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0. ABSTRACT

In this paper we study the decomposability of some matrix operators as well as other special properties of theirs. These matrix operators are derived from non-analytic functional calculi. As by-products, we obtain statements concerning the existence of (non-trivial) hyperinvariant subspaces.

1. INTRODUCTION

Let X be a complex Banach space and let $\mathcal{L}(X)$ be the algebra of all bounded linear operators acting on X. For each $S \in \mathcal{L}(X)$ we denote by $\sigma(S)$ its spectrum.

Let us fix an integer $n \geq 1$ and let us denote by X^n the Banach space $X \oplus \ldots \oplus X$ (n times). Every operator $T \in \mathcal{L}(X^n)$ can be represented as a matrix $(T_{jk})_{j,k=1}^{n}$, where $T_{jk} \in \mathcal{L}(X)$ for each pair of indices j,k. We shall study in the sequel a class of operators $T \in \mathcal{L}(X^n)$ with the property that the operators T_{jk} from the matrix representation of T mutually commute. To present this class, we need some preliminaries.

Let Ω be a compact topological space, let $C(\Omega)$ be the algebra of all complex-valued continuous functions on Ω and let $A \subset C(\Omega)$ be a (not necessarily closed) subalgebra. We recall that A is said to be <u>normal</u> if for every open cover

$\{G_1,\dots,G_m\}$ of Ω there are positive functions $f_1,\dots,f_m$ in A such that supp $(f_p) \subset G_p (p=1,\dots,m)$ and $f_1(\omega)+\dots+f_m(\omega)=1$ for all $\omega \in \Omega$. In particular, $1 \in A$ (the positivity of the functions $f_1,\dots,f_m$ will play no rôle in what follows).

1.1. DEFINITION. For an algebra $A \subset C(\Omega)$ we shall consider the following properties:

(i) A is a normal algebra;

(ii) for every pair $f,h \in A$ such that $\mathrm{supp}(h) \subset \{\omega \in \Omega : f(\omega) \neq 0\}$ the function $\omega \to h(\omega)/f(\omega)$, extended with zero outside the set supp (h), is an element of A;

(iii) A has a Banach algebra structure which makes the inclusion $A \subset C(\Omega)$ continuous.

We shall indicate at the beginning of each section which of these hypotheses on the algebra A are going to be used.

It is plain that $C(\Omega)$ has the properties (i), (ii) and (iii). If Ω is the closure of a relatively compact open subset of $\mathbb{R}^m$, then the algebra $C^r(\Omega)$ of all r-times differentiable functions in the interior of Ω, whose partial derivatives up to order r have continuous extensions to Ω, also has the properties (i) (ii) and (iii). These are, in fact, the most significant examples that we have in mind.

If A is an arbitrary commutative unital algebra, we denote by $M_n(A)$ the algebra of n x n-matrices whose entries are elements of A. The algebra $M_n(A)$ will sometimes be regarded as an A-module. Every unital algebra morphism $\Phi : A \to \mathcal{L}(X)$ induces a unital algebra morphism $\Phi_n : M_n(A) \to \mathcal{L}(X^n)$, defined by the equality

$$\Phi_n(\alpha) = (\Phi(\alpha_{jk}))^n_{j,k=1}, \text{ where } \alpha = (\alpha_{jk})^n_{j,k=1} \in M_n(A).$$

1.2. DEFINITION. Let $A \subset C(\Omega)$ be an algebra with the properties (i) and (ii) from Definition 1.1. An operator $T \in \mathcal{L}(X^n)$ ... scalar if there exists a unital algebra morphism $\Phi : A \longrightarrow \mathcal{L}(X)$ and an element $\tau \in M_n(A)$ such that $T = \Phi_n(\tau)$.

This concept extends the concept of n-spectral operator, introduced in [8] , which in turn extends that of n-normal operator [6] . When A is an admissible algebra, then Definition 1.2 also provides an extension of the concept of A-scalar operator [4] , [11] .

One of the main purposes of this paper is to prove that every (A,n)-scalar operator is decomposable (details concerning decomposable operators can be found in [4] or [11]). As a matter of fact, we shall prove a stronger result. Specifically, we shall show that if $\{U_1, U_2\}$ is an open cover of $\sigma(T)$, then there exists an operator $R \in \mathcal{L}(X^n)$ such that $RT = TR$, $\sigma(T \mid \overline{R(X^n)}) \subset \overline{U}_1$ and $\sigma(T \mid \overline{(1_n - R)(X^n)}) \subset \overline{U}_2$ (where 1_n is the identity of X^n; we use the same notation for the identity of $M_n(A)$). With the terminology of [7] , we therefore show that every (A,n)-scalar operator is super-decomposable (see Theorem 3.8).

The decomposability of an (A,n)-scalar operator $T \in \mathcal{L}(X^n)$ can be used to derive the existence of a proper hyperinvariant subspace (i.e. invariant under each operator commuting with T), when $\sigma(T)$ contains at least two points. This explains one of the main results of [8] (which is also extended by our Corollary 3.6).

By analyzing the spectrum of an (A,n)-scalar operator T (Theorem 4.6), we shall obtain the existence of hyperinvariant subspaces of T, even if $\sigma(T)$ contains only one point, provided T is not a multiple of the identity, i.e. a complete extension of Theorem 5.3 from [6] (see Corollary 4.8).

In connection with this subject, we also refer to [5] , [9] and [10] . Unlike in most of these works, we shall not use the concept of spectral measure (or related notions).

We can apply our methods to a large enough class of matrix operators, including matrices of generalized scalar operators given by a spectral distribution [4].

2. A SPECTRAL CAPACITY

Let $A \subset C(\Omega)$ be a normal algebra. We also fix a unital algebra morphism $\Phi : A \to \mathcal{L}(X)$ and denote by Φ_n the corresponding morphism of $M_n(A)$ into $\mathcal{L}(X^n)$ induced by Φ.

Since a matrix $\alpha = (\alpha_{jk})^n_{j,k=1} \in M_n(A)$ can be regarded as a function $\alpha : \Omega \to M_n$ (where $M_n = M_n(\mathbb{C}) \subset M_n(A)$), the notation $\alpha(\omega) = (\alpha_{jk}(\omega))^n_{j,k=1}$ $(\omega \in \Omega)$ and supp (α) makes sense. Moreover, supp $(\alpha' . \alpha'') \subset$ supp $(\alpha') \cap$ supp(α'') for each pair $\alpha', \alpha'' \in M_n(A)$.

For every $f \in A$ we denote by $\delta(f) \in M_n(A)$ the matrix $\delta(f) = (\delta_{jk} f)^n_{j,k=1}$, where δ_{jk} is the Kronecker symbol. Notice that δ is, in fact, a unital algebra morphism of A into $M_n(A)$ and that $\delta(A)$ is in the center of $M_n(A)$.

The set supp(Φ) (i.e. the <u>support</u> of Φ) is defined as the intersection of all closed sets $F \subset \Omega$ such that $\Phi(f) = 0$ whenever supp $(f) \subset \Omega \setminus F$ $(f \in A)$. The set supp (Φ_n) is defined in a similar way. It is easily seen that supp (Φ_n)=supp$(\Phi$ (Note that supp$(\alpha) = \cup\{$supp$(\alpha_{jk}) : 1 \leqslant j,k \leqslant n\}$ for each $\alpha = (\alpha_{jk})^n_{j,k=1} \in M_n(A)$).

2.1. PROPOSITION. <u>For every closed set</u> $F \subset \Omega$ <u>we define the space</u>

$$(2.1) \quad X^n_\Phi(F) = \bigcap \{ \ker(\Phi_n(\alpha)) : \text{supp}(\alpha) \cap F = \emptyset \}$$

<u>Then the assignment</u> $F \to X^n_\Phi(F)$ <u>is a spectral capacity</u> [2], [11].

<u>PROOF</u>. We follow some lines from the proof of Theorem IV.7.3 in [11] (see also [1]).

It is plain that $X^n_\Phi(F)$ is a closed linear subspace of X^n. The fact that $X^n_\Phi(\emptyset) = \{0\}$, $X^n_\Phi(\Omega) = X^n$ and that $X^n_\Phi(F_1) \subset X^n_\Phi(F_2)$ whenever $F_1 \subset F_2$ can be easily seen.

Let $\{G_1,\dots,G_m\}$ be an open cover of Ω. Since A is normal, we can find functions $f_1,\dots,f_m$ in A such that $\text{supp}(f_p) \subset G_p$ $(p=1,\dots,m)$ and $f_1+\dots+f_m = 1$. Let $\alpha_p = \delta(f_p)$; therefore $\text{supp}(\alpha_p) \subset G_p$ and $\alpha_1+\dots+\alpha_m = 1_n$. It is then clear that

$$X^n = \Phi_n(\alpha_1)X^n+\dots+\Phi_n(\alpha_m)X^n.$$

We have only to note that

$$\Phi_n(\alpha_p)X^n \subset X^n_\Phi(\text{supp}(\alpha_p)) \subset X^n_\Phi(\bar{G}_p)$$

for every p, and therefore

(2.2) $X^n = X^n_\Phi(\bar{G}_1)+\dots+X^n_\Phi(\bar{G}_m)$.

Now, let $\{F_\gamma\}_{\gamma\in\Gamma}$ be an arbitrary family of closed subsets of Ω. We shall prove that

(2.3) $X^n_\Phi(\bigcap_{\gamma\in\Gamma} F_\gamma) = \bigcap_{\gamma\in\Gamma} X^n_\Phi(F_\gamma)$

Since the mapping $F \to X^n_\Phi(F)$ is increasing, it suffices to prove that the right hand side of (2.3) is contained in the left hand side. Let $x \in X^n_\Phi(F_\gamma)$ for all $\gamma\in\Gamma$ and let $F_0 = \bigcap\{F_\gamma : \gamma\in\Gamma\}$. Let also $\alpha \in M_n(A)$ be such that $\text{supp}(\alpha) \cap F_0 = \emptyset$. Since $\text{supp}(\alpha)$ is compact, we can choose open sets $H_q = \Omega \setminus F_{\gamma_q}$ $(q=1,\dots,r)$ such that $\text{supp}(\alpha) \subset H_1\cup\dots\cup H_r$. If $H_0 = \Omega \setminus \text{supp}(\alpha)$, there are functions $h_0,h_1,\dots,h_r$ in A such that $h_0+h_1+\dots+h_r = 1$ and $\text{supp}(h_q) \subset H_q$ $(q=0,1,\dots,r)$. Let $\beta_q = \delta(h_q) \in M_n(A)$. Then

$$\Phi_n(\alpha)x = \Phi_n(\alpha\beta_0)x+\Phi_n(\alpha\beta_1)x+\dots+\Phi_n(\alpha\beta_r)x.$$

Since $\text{supp}(\alpha) \cap \text{supp}(\beta_0) = \emptyset$ and $\text{supp}(\alpha\beta_q) \cap F_{\gamma_q} = \emptyset$ $(1 \leqslant q \leqslant r)$, we have $\Phi_n(\alpha\beta_q)x=0$ for all $q = 0,1,\dots,r$.

Consequently $\Phi_n(\alpha)x = 0$, so that x is contained in the left hand side of (2.3). The proof of Proposition 2.1 is complete.

2.2. COROLLARY. *Let $f \in A$ be such that* $\mathrm{supp}(f) \cap \mathrm{supp}(\Phi) = \emptyset$. *Then* $\Phi(f) = 0$.

PROOF. Consider first a closed subset $F \subset \Omega$ such that if $h \in A$ and $\mathrm{supp}(h) \cap F = \emptyset$, then $\Phi(h) = 0$. In this case we must have $X^n_\Phi(F) = X^n$, by (2.1). Indeed, if $\alpha \in M_n(A)$ and $\mathrm{supp}(\alpha) \cap F = \emptyset$, then $\Phi_n(\alpha) = 0$, i.e. $\ker \Phi_n(\alpha) = X^n$.

Now, let $\{F_\gamma\}_{\gamma \in \Gamma}$ be the family of all closed subsets of Ω sharing the property of F. Then $\bigcap \{F_\gamma : \gamma \in \Gamma\} = \mathrm{supp}(\Phi)$. Since $\mathrm{supp}(\delta(f)) \cap \mathrm{supp}(\Phi) = \emptyset$ it follows that

$$\ker \Phi_n(\delta(f)) \supset X^n_\Phi(\mathrm{supp}(\Phi)) = \bigcap_{\gamma \in \Gamma} X^n_\Phi(F_\gamma) = X^n,$$

by (2.3) and the first part of the proof. Consequently $\Phi(f) = 0$.

2.3. COROLLARY. *For every closed* $F \subset \Omega$ *we have the equality*

$$X^n_\Phi(F) = X^n_\Phi(F \cap \mathrm{supp}(\Phi))$$

PROOF. As we have noted in the proof of Corollary 2.2, $X^n_\Phi(\mathrm{supp}(\Phi)) = X^n$. Therefore

$$X^n_\Phi(F) = X^n \cap X^n_\Phi(F) = X^n_\Phi(F \cap \mathrm{supp}(\Phi)),$$

by (2.3).

2.4. REMARK. We have not used so far the fact that the functions of A are continuous.

A supplementary condition on the algebra $A \subset C(\Omega)$ makes the mapping $\Phi_n : M_n(A) \longrightarrow \mathcal{L}(X^n)$ injective on its support.

2.5. LEMMA. *Assume that the algebra A also has the property (ii) from Definition* 1.1. *If* $\Phi_n(\alpha) = 0$ *for some* $\alpha \in M_n(A)$, *then* $\alpha(\omega) = 0$ *for every* $\omega \in \mathrm{supp}(\Phi)$.

PROOF. Note first that if $\Phi(f) = 0$ for some $f \in A$, then $f(\omega) = 0$ for every $\omega \in \mathrm{supp}(\Phi)$.

Indeed, if $h \in A$ is such that $\mathrm{supp}(h) \subset G = \{\omega \in \Omega : f(\omega) \neq 0\}$ then the extension h_1 of the function $\omega \longrightarrow h(\omega)/f(\omega)$ belongs to A and we have $\Phi(h) = \Phi(h_1)\Phi(f) = 0$. Therefore $\mathrm{supp}(\Phi) \cap G = \emptyset$.

Now, if $\Phi_n(\alpha) = 0$ and $\alpha = (\alpha_{jk})_{j,k=1}^n$, then $\Phi(\alpha_{jk}) = 0$ for each pair (j,k). By the previous remark, it follow that $\alpha(\omega) = 0$ for all $\omega \in \mathrm{supp}(\Phi)$.

3. DECOMPOSABILITY

In this section A will be a subalgabra of $C(\Omega)$ with the properties (i) and (ii) from Definition 1.1. Let $\Phi : A \longrightarrow \mathcal{L}(X)$ be a fixed unital algebra morphism.

We also fix an element $\tau = (\tau_{jk})_{j,k=1}^n \in M_n(A)$. Let $T = \Phi_n(\tau) \in \mathcal{L}(X^n)$, i.e. T is (A,n)-scalar. From the defining relation (2.1), it follows easily that $T X^n_\Phi(F) \subset X^n_\Phi(F)$ for all closed subsets $F \subset \Omega$.

3.1. LEMMA . <u>For every closed</u> $F \subset \Omega$ <u>we have the inclusion</u>

$$\sigma(T \mid X^n_\Phi(F)) \subset \bigcup_{\omega \in F} \sigma(\tau(\omega))$$

<u>and the set from the right hand side is closed.</u>

PROOF. We use the straightforward equality

$$\sigma(\tau(\omega)) = \{ z \in \mathbb{C} : \det(z 1_n - \tau(\omega)) = 0 \}, \omega \in \Omega,$$

where "det" stands for <u>determinant</u>. It is also an elementary fact the existence of a matrix $\tau_*(z) \in M_n(A)$ such that

$$(3.1) \quad (z 1_n - \tau)\tau_*(z) = \tau_*(z)(z 1_n - \tau) = \delta(\det(z 1_n - \tau))$$

for each $z \in \mathbb{C}$.

Now, let $z \in \mathbb{C}$ be such that $\det(z 1_n - \tau(\omega)) \neq 0$ for all $\omega \in F$. We take a function $h \in A$ such that $h=1$ in a neighbourhood of F and

$$\mathrm{supp}(h) \subset \{ \omega \in \Omega : \det(z 1_n - \tau(\omega)) \neq 0 \}.$$

Since $\det(z 1_n - \tau) \in A$ and A has the property (ii) from

Definition 1.1, the function $g(\omega) = h(\omega)(\det(z1_n - \tau(\omega)))^{-1}$ (equal to zero outside the set supp(h)) is an element of A. From (3.1) we deduce that

$$(z1_n - T)\,\Phi_n(g\,\tau_*(z)) = \Phi_n(g\,\tau_*(z))(z1_n - T) = \Phi_n(\delta(h)).$$

As we have $\mathrm{supp}(1-h) \cap F = \emptyset$, it is clear that $\Phi_n(\delta(h)) \mid X^n_\Phi(F)$ is the identity on $X^n_\Phi(F)$. Therefore

$$\Phi_n(g\,\tau_*(z)) \mid X^n_\Phi(F) = ((z1_n - T) \mid X^n_\Phi(F))^{-1},$$

i.e. $z \notin \sigma(T \mid X^n_\Phi(F))$.

Finally, if $\det(z1_n - \tau(\omega)) \neq 0$ for all $\omega \in F$, then there exists a neighbourhood V of z such that if $w \in V$, then $\det(w1_n - \tau(\omega)) \neq 0$ for all $\omega \in F$.

Consequently, the set $\cup\{\sigma(\tau(\omega)) : \omega \in F\}$ is closed.

3.2. REMARK. The inclusion in the statement of Lemma 3.1 can be written as

$$\sigma(T \mid X^n_\Phi(F)) \subset \cup\{\sigma(\tau(\omega)) : \omega \in F \cap \mathrm{supp}(\Phi)\} ,$$

via Corollary 2.3.

3.3. LEMMA. <u>Let</u> $L \subset \mathbb{C}$ <u>be a closed subset and let</u>

$$\theta(L) = \{\omega \in \Omega : \sigma(\tau(\omega)) \cap L \neq \emptyset\}.$$

<u>Then</u> $\theta(L)$ <u>is a compact subset of</u> Ω <u>with the property that</u> $\sigma(T \mid X^n_\Phi(F)) \cap L = \emptyset$ <u>whenever</u> $\theta(L) \cap F = \emptyset$, F <u>closed in</u> Ω.

PROOF. If $\omega_0 \notin \theta(L)$, then $\sigma(\tau(\omega_0)) \cap L = \emptyset$. Thus, by the upper semicontinuity of the spectrum, there exists a neighbourhood W_0 of ω_0 such that $\sigma(\tau(\omega)) \cap L = \emptyset$ for each $\omega \in W_0$. Hence $\Omega \setminus \theta(L)$ is open.

Now, let $F = \bar{F} \subset \Omega$ be such that $\theta(L) \cap F = \emptyset$. If z were a point of $\sigma(T \mid X^n_\Phi(F)) \cap L$, then, by virtue of Lemma 3.1, there would exists a point $\omega \in F$ such that $z \in \sigma(\tau(\omega))$. Therefore $\omega \in \theta(L) \cap F$, which contradicts the choice of F.

3.4. LEMMA. The operator T satisfies the condition β of Bishop [3] .

PROOF. We have to show that if $U \subset \mathbb{C}$ is an arbitrary open set and $\{g_p\}_{p=1}^{\infty}$ is a sequence of X^n-valued functions, analytic in U , such that $(z1_n - T)g_p(z) \to 0$ $(p \to \infty)$ uniformly on the compact subsets of U , then it follows that $g_p(z) \to 0$ $(p \to \infty)$ uniformly on the compact subsets of U.

Let $\{g_p\}_{p=1}^{\infty}$ be a sequence as above and let Δ be a fixed closed disc. Let us show that $g_p(z) \to 0$ $(p \to \infty)$ uniformly on Δ .

We consider the set $\theta(\Delta) \subset \Omega$ (defined in Lemma 3.3) and fix a point $\omega_0 \in \theta(\Delta)$. Let $D_0 \subset \bar{D}_0 \subset U$ be an open disc containing Δ and let $V_0 \subset \mathbb{C}$ an open set such that $\bar{D}_0 \cap \bar{V}_0 = \emptyset$ and $\sigma(\tau(\omega_0)) \subset D_0 \cup V_0$, which is obviously possible. By the upper semicontinuity of the spectrum, we infer the existence of an open neighbourhood W_0 of ω_0 in Ω such that if $\omega \in W_0$, then one has $\sigma(\tau(\omega)) \subset D_0 \cup V_0$. This procedure can be applied to any point ω of $\theta(\Delta)$. By the compactness of $\theta(\Delta)$ (Lemma 3.3), we obtain a finite open cover $\{W_1,\dots,W_m\}$ of $\theta(\Delta)$, open discs $D_1,\dots,D_m$ whose closures are in U and open sets $V_1,\dots,V_m$ in $\mathbb{C}$ such that $D_q \supset \Delta$, $\bar{D}_q \cap \bar{V}_q = \emptyset$ and $\sigma(\tau(\omega)) \subset D_q \cup V_q$ for every $\omega \in W_q$ $(q = 1,\dots,m)$. Let $W_{m+1} = \Omega \setminus \theta(L)$. We take the functions $h_1,\dots,h_m, h_{m+1}$ from A such that $h_1+\dots+h_m + h_{m+1} = 1$ and $\mathrm{supp}\,(h_q) \subset W_q$ $(q=1,\dots,m+1)$. Then consider the matrices $\alpha_q = \delta(h_q)$. Note that

$$\Phi_n(\alpha_q)g_p(z) \in X^n_{\Phi}(\mathrm{supp}\,(\alpha_q)),\ q=1,\dots,m,m+1,$$

and that $\sigma(T \mid X^n_{\Phi}(\mathrm{supp}(\alpha_q)) \subset \bar{D}_q \cup \bar{V}_q$ $(q = 1,\dots,m)$. Since

$\overline{D}_q \cap \overline{V}_q = \emptyset$, we can take another open disc $D'_q \supset \overline{D}_q$ in U such that $D'_q \cap \overline{V}_q = \emptyset$ $(1 \leqslant q \leqslant m)$. Note that the operator $(z1_n - T) \mid X^n_{\Phi}$ $(\mathrm{supp}\,(\alpha_q))$ is invertible for $z \in D'_q \setminus \overline{D}_q$ and that $\Phi_n(\alpha_q)$ commutes with T. Therefore the sequence $\Phi_n(\alpha_q)g_p(z) \to 0$ $(p \to \infty)$ uniformly on the compact subsets of $D'_q \setminus \overline{D}_q$. By the maximum principle, we deduce that $\Phi_n(\alpha_q)g_p(z) \to 0$ $(p \to \infty)$ uniformly on $\overline{D}_q$, in particular on Δ, for every $q=1,\ldots,m$.

From Lemma 3.3 we obtain that

$$\sigma(T \mid X^n_{\Phi}(\mathrm{supp}(\alpha_{m+1})) \cap \Delta = \emptyset.$$

Hence $\Phi_n(\alpha_{m+1})g_p(z) \to 0$ $(p \to \infty)$ uniformly on Δ, and therefore

$$g_p(z) = \Phi_n(\alpha_1)g_p(z) + \ldots + \Phi_n(\alpha_{m+1})g_p(z) \to 0 \quad (p \to \infty)$$

uniformly for $z \in \Delta$.

The general assertion now follows by convering an arbitrary compact subset $L \subset U$ with a finite number of closed discs and applying the previous argument to each of these discs.

Since T satisfies the condition β, then T has the single valued extension property. Particularly, we can speak about the spectral spaces

(3.2) $\quad X^n_T(L) = \{x \in X^n : \gamma_T(x) \subset L\}$,

where $L \subset \mathbb{C}$ is an arbitrary closed set and $\gamma_T(x)$ is the local spectrum of T at x (see [4] or [11] for details). In addition, the space $X^n_T(L)$ is closed (which is an easy consequence of the condition β), $X^n_T(L)$ is invariant under every operator that commutes with T(i.e. $X^n_T(L)$ is hyperinvariant) and $\sigma(T \mid X^n_T(L)) \subset L$([4] , [11]).

3.5. LEMMA. <u>The operator T is decomposable.</u>

PROOF. Let $\{U_1, U_2\}$ be an open cover of $\mathbb{C}$, and let us fix a point $\omega_0 \in \Omega$. Then we can choose two open sets V_0^1 and V_0^2 in $\mathbb{C}$ such that $\sigma(\tau(\omega_0)) \subset V_0^1 \cup V_0^2$, $\overline{V}_0^q \subset U_q$ $(q=1,2)$ and $\overline{V}_0^1 \cap \overline{V}_0^2 = \emptyset$. Let $W_0 \subset \Omega$ be an open set that $\omega \in \overline{W}_0$ implies $\sigma(\tau(\omega)) \subset V_0^1 \cup V_0^2$. Since Ω is compact, the previous remark shows that we can find an open cover $\{W_1, \dots, W_m\}$ of Ω and open sets $\{V_p^q : 1 \leq p \leq m,\ q=1,2\}$ in $\mathbb{C}$ such that

(a) $\overline{V}_p^q \subset U_q$, $\overline{V}_p^1 \cap \overline{V}_p^2 = \emptyset$;

(b) $\omega \in \overline{W}_p \Rightarrow \sigma(\tau(\omega)) \subset \overline{V}_p^1 \cup \overline{V}_p^2$

for all $p = 1, \dots, m$ and $q = 1,2$. From Lemma 3.1 and the property (b) we deduce that

$$\sigma(T \mid X_\Phi^n(\overline{W}_p)) \subset \bigcup_{\omega \in \overline{W}_p} \sigma(\tau(\omega)) \subset \overline{V}_p^1 \cup \overline{V}_p^2$$

Therefore

$$X_\Phi^n(\overline{W}_p) \subset X_T^n(\overline{V}_p^1 \cup \overline{V}_p^2)$$

by the fact that the space from the right hand side is spectral maximal [4], [11]. Let us also note the decomposition

$$(3.3) \qquad X_T^n(\overline{V}_p^1 \cup \overline{V}_p^2) = X_T^n(\overline{V}_p^1) + X_T^n(\overline{V}_p^2),$$

which follows from (a), the decomposition of the space with respect to separate parts of the spectrum (see, for instance, [11], Theorem III 3.11) and the fact that all involved spaces are spectral maximal.

According to (2.2) and the above considerations, we can write that

$$X^n = \sum_{p=1}^m X_\Phi^n(\overline{W}_p) = \sum_{p=1}^m X_T^n(\overline{V}_p^1) + \sum_{p=1}^m X_T^n(\overline{V}_p^2) =$$

$$= X_T^n(\overline{U}_1) + X_T^n(\overline{U}_2),$$

which proves the decomposability of T, by virtue of Theorem IV. 4.28 from [11] .

3.6. COROLLARY. *If $\sigma(T)$ contains more than one point, then T has at least one proper hyperinvariant subspace.*

This fact is well known in the theory of decomposable operators and is based on the existence of a compact subset $L \subset \sigma(T)$ such that $X_T^n(L)$ is neither zero nor the whole space. As we have already mentioned, $X_T^n(L)$ is a hyperinvariant subspace of T.

3.7. REMARK. If $A = C(\Omega)$ and Φ is obtained via a spectral measure on Ω , then the operator T is n-spectral [8] . If $\sigma(T)$ contains more than one point, then T has a proper hyperinvariant subspace, as proved in [8] . Consequently, Corollary 3.6 provides an extension of this result.

3.8. THEOREM. *Every (A,n)-scalar operator is super-decomposable.*

PROOF. We use the notation and the considerations from the proof of Lemma 3.5.

Let $\{f_1,\ldots,f_m\} \subset A$ be such that $f_1+\ldots+f_m = 1$ and $supp(f_p) \subset W_p$ $(p=1,\ldots,m)$. Let also Q_p^q be the spectral projection of the space $X_T^n(\overline{V}_p^1 \cup \overline{V}_p^2)$ onto $X_T^n(\overline{V}_q^p)$ $(q=1,2;\ p=1,\ldots,m)$, which is obtained from the decomposition (3.3), via the analytic functional calculus of the restriction of T to $X_T^n(\overline{V}_p^1 \cup \overline{V}_p^2)$ (see [11] , Theorem III.3.11). Since

$$\Phi_n(\delta(f_p))X^n \subset X_\Phi^n(supp(f_p)) \subset X_\Phi^n(\overline{W}_p) \subset X_T^n(\overline{V}_p^1 \cup \overline{V}_2^p),$$

we may define the operators

$$R_q = \sum_{p=1}^{m} Q_p^q \Phi_n(\delta(f_p)) \in \mathcal{L}(X^n), \quad q=1,2.$$

It is straighforward that $R_1 + R_2 = 1_n$. Moreover,

$$TR_q = T\left(\sum_{p=1}^{m} \varrho_p^q \Phi_n(\delta(f_p))\right) =$$

$$= \sum_{p=1}^{m} (T \mid X_T^n(\overline{V}_p^1 \cup \overline{V}_p^2))\, \varrho_p^q \Phi_n(\delta(f_p)) =$$

$$= \sum_{p=1}^{m} \varrho_p^q (T \mid X_T^n(\overline{V}_p^1 \cup \overline{V}_p^2))\, \Phi_n(\delta(f_p)) =$$

$$= \sum_{p=1}^{m} \varrho_p^q T \Phi_n(\delta(f_p)) = R_q T,$$

since every operator commutes with its analytic functional calculu and $\delta(f_p)$ is in the center of $M_n(A)$.

In particular, the space $R_q(X^n)$ is invariant under T.

To end the proof that T is super-decomposable, it remains to show that $\sigma(T \mid \overline{R_q(X^n)}) \subset \overline{U}_q$ $(q=1,2)$.

Let $z \notin \overline{U}_q$. We choose two open sets U_q' and U_q'' such that $U_q' \supset \overline{U}_q$, $z \notin \overline{U}_q'$, $\overline{U}_q'' \cap \overline{U}_q = \emptyset$ and $U_q' \cup U_q'' = \mathbb{C}$. Then $X^n = X_T^n(\overline{U}_q') + X_T^n(\overline{U}_q'')$, since T is decomposable (Lemma 3.5). If $x \in X^n$ is a fixed vector, then $x = x' + x''$, where $x' \in X_T^n(\overline{U}_q')$ and $x'' \in X_T^n(U_q'')$. Note that $R_q x'' = 0$. Indeed, on one hand,

$$R_q(X^n) \subset \sum_{p=1}^{m} X_T^n(\overline{V}_p^q) \subset X_T^n(\overline{U}_q),$$

which shows that $\gamma_T(R_q x'') \subset \overline{U}_q$, by (3.2). On the other hand, since $x'' \in X_T^n(\overline{U}_q'')$, then $\gamma_T(R_q x'') \subset \gamma_T(x'') \subset \overline{U}_q''$ [4]. Therefore $\gamma_T(R_q x'') = \emptyset$, so that $R_q x'' = 0$ because of the single valued extension property of T. This establishes the equality

(3.4) $R_q(X^n) = R_q(X_T^n(\overline{U}_q'))$.

Let us show that the space $R_q(X^n)$ is invariant under $((z 1_n - T) \mid X_T^n(\overline{U}_q'))^{-1}$. Indeed, the operator R_q commutes with

T and the space $X^n_T(\bar{U}'_q)$ is invariant under R_q. Consequently

$$R_q((z\,1_n - T) \mid X^n_T(\bar{U}'_q))^{-1} x' =$$

$$= ((z\,1_n - T) \mid X^n_T(\bar{U}'_q))^{-1} R_q x'$$

for every $x' \in X^n_T(\bar{U}'_q)$. This shows, in particular, that $\overline{R_q(X^n)}$ is invariant under $((z\,1_n - T) \mid X^n_T(\bar{U}'_q))^{-1}$, via (3.4). In other words, $(z\,1_n - T) \mid \overline{R_q(X^n)}$ is invertible, i.e. $\sigma(T \mid \overline{R_q(X^n)}) \subset \bar{U}_q$.

The proof of the theorem is complete.

3.9. REMARK. Let $\Omega_\tau = \tau(\Omega)$ and set

$$A_\tau = \{ f \in C(\Omega_\tau) : f \circ \tau \in A \},$$

which is a subalgebra of $C(\Omega_\tau)$. Then the map $\Phi_\tau : A_\tau \longrightarrow \mathcal{L}(X)$ given by $\Phi_\tau(f) = \Phi(f \circ \tau)$ is a unital algebra morphism. Suppose that A_τ has the properties (i) and (ii) from Definition 1.1 (this happens, for instance, when $A = C(\Omega)$). Then the morphism Φ_τ can be used instead of Φ. In this case there is no loss of generality in assuming that Ω is a compact subset of $\mathbb{C}^m$, where $m = n^2$, and that τ is the matrix of coordinate functions on $\mathbb{C}^m$, restricted to Ω.

4. MORE ABOUT THE SPECTRUM

In this section we assume that $A \subset C(\Omega)$ has the properties (i), (ii) and (iii) from Definition 1.1. As in the previous section, we fix a unital algebra morphism $\Phi : A \longrightarrow \mathcal{L}(X)$, an element $\tau = (\tau_{jk})^n_{j,k=1} \in M_n(A)$, and consider the (A,n)--scalar operator $T = \Phi_n(\tau) \in \mathcal{L}(X^n)$.

For every closed $F \subset \Omega$ we define the set

(4.1) $$S_{\tau,F} = \bigcup_{\omega \in F} \sigma(\tau(\omega)) \subset \mathbb{C}.$$

The set $S_{\tau,F}$ is closed (in fact compact), by Lemma 3.1. When $F = \text{supp}(\Phi)$, the set $S_{\tau,F}$ will be denoted simply by S_τ.

4.1. LEMMA. <u>For every</u> $h \in A$ <u>there exists an analytic function</u> $\varphi_h : \mathbb{C} \setminus S_{\tau,F} \to M_n(A)$ <u>such that</u> $(z 1_n - \tau)\varphi_h(z) = \delta(h)$ <u>for all</u> $z \notin S_{\tau,F}$, <u>where</u> $F = \text{supp}(h)$.

PROOF. Consider the Banach space $Y = A^n$ and the map $\Psi : A \to \mathcal{L}(Y)$ given by

$$\Psi(h) f_1 \oplus \ldots \oplus f_n = hf_1 \oplus \ldots \oplus hf_n,\ h, f_1, \ldots, f_n \in A.$$

Plainly, Ψ is a unital algebra morphism. Let $\Psi_n : M_n(A) \to \mathcal{L}(Y^n)$ be the unital algebra morphism induced by Ψ. If we indentify Y^n with $M_n(A)$, then, with this identification, $\Psi_n(\alpha)\beta = \alpha\beta$ for all $\alpha, \beta \in M_n(A)$. In particular, $\Psi_n(\tau)$ is the multiplication by the matrix τ, which will be also denoted by τ. The operator τ is (A,n)-scalar, and therefore it has the properties described in the previous section.

It is easily seen that $\delta(h) \in Y^n_\Psi(\text{supp}(h))$ (which is defined by (2.1)). According to Lemma 3.1, $\sigma(\tau \mid Y^n_\Phi(F)) \subset S_{\tau,F}$, where $F = \text{supp}(h)$. Consequently, we may take

$$\varphi_h(z) = ((z 1_n - \tau) \mid Y^n_\Psi(F))^{-1}\, \delta(h),\ z \notin S_{\tau,F}.$$

4.2. LEMMA. <u>Assume that there exists a compact subset</u> $L \subset S_\tau \setminus \sigma(T)$ <u>such that</u> $S_\tau \setminus L$ <u>is also compact. Then</u> $L = \emptyset$

PROOF. Let us assume that $L \neq \emptyset$. Let $V_1 \supset L$ and $V_2 \supset S_\tau \setminus L$ be open sets such that $\overline{V_1} \cap \overline{V_2} = \emptyset$. Then there is an open neighbourhood W of $\text{supp}(\Phi)$ such that $S_{\tau,\overline{W}} \subset V_1 \cup V_2$. We may also assume that $\Gamma = \partial V_1$ is a finite system of Jordan rectifiable curves, positively oriented.

Let $h \in A$ be such that $h = 1$ in a neighbourhood of supp (Φ) and supp (h) $\subset$ W. Let also φ_h be the analytic function given by Lemma 4.1, which is defined outside the set $S_{\tau,\overline{W}}$. Then we may consider the element

$$e = \frac{1}{2\pi i} \int_{\Gamma} \varphi_h(z)dz \in M_n(A).$$

Set $F_1 = \{\omega \in \Omega : h(\omega) = 1\}$. Since $\delta(h)(\omega) = 1_n$ for $\omega \in F_1$, then $\varphi_h(z)(\omega) = (z 1_n - \tau(\omega))^{-1}$. It follows from our assumption on the algebra A (Definition 1.1(iii)) that the point evaluations are continuous. Hence

$$e(\omega) = \frac{1}{2\pi i} \int_{\Gamma} (z 1_n - \tau(\omega))^{-1} dz, \qquad \omega \in F_1,$$

which shows that $e(\omega)^2 = e(\omega)$ ($e(\omega)$ is, in fact, a spectral projection of $\tau(\omega)$). Since F_1 is a neighbourhood of supp(Φ), it follows that $\Phi_n(e)$ is an idempotent. In addition, $\Phi_n(e)$ commutes with T because of the equality $\tau(\omega)e(\omega)=e(\omega)\tau(\omega)$ ($\omega \in F_1$).

Consider now the integral

$$e_w = \frac{1}{2\pi i} \int_{\Gamma} (w-z)^{-1} \varphi_h(z)dz, \qquad w \notin \overline{V}_1.$$

It is clear that

(4.2) $\quad (w 1_n - \tau(\omega))e_w(\omega)=e_w(\omega)(w 1_n - \tau(\omega)) = e(\omega)$

for all $\omega \in F_1$ and $w \notin \overline{V}_1$.

Since $\Phi_n(e)$ is idempotent , then $Z = \Phi_n(e)(X^n)$ is a closed subspace of X^n, invariant under T and under $\Phi_n(e_w)$ as well. Moreover, from (4.2) we deduce that

$$((w 1_n - T)|Z)(\Phi_n(e_w)|Z) = (\Phi_n(e_w)|Z)((w 1_n - T)|Z) = 1_Z$$

where 1_Z is the identity of Z. This shows that $\sigma(T|Z) \subset \overline{V}_1$. On the other hand, $\sigma(T) \subset \overline{V}_2$, by Remark 3.2 and the property of L.

Therefore $\sigma(T) \cap \sigma(T \mid Z) = \emptyset$, which is not possible unless $Z = \{0\}$. This shows that $\Phi_n(e) = 0$, so that $e(\omega) = 0$ for each $\omega \in \operatorname{supp}(\Phi)$, in virtue of Lemma 2.5, which contradicts our assumption. Indeed, if $z_0 \in L$, then there exists $\omega_0 \in \operatorname{supp}(\Phi)$ such that $z_0 \in \sigma(\tau(\omega_0))$. Then V_1 contains at least one point from the spectrum of the matrix $\tau(\omega_0)$, whence $e(\omega_0) \neq 0$.

Consequently we must have $L = \emptyset$.

4.3. LEMMA. Let $F \subset \Omega$ be closed and let

$$X_\Phi(F) = \bigcap \{\ker(\Phi(f)) : \operatorname{supp}(f) \cap F = \emptyset\}.$$

Then the space $X_\Phi(F)^n$ is invariant under T and the restriction $T \mid X_\Phi(F)^n$ is (A,n)-scalar.

PROOF. It is easily seen that $X_\Phi(F)^n = X_\Phi(F) \oplus \ldots \oplus X_\Phi(F)$ (n times) is invariant under T.

Since $X_\Phi(F)$ is invariant under $\Phi(f)$ for every $f \in A$, we may define the map

$$(4.3) \qquad A \ni f \to \Phi_F(f) = \Phi(f) \mid X_\Phi(F) \in \mathcal{L}(X_\Phi(F)),$$

which is a unital algebra morphism. If $\Phi_{F,n}$ is the unital algebra morphism from $M_n(A)$ into $\mathcal{L}(X_\Phi(F)^n)$ induced by Φ_F, then $T \mid X_\Phi(F)^n = \Phi_{F,n}(\tau)$, which is precisley our assertion.

4.4. REMARK. With the notation of Lemma 4.3, we have the inclusion $\sigma(T \mid X_\Phi(F)^n) \subset \sigma(T)$.

Indeed, if $z \notin \sigma(T)$, then the space $X_\Phi(F)^n$ is invariant under $(z 1_n - T)^{-1}$ since

$$\Phi_n(\delta(f))(z 1_n - T)^{-1} x = (z 1_n - T)^{-1} \Phi_n(\delta(f)) x = 0$$

for every $f \in A$ with $\operatorname{supp}(f) \cap F = \emptyset$ and each $x \in X_\Phi(F)^n$.

4.5. LEMMA. The morphism Φ_F from (4.3) has the following property:

$$\text{int}(F) \cap \text{supp}(\Phi) \subset \text{supp}(\Phi_F) \subset F \cap \text{supp}(\Phi).$$

for each closed F .

PROOF. Let X_F be the space $X_\Phi(F)$, defined in the preceding lemma. Let also $f \in A$ be such that $\text{supp}(f) \cap F \cap \text{supp}(\Phi) = \emptyset$. By using the normality of the algebra A, we can write $f = f_1 + f_2$, where $f_1, f_2 \in A$, $\text{supp}(f_1) \cap F = \emptyset$ and $\text{supp}(f_2) \cap \text{supp}(\Phi) = \emptyset$. Then $\Phi_F(f) = \Phi(f_1) | X_F + \Phi(f_2) | X_F = 0$, which shows that $\text{supp}(\Phi_F) \subset F \cap \text{supp}(\Phi)$.

Conversely, let $\omega_0 \in \text{int}(F) \cap \text{supp}(\Phi)$, let W_0 be an open neighbourhood of ω_0 such that $\overline{W}_0 \subset \text{int}(F)$, let $W_1 = \text{int}(F)$ and let $W_2 \subset \Omega$ be open such that $\overline{W}_2 \cap \overline{W}_0 = \emptyset$ and $W_1 \cup W_2 = \Omega$. Then, by Proposition 2.1 (with n=1), $X = X_{\overline{W}_1} + X_{\overline{W}_2} = X_F + X_{\overline{W}_2}$. If $f \in A$ and $\text{supp}(f) \subset W_0$, then $\Phi(f) | X_{\overline{W}_2} = 0$. Since $\omega_0 \in \text{supp}(\Phi)$, this shows that $\omega_0 \in \text{supp}(\Phi_F)$.

4.6. THEOREM. Let $T \in \mathcal{L}(X^n)$ be a (A,n)-scalar operator such that $T = \Phi_n(\tau)$. Then one has the equality

$$\sigma(T) = \bigcup \{ \sigma(\tau(\omega)) : \omega \in \text{supp}(\Phi) \} .$$

PROOF. The inclusion $\sigma(T) \subset S_\tau$ has been already noticed (see Remark 3.2).

Conversely, assume that there exists a point $z_0 \in S_\tau \setminus \sigma(T)$. Let $\omega_0 \in \Omega$ be such that $z_0 \in \sigma(\tau(\omega_0))$. Let V_1, V_2 be open sets in $\mathbb{C}$ such that $V_1 \ni z_0$, $V_2 \supset \sigma(T)$, $\overline{V}_1 \cap \overline{V}_2 = \emptyset$ and $\sigma(\tau(\omega_0)) \subset V_1 \cup V_2$. Then there exists an open set $W_0 \ni \omega_0$ in Ω such that $\sigma(\tau(\omega)) \subset V_1 \cup V_2$ for every $\omega \in F = \overline{W}_0$. According to Remark 4.4, we have the inclusion $\sigma(T_F) \subset \sigma(T) \subset V_2$, where $T_F = T | X_\Phi(F)^n$. On the other hand,

$$\cup \{\sigma(\tau(\omega)) : \omega \in \text{supp}(\Phi_F) \subset S_{\tau,F} \subset V_1 \cup V_2,$$

by virtue of Lemma 4.5. From the same lemma it also follows that $\omega_0 \in \text{supp}(\Phi_F)$. This shows that the set

$$L = \cup \{\sigma(\tau(\omega) : \omega \in \text{supp}(\Phi_F)\} \cap \overline{V_1}$$

is nonempty, which contradicts Lemma 4.2, applied to T_F. Therefore $S_\tau \setminus \sigma(T) = \emptyset$.

4.7. DEFINITION. The map $\Phi_n : M_n(A) \longrightarrow \mathcal{L}(X^n)$ is said to be of finite algebraic order if there exists an integer $m \geq 1$ such that from the fact that $\alpha(\omega) = 0$ for all $\omega \in \text{supp}(\Phi_n)$ and a certain $\alpha \in M_n(A)$, it follows that $\Phi_n(\alpha^m) = 0$.

If $A = C^r(\Omega)$ and $\Phi_n : M_n(A) \longrightarrow \mathcal{L}(X^n)$ is continuous, then for every $\beta \in M_n(A)$ which is null on $\text{supp}(\Phi_n)$ together wit[h] its partial derivatives up to order r, we have $\Phi_n(\beta) = 0$. This fact is well-known for scalar distributions and can be extended to vector distributions as well; an outline of proof can be found in [11], Lemma IV.8.8.

This fact shows, in particular, that Φ_n is of finite algebraic order $\leq r + 1$.

We can complete now Corollary 3.6 with the following statement.

4.8. COROLLARY. If $\sigma(T) = \{z_0\}$ and the morphism $\Phi_n : M_n(A) \longrightarrow \mathcal{L}(X^n)$ is of finite algebraic order, then $z_0 1_n - T$ is nilpotent.

In particular, if T is not a multiple of the identity, then T has a proper hyperinvariant subspace.

PROOF. If follows from Theorem 4.6 that $\sigma(\tau(\omega)) = \{z_0\}$ for every $\omega \in \text{supp}(\Phi)$. In other words, the matrix $z_0 1_n - \tau(\omega$ is nilpotent for each $\omega \in \text{supp}(\Phi)$, i.e. $(z_0 1_n - \tau(\omega))^n = 0$

($\omega \in \operatorname{supp}(\Phi)$).

Since the map Φ_n has finite algebraic order, then $\Phi_n((z_0 1_n - \tau)^{mn}) = 0$ for some integer $m \geqslant 1$, i.e. $z_0 1_n - T$ is nilpotent.

If T is not a multiple of the identity, then $\ker(z_0 1_n - T)$ is a proper hyperinvariant subspace of T.

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