = e^{-\frac{x}{2}}(-xdz + dt)$$

$$\omega^4 = e^{2py - ax}(du - zdt - \tfrac{1}{2}\, e^{x - 2py}\, dx)$$

The system $\omega^i = 0$ is equivalent with $dx = dy = dt - xdz = du - zdt = 0$, and has the prime integrals

$$x^1 = x,\ x^2 = y,\ x^3 = t - xz,\ x^4 = u - \tfrac{1}{2}\, xz^2$$

$$(19)\quad g = (dx^2)^2 + e^{2px^2 - x^1}((dx^3)^2 + 2dx^1\, dx^4)$$

$\underline{X_o.1.1.5.}$ $q = -p \neq 0 = m = h$, $\beta = pc - p^2 - \alpha$. As $\tilde{\Omega}$ does not depend on c, we may take $c = 0$. The structure equations have the following local solution :

$$\omega'^1 = dx,\ \omega^3 = dy,\ \omega^2 = e^{py}\,\theta^2,\ \omega^1_2 = e^{py}\,\theta^1 \quad \text{where}$$

$$d\,\theta^2 + dx\,\theta^1 = d\,\theta^1 - dx\,\theta^2 = 0.$$

If $\underline{\beta = 0}$, $\omega^4 = e^{2py}\theta$, then; $\theta^1 = dz$, $\theta^2 = zdx + du$

$$\theta = -zdu - px\, e^{2py}\, dy + dt$$

The system $\omega^i = 0$, $i = \overline{1,4}$, equivalent to $dx = dy = \theta^2 = \theta = 0$ has $x^1 = x$, $x^2 = y$, $x^3 = u$, $x^4 = t$ as prime integrals.

$$(20)\quad g = (dx^1)^2 + (dx^2)^2 + e^{2px^2}((dx^3)^2 + 2dx^1(dx^4 - px^1dx^2))$$

If $\underline{\beta > 0}$, we have successively :

$$i\theta^1 + \sqrt{\beta}\,\theta^2 = e^{i\sqrt{\beta}\,x}(dz^1 - i dz^2),\ \omega^4 = e^{2py}\theta,$$

$$d\theta = -\beta^{-\frac{1}{2}} dz^1 dz^2 + p\, e^{-2py} dy dx,$$

$$\theta = -\beta^{-\frac{1}{2}} z^1 dz^2 - p e^{-2py} x dy + dt$$

The prime integrals of $\omega^i = 0$, $i = \overline{1,4}$ are

$$(X_o.19)\quad x^1 = x,\ x^2 = y,\ x^3 = z^1 \cos(x\sqrt{\beta}) + z^2 \sin(x\sqrt{\beta})$$

$$x^4 = t - \beta^{-\frac{1}{2}}\left(x^3 \cos^{-1}(x\sqrt{\beta}) - \tfrac{1}{2}(z^2)^2\, tg(x\sqrt{\beta})\right)$$

$$(21)\quad g = (dx^1)^2 + (dx^2)^2 + e^{2px^2}\left(\beta^{-\frac{1}{2}} dx^3 + x^3 tg(x^1\sqrt{\beta})\, dx^1\right)^2 +$$

$$+ 2e^{2px^2} dx^1 dx^4 - 2px^1 dx^1 dx^2$$

If $\underline{\beta < 0}$, we successively find :

$$\theta^1 - \sqrt{|\beta|}\,\theta^2 = e^{\sqrt{|\beta|}\,x}(dy^1 - dy^2)$$

$$\theta^1 + \sqrt{|\beta|}\,\theta^2 = e^{-\sqrt{|\beta|}\,x}(dy^1 + dy^2)$$

$$\omega = e^{2py}\theta,\ \text{where}\ \theta = |\beta|^{-\frac{1}{2}} y^2 dy^1 - p e^{-2py} x dy + dt$$

The system $\omega^i = 0$, $i = \overline{1,4}$ equivalent to $\omega'^1 = \omega^3 = \theta^2 = \theta = 0$ has as prime integrals :

$$(X_o.20)\quad x^1 = x,\ x^2 = y,\ x^3 = -y^1 sh(x^1\sqrt{|\beta|}) + y^2 ch(x^1\sqrt{|\beta|})$$

$$x^4 = t + |\beta|^{-\frac{1}{2}} ch^{-1}(x^1\sqrt{|\beta|})\, x^3 y^1 + \tfrac{1}{2}(y^1)^2\, th(x^1\sqrt{|\beta|})$$

$$(22)\quad g = (dx^1)^2 + (dx^2)^2 + e^{2px^2}\left(|\beta|^{-\frac{1}{2}} dx^3 - x^3 th(\sqrt{|\beta|}\,x^1) dx^1\right)^2 +$$

$$+ 2\, e^{2px^2} dx^1 dx^4 - 2px^1 dx^1 dx^2$$

$\underline{X_o.1.1.6.}$ $-q \neq p \neq m = h = 0 = \alpha + p^2 - pc$

$$\omega^3 = dx,\ \omega'^1 = e^{(p+q)x}\, dy, \omega^1_2 = e^{-qx}\, dz$$

$(X_o.21)$ $\omega^2 = e^{px}(zdy + du$

$$\omega^4 = e^{(p-q)x}\,(dt - zdu + (c+q)e^{2qx}\, ydx - \frac{z^2}{2}\, dy)$$

$x^1 = x,\ x^2 = y,\ x^3 = u,\ x^4 = t$ are prime integrals of $\omega^i = 0,\ i = \overline{1,4}$.

(23) $g = (dx^1)^2 + e^{2(p+q)x^1}((dx^2)^2 + 2(q + c)\, x^2dx^1dx^2) +$

$$+ e^{2px^1}\,((dx^3)^2 + 2\, dx^2\, dx^4)$$

<u>$X_o.1.1.7.$</u> $q \neq \alpha = p = m + h = 0.$

$\widetilde{\Omega}$ depends only on $cq - h^2$ we may suppose $h = 0,\ q=1$

$(X_o.22)$ $\omega^3 = dx,\ \omega'^1 = e^x dy, \omega^1_2 = e^{-x}\, dz,\ \omega^2 = zdy + du$

$$\omega^4 = e^{-x}(dt - \frac{z^2}{2}\, dy - zdu + (c + 1)\, e^{2x}\, ydx)$$

$x^1 = x,\ x^2 = y,\ x^3 = u,\ x^4 = t$ are prime integrals of $\omega^i = 0,\ i = \overline{1,4}$

(24) $g = (dx^1)^2 + (dx^3)^2 + 2dx^2dx^4 + e^{2x^1}((dx^2)^2 +$

$$+ 2(c+1)\, x^2\, dx^1\, dx^2)$$

<u>$X_o.1.1.8.$</u> $p = q = 0.$ $\widetilde{\Omega}$ depends only on α and $h(m-h)$. We may take $m = h + 1$, $c = 0$.

The structure equations are

$$d\omega'^1 = 0,\quad d\omega^3 = -h\,\omega'^1\omega^3,\ d\omega^2 = -\omega'^1\omega^1_2$$

$(X_o.23)$ $d\omega^1_2 = -m\,\omega'^1\omega^1_2 + \alpha\,\omega^1\omega^2$

$$d\omega^4 = \omega^2\omega^1_2 - m\,\omega'^1\omega^4$$

Then $\omega'^1 = dx,\ \omega^3 = e^{-hx}\, dy$,

$(X_o.24)$ $d\begin{pmatrix}\omega^2\\ \omega^1_2\end{pmatrix} = dx\begin{pmatrix}0 & -1\\ \alpha & -m\end{pmatrix}\begin{pmatrix}\omega^2\\ \omega^1_2\end{pmatrix}$

In order to find the solutions of $(X_o.26)$; we note by Δ the discriminant of $\lambda^2 + m\lambda + \alpha = 0$ (E)

If $\underline{\Delta < 0,}$ λ is a root of (E), then by L.2.2.

$(m + \lambda)\omega^2 - \omega^1_2 = e^{\lambda x}\, d\mathfrak{z}$. We take $\mathfrak{z} = (\lambda - \bar{\lambda})(z^1 + iz^2)$

$\omega^2 = 2\mathrm{Re}(e^{\lambda x}dz)$. If, $\omega^4 = e^{-mx}$, then

$\Lambda = 4\ \mathrm{Im}\ \ z^1dz^2 + dt$.

The system $\omega^i = 0$, $i = \overline{1,4}$ has as prime integrals :

$$(X_o.25) \qquad x^1 = x,\ x^2 = y,\ x^3 = \cos(x\ \mathrm{Im}\,\lambda)z^1 - \sin(x\ \mathrm{Im}\lambda)\ z^2$$

$$x^4 = t + 4\mathrm{Im}\,\lambda \sec(x\ \mathrm{Im}\lambda)x^3z^2 + 2\ \mathrm{Im}\lambda\ \mathrm{tg}(x\ \mathrm{Im}\lambda)(z^2)^2$$

$$(25) \qquad g = (dx^1)^2 + e^{-2hx^1}(dx^1)^2 + 4e^{-(h+1)x^1}(dx^3 + (\mathrm{Im}\,\lambda)x^3$$
$$\mathrm{tg}(x^1\mathrm{Im}\,\lambda)dx^1)^2 + 2\ e^{-(h+1)x^1}\ dx^1\ dx^4$$

$\underline{\text{If} \qquad 0}$ and $\ _1,\ _2$ are the roots of (E), we have

$(m + \lambda_i)\omega^2 - \omega^1_2 = e^{\lambda_i x}du^i$, $i = 1,2$

We take $u^i = (\lambda_2 - \lambda_1)t^i$, and $\omega^4 = e^{-mx}\theta$, then

$$(X_o.26) \qquad \begin{aligned} \omega^2 &= e^{\lambda_2 x}\ dt^2 - e^{\lambda_1 x}\ dt^1 \\ \theta &= (\lambda_2 - \lambda_1)\ t^1\ dt^2 + dt \end{aligned}$$

The Pfaff system $\omega^i = 0$, $i = \overline{1,4}$. has the prime integrals

$$(X_o.27) \qquad \begin{aligned} & x^1 = x,\ x^2 = y,\ x^3 = e^{(\lambda_2 - \lambda_1)x}\ t^2 - t^1\ , \\ & x^4 = t + (\lambda_2 - \lambda_1)(e^{(\lambda_2 - \lambda_1)x}\ \frac{(t^2)^2}{2} - x^3t^2). \end{aligned}$$

$$(26) \qquad g = (dx^1)^2 + e^{-2hx^1}(dx^2)^2 + e^{2\lambda_1 x^1}(dx^3)^2 +$$
$$+ 2\ e^{-(h+1)x^1}\ dx^1\ dx^4$$

If $\underline{\Delta = 0}$, we note $m = 2A$, and using L.2.2. we find :

$$A\omega^2 - \omega^1_2 = e^{-Ax}\,dz$$

$$(X_0.28) \qquad \omega^2 = e^{-Ax}(zdx + du)$$

$$\omega^4 = e^{mx}\left(-\frac{z^2}{2}dx - zdu + dt\right)$$

$\omega^i = 0$, $i = \overline{1,4}$ has $x^1 = x$, $x^2 = y$, $x^3 = y$, $x^4 = t$ as prime integrals.

$$(27) \quad g = (dx^1)^2 + e^{-(h+1)x}((dx^3)^2 + 2dx^1\,dx^4) + e^{-2hx^1}(dx^2)^2$$

$\underline{X_0 1.2}$ $f = 0 \neq b$. The symmetry of $\widetilde{\Omega}$ implies $h = q + 3p = 0$. The condition $d^2\omega^1_2 = 0$ modulo the vector space generated by $\{\omega^i\omega^j\omega^k\}_{1\leq i<j<k\leq 4}$ is $bp\omega^2\omega^1_2 = 0$. Then $p = 0$ and $d^2\omega^1_2 \equiv 0$.

As $\widetilde{\Omega}$ does not depend on c, we may take $c = 0$. The structure equations become :

$$d\omega'^1 = d\omega^3 = 0 \quad d\omega^2 = \omega'^1(b\omega^3 - \omega^1_2)$$

$$(X_0.29) \qquad d\omega^4 = -m\,\omega'^1\omega^4 + (b\omega^3 - \omega^1_2)\omega^2$$

$$d\omega^1_2 = \omega'^1(-m\,\omega^1_2 + bm\,\omega^3 + (\alpha + b^2)\omega^2)$$

Then $\omega'^1 = dx$, $\omega^3 = dy$, and if $\omega = b\omega^3 - \omega^1_2$,

$$d\begin{pmatrix}\omega^2\\ \omega\end{pmatrix} = dx\begin{pmatrix}0 & 1\\ -(\alpha+b^2) & -m\end{pmatrix}\begin{pmatrix}\omega^2\\ \omega\end{pmatrix}$$

We note Δ the discriminant of

(E') : $\lambda^2 + m\lambda + \alpha + b^2$

If $\underline{\Delta > 0}$ $(\lambda_i + m)\omega^2 + \omega = e^{\lambda_i x}\,du^i$, $i = 1,2$, where λ_i are roots of (E'). We find :

$$\omega^4 = e^{-mx}(dt + (\lambda_2 - \lambda_1)^{-1}\,u^1\,du^2)$$

The system $\omega^i = 0$, $i = \overline{1,4}$, has the prime integrals :

$$(X_0.30)\quad \begin{aligned} & x^1 = x,\ x^2 = y,\ x^3 = e^{\lambda_1 x} u^1 - e^{\lambda_2 x} u^2 \\ & x^4 = t - \tfrac{1}{2}(\lambda_1 - \lambda_2)^{-1} (u^1)^2 e^{(\lambda_1 - \lambda_2)x} \end{aligned}$$

$$(28)\quad g = (dx^1)^2 + (dx^2)^2 + 2e^{-mx^1} dx^1 dx^4 + \Delta^{-1}(dx^3 - \lambda_2 x^3 dx^1)^2$$

If $\Delta < 0$. We note $m = 2r$, $\sqrt{-\Delta} = 2s$, then from L.2.2. we find $(r+is)\omega^2 + \omega = e^{-rx}(\cos(sx) + i\sin(sx))(du^1 - idu^2)$

$\omega^4 = e^{-mx}(dt - s^{-1} u^1 du^2)$.

The system $\omega^i = 0$ has the following prime integrals

$$(X_0.31)\quad \begin{aligned} & x^1 = x,\ x^2 = y \\ & sx^3 = \sin(sx)u^1 - \cos(sx)u^2 \\ & 2x^4 = 2t - s^{-1}\ \ tg(sx)(u^1)^2 \end{aligned}$$

$$(29)\quad g = (dx^1)^2 + (dx^2)^2 + e^{-mx^1}\left(2dx^1dx^4 + \left(dx^3 + \frac{\sqrt{|\Delta|}}{2} x^3\ tg\left(\frac{x^1\sqrt{|\Delta|}}{2}\right) dx^1\right)^2\right)$$

If $\Delta = 0$ $\quad \omega = e^{-mx} du^1, \omega^2 = -u^1 e^{-mx} dx + du^2$,

$$\omega^4 = e^{-mx}(u^1 du^2 + dt - \tfrac{1}{2}(u^1)^2 e^{-mx} dx)$$

If we note $x^1 = x$, $x^2 = y$, $x^3 = u^2$, $x^4 = t$, then

$$(30)\quad g = (dx^1)^2 + (dx^2)^2 + (dx^4)^2 + 2e^{-mx^1} dx^1 dx^4$$

2. Case $f \neq 0$. Then $p = h = 0$

$$d^2\omega^2 = \omega'^1(2bf\omega^3\omega^2 + 2f^2\omega^4\omega^2 + qf\omega^4\omega^3 + \Omega^1_2 + f^2\omega^2\omega^4 - bf\omega^3\omega^2 + qf\omega^3\omega^4)$$

$$d^2\omega^2 = 0 \Leftrightarrow \Omega^1_2 = bf\omega^2\omega^3 + f^2\omega^2\omega^4 + \sum_{i=2}^{4} \Omega'_{1i}\omega'^1\omega^i.$$

$$d^2\omega^4 = \omega'^1((3cf + bm - \overset{\circ}{\Omega}'_{13})\omega^3\omega^2 + (4mf + \Omega_{14})\omega^2\omega^4 + (mq + bf)\omega^3\omega^4) \qquad d^2\omega^4 = 0 \Leftrightarrow$$

$$\Leftrightarrow \Omega'_{13} = bm + 3cf,\ \Omega'_{14} = -4mf,\ mq + bf = 0.$$

$$d\omega^1_2 = \omega^1_2(m\omega'^1 + f\omega^2 + q\omega^3) + \omega'^1((b^2 - f^2 + \Omega'_{12})\omega^2 + (bm + 3cf - fq)\omega^3 - 4mf\omega^4) + f\omega^2(b\omega^3 + f\omega^4).$$

We note $\Omega'_{12} = \alpha$

$$d^2\omega^1_2 = \Big[\omega'^1((b^2-f^2+\alpha)\omega^2 + (bm+3cf-fq)\omega^3 - 4mf\omega^4) + f\omega^2(b\omega^3+f\omega^4)\Big](m\omega'^1+f\omega^2+q\omega^3) - m\omega^1_2(2f\omega^2+q\omega^3)q\omega'^1 - f\omega^1_2\omega'^1(b\omega^3+f\omega^4) + (2f\omega^2+q\omega^3)\omega'^1(b^2-f^2+\alpha)\omega^2 + (bm+3cf-fq)\omega^3 - 4mf\omega^4\Big] + 4mf\omega'^1\Big[\omega^2(\omega^1_2 - f\omega^4) + \omega^3(b\omega^2 - q\omega^4)\Big] + f\omega'^1(b\omega^3+f\omega^4)\omega^1_2 - f^2\omega^2\Big[\omega'^1((q+c)\omega^3 - m\omega^4) - q\omega^3\omega^4\Big] = 0.$$

The last integrability conditions give us $m = q = b = c = 0$. From $(X_o.4)$ there results that M is locally the product of the Euclidian line with a space of type $M^3_{1,o}(f,\alpha)$.

The structure equations of the transitive group on $M^3_{1,o}(f,\alpha)$ are

$$(X_o.32)\begin{cases} d\theta^1 = -2f\theta^1\theta^2 \quad d\theta^2 = \theta^1(-\theta^1_2 + f\theta^3) \\ d\theta^3 = \theta^2(-2f\theta^1 + \theta^1_2 - f\theta^3) \\ d\theta^1_2 = \Big(f(\theta^1_2 - f\theta^3) + (\alpha - f^2)\theta^1\Big)\theta^2 \end{cases}$$

where $\theta^1 = \omega^1 - \omega^3,\ \theta^2 = \omega^2,\ \theta^3 = \omega^3,\ \theta^1_2 = \omega^1_2$. If we take

$\lambda^i = \theta^i,\ i = \overline{1,3},\ \lambda^4 = \theta^1_2 - f\theta^3$, then

$\theta = 4f^2\lambda^3 + 2f\lambda^4 + (\alpha + f^2)\lambda^1$ is an exact 1-form.

We take $\mu^1 = (3f^2-\alpha)\lambda^1 - 2f\lambda^4,\ \mu^4 = 2f\lambda^4;\ \mu^2 = 2f\lambda^2$

$(X_o.32)$ is equivalent to :

$$(X_o.33) \quad \begin{aligned} \theta &= ds \\ d\mu^1 &= \mu^2\mu^4 \\ d\mu^4 &= \mu^2\mu^1 \\ d\mu^2 &= (\alpha - 3f^2)^{-1}\mu^1\mu^4 \end{aligned}$$

If $\underline{\alpha - 3f^2 > 0}$ We note $s = (\alpha-3f^2)^{\frac{1}{2}}t, \mu^1 = (\alpha-3f^2)^{\frac{1}{2}}\gamma^1, \mu^2 = \gamma^3$

$\mu^4 = (\alpha-3f^2)^{\frac{1}{2}}\gamma^2$, then γ^i, $i = \overline{1,3}$ are given by L.2.1. ($\varepsilon = -1$). The Pfaff system $\omega^i = 0$, $i = \overline{1,3}$ is equivalent with $\gamma^1 + \gamma^2 = \gamma^3 = \gamma^1 + dt = 0$, and has the prime integrals :

$$(X_o.34) \quad \begin{aligned} x^1 &= z + 2\,\mathrm{arctg}(e^y) \\ x^2 &= x + \ln(\mathrm{ch}y) \\ x^3 &= t - e^{-x^2}\,\mathrm{sh}y \end{aligned}$$

The tensor metric g' of $M^3_{1,o}(f,\alpha)$ is given by :

$$4f^2g' = 12\,f^2(3f^2-\alpha)^{-1}(\gamma^1+\gamma^2)^2 + (\gamma^3)^2 - 2(\gamma^1+\gamma^2)(dt+\gamma^1)$$

We may take $4f^2g = 4f^2g' + (dx^4)^2$,

$$(31) \quad 4f^2g = ((9f^2+\alpha)(3f^2-\alpha)^{-1}\,e^{2x^2} - 1)(dx^1)^2 + (dx^2)^2 - 2\,e^{x^2}dx^1dx^3 + (dx^4)^2$$

If $\underline{\alpha - 3f^2 < 0}$ We note $s = (3f^2-\alpha)^{\frac{1}{2}}t, \mu^1 = (3f^2-\alpha)^{\frac{1}{2}}\gamma^2$,

$^4 = (3f^2- \ )^{\frac{1}{2}} \ ^1, \ ^2 = \ ^3, \ ^i$, $i = \overline{1,3}$ are given by L.2.1. ($\varepsilon = -1$).

The system $\omega^i = 0$, $i = \overline{1,3}$ is equivalent with : $\gamma^1 + \gamma^2 = \gamma^3 = \gamma^1 - dt = 0$. Then use have the prime integrals :

$$(X_o.35) \quad \left\{ \begin{aligned} x^1 &= z + 2\,\mathrm{arctg}\,(e^y) \\ x^2 &= x + \ln(\mathrm{ch}y) \\ x^3 &= t + e^{-x^2}\,\mathrm{sh}y \end{aligned} \right.$$

The tensor metric g' of $M^3_{1,o}(f,\alpha)$ is given by

$$4\, f^2 g' = 2(f^2-\alpha)(3f^2-\alpha)^{-1}(\gamma^1+\gamma^2)^2 + (\gamma^3)^2 + 2(\gamma^1+\gamma^2)(dt-\gamma^1).$$

We may take $4f^2g = 4f^2g' + (dx^4)^2$

$$(32) \quad 4f^2 g = -(1 + (\alpha+f^2)(3f^2-\alpha)^{-1}\, e^{2x^2})(dx^1)^2 + (dx^2)^2 + $$
$$+ 2\, e^{x^2}\, dx^1\, dx^3 + (dx^4)^2$$

If $\alpha = 3f^2$. $\tilde{\Omega}$ depends only on f^2, then we may suppose that $f < 0$.

We take $\theta = 2f(\lambda^4 + 2f(\lambda^3+\lambda^1))$, $\lambda^1 - \lambda^4 = |f|^{-\frac{1}{2}}\gamma^1$, $\lambda^1 + \lambda^4 = |f|^{-\frac{1}{2}}\gamma^2$, $2f\lambda^2 = \gamma^3$.

Then γ^i, $i = \overline{1,3}$ are given by L.2.1 ($\varepsilon = -1$).

If we note $s = |f|^{-\frac{1}{2}}\, t$, the Pfaff system $\omega^i = 0$, $i=\overline{1,3}$ is equivalent with $\gamma^1 + \gamma^2 = \gamma^3 = dt + \gamma^1 - \gamma^2 = 0$ and we may take the prime integrals :

$$(X_o.36) \quad \begin{aligned} x^1 &= z + 2\,\mathrm{arctg}(e^y) \\ x^2 &= x + \ln(\mathrm{ch}y) \\ x^3 &= t - 2\, e^{-x^2}\, \mathrm{sh}y \end{aligned}$$

The tensor metric g' of $M^3_{1,o}(f,\, 3f^2)$ is given by

$$-2fg' = -\frac{1}{2}(\gamma^1+\gamma^2)^2 - (2f)^{-1}(\gamma^3)^2 - (\gamma^1+\gamma^2)(dt + \gamma^1 - \gamma^2)$$

We may take $4f^2g = 4f^2g' + (dx^4)^2$ and then :

$$(33) \quad 4f^2 g = (f\, e^{2x^2} - 1)(dx^1)^2 + (dx^2)^2 - e^{x^2}\, dx^1 dx^3 + (dx^4)^2$$

$\mathfrak{h} = \mathfrak{C}_m$, $m > 0$. In this case we find

$\omega^1_2 - m\,\omega^3_4 = \omega^1_3 = \omega^1_4 = \omega^2_3 = \omega^2_4 = 0$, $\Omega = 0$, M is flat.

$\mathfrak{h} = \mathfrak{O}(2)$. We have $\omega^i_j = \theta^i_j = a^i_j\omega^1 + b^i_j\omega^2 + c^i_j\omega^3 +$

$+ d^i_j\omega^4$, $\forall (i,j)$, $i<j\leq 4$, $j\neq 2$. If $\theta \doteq A\omega^1 + B\omega^2 + C\omega^3 + D\omega^4$,

From the structure equations of K we find :

$$(\bigcirc(2).1) \qquad d\theta \sim (-B\omega^1 + A\omega^2)\omega^1_2$$

We use $(\bigcirc(2).1)$ taking $\theta = \theta^i_j$, then

$$(\bigcirc(2).2) \qquad \theta^1_3 = a\omega^1 + b\omega^2 \qquad \theta^2_3 = -b\omega^1 + a\omega^2$$

$$\theta^1_4 = \alpha\omega^1 + \beta\omega^2 \qquad \theta^2_4 = -\beta\omega^1 + \alpha\omega^2$$

$$\theta^3_4 = r\omega^3 + s\omega^4$$

If $\Omega^1_2 = \sum_{i<j}\Omega_{ij}\,\omega^i\omega^j$, the structure equations of K

are

$$d\omega^1 = -\omega^1_2\omega^2 - (a\omega^1 + b\omega^2)\omega^3 - (\alpha\omega^1 + \beta\omega^2)\omega^4$$

$$d\omega^2 = \omega^1_2\omega^1 + (b\omega^1 - a\omega^2)\omega^3 + (\beta\omega^1 - \alpha\omega^2)\omega^4$$

$$d\omega^3 = -2b\,\omega^1\omega^2 - r\omega^3\omega^4$$

$$d\omega^4 = 2\beta\omega^1\omega^2 + s\omega^3\omega^4$$

$$d\omega^1_2 = (a^2 + b^2 - \alpha^2 - \beta^2 + \Omega_{12})\omega^1\omega^2 + \sum_{i<j\neq 2}\Omega_{ij}\;{}^{i}\;{}^{j}$$

We impose the integrability conditions, and find the system

$$S \left\{ \begin{array}{l} \alpha s = ar \quad \Omega_{34} = \beta s - br \quad \Omega_{i3} = {}_{i4} = 0, \quad i = 1,2 \\ b(r-2\alpha) = 0, \quad \beta(s-2a) = 0, \quad 2ab - \beta r = 2\alpha\beta - bs = 0 \\ a(a^2+b^2-\alpha^2-\beta^2+\Omega_{12}) - \Omega_{34} = 0 \\ (a^2+b^2-\alpha^2-\beta^2+\Omega_{12}) - \Omega_{34}\,b = 0 \end{array} \right.$$

The solutions of the system S are the following :

$\underline{\mathcal{O}(2).1}$ $\beta = b = ar - \alpha s = \Omega_{12} - \alpha^2 + a^2 = \Omega_{34} = 0,$

$\alpha^2 + a^2 \neq 0.$

$$(\mathcal{O}(2).4) \quad \widetilde{\Omega} = \begin{pmatrix} \alpha^2-a^2 & 0 & 0 & 0 & 0 & 0 \\ & \alpha r-a^2 & \alpha(s-a) & 0 & 0 & 0 \\ & & as-\alpha^2 & 0 & 0 & 0 \\ & & & \alpha r-a^2 & \alpha(s-a) & 0 \\ & & & & as-\alpha^2 & 0 \\ & & & & & s^2-r^2 \end{pmatrix}$$

$\underline{\mathcal{O}(2).1.1.}$ $a \neq 0.$ Then

$$(\mathcal{O}(2).5) \quad \begin{aligned} & a\omega^3 + \alpha\omega^4 = dx \\ & \omega^4 = a\, e^{sx}\, dt \\ & \omega^1 + i\omega^2 = e^{ax+iy}\,(du + idv) \\ & \omega_2^1 = dy \end{aligned}$$

If we note $x = x^1$, $u = x^2$, $v = x^3$, $t = x^4$,

$$(34) \quad g = e^{2ax^1}((dx^2)^2 + (dx^3)^2) + (dx^1)^2 - 2\alpha\, e^{sx^1}\, dx^1\, dx \; +$$

$$+ \; e^{2sx^1}(\alpha^2 - a^2)(dx^4)^2$$

$\underline{\mathcal{O}(2).1.2.}$ $a = 0 \neq \alpha \Rightarrow s = 0.$ Then

$$(\mathcal{O}(2).6) \quad \begin{aligned} & \omega^1 + i\omega^2 = e^{\,y+it}\,(du + idv) \\ & \omega^3 = dx, \; \omega^4 = dy, \; \omega_2^1 = dt \end{aligned}$$

If we note $x = x^1$, $u = x^2$, $v = x^3$, $y = x^4$, we find

$$(35) \quad g = (dx^1)^2 + e^{2\alpha x^4}((dx^2)^2 + (dx^3)^2) - (dx^4)^2$$

$\underline{\mathcal{O}(2).2.}$ $b = \beta = a = \alpha = 0.$ M is locally the product of two surfaces of constant curvature.

$\Diamond(2).3$ $b \neq 0 = a = \alpha = \beta = r = s = \Omega_{34} = 0$

$$\tilde{\Omega} = \begin{pmatrix} \Omega_{12} & 0 & 0 & & & \\ & b^2 & 0 & & O_{3,3} & \\ & & 0 & & & \\ & & & b^2 & 0 & 0 \\ & & & & & 0 \end{pmatrix}$$

M is locally the Lorentz product of $M^3_{o,1}(b, \Omega_{12})$ with the Euclidean line.

The tensor metric g' of $M^3_{o,1}(b, \Omega_{12})$ is given in [Car]. If $L = \Omega_{12} + 3b^2$,

$$F = (1 + \frac{L}{4}((x^1)^2 + (x^2)^2))^{-1}, \text{ then}$$

$$(\Diamond(2).7) \qquad g' = \begin{cases} F^2((dx^1)^2 + (dx^2)^2) + (dx^3 + bF(x^1dx^2 - x^2dx^1))^2, & \text{if } L \neq 0 \\ (dx^1)^2 + (dx^2)^2 + (dx^3 - 2bx^1dx^2)^2, & \text{if } L = 0 \end{cases}$$

(36) $g = g' - (dx^4)^2$, where g' is given in $(\Diamond(2).7)$.

$\Diamond(2).4$. $b \neq 0 = \beta = a = s = \Omega_{12} - \alpha^2 + 3b^2 = \Omega_{34} + 2\alpha b =$
$= r - 2\alpha \neq \alpha$

$$(\Diamond(2).8) \qquad \tilde{\Omega} = \begin{pmatrix} \alpha^2 - 3b^2 & 0 & 0 & 0 & 0 & -2\alpha b \\ & b^2 + 2\alpha^2 & 0 & 0 & -\alpha b & 0 \\ & & -\alpha^2 & \alpha b & 0 & 0 \\ & & & b^2 + 2\alpha^2 & 0 & 0 \\ & & & & -\alpha^2 & 0 \\ & & & & & -4\alpha^2 \end{pmatrix}$$

$$(\Diamond(2).9) \qquad \begin{aligned} \omega^1 + i\omega^2 &= e^{ix + \alpha t}(du + idv) \\ \omega^3 &= e^{2\alpha t}(2bvdu + dy) \\ \omega^4 &= dt, \quad \omega^1_2 - b\omega^3 = dx \end{aligned}$$

We note $x^1 = u$, $x^2 = v$, $x^3 = y$, $x^4 = t$

(37) $g = e^{2\alpha x^4}((dx^1)^2 + (dx^2)^2) + e^{4\alpha x^4}(2bx^2dx^1 + dx^3)^2 - (dx^4)^2$

$\bigcirc(2).5$ $\beta \neq 0 = b = r = \alpha = a = s = \Omega_{34}$. The form of shows us that M is locally the product of the Euclidean line with $M^3_{1,1}(\beta,\Omega_{12})$. The metric tensor g' of $M^3_{1,1}(\beta,\Omega_{12})$ is given in [Pă]. If $L = 2\beta^2 + \Omega_{12}$, then

($\bigcirc$(2).10) $g' = (dx^1)^2 + (dx^2)^2 - \beta^2(2x^1dx^2 + dx^3)^2$, if $L = 0$

($\bigcirc$(2).11) $$L^2g' = \begin{cases} L((dx^1)^2 + \cos^2(x^1)(dx^2)^2) - \\ \quad -\beta^2(2\sin(x^1)dx^2 + dx^3)^2, & \text{if } L > 0 \\ -L((dx^1)^2 + ch^2(x^1)(dx^2)^2) - \\ \quad -\beta^2(2sh(x^1)dx^2 + dx^3)^2, & \text{if } L < 0 \end{cases}$$

(38) $g = g' + (dx^4)^2$, where g' is given by ($\bigcirc$(2).10) or in ($\bigcirc$(2).11)

$\bigcirc(2).6$ $\beta \neq 0 = b = r = \alpha = s - 2a = \Omega_{12} + a^2 - 3\beta^2 = \Omega_{34} - 2\beta \neq a$

($\bigcirc$(2).12) $$\tilde{\Omega} = \begin{pmatrix} 3\beta^2 - a^2 & 0 & 0 & 0 & 0 & 2a\beta \\ & -a^2 & 0 & 0 & a\beta & 0 \\ & & 2a^2+\beta^2 & -a\beta & 0 & 0 \\ & & & -a^2 & 0 & 0 \\ & & & & 2a^2+\beta^2 & 0 \\ & & & & & 4a^2 \end{pmatrix}$$

($\bigcirc$(2).13) $$\begin{aligned} \omega^3 &= dx, \quad \omega^{\frac{1}{2}} - \beta\omega^4 = dt \\ \omega^1 + i\omega^2 &= e^{ax+it}(du + idv) \\ \omega^4 &= e^{2ax}(2\beta u dv + dy) \end{aligned}$$

If $x^1 = u$, $x^2 = v$, $x^3 = x$, $x^4 = y$,

(39) $g = e^{2ax^3}((dx^1)^2 + (dx^2) - e^{2ax^3}(dx^4 + 2\beta x^1dx^2)^2) + (dx^3)^2$

⬡(2).7 $b \neq 0 = r - 2\alpha = s - 2a = ab - \alpha\beta = \Omega_{34} - 2(a\beta - \alpha b) =$

$= a(\Omega_{12} - 3(\beta^2 - b^2; + a^2 - \alpha^2) \neq \beta$.

(⬡(2).14)

$$\tilde{\Omega} = \begin{pmatrix} \Omega_{12} & 0 & 0 & 0 & 0 & 2(a\beta - \alpha b) \\ & 2\alpha^2 - a^2 + b^2 & \alpha a + b\beta & 0 & a\beta - b\alpha & 0 \\ & & 2a^2 - \alpha^2 + \beta^2 & -a\beta + b\alpha & 0 & 0 \\ & & & 2\alpha^2 + b^2 - a^2 & 0 & 0 \\ & & & & 2a^2 - \alpha^2 + \beta^2 & 0 \\ & & & & & 4(a^2 - \alpha^2) \end{pmatrix}$$

⬡(2).7.1 $a \neq 0 \Leftrightarrow \alpha \neq 0$. The solution of (⬡(2).3) is

(⬡(2).15)

$$a\omega^3 + \alpha\omega^4 = a\,dx$$

$$\omega_2^1 - b\omega^3 - \beta\omega^4 = dt$$

$$\omega^1 + i\omega^2 = e^{ax+it}\,(du + i\,dv)$$

$$\omega^4 = e^{2ax}\,(2\beta u\,dv + dy)$$

If we note $x = x^1$, $u = x^2$, $v = x^3$, $y = x^4$,

$$g = e^{2ax^1}((dx^2)^2 + (dx^3)^2) + (dx^1 - \frac{\alpha}{a}e^{2ax^1}(2\beta x^2 dx^3 + dx^4))^2 - \tag{40}$$

$$- e^{4ax^1}(2\beta x^2 dx^3 + dx^4)^2$$

⬡(2).7.2 $a = \alpha = 0 \Rightarrow \Omega_{34} = 0$. (⬡(2).3) becomes :

(⬡(2).16)

$$\begin{cases} d(\omega^1 + i\omega^2) = i(\omega_2^1 - (b\omega^3 + \beta\omega^4))(\omega^1 + i\omega^2) \\ d\omega^3 = -2b\omega^1\omega^2 \\ d\omega^4 = 2\beta\omega^1\omega^2 \\ d\omega_2^1 = (b^2 - \beta^2 + \Omega_{12})\omega^1\omega^2 \end{cases}$$

$\beta\omega^3 + b\omega^4 = b\,dt$, $2b\omega_2^1 + (b^2 - \beta^2 + \Omega_{12})\omega^3 = 2b\,dy$

If we note $3(\beta^2 - b^2) - \Omega_{12} = 2bc$, then

$$(\omega^1 + i\omega^2) = e^{i(y+\beta t)}(\theta^1 + i\theta^2), \text{ where}$$
$$d(\theta^1 + i\theta^2) = ic\omega^3(\theta^1 + i\theta^2), \quad d\omega^3 = -2b\,\theta^1\theta^2$$

<u>$O(2).7.2.1$</u> $c = 0$ $\quad \theta^1 + i\theta^2 = du + idv$,

$\omega^3 = \beta bvdu + bdz$. We note $x^1 = u$, $x^2 = v$, $x^3 = z$, $x^4 = t$,

(41) $$g = (dx^1)^2 + (dx^2)^2 + b^2(\beta x^2dx^1 + dx^3)^2 - \\ - (dx^4 - \beta^2x^2dx^1 - \beta dx^3)^2$$

<u>$O(2).7.2.2.$</u> $bc < 0$. We take $\gamma^j = \sqrt{-2bc}\,\theta^j$, $j = 1,2$, $\gamma^3 = c\omega^3$. Then $\gamma^1, \gamma^2, \gamma^3$ are given by L.2.1. $(\varepsilon = 1)$

$$\gamma^1 + i\gamma^2 = e^{ix}(dy + i\cos y dz)$$
$$\gamma^3 = -dx + \sin y dz$$

If we note $x^1 = y$, $x^2 = z$, $x^3 = x$, $x^4 = t$, then

(42) $$g = \frac{1}{2|bc|}((dx^1)^2 + \cos^2x^1(dx^2)^2) + \frac{1}{c^2}(dx^3 - \sin(x^1)dx^2)^2 - \\ - (dx^4 - \frac{\beta}{|bc|}(dx^3 - \sin(x^1)\,dx^2))^2$$

<u>$O(2).7.2.3$</u> $bc > 0$. We take $\gamma^j = \sqrt{2bc}\,\theta^j$, $j = 1,2$, $\gamma^3 = c\omega^3$. Then γ^i, $i = \overline{1,3}$ are given by

$\gamma^1 + i\gamma^2 = e^{ix}(dy + i\,\mathrm{ch}y dz)$, $\gamma^3 = dx - \mathrm{sh}y dz$

If we note $y = x^1$, $z = x^2$, $x = x^3$, $t = x^4$, then

(43) $$g = \frac{1}{2bc}((dx^1)^2 + \mathrm{ch}^2x^1(dx^2)^2) + \frac{1}{c^2}(dx^3 - \mathrm{sh}x^1dx^2)^2 - \\ - (dx^4 - \frac{\beta}{bc}(dx^3 - \mathrm{sh}x^1dx^2))^2$$

<u>$\mathfrak{h} = O_1(2)$.</u> We have $\omega^i_j = \theta^i_j = a^i_j\omega^1 + b^i_j\omega^2 + c^i_j\omega^3 + d^i_j\omega^4$, $1 \leq i < j \leq 4$, $i \neq 3$. If $\theta = A\omega^1 + B\omega^2 + C\omega^3 + D\omega^4$, the structure equations of K imply

$(O_1(2).1)$ $d\theta \sim (D\omega^3 + C\omega^4)\omega^3_4$

We use (O_1(2).1) taking $\theta = \theta^i_j$, then

$$\theta^1_2 = a\omega^1 + b\omega^2$$

$$(O_1(2).2) \qquad \theta^1_3 = c\omega^3 + r\omega^4 \qquad \theta^1_4 = -r\omega^3 - c\omega^4$$

$$\theta^2_3 = h\omega^3 + f\omega^4 \qquad \theta^2_4 = -f\omega^3 - h\omega^4$$

Let us note $\omega'^3 = \omega^3 - \omega^4, \omega'^4 = \omega^3 + \omega^4$

The structure equations of K are :

$$(O_1(2).3) \qquad \begin{aligned} d\omega^1 &= -a\omega^1\omega^2 + r\omega'^3\omega'^4 \\ d\omega^2 &= -b\omega^1\omega^2 + f\omega'^3\omega'^4 \\ d\omega'^3 &= \omega'^3((c-r)\omega^1 + (h-f)\omega^2 - \omega^3_4) \\ d\omega'^4 &= \omega'^4((c+r)\omega^1 + (h+f)\omega^2 + \omega^3_4) \\ d\omega^3_4 &= \tfrac{1}{2}(r^2-c^2+f^2-h^2)\omega'^3\omega'^4 + \Omega^3_4 , \end{aligned}$$

$$\Omega^3_4 = \alpha_{12}\omega^1\omega^2 + \tfrac{1}{2}\alpha_{34}\omega'^3\omega'^4 + \sum_{\substack{i=1,2\\ j=3,4}} \alpha_{ij}\omega^i\omega'^j$$

We impose the integrability conditions and we find the system :

$$S_\eta : \quad \begin{aligned} & r(a+2h) = 0 \qquad br + 2hf = 0 \\ & f(b-2c) = 0 \qquad bh + ac = 0 \\ & af - 2cr = 0 \qquad bf + ar - \alpha_{12} = 0 \\ & \alpha_{12}r - (r^2 - c^2 + f^2 - h^2 + \alpha_{34})\, h = 0 \\ & \alpha_{12}f + (r^2 - c^2 + f^2 - h^2 + \alpha_{34})\, c = 0 \\ & \alpha_{ij} = 0 , \qquad i = 1,2; \quad j = 3,4 \end{aligned}$$

$$\tilde{\Omega} = \begin{pmatrix} -(a^2+b^2) & 0 & 0 & 0 & 0 & ar+bf \\ & ah-c^2-r^2 & 0 & bh-ch-rf & bf-cf-rh & 0 \\ & & c^2+r^2-ah & rh+cf-bf & rf+ch-bh & 0 \\ & & & -h^2-f^2-bc & 0 & 0 \\ & & & & f^2+h^2+bc & 0 \\ (O_1(2).4) & & & & & \alpha_{34} \end{pmatrix}$$

The solutions of the system S_h are the following :

<u>$O_1(2).1$</u> <u>$r = f = 0$</u> $bh + ac = \alpha_{12} = (c^2+h^2-\alpha_{34})h =$ $= (c^2 + h^2 - \alpha_{34})c = 0$

<u>$O_1(2).1.1$</u> $c = h = 0$. M is locally the product of two surfaces of constant curvature one riemannian, and the other a Lorentz one.

<u>$O_1(2).1.2$</u> $h \neq 0 = \alpha_{34} - c^2 - h^2$. The solution of $(O_1(2).3)$ is

$$(O_1(2).5) \quad \begin{aligned} &\omega_4^3 = dy;\ c\omega^1 + h\omega^2 = dx \\ &\omega'^3 = e^{y-x} du,\ \omega'^4 = e^{-x-y}\, dv,\ \omega^1 = e^{\frac{a}{h}x}\, dt \end{aligned}$$

If we note $x = hx^2$, $u = x^3$, $v = x^4$, $t = hx^1$, then

$$(44) \quad g = (c^2+h^2)\, e^{2ax^2} (dx^1)^2 - 2ce^{ax^2}\, dx^1 dx^2 + (dx^2)^2 + $$
$$+ e^{-2hx^2}\, dx^3 dx^4$$

<u>$O_1(2).1.3$</u> $h = 0 = \alpha_{34} - c^2 \neq c \Rightarrow a = 0$. We find

$$(O_1(2).6)\ \omega_4^3 = dy,\ \omega'^3 = e^{y-cx}\, du,\ \omega'^4 = e^{-y-cx}\, dv,$$
$$\omega^1 = dx,\ \omega^2 = e^{-bx}\, dz$$

If we note $x = x^1$, $z = x^2$, $u = x^3$, $v = x^4$,

$$(45) \quad g = (dx^1)^2 + e^{-2bx^1} (dx^2)^2 + e^{-2cx^1}\, dx^3\, dx^4.$$

$\underline{\mathcal{O}_1(2).2}$ $\underline{f \neq 0 \neq r}$ $\Rightarrow b = 2c,\ a = -2h$. We may write $h = \lambda r,\ c = -\lambda f,\ b = -2\lambda f,\ a = -2\lambda r,\ \alpha_{12} = -2\lambda(r^2 + f^2)$,
$\lambda(\alpha_{34} + (3-\lambda^2)(r^2 + f^2)) = 0$

If $\underline{\lambda \neq 0}$,

the structure equations become

$$(\mathcal{O}_1(2).7)\qquad \begin{aligned} d\omega^1 &= r(2\lambda\omega^1\omega^2 + \omega'^3\omega'^4) \\ d\omega^2 &= f(2\lambda\omega^1\omega^2 + \omega'^3\omega'^4) \\ d\omega'^3 &= \omega'^3(-(\lambda f+r)\omega^1 + (\lambda r-f)\omega^2 - \omega^3_4) \\ d\omega'^4 &= \omega'^4((r-\lambda f)\omega^1 + (\lambda r+f)\omega^2 + \omega^3_4) \\ d\omega^3_4 &= -(r^2+f^2)(2\omega^1\omega^2 + \omega'^3\omega'^4) \end{aligned}$$

The solution of $(\mathcal{O}_1(2).7)$ is

$$(\mathcal{O}_1(2).8)\qquad \begin{aligned} &-(\lambda f+r)\omega^1 + (\lambda r-f)\omega^2 - \omega^3_4 = -2\lambda\, dx \\ &(r-\lambda f)\omega^1 + (\lambda r+f)\omega^2 + \omega^3_4 = -2\lambda\, dy \\ &\omega'^3 = e^{2\lambda x}\, du\ ,\quad \omega'^4 = e^{2\lambda x}\, dv \\ &\omega^2 = f e^{2\lambda(x+y)}\,(u\,dv + dt) \end{aligned}$$

We may take the prime integrals of $\omega^i = 0 \quad i = \overline{1,4}$

$$x^1 = x + y,\quad x^2 = u,\quad x^3 = v,\quad x^4 = t.$$

$$(46)\quad g = (r^2+f^2)e^{4\lambda f x^1}(x^2dx^3+dx^4)^2 + 2re^{2\lambda f x^1}dx^1(x^2dx^3+dx^4) + (dx^1)^2 + e^{2\lambda f x^1}dx^2\,dx^3$$

$\underline{\mathcal{O}_1(2).2.2.}$ $\lambda = 0$, we write $L = \alpha_{34} + 3(r^2 + f^2)$, $= f\omega^2 + r\omega^1 + \omega^3_4$.

The system $(\mathcal{O}_1(2).3)$ becomes :

$$(\mathcal{O}_1(2).9)\qquad \begin{aligned} &d\omega^3 = -\omega\omega^4 \qquad d\omega^4 = -\omega\omega^3 \\ &d\omega = L\omega^3\omega^4 \quad d\omega^1 = 2r\omega^3\omega^4 \quad d\omega^2 = 2f\omega^3\omega^4 \end{aligned}$$

If $\underline{L = 0}$

$$(\mathcal{O}_1(2).10) \quad \begin{aligned} &\omega = du,\ \omega^3 = \mathrm{ch}u dx + \mathrm{sh}u dy,\ \omega^4 = \mathrm{sh}u dx + \mathrm{ch}u dy \\ &\omega^1 = 2rxdy + dz,\ \omega^2 = 2fxdy + dt \end{aligned}$$

If we note $x = x^3$, $y = x^4$, $z = x^1$, $t = x^2$, then

$$(47) \quad g = (2rx^3dx^4 + dx^1)^2 + (2fx^3dx^4 + dx^2)^2 + (dx^3)^2 - (dx^4)^2$$

$\underline{\text{If } L < 0}$ we take $\gamma^1 = |L|^{\frac{1}{2}}\omega^3, \gamma^2 = |L|^{\frac{1}{2}}\omega^4,\ \gamma^3 = -\omega$
Then γ^i, $i = \overline{1,3}$ are given by L.2.1. ($\varepsilon = -1$). We have

$$(\mathcal{O}_1(2).11) \quad \begin{aligned} &\omega^1 = 2r|L|^{-1}\gamma^3 + |L|^{-1} du \\ &\omega^2 = 2f\,|L|^{-1}\gamma^3 + |L|^{-1} dv \end{aligned}$$

The system $\omega^i = 0$, $i = \overline{1,4}$ has the prime integrals :

$x^1 = u + 2rx, \quad x^2 = v + 2fx, \quad x^3 = y, \quad x^4 = z$

$$(48) \quad L^2g = (dx^1 - 2r\mathrm{sh}x^3dx^4)^2 + (dx^2 - 2f\mathrm{sh}x^3dx^4)^2 + \\ + |L|\,(-(dx^3)^2 + \mathrm{ch}^2x^3(dx^4)^2)$$

$\underline{\text{If } L > 0}$, we take $\gamma^1 = L^{\frac{1}{2}}\omega^4, \gamma^2 = L^{\frac{1}{2}}\omega^3$, $\gamma^3 = -\omega$,
and we use L.2.1 ($\varepsilon = -1$). Then

$$(\mathcal{O}_1(2).12) \quad \omega^1 = -2rL^{-1}\gamma^3 + L^{-1}du \quad \omega^2 = -2f^{-1}\,L^{-1}\gamma^{-3} + L^{-1}dv$$

We may use the prime integrals of the system $\omega^i = 0$,
$i = \overline{1,4}$, given by

$$x^1 = u - 2rx, \quad x^2 = v - 2fx, \quad x^3 = y, \quad x^4 = z$$

$$(49) \quad L^2g = (dx^1 + 2r\mathrm{sh}x^3dx^4)^2 + (dx^2 + 2f\mathrm{sh}x^3dx^4)^2 + \\ + L((dx^3)^2 - \mathrm{ch}^2x^3(dx^4)^2)$$

$\underline{\mathcal{O}_1(2).3}$ $f \neq 0 = r \Rightarrow b = 2c$, $a = h = 0$, $\alpha_{12} = 2cf$,
$c(3f^2 - c^2 + \alpha_{34}) = 0$.

$\underline{O_1(2).3.1}$ If $c = 0$ $\quad \Omega^1_i = 0$, $i = \overline{2,4}$,

$\Omega^2_3 = -f^2\omega^2\omega^3$, $\Omega^2_4 = f^2\omega^2\omega^4$, $\Omega^3_4 = \alpha_{34}\omega^3\omega^4$. Then, due to T.1.1., M is locally the product of the Euclidean line with $M^3_{1,-1}(f, -f^2 - \alpha_{34})$. The structure equations of K are $d\omega^1 = 0$, $d\omega^3 = -\omega\omega^4$, $d\omega^4 = -\omega\omega^3$, $d\omega = (3f^2 + \alpha_{34})\omega^3\omega^4$, $d\omega^2 = 2f\omega^3\omega^4$, where $\omega = f\omega^2 + \omega^3_4$. We write $3f^2 + \alpha_{34} = L$.

If $\underline{L > 0}$, we note $\sqrt{L}\omega^4 = \gamma^1$, $\sqrt{L}\omega^3 = \gamma^2$, $\omega = -\gamma^3$.

If $\underline{L < 0}$ we note $\sqrt{|L|}\,\omega^3 = \gamma^1$, $\sqrt{|L|}\omega^4 = \gamma^2$, $\omega = -\gamma^3$.

In both cases γ^i, $i = \overline{1,3}$ are given by L.2.1 $(\varepsilon = -1)$, $|L|\omega^2 = 2f(\gamma^3 + dt)$, $L\omega^1 = dx^1$

If we note $x+t = -x^2$, $y = x^3$, $z = x^4$,

$$(50) \quad L^2 g = (dx^1)^2 + 4f^2(dx^2 + \mathrm{sh}\,x^3 dx^4)^2 - L(dx^3 - \mathrm{ch}^2 x^3 (dx^4)^2)$$

If $\underline{L = 0}$, $\omega = dt$, $\omega'^3 = e^t dy$, $\omega'^4 = e^{-t}\,dz$, $\omega^2 = 2f(ydz + dv)$, $\omega^1 = du$

We write $u = x^1$, $v = x^2$, $y = x^3$, $z = x^4$, then

$$(51) \quad g = (dx^1)^2 + 4f^2(x^3dx^4 + dx^2)^2 + dx^3dx^4$$

$\underline{O_1(2).3.2.}$ $c \neq 0 \Rightarrow \alpha_{34} = c^2 - 3f^2$. If we write $c = -\lambda f$, $\lambda \neq 0$ and the structure equations can be integrated in the some manner as in the case

$\underline{O_1(2).2.1.}$ The metric tensor is given by (45), with $r = 0$.

$\underline{O_1(2).4.}$ $f = 0 \neq r$. If we look at the form of $\widetilde{\Omega}$, we see that M is locally equivalent to one m-space belonging to the family $\underline{O_1(2).3}$.

Theorem 2.4. Any $\Psi.R_{4,1}$ m-space is locally equivalent with one of the following m-spaces :

- a $\Upsilon.R_{4,1}$ manifold of constant curvature
- a Lorentz product of two $\Upsilon.R_{n,q}$ manifolds of constant curvature
- a $\Upsilon.R_{4,1}$ manifold whose metric g is given by one of the formulas (1) - (51).

In order to show that the list of $\Upsilon.R_{4,1}$ 5-spaces given by A.Z. Petrov is not complete, we shall use a Sard argument.

The list of metrics of 5-spaces due to Petrov contains families of metrics that depend at most on 5 parameters; but the greatest number of essential parameters that such a family depends on, is 3. More precisely the only families which may depend on 3 essential parameters are (33.5), (33.12) and (33.13) in [Pe]. We shall note the Lie algebras of the corresponding Lie groups respectively by $\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3$. Their structure equations are given by :

$$(\mathfrak{F}_1): [X_1,X_4] = \frac{2k}{k-1} X_1,\ [X_1,X_5] = (\alpha+\beta)X_1,\ [X_2,X_3] = X_1,$$

$$[X_2,X_4] = X_2,\ [X_2,X_5] = \alpha X_2,\ [X_3,X_4] = \frac{k+1}{k-1} X_3$$

$$[X_3,X_5] = \beta X_3.$$

$$(\mathfrak{F}_2): [X_1,X_4] = \frac{2k}{k-1} X_4,\ [X_2,X_3] = X_1,\ [X_2,X_4] = X_2\ ,$$

$$[X_2,X_5] = -X_3,\ [X_3,X_4] = \frac{k+1}{k-1} X_3,\ [X_4,X_5] = \frac{2}{1-k} X_5$$

$$(\mathfrak{F}_3): [X_1,X_4] = \frac{2k}{k-1} X_4,\ [X_2,X_3] = X_1,\ [X_2,X_4] = X_2$$

$$[X_3,X_4] = \frac{k+1}{k-1} X_3,\ [X_4,X_5] = \frac{2}{k-1} X_5$$

Let us note by $\check{M}(a,b,c)$ the local equivalence class of the $\Upsilon.R_{4,1}$ 5-space with g given by (44), and let $\mathcal{Y}$ be the set $\{ \check{M}(a,b,c),\ (a,b,c) \in (\mathbb{R}^*)^3 \}$.

<u>Proposition 2.5</u>. These is a neighbourhood N of $(2,2,1)$ such that $\Psi : N \to \mathcal{S}$, $\Psi(a,b,c) = \check{M}(a,b,c)$ is injective.

<u>Proof</u>. The Ricci form associated to the punctual curvature tensor of $M(a,b,c)$ is

$$\rho_{a,b,c}(u) = (2ah-2c^2-a^2-b^2)(u^1)^2 - (2bc+2h^2+a^2+c^2)(u^2)^2 +$$
$$+ (ah-bc-2h^2-2c^2)((u^3)^2 - (u^4)^2) - 2h(b-c)(2u^1u^2-u^3u^4),$$ where $ac + bh = 0$.

The eigenvalues of $\rho_{2,2,1}$, $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$ are the roots of $(\lambda^2-8\lambda+14)(\lambda^2-9\lambda+17) = 0$. In order to proof P.2.5. it is enough to show that the map $\Lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ is imersive at $(2,2,1)$.

But a straightforward calculation shows us that if $F(t_1,t_2,t_3,t_4) = (t_1+t_3, t_2+t_4, t_1t_3, t_2t_4)$, $F\circ\Lambda$ is imersive at $(2,2,1)$.

<u>Corrolary 2.6</u>. There are 5-spaces members of the familly (44) that are not marked in the list of A.Z. Petrov.

<u>Proof</u>. In the case (44) the structure equations of $\mathfrak{b}_3$ are $[X_1,X_2] = aX_1 + bX_2$, $[X_1,X_3] = -cX_3$, $[X_1,X_4] = cX_4$, $[X_2,X_4] = -\frac{ac}{b}X_3$. Then the second derived algebra $D^2(\mathfrak{b}_3) = Sp(X_3)$, and $\mathrm{im}\,(\mathrm{ad}X_3) = Sp(X_3,X_4)$.

On the other part, $\dim(D^2(\mathfrak{b}_2)) = 2$, and $D^2(\mathfrak{b}_i) = Sp(X_i)$, but $\mathrm{im}\,(\mathrm{ad}X_1) = Sp(X_1)$, for $i = 1,3$

We must analyse if the spaces given by (44) posses a higher dimensional group of automorphisms.

The only family of spaces that may depend on 3 essential parameters, and a larger group of automorphisms is that of the case <u>$\mathfrak{b}$.1</u>; but every 1-dimensional subspace of $\mathfrak{b}$ is conjugated with $Sp(X_0)$, which is not conjugated with $\mathfrak{O}_1(2)$ q.e.d.

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