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SOME DEVELOPMENTS OF THE CVBEM - APPLICATION TO THE MIXED BOUNDARY VALUE PROBLEM FOR THE LAPLACE EQUATION -

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ABSTRACT

The main developments of the CVBEM given in the paper consists in removing the need for approximating the boundary curve by a poligonal line and the introduction of the singular boundary elements for the case of re-entrant corners or collision points on the boundary. The paper gives also explicit formulae for solving the mixed Dirichlet- Neumann boundary value problem for two-dimmensional Laplace equation by CVBEM. The Fortran program MIXED coded by this ideas is applied at some test problems.

1. INTRODUCTION

Last years the boundary element method (BEM) proved to be a very efficient tool for solving many boundary-value problems (BVP) involved in engineering analysis. It was applied to an incrising number and variety of practical problems and has formed the basis of several library programs /1/.

Very recently, using the ideas of the BEM, Hromadka /2/ presented a complex variable boundary element method (CVBEM). This is a specialised method to solve BVP for Laplace equation in two dimmensions. As is well known, every function $u(x,y)$, harmonic into a simple connected plane domain Ω, has a harmonic conjugate function $v(x,y)$ such that the complex potential function

$$f(z) = u(x,y) + i\, v(x,y)$$

is holomorphic in Ω. In many problems the function $v(x,y)$ has also a definite physical meaning (potential function, stream function, flux function, etc.). In order to approximate the function $f(z)$

inside Ω, Hromadka starts with considering a polygonal approximation for the boundary curve and then uses a piecewise Lagrange interpolation function to approximate the values of the complex function on the boundary. The representation formula for the function f(z) inside Ω results now by using the Cauchy's integral formula. The system of equations for the nodal values of the complex function is obtained by collocation.

In this paper we give some developments of the CVBEM. First of all we remove the approximation of the boundary curve by a polygonal line. Consequently, the only needed approximation is that of the complex function f(z) on the boundary by means of some algebraic interpolation functions; these are chosen such that the quadratures implied by Cauchy's integral formula can be performed exactly.

In the case of nonsmooth boundary curves or the mixed BVP, the precision of the method is altered by boundary singularities (re-entrant corners or collision points). To improve the accuracy of computation and the order of convergence, we included some singular boundary elements, where the interpolation functions took into account the real function behaviour in the neighborhood of the singularity. In section 3 explicit formulae for the singular elements are given in the case of rational order singularities. These include, also, the collision points (Dirichlet - Neumann mixed boundary-value conditions) where the singularity order is 1/2.

In section 4 we solve the mixed Dirichlet- Neumann BVP, for Laplace equation by means of the CVBEM. To this purpose the problem is first converted into a modified Volterra problem: i.e. the problem of determining the complex function f(z), when we know the values of the real part u, on some boundary arcs and the value of the imaginary part v, on the remaining arcs. Formulae for explicit calculation of the solution are given.

In sectin 5 two examples are worked in order to show how the method works. The first application consists of a heat conduction problem. We emphasise the influence of the singular boundary elements on the precision of the solution. The next application shows how the method works in the case of a domain with four boundary electrodes (and four insulated arcs) on a resistive distributed structure; this type of problem has important applications in microelectronics.

2. CVBEM DEVELOPMENT

Let Γ be a rectifiable Jordan curve in the complex z-plane, Ω-the region interior to Γ and let $z_1, z_2, \ldots, z_n$ be nodal points on Γ, ordered counterclockwise, separating the curve into boundary elements Γ_j $(j=1,\ldots,n)$ where Γ_j is the arc from z_{j-1} to z_j $(z_0 \equiv z_n)$ (Fig.1).

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Assume that the function $f(z)=u(x,y)+iv(x,y)$ is analytic into the domain Ω and continuous in $\overline{\Omega} = \Omega \cup \Gamma$. By Cauchy integral formula we can write

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta , \quad z \in \Omega \tag{2.1}$$

This formula determines the complex function $f(z)$ inside the domain Ω by means of its values on the boundary curve. The troubles arise from the fact that the values of the complex function $f(z)$ are unknown on the boundary. We know either the values of the real part or those of the imaginary part, or a combination of them. To obtain the function $f(\zeta)$ on the boundary we must solve an integral equation whose solution cannot be performed analitically. Next, it remaines the problem of computation of the solution in Ω by means of formula (2.1).

To overcome all these difficulties, we shall approximate the boundary values $f(\zeta)$ by means of some picewise algebraic interpolation function

$$F(\zeta) = \sum_{j=1}^{n} f_j \, L_j(\zeta) \tag{2.2}$$

where $f_j = f(z_j)$. A simple choice for the basis $L_j(\zeta)$ $(j=1,\ldots,n)$ is the picewise linear Lagrange one

$$L_j(\zeta) = \begin{cases} \dfrac{\zeta - z_{j-1}}{z_j - z_{j-1}} & \text{for } \zeta \in \Gamma_j \\[2ex] \dfrac{\zeta - z_{j+1}}{z_j - z_{j+1}} & \text{for } \zeta \in \Gamma_{j+1} \\[2ex] 0 & \text{otherwise} \end{cases} \quad (j=1,\ldots,n) \tag{2.3}$$

Substituting function (2.2) into the Cauchy integral formula (2.1), we obtain the approximation $F(z)=U(x,y)+iV(x,y)$ of the analytic function $f(z)$ in the domain Ω

$$F(z) = \sum_{j=1}^{n} f_j \hat{L}_j(z) \tag{2.4}$$

The functions

$$\hat{L}_j(z) = \frac{1}{2\pi i}\int_{\Gamma} \frac{L_j(\zeta)}{\zeta - z}\, d\zeta \tag{2.5}$$

can be expressed in terms of elementary functions only. Thus, in the case of the basis (2.3) we have

$$\hat{L}_j(z) = \frac{1}{2\pi i}\left\{\frac{z - z_{j-1}}{z_j - z_{j-1}} \ln\frac{z - z_j}{z - z_{j-1}} + \frac{z - z_{j+1}}{z_j - z_{j+1}} \ln\frac{z - z_{j+1}}{z - z_j}\right\} \tag{2.6}$$

Here the complex logarithm determination writes

$$\ln\frac{z - z_j}{z - z_{j-1}} = \ln\left|\frac{z - z_j}{z - z_{j-1}}\right| + i\,\hat{\gamma}_j$$

where $\hat{\gamma}_j$ is the angle at the point z between the direction zz_{j-1} and the direction zz_j (fig.1). The function $\hat{L}_j(z)$ holomorphic in the whole complex z-plane with a cut along the boundary (curved) element Γ_j. As far as the computation is concerned one has to remark that in the case of the stright-line are Γ_j the determination of the logarithm in (2.6) is just the principal determination; in the case of the curved arcs it differs from this in the convex part of the Γ_j arc.

Thus if z_0 approches Γ_j from the convex part of the arc, we have

$$\ln\frac{z_0 - z_j}{z_0 - z_{j-1}} = \ln\left|\frac{z_0 - z_j}{z_0 - z_{j-1}}\right| + i\hat{\gamma}_j, \qquad \pi < \hat{\gamma}_j < 2\pi$$

Now, equation (2.4) can be used to solve different boundary value problems for the domain Ω. To this aim the most convenient way is to obtain, by collocation, a system of linear equations for the unknown coefficient f_j (in fact u_j or v_j) and to consider the function $F(z)$ as the approximation of the real solution $f(z)$.

During collocation we must compute the values $L_{kj} = \hat{L}_j(z_k)$. For $k \neq j-1, j, j+1$ these can be obtained straight-forward by equation (2.6). The values $L_{j-1,j}$ and $L_{j+1,j}$ result also from (2.6) by taking into account formula

$$\lim_{z\to z_j} (z - z_j)\ln(z - z_j) = 0$$

To compute $L_{jj} = \hat{L}_j(z_j)$ we put (2.6) in the form

$$\hat{L}_j(z) = \frac{1}{2\pi i}\left\{\frac{z-z_j}{z_j-z_{j-1}}\ln\frac{z-z_j}{z-z_{j-1}} + \frac{z-z_j}{z_j-z_{j+1}}\ln\frac{z-z_{j+1}}{z-z_j} + \ln^*\frac{z-z_{j+1}}{z-z_{j-1}}\right\}$$

where the star indicates that we must take the positive determination (smaller than 2π) of the imaginary part.

Let us write

$$L_{kj} \equiv M_{kj} + i\,N_{kj} = \hat{L}_j(z_k)$$

$$F_k \equiv U_k + i\,V_k = F(z_k) \qquad (2.7)$$

$$f_j = u_j + i\,v_j$$

Formula (2.4) gives, on the boundary

$$U_k = \sum_{j=1}^{n} M_{kj}\,u_j - \sum_{j=1}^{n} N_{kj}\,v_j$$

$$\text{for } k=1,\dots,n \qquad (2.8)$$

$$V_k = \sum_{j=1}^{n} M_{kj}\,v_j + \sum_{j=1}^{n} N_{kj}\,u_j$$

At the end of this section we note that the only approximation used in formula (2.4), was the interpolation of the boundary values of the function $f(z)$ by (2.2). Further on, we did not need any approximation of the curve Γ or any other numerical quadrature formula.

3. TREATMENT OF SINGULARITIES

Let z_k be a boundary singularity point of the function $f(z)$. It can be a corner point of the curve Γ or a collision point, that is a point where the boundary condition changes its type (e.g. we have a Dirichlet boundary condition for the function $u(x,y)$ on Γ_k and a Neumann type boundary condition on Γ_{k+1}). In a neighborhood of the point z_k the complex function $f(z)$ has the development

$$f(z) = f(z_k) + a_1(z-z_k)^{\mu} + a_2(z-z_k)^{2\mu} + \dots \qquad (3.1)$$

where $a_1, a_2, \dots$ are complex constants. The exponent μ equals π/θ_k at a corner singularity, equals 1/2 at a point where we have a change from Dirichlet to Neumann (or conversely) boundary conditions and equals $\pi/(2\theta_k)$ at a corner point, where, at the same time, the boundary condition type changes. θ_k denotes the angle between direction $z_k z_{k+1}, z_k z_{k-1}$ inside the domain Ω (Fig.2).

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In fact, the function f(z) has a finite value at the point z_k, but in the case $0<\mu<1$ his derivative becomes infinite as z approaches z_k. (Physicaly, we have a finite value for the potential function at the point z_k, but infinite values for the field quantities). The case $0<\mu<1$ corresponds to the collision points or to that of re-entrant corner and these are the cases we are dealing with here.

We choose the interpolant function which take into account the behavior (3.1) of the complex potential function in the neighborhood of the singularity. Thus, we write

$$L_{k-1}(\zeta) = \begin{cases} \dfrac{\zeta - z_{k-2}}{z_{k-1}-z_{k-2}} & \text{for } \zeta \in \Gamma_{k-1} \\ \left(\dfrac{\zeta - z_k}{z_{k-1}-z_k}\right)^{\mu} & \text{for } \zeta \in \Gamma_k \\ 0 & \text{otherwise} \end{cases} \tag{3.2}$$

$$L_k(\zeta) = \begin{cases} 1 - \left(\dfrac{\zeta - z_k}{z_{k-1}-z_k}\right)^{\mu} & \text{for } \zeta \in \Gamma_k \\ 1 - \left(\dfrac{\zeta - z_k}{z_{k+1}-z_k}\right)^{\mu} & \text{for } \zeta \in \Gamma_{k+1} \\ 0 & \text{otherwise} \end{cases} \tag{3.3}$$

$$L_{k+1}(\zeta) = \begin{cases} \dfrac{\zeta - z_{k+2}}{z_{k+1}-z_{k+2}} & \text{for } \zeta \in \Gamma_{k+2} \\ \left(\dfrac{\zeta - z_k}{z_{k+1}-z_k}\right)^{\mu} & \text{for } \zeta \in \Gamma_{k+1} \\ 0 & \text{otherwise} \end{cases} \tag{3.4}$$

Equation (2.5) gives

$$\hat{L}_{k-1}(z) = \frac{1}{2\pi i}\left\{\frac{z - z_{k-2}}{z_{k-1}-z_{k-2}} \ln\frac{z-z_{k-1}}{z-z_{k-2}} + 1 - F_{\mu}\left(\frac{z - z_k}{z_{k-1}-z_k}\right)\right\} \tag{3.5}$$

$$\hat{L}_k(z) = \frac{1}{2\pi i}\left\{\ln\frac{z-z_{k+1}}{z-z_{k-1}} + F_{\mu}\left(\frac{z - z_k}{z_{k-1}-z_k}\right) - F_{\mu}\left(\frac{z - z_k}{z_{k+1}-z_k}\right)\right\} \tag{3.6}$$

$$\hat{L}_{k+1}(z) = \frac{1}{2\pi i}\left\{\frac{z - z_{k+2}}{z_{k+1}-z_{k+2}} \ln\frac{z-z_{k+2}}{z-z_{k+1}} - 1 + F_{\mu}\left(\frac{z - z_k}{z_{k+1}-z_K}\right)\right\} \tag{3.7}$$

where

$$F_\mu(z) = \int_0^1 \frac{t^\mu}{t-z}\, dt \tag{3.8}$$

The integration in equation (3.8) can be analitically performed in the case of the rational index $\mu=m/n$ (where $m<n$). We have

$$F_{m/n}(z) = \frac{n}{m} + (\sqrt[n]{z})^m \sum_{j=0}^{n-1} \omega^{jm} \ln \frac{\sqrt[n]{z}\ \omega^{j-1}}{\sqrt[n]{z}\ \omega^{j}} \tag{3.9}$$

$\sqrt[n]{z}$ being the n-th root of the complex number z and $\omega = \exp\{2\pi i/n\}$ - the n-th root of the unity.

The computation of the nodal values of the functions (3.5) ÷ (3.7) requires the values $F_\mu(0)$, $F_\mu(1)$.

Equation (3.9) gives

$$F_{m/n}(0) = \frac{n}{m} \tag{3.10}$$

In the neighborhood of the point z=1 we have also

$$F_\mu(z) = \int_0^1 \frac{t^\mu-1}{t-z}\, dt + \int_0^1 \frac{dt}{t-z} = F^*_\mu(z) + \ln\frac{z-1}{z} \tag{3.11}$$

The value $F^*_{m/n}(1)$ can be computed by means of the Ψ function /3/

$$F^*_{m/n}(1) = 2\sum_{j=1}^{\left[\frac{n+1}{2}\right]-1} \cos\left(2\pi j\frac{m}{n}\right)\cdot \ln\left(\sin\frac{j\pi}{n}\right) - \frac{\pi}{2}\operatorname{ctg}\frac{m\pi}{n} - \ln(2n) \tag{3.12}$$

In the particular case of the Dirichlet- Neumann collision points we have $\mu=1/2$ and equations (3.9), (3.10) become

$$F_{1/2}(z) = 2 + \sqrt{z}\cdot \ln\frac{\sqrt{z}-1}{\sqrt{z}+1} \tag{3.13}$$

$$F^*_{1/2}(1) = -\ln 4 \tag{3.14}$$

4. SOLUTION OF THE MIXED BVP BY CVBEM

Let the boundary Γ of the simply connected domain Ω be divided into 2p arcs by the points $z(I(1)), z(J(1)), \ldots, z(I(p)), z(J(p))$. We denote by Γ_u the set of the arcs $z(I(1))z(J(1)), \ldots, z(I(p))z(J(p))$ and we put $\Gamma_v = \Gamma - \overline{\Gamma_u}$. We want to determine the function $u(x,y)$ which satisfies the conditions

$$\Delta u(x,y) = 0 \quad \text{in } \Omega \tag{4.1}$$

$$u(x,y) = \tilde{u}(\zeta) \equiv \tilde{u}(s) \quad \text{on } \Gamma_u \tag{4.2}$$

$$\frac{\partial u}{\partial n}(x,y) = \tilde{v}_1(\zeta) \equiv \tilde{v}_1(s) \quad \text{on } \Gamma_v \tag{4.3}$$

where $\tilde{u}(s)$ and $\tilde{v}_1(s)$ are two given functions, $\partial/\partial n$ is the derivative in the direction of the external normal and s is the curvilinear abscissa on the Γ curve. (The unit vectors $\vec{n}$, $\vec{s}$ are directed as an fig.1).

We denote by $v(x,y)$ the harmonic conjugate function of the $u(x,y)$. If the function u is given, then the function v is determined up to a real constant. The complex function

$$f(z) = u(x,y) + i\, v(x,y)$$

is a holomorphic function on the complex variable $z=x+iy$ in the domain Ω.

Taking into account the Cauchy- Riemann conditions, equation (4.3) can also be written in the form

$$\frac{\partial v}{\partial s} = \tilde{v}_1(s) \quad \text{on } \Gamma_v \tag{4.4}$$

and hence

$$v(s) = v(J(r)) + \int_{s(J(r))}^{s} \tilde{v}_1(\hat{s})\, d\hat{s} \equiv v(J(r)) + \tilde{v}^*(s) \tag{4.5}$$

on the arc $z(J(r))z(I(r+1))$ $(r=1,\dots,p)$

The constants $v(J(r))$ are unknown. Since the function $v(x,y)$ is determined up to a real constant, we can put

$$v(J(p)) = 0$$

According to the above considerations the solution of the mixed BVP is equivalent with the determination of the complex function $f(z)$, holomorphic in Ω and whose real and immaginary parts satisfy the boundary conditions (4.2) and (4.5) respectivly. The problem of determining the function $f(z)$ and implicity of the constants $v(J(r))$, $r=1,\dots,p-1$ is called the modified Volterra problem. In /4/ it is shown that this problem has an unique solution.

Let a division of the curve Γ, be given by the set of points $z(1),\dots,z(n)$ which include the points $z(I(r)),z(J(r))$ $r=1,\dots,p$. The boundary conditions (4.2), (4.5) written in the mesh points, give

$$u(k)=\tilde{u}(k) \quad \text{for } k\in[I(r),J(r)], \ (r=1,\dots,p) \tag{4.6}$$

$$v(k)=v(J(r))+\tilde{v}^*(k) \quad \text{for } k\in[J(r),I(r+1)], \ (r=1,\dots,p-1)$$

$$v(k)=\tilde{v}^*(k) \quad \text{for } k\in[J(p),I(p+1)], \ (I(p+1)\equiv n+1)$$

where $u(k)\equiv u(z_k)$, $v(k)\equiv v(z_k)$, $\tilde{u}(k)\equiv\tilde{u}(z_k)$, $\tilde{v}^*(k)\equiv\tilde{v}^*(z_k)$ and we denoted by $[a,b]$ the integers between a and b.

The values $\tilde{u}(k)$, $\tilde{v}^*(k)$ are known ($\tilde{v}^*(J(r))=0$, $r=1,\dots,p$).

In fact the unknowns of the problem are the quantities

$$\begin{aligned} &v(r) \quad \text{for} \quad r\in[I(r)+1,J(r)], \quad (r=1,\dots,p-1)\\ &v(r) \quad \text{for} \quad r\in[I(p)+1,J(p)-1] \\ &u(r) \quad \text{for} \quad r\in[J(r)+1,I(r+1)-1], \quad (r=1,\dots,p) \end{aligned} \tag{4.7}$$

We shall now impose $U(r)=u(r)$ at the points where the real part of the potential function is to be determined and the condition $V(r)=v(r)$ at the points z_r, $r\in\bigcup_{r=1}^{p-1}[I(r)+1,J(r)]\cup$ $\cup[I(p)+1,J(p)-1]$.

It follows a system of n-p-1 linear equations for the above mentioned unknown nodal values

$$\begin{aligned} &\sum_{s=1}^{p-1}\Big\{\sum_{j=I(s)+1}^{J(s)-1}\big[\delta(k,j)-M(k,j)\big]v(j)+\Big[\delta(k,J(s))-\sum_{r=J(s)}^{I(s+1)}M(k,r)\Big]v(J(s))-\\ &-\sum_{j=J(s)+1}^{I(s+1)-1}N(k,j)u(j)\Big\}+\sum_{j=I(p)+1}^{J(p)-1}\big[\delta(k,j)-M(k,j)\big]v(j)-\\ &-\sum_{j=J(p)+1}^{I(p+1)-1}N(k,j)u(j)=\sum_{s=1}^{p}\Big\{\sum_{j=I(s)}^{J(s)}N(k,j)\tilde{u}(j)+\sum_{j=J(s)+1}^{I(s+1)}M(k,j)\tilde{v}^*(j)\Big\}\\ &\text{for } k\in[I(l)+1,J(l)] \ (l=1,\dots,p-1) \text{ and } k\in[I(p)+1,J(p)-1] \end{aligned} \tag{4.8}$$

$$\begin{aligned} &\sum_{s=1}^{p-1}\Big\{\sum_{j=I(s)+1}^{J(s)-1}N(k,j)v(j)+\Big[\sum_{r=J(s)}^{I(s+1)}N(k,r)\Big]v(J(s))+\sum_{j=J(s)+1}^{I(s+1)-1}\big[\delta(k,j)-\\ &-M(k,j)\big]u(j)\Big\}+\sum_{j=I(p)+1}^{J(p)-1}N(k,j)v(j)+\sum_{j=J(p)+1}^{I(p+1)-1}\big[\delta(k,j)-M(k,j)\big]u(j)=\\ &=\sum_{s=1}^{p}\Big\{\sum_{j=I(s)}^{J(s)}M(k,j)\tilde{u}(j)-\sum_{j=J(s)+1}^{I(s+1)}N(k,j)\tilde{v}^*(j)\Big\}\\ &\text{for } k\in[J(l)+1,I(l+1)-1], \ (l=1,\dots,p) \end{aligned} \tag{4.9}$$

We have written an equation for every node, except for those with indices $I(l)$ $(l=1,\dots,p)$ and $J(p)$. The rate of convergence of the resulting approximation to the real solution (as the number of mesh points is increased) is altered by the presence of the boundary singularities. Even in the case of the smooth domains, we have algebraic singularities of 1/2 order at all collision points. Therefore, we must consider singular interpolants of the type (3.2)÷(3.4) at all points $z(I(r))$, $z(J(r))$ $(r=1,\dots,p)$ and, consequently, the influence of these points over approximative potential complex functions $F(z)$ enters by weights given by equations (3.5)÷(3.7).

Our method for solving the mixed BVP by means of CVBEM requires only n-p-1 equations of system (2.8). The remaining equations can be used to estimate the error in matching the

given boundary condition. In the neighborhood of the points where there are high errors, additional nodal pionts can be added and another approximation function is generated. This is in fact the CVBEM strategy to reduce the approximation error given in /5/.

5. NUMERICAL EXAMPLES

The above outlined method was implemented numerically. The resulting Fortran program MIXED is applied to two test problems.

Example 1. We consider a heat conduction problem analysed previously by Symm /6/. The domain Ω is the rectangle in fig.3 where the corner coordinates are $A_1(-7,0)$, $A_2(7,0)$, $B_2(7,7)$, $E(-7,7)$. The steady temperature $u(x,y)$ is a harmonic function inside the rectangle and satisfies the boundary conditions

$$\begin{aligned} u(x,0) &= 500 && \text{on } A_1B_1 \\ u(0,y) &= 1000 && \text{on } A_2B_2 \\ \frac{\partial u}{\partial n}(x,y) &= 0 && \text{on the remainder of the boundary.} \end{aligned} \qquad (5.1)$$

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The problem has a collision singularity point at the origin $B_1(0,0)$. Numerical results at three points near the origin are shown on table 1. The results in column (1) correspond to an uniform mesh with 48 points of the boundary without considering singular elements. In column (2) we give the results obtained by using singular elements, around the origin. The last two columns show the results obtained by Symm (3) and the analytical solution (4).

x	y	1	2	3	4
-1.0	1.0	572.83	561.45	561.93	561.95
0.0	1.0	616.37	603.35	603.76	603.77
1.0	1.0	682.18	669.50	669.54	669.54

T A B L E 1

By considering singular boundary elements we have improved greately upon the results obtained in column (1). Nevertheless, the results obtained by Symm's method are slightly better. His

method consists in subtraction of singularities and the numerical procedure it needs a more detalied analytical analysis of the problem. In the case of many singularities this can be a tedious work.

Example 2. Let Ω be the resistive distributed structure, given in fig. 4. It consists of a thin-film area of resistive material, which has on its boundary four conductor terminals $A_1B_1,\ldots,A_4B_4$ separated by insulated arcs $B_1A_2,\ldots,B_4A_1$. Such structures have been considered in order to improve the performances of the integrated circuits in microelectronics /7/.

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The potential function $u(x,y)$ is harmonic in domain Ω and takes constant values on the electrodes. On the insulated arcs, we have $\partial u/\partial n=0$, hence the stream function $v(x,y)$ is constant on the B_kA_{k+1} arcs. The complex potential function $f(z)=u+iv$ is holomorphic in the domain Ω and its determination requires to solve a modified Volterra BVP.

The use of distributed resistive structures in electronic circuits, require the determination of the indefined admitance matrix defined by relations

$$R_{\square}Y_{jk} = \frac{v(B_k)-v(A_k)}{u(A_j)}, \quad u(A_k)=0 \quad \begin{matrix} k \neq j \\ j,k=1,2,3,4 \end{matrix} \qquad (5.2)$$

Here the "square-resistence" $R_{\square}$ is a physical characteristic of the resistive structure and $u(A_j)$, $v(A_k)$ are the values of the potential and respectively of the stream function at the nodal points A_j, A_k.

To determine the admitance matrix of the given structure we must solve four mixed BVP for the domain Ω where, in succession, we have the value 1 for the potential function on an electrode and 0 values on the others.

To test how the method works in the case of the more complicated problems, we considered the structure given in Fig.4 where the geometrical dimensions are:

$$A_1B_1=B_3A_4=11.5,\ B_1A_2=A_3B_3=7.5,\ A_2B_2=B_2A_3=A_4B_4=B_4A_1=4.75$$

This is a dual-symmetrical structure whose admitance matrix was determinated "exactly" by analytical methods in /8/.

$$R_{\Omega}Y = \begin{vmatrix} 1.70862 & -0.35362 & -0.53727 & -0.81771 \\ -0.35362 & 1.07318 & -0.68183 & -0.03772 \\ -0.53727 & -0.68183 & 1.38311 & -0.16400 \\ -0.81771 & -0.03772 & -0.16400 & 1.01944 \end{vmatrix}$$

We applied the program MIXED for 48 points in an non-uniform subdivision (the interval length decreasing towards the points B_j) and with collision boundary singularities at all B_j-points. The numerical results give the matrix Y_a

$$R_{\Omega}Y_a = \begin{vmatrix} 1.70575 & -0.35442 & -0.53372 & -0.81761 \\ -0.35442 & 1.07595 & -0.68394 & -0.03759 \\ -0.53372 & -0.68394 & 1.37969 & -0.16203 \\ -0.81761 & -0.03759 & -0.16203 & 1.01723 \end{vmatrix}$$

The obtained results can be considered as good ones, especially if one take into account that we have hold four boundary singular points and the determination of the indefinite admitance matrix requires the values of the stream function just at the singular points.

6. CONCLUSION

The advantages of the given procedure for solving the mixed BVP over other BEM are:

1. The method yields an approximation for the full complex potential function in terms of the some easily computable functions.

2. It does not requires any approximation of the boundary curve or any numerical quadrature. Consequently, equation (2.2) can be used in any point z inside domain Ω now matter how close it is to the boundary Γ. (In usual BEM programs the evaluation of the solution near boundary curve gives some troubles due to numerical quadrature formulae used).

3. The method takes into account stright forwardly the boundary singularities of the solution at the re-entrant corners or at the collision points.

4. It is possible to obtain "a mesure" of the error in matching the given boundary condition and hence to develop an error reduction algorithm by adding new points in the boundary region where the error is large.

7. REFERENCES

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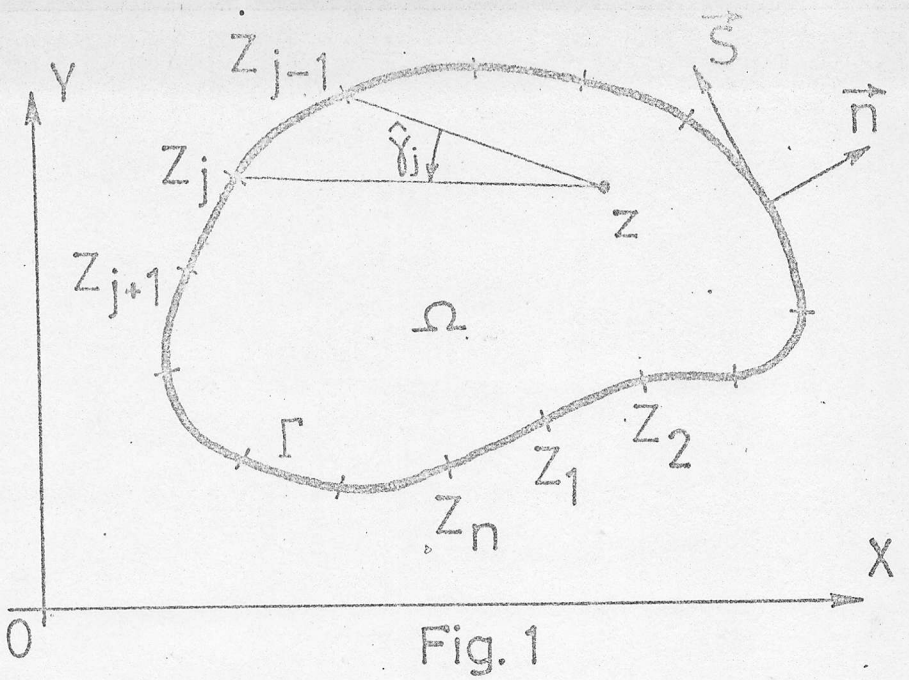

Fig. 1

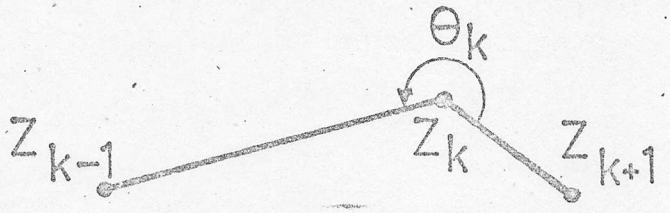

Fig. 2

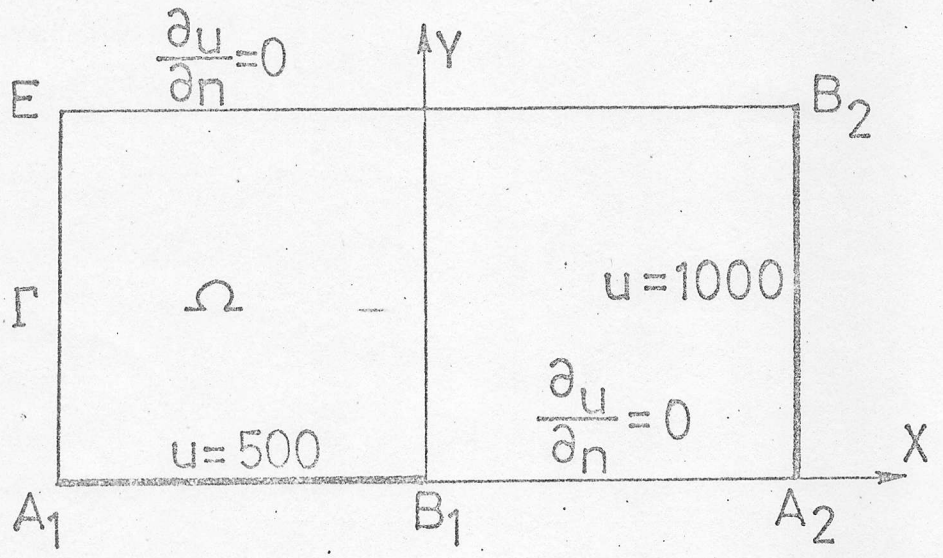
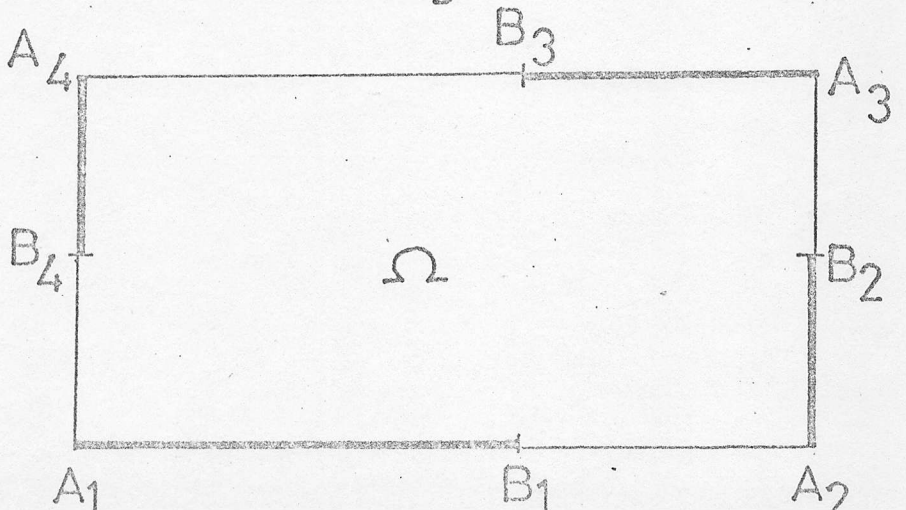
Fig. 3