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C O N T E N T S

INTRODUCTION

This work may be placed in the area of computational complexity using van Wijngaarden or W grammars or also two level grammars.

In chapter 1 is defined the notion of a W grammar. Some properties of W grammars are given.

In chapter 2 we introduce the notion of W system, closely related to the notion of W grammar. These systems play a similar role for recursively enumerable sets as the Ginsburg-Rice-Schutzenberger systems of equations for the description of context - free languages. Chapter 2 is optional.

The notion of a generative Blum space is introduced in chapter 3. These spaces play the same role for grammars as the Blum spaces for algorithms. Some new classes of computational complexity measures are considered.

In chapter 4 is proved a linear speed-up theorem for the complexity measure time W.

Chapter 5 begin by introducing a special class of functions, so called TW-self-bounded. This class is closed under sum, product and exponentation. This class is also closed under inverse mappings and composition of functions. The class of TW-self-bounded functions is rich.

In chapter 5 one also proves a special property of separation between complexity classes. Using this separation property some new hierarchies of recursively enumerable sets are obtained. A dense hierarchy of recursively enumerable sets is also found. The work finishes with final remarks and bibliographical references.

CHAPTER 1

W GRAMMARS

§1.1. *Preliminary*

In this paragraph we introduce the notion of *W grammar* or *van Wijngaarden grammar* or also *two-level grammar*.

If A is a set, then A^* is the set of all words over A including the empty word λ and $A^+=A^*-\{\lambda\}$.

The set of natural numbers is ω.

A *grammar without axiom* is an ordered system (V_N, V_T, P) such that for any S, $S\in V_N$, the system (V_N, V_T, S, P), is a Chomsky grammar.

A *context - free grammar without axiom* (*cfga*) is a grammar without axiom, (V_N, V_T, P) with $P\subset V_N \times (V_N \cup V_T)^*$.

The following definition of a W grammar is, with litle change, the same as definitions from (8), (32), (33) or (34):

Definition 1. A *W grammar* is an ordered system $G=(G_1,G_2,z)$, where $G_1=(M,V,R_M)$, $G_2=(H,\Sigma,R_H)$ are cfga's with the property that the elements of H are of the form $\langle h\rangle$, where $h\in(M\cup V)^+$, $(M\cup V\cup\Sigma)\cap\{<,>\}=\emptyset$ and $z\in H$, $z=\langle v\rangle$ with $v\in V^+$.

Terminology G_1 is the *metalevel of* G and the elements of the sets M,V,R_M are called the *metanotions*, *metaterminals* and *metarules*. G_2 is the *hyperlevel of* G and the elements of the sets H,Σ,R_H are called the *hypernotions*, *terminals* and *hyperrules*. The symbol z is the *start symbol*.

We define now the language generated by a W grammar.

For each X, $X\in M$, let L_X be the language generated by the context-free grammar $G_X=(M,V,X,R_M)$. L_X is *the language of the metanotion* X.

$G_1=(M,V,R_M)$ and $G_2=(H,Z,R_H)$. The language of the metanotion A is $L_A=\{v\}^+$. The set of strict rules, R_s, is:

$$\langle s\rangle \longrightarrow \langle v^i\rangle,\ i\geqslant 1$$
$$\langle v^{2j}\rangle \longrightarrow \langle v^j\rangle\langle v^j\rangle,\ j\geqslant 1$$
$$\langle v\rangle \longrightarrow a$$

and the set of strict hypernotions is:

$$H_s=\{\langle v^i\rangle \mid i\geqslant 1\}\cup\{\langle s\rangle\}$$

Note that, if a derivation begin with a rule $\langle v\rangle\longrightarrow\langle v^i\rangle$, where i is not 2^n for some $n\in\omega$, then this derivation terminate without producing a terminal word. Therefore, the language generated by G is:

$$L(G)=\{a^{2^n} \mid n\in\omega\}$$

§1.2. <u>Posets and systems</u>

If A and B are sets, then:

$$B^A=\{f \mid f:A\longrightarrow B,\ f \text{ function}\}.$$

If $(B,\leqslant)$ is a partial ordered set (poset), then B^A is a poset with the realtion:

$$f\leqslant g \text{ iff for any } x,\ x\in A,\ f(x)\leqslant g(x).$$

If A is a set, then P(A) is the set of all subsets of A and P(A) is a poset with the relation of inclusion.

The same symbol denotes the polynom and its associated polynomial function.

Let $G=(V_N, V_T, P)$ be a cfga and define for any x, $x \in V_N$, the set $D_x = \{u \mid x \rightarrow u \in P\}$. Obviously, $D_x \subset V_T^*[V_N]$, for all $x \in V_N$.

Consider now the system of equations:

$$E = \{x = D_x \mid x \in V_N\}$$

A function $f: V_N \rightarrow P(V_T^*)$ is a solution of E, iff $f(x) = \bigcup_{u \in D_x} u(f)$, for any $x \in V_N$.

Define the grammar function associated to G:

$$F : P(V_T^*)^{V_N} \longrightarrow P(V_T^*)^{V_N}$$

$$F(f)(x) = \bigcup_{u \in D_x} u(f).$$

Any fixed-point of F is a solution of E and conversely. We consider only the minimal solutions.

One should note that $P(V_T^*)^{V_N}$ is an ω-complete poset with bottom element, $\bot$, where $\bot : V_N \rightarrow P(V_T^*)$, $\bot(x) = \emptyset$ for any $x \in V_N$. F is an ω- continuous function. Thus, from Kleene's theorem, F has a least fixed-point, $f_o = \bigvee_{n \in \omega} F^{(n)}(\bot)$.

For any $x \in V_N$, consider the context-free grammar $G_x = (V_N, V_T, x, P)$ and let L_x be the language generated by G_x.

According to Ginsburg-Rice-Schutzenberger theorem, $f_o(x) = L_x$, for any $x \in V_N$.

Notation. For any $n \in \omega$ and $x \in V_N$:

$$L_{n,x} = F^{(n)}(\bot)(x)$$

Notation. For any $n\in\omega$, $K_n=L_{n,n,z}$.

We denote also $K_n(\langle u\rangle)$ the set $L_{n,n,\langle u\rangle}$, where $u\in V^*$.

Exemple 6. Consider again the W-grammar from exemple 3. The grammar function associated to G_1 is:

$$F : P(\{s,v\}^*)^{\{A\}} \longrightarrow P(\{s,v\}^*)^{\{A\}}$$

$$F(f)(A)=\{v\}f(A)\cup\{v\}$$

Observe that $F^{(0)}(\perp)(A)=\perp(A)=\emptyset$, $F^{(1)}(\perp)(A)=\{v\}$, $F^{(2)}(\perp)(A)=\{v,v^2\}$ and in general $F^{(n)}(\perp)(A)=\{v^i \mid 1\leq i\leq n\}$, $n\geq 1$. Thus, $L_{n,A}=\{v^i \mid 1\leq i\leq n\}$ and the context-free grammar $G_{2,n}$ has the rules:

$$\langle s\rangle\rightarrow\langle v^i\rangle,\quad 1\leq i\leq n$$

$$\langle v^{2j}\rangle\rightarrow\langle v^j\rangle\langle v^j\rangle,\quad 1\leq j\leq n$$

$$\langle v\rangle\rightarrow a$$

Moreover, it is easy to prove, that the sets $K_n=F_{2,n}^{(n)}(\perp)(\langle s\rangle)$ are: $K_0=K_1=\emptyset$, $K_2=\{a\}$ and $K_n=\{a^{2^i} \mid 0\leq i\leq \log_2 n\}$ for any $n\geq 3$.

Remark 7. Note that for any n,m and x, $L_{n,m,x}$ is a finite set and thus K_n is finite. Moreover, it is easy to show that $L(G)=\bigcup_{n\in\omega} K_n$.

Sintzoff (28) obtained that for any recursively enumerable set E, there exists a W grammar G, such that L(G)=E.

This remark is the key to define some new complexity measures (see Chapter 3).

CHAPTER 2

W SYSTEMS

We introduce the notion of W systems. No detailed knowledge of this chapter is required to understand chapters 3-5.

§2.1. Preliminary

Definition 1. A context-free system (cfs) is a 3-ordered system $S_1=(X, V, F)$, where X is a set of variables, V is a monoid of coefficients, $X \cap V=\emptyset$ and:

$$F : P(V)^X \longrightarrow P(V)^X$$

$$F(f)(x) = \bigcup_{u \in D_x} u(f),$$

where for any $x \in X$, $D_x \subset V[X]$, D_x is fixed.

Remark 2. X or D_x , $x \in X$, can be infinite sets. The set of equations is $E_1=\{x=D_x \mid x \in X\}$. The least fixed-point of F is the solution of S_1. One should note that for any cfs, S_1; there exists $f_o=\bigvee_{n\in\omega} F^{(n)}(\bot)$ and f_o is the solution of S_1.

Definition 3. Let $S_1=(X, V, F)$ be a cfs. We say that $S_2=(H,\Sigma,R)$ is a parametric context-free system (pcfs) over S_1 iff S_2 is a cfs, $H \subset V[X]$ and $\Sigma \cap (H \cup X \cup V)=\emptyset$.

The set of equations of S_2 is E_2, i.e. $E_2=\{h=D_h \mid h \in H\}$, where for any $h \in H$, $D_h \subset \Sigma[H]$, and D_h is fixed.

Note that, if $f\in P(V)^X$ and $f(x)=\emptyset$ for some $x\in X$, then $T_f=\emptyset$, $E_{2,f}=\emptyset$ and $S_{2,f}$ is the empty system, denoted $S_{2,\bot}$.

If $f_o\in P(V)^X$ is the solution of S_1, then the f_o-strict system associated to S_2 is the <u>strict system</u> and is denoted $S_{2,s}$.

<u>Definition 7</u>. If $S=(S_1, S_2)$ is a W system, then the <u>solution of S</u> is the solution of the strict system $S_{2,s}$.

<u>Definition 8</u>. If $S=(S_1, S_2)$ is a W system, then $\mathcal{T}=\{S_{2,f}|f\in P(V)^X\}$ is the <u>space of strict systems</u> and $\mathcal{P}=\{g|g\in \in P(\Sigma)^{H_f}, f\in P(V)^X\}$ is <u>the space of strict solutions</u>.

Note that, if $f,g\in P(V)^X$ and $f\leq g$, then $T_f\subset T_g$, $H_f\subseteq H_g$, for any $v\in H_f$, $D_{v,f}\subseteq D_{v,g}$, and for any $v\in H_f$ and for any $j\in P(\Sigma)^{H_f}$ $R_f(j)(v)\subseteq R_g(j)(v)$. It is natural to introduce the relation $S_{2,f}\leq S_{2,g}$ iff $f\leq g$.

Observe that, as defined above, $(\mathcal{T}, \leq)$ is a poset, ω-complete with bottom element $S_{2,\bot}$, the empty system.

If $f\in P(V)^X$ and $j\in P(\Sigma)^{H_f}$, then, since $H_f\subseteq V$, we can consider $j'\in P(\Sigma)^V$, the natural extension of j, $j'(v)=\emptyset$ for any $v\in V-H_f$ and $j'(v)=j(v)$ for any $v\in H_f$.

If $f,g\in P(V)^X$, $j\in P(\Sigma)^{H_f}$ and $l\in P(\Sigma)^{H_g}$, then we consider $j\leq l$ iff $j'\leq l'$ as elements of $P(\Sigma)^V$. With this relation, $(\mathcal{P},\leq)$ is a poset, ω- complete with bottom element $\bot$, the bottom element of $P(\Sigma)^V$.

§2.2. <u>Strategies for solving W systems</u>

<u>Definition 9</u>. Let $S=(S_1, S_2)$ be a W system and $\mathcal{T}$ the corresponding space of strict systems. The <u>transfer function</u>

$K^{(n,m)}(\perp)\leqslant K^{(n',m)}(\perp)$. Note that $R^{(m'-m)}Q'(\perp)$ and that $R_{Q'}^{(m)}$ is monotonic, so $K^{(n',m)}(\perp)=R_{Q'}^{(m)}(\perp)\leqslant R_{Q'}^{(m)}(R_{Q'}^{(m'-m)}(\perp))=$ $=R_{Q'}^{(m')}(\perp)=K^{(n',m')}(\perp)$.

Definition 13. A chain is a set I, $I\subset\omega^2$, such that I is totally ordered. An unbounded chain is a chain I such that for any $(n,m)\in\omega^2$ there exists $(n',m')\in I$, with $(n,m)\leqslant$ $\leqslant(n',m')$.

Remark. If I is a chain, then $\{K^{(n,m)}(\perp)\}_{(n,m)\in I}$ is an increasing sequence in $\mathcal{P}$(proposition 12). Moreover, $\mathcal{P}$ is ω-complete and therefore $\bigvee_{(n,m)\in I} K^{(n,m)}(\perp)\in\mathcal{P}$.

Some strategies for finding the solution of a W system are given by the next:

Theorem 14. If $S=(S_1,S_2)$ is a W system with the solution f_S, then:

$$f_S=\bigvee_{m\in\omega}(\bigvee_{n\in\omega}K^{(n)}(R_\perp))^{(m)}(\perp)=\bigvee_{n\in\omega}(\bigvee_{m\in\omega}(K^{(n)}(R_\perp)^{(m)}(\perp))=$$

$$=\bigvee_{(n,m)\in I}K^{(n,m)}(\perp),\text{ for any unbounded chain I.}$$

(see notation 11).

Proof. Let I be a fixed unbounded chain and denote

$$\alpha=\bigvee_{m\in\omega}(\bigvee_{n\in\omega})K^{(n)}(R_\perp))^{(m)}(\perp),$$

$$\beta=\bigvee_{n\in\omega}(\bigvee_{m\in\omega}(K^{(n)}(R_\perp))^{(m)})(\perp)\text{ and}$$

There exists $m\in\omega$, such that, $u\in(\bigvee_{n\in\omega} K^{(n)}(R_\perp))^{(m)}(\perp)(v)$.

By induction on m, there is a $(p,q)\in I$ such that $u\in K^{(p,q)}(\perp)(v)$.

Since $K^{(p,q)}(\perp)(v)\subseteq \bigvee_{(p,q)\in I} K^{(p,q)}(\perp)(v)=\gamma(v)$,

we deduce that $u\in\gamma(v)$ and therefore $\gamma\geq\alpha$.

Comments. This theorem says that the solution of a W system (the function α) is equal to the least upper bound of all solutions of partial systems $S_2, F^{(n)}(\perp)$ (the function β) and it is also equal to the least upper bound of all m-partial solutions of the n-partial systems, when (n,m) belongs to an arbitrary unbounded chain I.

§2.3. W systems and recursive enumerable sets

In this section we prove that for every recursively enumerable set, E, there exists a W-system S, such that E is a component of the solution of S.

Theorem 15. If $G=(G_1,G_2,z)$ is a W grammar, there exists a W system S, such that $f_S(z)=L(G)$, where f_S is the solution of S.

Proof. Let $G_1=(M,V,R_M)$ and $G_2=(H,\Sigma,R_H)$ by the metalevel respective the hyperlevel of G. We define the W system $S=(S_1,S_2)$, where the metasystem is $S_1=(M,V^*,F)$ with:

$$F : P(V^*)^M \longrightarrow P(V^*)^M$$

$$F(f)(m)=\bigcup_{u\in D_m} u(f) \quad \text{where}$$

$D_m=\{u \mid m\rightarrow u\in R_M\}$ for any $m\in M$.

Proposition 19. For any unbounded chain I, it is undecidable for an arbitrary W system and for a fixed $v \in V$, if the sequence $\{K^{(n,m)}(1)(v)\}_{(n,m)\in I}$ contains only a finite number of distinct terms.

Proof. From remark 18, the decidability of this problem imply the decidability of the problem if a recursively enumerable set is finite or not. But this problem is undecidable.

Many other problems like: the emptyness problem, membership problem, inclusion problem, equivalence problem etc., are undecidable for W systems.

$D=(d_i)_{i\in\omega}$ is a Gödel enumeration of all re languages over Σ using generative devices of type $\mathcal{D}$, $A=\{A_i | _{i\in\omega}\ A_i:\omega\to\omega,$ A_i partial recursive function$\}$ is the set of the cost functions and $C=(C_n)_{n\in\omega}$ is a criterion over Σ such that are satisfied the following two axioms:

GB1) $A_i(n)<\infty$ iff $L(d_i)\cap C_n\neq\emptyset$

GB2) The predicate:

$$R(i,x,y)=\begin{cases}1 & \text{if } A_i(x)=y \text{ and} \\ 0 & \text{otherwise}\end{cases}$$

is totaly recursive.

Note that GB2 is the same as the classical axiom of Blum (3) but GB1 say that the cost function A_i is defined in n iff $L(d_i)$ "satisfy" the criterion C in n, i.e. $L(d_i)\cap C_n\neq\emptyset$.

§3.2. Criterions, measures and complexity classes

Next we consider some exemples.

Exemple 3. Let Σ be a fixed alphabet and consider the sequence $C=(C_n)_{n\in\omega}$, where $C_n=\Sigma^n$ for all $n\in\omega$. Obviously, C is a criterion over Σ, called the equal to n criterion and denoted by "=n".

The sequence $C=(C_n)_{n\in\omega}$, where $C_0=\Sigma^*$ and $C_n=\Sigma^*-\bigcup_{i=0}^{n-1}\Sigma^i$, $n\geq 1$ is also a criterion over Σ, called the greater than or equal to n criterion and denoted by "$\geq$n".

For the criterions "=n" and "$\geq$n" the value of the cost function $A_i(n)$ measure the resources necessary for the i-th device to generate a word of length equal to n, respective of length greater than or equal to n.

Observe that the axiom GB1 is satisfied, because $TW_i(n)<\infty$ iff there exists $r\in\omega$ such that $K^i_r \cap C_n \neq \emptyset$ iff $(\bigcup_{j\in\omega} K^i_j)\cap C_n \neq \emptyset$ iff $L(w_i)\cap C_n \neq \emptyset$.

Also, the axiom GB2 is satisfied. In order to compute the value of $R(i,x,y)$ we consider the following algorithm:

<u>Step 0</u> Input i, x and y

<u>Step 1</u> From i determine the W-grammar w_i (or the W-system S_i)

<u>Step 2</u> If there exists j, $0\leq j<y$ such that $K^i_j \cap C_x \neq \emptyset$ then $R(i,x,y)=0$ STOP else:

<u>Step 3</u> If $K^i_y \cap C_x \neq 0$ then $R(i,x,y)=1$ STOP else $R(i,x,y)=0$ STOP.

Note that $\{K^i_j\}_{j\in\omega}$ is an increasing sequence of finite sets and for any i and j, K^i_j may be effectivelly computed. Moreover, C is a criterion and therefore Step 2 and Step 3 may be executed.

In view of the preceding remarks, the ordered system $\mathcal{G}_{WT}=(\Sigma,W,TW,C)$ is a GBS, called the <u>W-time GBS</u>.

We can also consider the cost functions $SW=\{SW_i | SW_i:\omega\to\omega$ where:

$$SW_i(n)=\begin{cases}\text{card}(K^i_m), \text{ where } m=TW_i(n), \text{ iff } TW_i(n)<\infty \text{ and}\\ \infty, \text{ otherwise}\end{cases}$$

The value $SW_i(n)$, where $SW_i(n)<\infty$, is equal to the number of words, (over Σ) involving in the process of Kleene iterations associated to W_i, necessary to generate a first word from C_n.

Intuitivelly, SW_i measure the "space" or the "memory"

Define the cost functions MT like in exemple 6 and observe again that we obtain a GBS.

Next we define the notion of generative complexity class in a classical manner.

Definition 8. If $\mathcal{G}=(\Sigma,D,A,C)$ is a GBS and $f:\omega\rightarrow\omega$ is a recursive function, then the generative complexity class bounded by f is:

$$C_{\mathcal{G}}(f(n))=\{L \mid L\subseteq\Sigma^*,\ \exists i\in\omega,\ L(d_i)=L \text{ and } A_i(x)\leq f(x) \text{ almost everywhere (ae)}\}.$$

Remark 9. For the generative Blum spaces $\mathcal{G}_{WT}$ respective $\mathcal{G}_{WS}$ the generative classes bounded by f will be denoted by TIMEW(f(n)) respective SPACEW(f(n)).

Exemple 10. Consider the GBS, $\mathcal{G}=(\{a\},W,TW,"\geq n")$ and consider also the W grammar G from exemple 3 (6) Ch.I. For this grammar $TW_G(0)=TW_G(1)=2$, $TW_G(2)=3$, $TW_G(3)=4$ etc. and observe that, for exemple $TW_G(n)\leq 2n$ a.e. Therefore $L(G)=\{a^{2^n} \mid n\in\omega\}$ is in TIMEW(2n).

REMARK. For $L=\{a^{2^n} \mid n\in\omega\}$ we can find a faster W-grammar. For exemple consider $R_H=\{A\rightarrow AA,\ A\rightarrow v^2,\ A\rightarrow v\}$ and R_H:

$\langle s\rangle\rightarrow\langle A\rangle$

$\langle AA\rangle\rightarrow\langle A\rangle\langle A\rangle$

$\langle v^4\rangle\rightarrow a^4$

$\langle v^2\rangle\rightarrow a^2$

$\langle v\rangle\rightarrow a$

$z \rightarrow z$ if $L=\emptyset$).

Clearly, we can decide if a W-grammar is trivial or not (of course this not implies the decidability of emptiness or finitess problem for W-grammars!).

In what follows, we assume that, (for D=W) the empty language and the finite languages are in the complexity class $C_{\mathcal{G}}(1)$, respective $C_{\mathcal{G}}(2)$, for any criterion C.

Of course this is a <u>convention</u>!

<u>Comments</u>. The works of Blum (3), Book (4) and Gladkij (11) (see also Salomaa (27)) inspired us the basic notion of GBS.

then we define $w_j=(G_1', G_2', z)$ with $G_1'=(M',V',R_M')$, $G_2'=(H',\Sigma,R_H')$, where: $M'=M\cup\{Y\}$, $V'=V\cup\{\#,?\}$ such that Y, # and ? are not in $M\cup V\cup\Sigma$.

Define $R_M'=R_M\cup R_M''$, where R_M'' contains all metarules which are obtained from R_M using the next method:

for each metarule $X_o\rightarrow\alpha_o X_1\alpha_1,\ldots,X_m\alpha_m\in R_M$ and for each m-tuplu, $(\pi_1,\ldots,\pi_m)$ of metarules from R_M, where $\pi_t:X_t\rightarrow\beta_t$, $t=1,\ldots,m$ with $X_t\in M, \alpha_t\in V^*$, $t=0,\ldots,m$ and $\beta_t\in(M\cup V)^*$, $t=1,\ldots,m$ we introduce in R_M'' the metarule:

$$X_o\rightarrow\alpha_o\beta_1\alpha_1\ldots\beta_m\alpha_m \tag{2}$$

Also, we add to R_M'' the metarules $Y\rightarrow v$ and $Y\rightarrow vY$ for all $v\in V^*$, with $|v|\leqslant p$ where $p=p_1\cdot p_2$, $p_1=\max\{|h|\;|\langle h\rangle\in H\}$ and $p_2=\max\{|\alpha|\;|\;X\rightarrow\alpha\in R_M\}$.

Observe that for any $X\in M$, G_1 and G_1' generate the same metalanguage, L_X.

Let F_i and F_j be the grammar functions associated to G_1 respective to G_1'.

By induction on ℓ, we prove that for any $X\in M$,

$$F_i^{(2\ell)}(\bot)(X)\subseteq F_j^{(\ell)}(\bot)(X) \tag{3}$$

For $\ell=0$, $F_i^{(0)}(\bot)(X)=\emptyset=F_j^{(0)}(\bot)(X)$, $X\in M$.

For the inductive step, let $u\in F_i^{(2\ell+2)}(\bot)(X)$ be an arbitrary element. But, $F_i^{(2\ell+2)}(\bot)(X)=F_i(F_i(F_i^{(2\ell)}(\bot)))(X)$ and therefore there is $X\rightarrow\alpha_o X_1\ldots X_n\alpha_n\in R_M$ and the words $u_1,\ldots,u_n$, $u_K\in F_i(F_i^{(2\ell)}(\bot))(X_K)$, $K=1,\ldots,n$ such that $u=\alpha_o u_1\ldots u_n\alpha_n$.

From $u_K\in F_i(F_i^{(2\ell)}(\bot))(X_K)$ we deduce that there exists the metarules $X_K\rightarrow\alpha_o^K X_1^K\ldots X_{nK}^K\alpha_{nK}^K\in R_M$ and the words $u_r^K\in F_i^{(2\ell)}(\bot)(X_r^K)$

where $\langle \bar{u}_t \rangle$ and $\bar{\gamma}_t$, t=1,...,n, are obtained from the hyper-rule $\langle u_t \rangle \to \gamma_t$ by renaming the metanotions such that they are distinct from those in $\langle h_o \rangle \to \sigma_o \langle h_1 \rangle \sigma_1 \ldots \langle h_n \rangle \sigma_n$ and also are distinct from those in $\langle \bar{u}_i \rangle \to \bar{\gamma}_i$, for any i<t.

Note that this new metanontions does not destroy the speed of metalevel.

We prove that $L(w_i)=L(w_j)$. Obviously, $L(w_i) \subseteq L(w_j)$, because $R_H \subset R'_H$ and the metalanguages L_X , $X \in M$, are the same.

For the contrary inclusion, assume that in w_j is used, in a terminal derivation, a strict rule:

$$\langle \varphi(h_o) \rangle \to \sigma_o \varphi(\bar{\gamma}_1) \sigma_1 \ldots \varphi(\bar{\gamma}_n) \sigma_n \langle \varphi(h_1) \# \varphi(\bar{u}_1) ? \rangle \ldots \ldots \langle \varphi(h_n) \# \varphi(\bar{u}_n) ? \rangle \qquad (8)$$

which is obtained from a hyperrule of type (7).

The derivation is terminal and therefore, $\langle \varphi(h_t) \# \varphi(\bar{u}_t) ? \rangle$ must derive in λ using the hyperrule $\langle Y \# Y ? \rangle$

Result that $\varphi(h_t)=\varphi(\bar{u}_t)$, t=1,..,n. We deduce that this step of derivation in w_j may be replaced in w_i using first the strict rule $\langle \varphi'(h_o) \rangle \to \sigma_o \langle \varphi'(h_1) \rangle \sigma_1 \ldots \langle \varphi'(h_n) \rangle \sigma_n$ with $\varphi' = \varphi|_{M \cup V}$ and after the strict rules $\langle \varphi_t(u_t) \rangle \to \varphi_t(\gamma_t)$, t=1,...,n, where $\varphi_t(u_t)=\varphi(\bar{u}_t)=\varphi(h_t)$ and thus $L(w_i)=L(w_j)$.

It is not difficult to prove, by induction on ℓ like for the metalevel, that for any $\ell \in \omega$,

$$K^i_{2\ell} \subseteq K^j_{\ell+1}$$

and therefore for any $\ell \in \omega$,

$$K^i_\ell \subseteq K^j_r \qquad (10)$$

$$f_{p'}(n) \leqslant \max\{[qf(n)],3\}$$

and thus TIMEW($f_{p'}(n)$) $\subseteq$ TIMEW($qf(n)$).

From (11) result that TIMEW($f(n)$) $\subseteq$ TIMEW($qf(n)$) and therefore TIMEW($f(n)$)=TIMEW($qf(n)$).

If $q>1$, then we put $q'=\frac{1}{q}$, $g(n)=qf(n)$ and therefore

$$\text{TIMEW}(qf(n))=\text{TIMEW}(g(n))=\text{TIMEW}(q'g(n))=\text{TIMEW}(f(n)).$$

Remark 2. For a concrete construction see (24).

Corollary 4. If g is a polynom, $g(n)=a_0n^K+a_1n^{K-1}+\ldots+a_K$, $a_i\in\omega$, $i=0,\ldots,K$, $K\geqslant1$, then

$$\text{TIMEW}(n^K) = \text{TIMEW}(g(n)).$$

§4.2. The linear speed-up of others measures

Comments. The study of Turing time complexity and the linear speed-up of this measure can be said to begin with Hartmanis and Stearns (13) (see also (16)). For the Turing space complexity (12), (16) was proved a similar result with our theorem 1. For the Turing time complexity are necessary some restrictions (see (16) Ch.12.2).

For Chomsky grammars, Book (4), Gladkij (11) (see also (27) Ch.IX) was introduced the time measure:

$$T_G(n)=\max\{t_G(P)\mid P\in L_S(G)\cap V^n\}$$

where $G=(V_N,V_T,X_o,\Pi)$, $V=V_N\cup V_T$, $L_S(G)=\{\alpha\mid\alpha\in V^*,\ X_o\overset{*}{\Rightarrow}\alpha\}$ and

$t_G(P)$ is the least integer m, $m \geq 1$ such that $S \overset{m}{\Longrightarrow} P$.

A linear speed-up theorem is proved for this measure and it is found that L(G) must to be a <u>recursive language</u>!

Therefore, $T_G(n)$ cannot be used as cost function asociated to G when L(G) is a re language which is not recursive.

CHAPTER 5

HIERARCHIES OF RECURSIVE ENUMERABLE SETS

We shall now introduce the term "TW-self-bounded function" which is to designate the ability to generate the graphic of a function f in TIMEW(f(n)) relative to the functional criterion "a^n". Also, the graphic of f, can be generate in TIMEW(logf(n)) (see remark 7.5.1). The family of this functions is closed under sum, product, exponent and with supplementary conditions under composition and inverse mapping.

The main result is in section 5.2, the TW-separation property. This permits to obtain some hierarchies of recursive enumerable sets (section 5.3). A dense hierarchy is also obtained.

§5.1. TW-self-bounded functions. Properties.

Definition 1. A W grammar G is related to the graphic (rg) of function f, $f:\omega\to\omega$ in case: $\{i,\#\}\subset V$, $L_X=\{i\}^*$, for any $X\in M$, $R_H\subset H\times H^*$ and for any $n,m\in\omega$:

$$z\xrightarrow[G]{*}\langle i^n\# i^m\rangle \text{ if and only if } f(n)=m. \quad (S).$$

Remark 2. Obviously, if G is rg to f, then $L(G)\subseteq\{\lambda\}$.

Definition 3. Let $f:\omega\to\omega$ be a function; then f is TW-self bounded (TWsb) if it is a constant function or there is a W grammar G rg to f and the constants $n_0, c_1, c_2 \in \omega$ such that for any n, $n \geq n_0 : f(n) > 1$ and if we add to the hyperrules of G, the hyperrule $\langle i^n \# i^{f(n)} \rangle \longrightarrow a$, then $K_{\mathfrak{f}} = \{a\}$, where $\mathfrak{f} = c_1 f(n) + c_2$.

Definition 4. Let $f : R_+ \to R_+$ be a function; then f is TW-self bounded if the function $[f]$, $[f] : \omega \to \omega$, $[f](n) = [f(n)]$ is TWsb.

Notation. The set of TWsb functions is denoted by Am.

Exemple 5. The function $f : \omega \to \omega$, $f(n) = n$ is TWsb. Consider the W grammar G, where: $M = \{N\}$, $V = \{i, \#\}$, $R_M : N \to iN | i | \lambda$ and $R_H : z \longrightarrow \langle N \# N \rangle$. Obviously G is a W grammar rg to f. Moreover for $n_0 = 2$, $c_1 = 1$ and $c_2 = 0$ we remark that the conditions of definition 3 are satisfied and therefore $f \in A_m$.

Theorem 6. If $f : \omega \to \omega$, $f \in A_m$ and $f(n) \geq \log_2 n$, for any $n \in \omega$, then in the GBS, $\mathcal{G} = (\Sigma, W, TW, "a^n")$, $L_f \in TIMEW(f(n))$, where $L_f = \{a^n f^{f(n)} | n \in \omega\}$.

Proof. Let G be the W grammar rg to f which exists cf definition 3. We define the W gramar G', where: $M' = M \cup \{N, N1\}$, $V' = V \cup \{\alpha, \beta\}$, $R'_M = R_M \cup \bigcup_{X \in M} \{N \to X,\ N1 \to X\}$, $z' = z$, and $R'_H = R_H \cup \{\Pi_i | i = 1, \ldots, 7\}$ where:

$\Pi_1)$ $\langle N \# N1 \rangle \to \langle \alpha N \rangle \langle \beta N1 \rangle$

$\Pi_2)$ $\langle \alpha NN \rangle \longrightarrow \langle \alpha N \rangle \langle \alpha N \rangle$

$\Pi_3)$ $\langle \alpha iNN \rangle \longrightarrow a \langle \alpha \mathring{N} \rangle \langle \alpha N \rangle$

$\Pi_4)$ $\langle \alpha \rangle \longrightarrow \lambda$

π_5) $\langle\beta NN\rangle \to \langle\beta N\rangle\langle\beta N\rangle$

π_6) $\langle\beta iNN\rangle \to b\langle\beta N\rangle\langle\beta N\rangle$

π_7) $\langle\beta\rangle \to \lambda$

Observe that in G', a terminal derivation, it is of the form:

$$z \overset{*}{\underset{R_H}{\Longrightarrow}} \langle i^n \# i^{f(n)}\rangle \underset{\pi_1}{\Longrightarrow} \langle\alpha i^n\rangle\langle\beta i^{f(n)}\rangle \overset{*}{\underset{\pi_2-\pi_4}{\Longrightarrow}} a^n\langle\beta i^{f(n)}\rangle \overset{*}{\underset{\pi_5-\pi_7}{\Longrightarrow}} a^n b^{f(n)}$$

From definition of the grammar G' and from condition (S), definition 1, $z \overset{*}{\underset{G'}{\Longrightarrow}} w$, $w\in\Sigma^*$ if and only if there is a $n\in\omega$, such that $w=a^n b^{f(n)}$ and therefore $L_f=L(G')$.

But f is TWsb and therefore for any $n\geqslant n_o$ if $u\in K'_p(\langle\alpha i^n\rangle)$ and $v\in K'_p(\langle\beta i^{f(n)}\rangle)$, then $uv\in K'_{p+\ell}(z)$, where $\ell=c_1 f(n)+c_2$.

The derivation of $\langle\alpha i^n\rangle$ and $\langle\beta i^m\rangle$ in terminal strings require $\log_2 n+1$ iterations and respectively $\log_2 m+1$ iterations

Thus, for $p\geqslant\max\{\log_2 n,\ \log_2 f(n)\}+1$, $a^n\in K'_p(\langle\alpha i^n\rangle)$ and $b^{f(n)}\in K'_p(\langle\beta i^{f(n)}\rangle)$.

We deduce that, for $n\geqslant n_o$, $a^n b^{f(n)}\in K'_q$ where:

$$q=c_1 f(n)+c_2+\max\{\log_2 n,\ \log_2 f(n)\}+1\leqslant(c_1+1)f(n)+c_2+1$$

The inequality is true because $f(n)\geqslant\log_2 n$.

Therefore, $TW_{G'}(n)\leqslant(c_1+1)f(n)+c_2+1$, for any $n\geqslant n_o$ and thus $L_f\in TIMEW((c_1+1)f(n)+c_2+1)$.

From the speed-up theorem result that $L_f\in TIMEW(f(n))$.

Remark 7. Moreover, if $f\in A_m$, then also the graphic of

$g(n)=2^{f(n)}$ is in TIMEW(f(n)) (see exemple 3.11 or the proof of theorem 5.2.14).

Next theorem has a long but not difficult proof.

Theorem 8. The family A_m is closed under sum, product and exponent. That means, if $f,g \in A_m$, then $f+g$, $f.g$, $f^g \in A_m$, where:

$$(f+g)(n)=[f(n)]+[g(n)],\ (f.g)(n)=[f(n)].[g(n)] \text{ and } f^g(n)= =[f(n)]^{[g(n)]}.$$

Proof. A_m is closed under sum. Let $f,g \in A_m$ be two non-constant functions and let $G_i=((M_i, V_i, R_M^i), (H_i,\Sigma,R_H^i),z_i)$ $i=1,2$ be the W grammars rg to f and respective to g which exists cf. definition 3. We define the W grammar $G_3=((M_3,V_3, R_M^3), (H_3,\Sigma,R_H^3),z_3)$ such that:

$$M_3=M_1 \cup M_2 \cup \{N, N1, N2, N3\},\ V_3=V_1 \cup V_2 \cup \{¢, \$, ?\}$$

where ¢, \$ and ? are new symbols,

$$R_M^3=R_M^1 \cup R_M^2 \cup \bigcup_{X \in M_1 \cup M_2} \{N \to X,\ N1 \to X,\ N2 \to X,\ N3 \to X\}$$

$$R_H^3 = \tilde{R}_H^1 \cup \tilde{R}_H^2 \cup \{\pi_1, \pi_2, \pi_3, \pi_4\} \text{ where:}$$

$\tilde{R}_H^1$ is obtained from R_H^1 by replacing all hypernotions $\langle h \rangle \in H_1$ with $\langle ¢h \rangle$, $\tilde{R}_H^2$ is obtained from R_H^2 by replacing all hypernotions $\langle h \rangle \in H_2$ with $\langle N \# N1\$h \rangle$ and:

π_1) $z_3 \longrightarrow \langle \cent v_1 \rangle$, iff $z_1 = \langle v_1 \rangle$

π_2) $\langle \cent N\#N1 \rangle \longrightarrow \langle N\#N1\$v_2 \rangle$, iff $z_2 = \langle v_2 \rangle$

π_3) $\langle N\#N1\$N2\#N3 \rangle \longrightarrow \langle N\#N1N3 \rangle \langle N?N2 \rangle$

π_4) $\langle N?N \rangle \longrightarrow \lambda$

We prove that G_3 is a W-grammar rg to f+g.

For any $n,m \in \omega$, if $(f+g)(n)=m$, then in G_3 is true that:

$$z_3 \underset{\pi_1}{\Longrightarrow} \langle \cent v_1 \rangle \overset{*}{\underset{\tilde{R}^1_H}{\Longrightarrow}} \langle \cent i^n \# i^{f(n)} \rangle \underset{\pi_2}{\Longrightarrow} \langle i^n \# i^{f(n)} \$ v_2 \rangle \overset{*}{\underset{\tilde{R}^2_H}{\Longrightarrow}}$$

$$\overset{*}{\underset{\tilde{R}^2_H}{\Longrightarrow}} \langle i^n \# i^{f(n)} \$ i^n \# i^{g(n)} \rangle \underset{\pi_3}{\Longrightarrow} \langle i^n \# i^{f(n)+g(n)} \rangle \langle i^n ? i^n \rangle$$

$$\underset{\pi_4}{\Longrightarrow} \langle i^n \# i^{f(n)+g(n)} \rangle \qquad (1)$$

Conversely, if $z_3 \overset{*}{\underset{G_3}{\Longrightarrow}} \langle i^n \# i^m \rangle$, then in this derivation are used π_3 and π_4.

Therefore, $z_3 \overset{*}{\underset{G_3}{\Longrightarrow}} \langle i^n \# i^p \$ i^n \# i^q \rangle \langle i^n ? i^n \rangle \underset{\pi_4}{\Longrightarrow}$

$\underset{\pi_4}{\Longrightarrow} \langle i^n \# i^{p+q} \rangle$, where $p+q=m$.

Finally, because $\cent$ and $\$$ are new symbols,

$$z_3 \Longrightarrow \langle \cent v_1 \rangle \overset{*}{\underset{R^1_H}{\Longrightarrow}} \langle \cent i^n \# i^p \rangle \underset{\pi_2}{\Longrightarrow} \langle i^n \# i^p \$ v_2 \rangle \overset{*}{\underset{R^2_H}{\Longrightarrow}}$$

$$\overset{*}{\underset{\tilde{R}^2_H}{\Longrightarrow}} \langle i^n \# i^p \$ i^n \# i^q \rangle \overset{*}{\underset{\pi_3 - \pi_4}{\Longrightarrow}} \langle i^n \# i^{p+q} \rangle$$

We deduce that, $z_1 \overset{*}{\underset{G_1}{\Longrightarrow}} \langle i^n \# i^p \rangle$ and

$z_2 \overset{*}{\underset{G_2}{\Longrightarrow}} \langle i^n \# i^q \rangle$ and therefore $f(n)=p$ and $g(n)=q$.

Moreover, the conditions of definition 3 are fulfilled.

Let n_o^i, c_1^i, c_2^i be the corresponding constants for the grammars G_i, $i=1,2$ and $n_o^3 = \max\{n_o^1, n_o^2\}$.

Consider $n \geqslant n_o^3$ and add the hyperrule (which is also a strict rule) $\langle i^n \# i^{f(n)+g(n)} \rangle \longrightarrow a$.

It is not difficult to observe that, $K_\ell^3 = \{a\}$, where $\ell = c_1^1 f(n) + c_2^1 + c_1^2 g(n) + c_2^2 + 4$.

Thus, we define $c_1^3 = \max\{c_1^1, c_1^2\}$, $c_2^3 = c_2^1 + c_2^2 + 4$ and result that $f+g \in A_m$.

Assume that g is a constant function $g(n)=t$, for a fixed $t \in \omega$.

Let G_1 be like above and define $G_3: M_3 = M_1 \cup \{N, N1\}$, $V_3 = V_1 \cup \{\not{c}\}$,

$$R_M^3 = R_M^1 \cup \bigcup_{X \in M_1} \{N \rightarrow X, N1 \rightarrow X\}, \quad R_H^3 = \tilde{R}_H^1 \cup \{\pi_1, \pi_2\},$$

where:

π_1) $z_3 \longrightarrow \langle \not{c} v_1 \rangle$, iff $z_1 = \langle v_1 \rangle$

π_2) $\langle \not{c} N \# N1 \rangle \longrightarrow \langle N \# N1 i^t \rangle$

Observe that the value t is added by π_2 at the end of computation of $f(n)$.

Analogous, result that $f+g \in A_m$ and therefore A_m is closed under sum.

A_m <u>is closed under product</u>.

Let $f, g \in A_m$ be two nonconstant functions. We repeat the above construction (for nonconstant functions) except that V_3 contains also two new symbols $*$, $=$ and the hyperrule π_3 is replaced with the next three hyperrules:

π_5) $\langle N\#N1\$N2\#N3\rangle \longrightarrow \langle N\#N1{*}N3{=}\rangle \langle N?N2\rangle$

π_6) $\langle N\#N1{*}iN2{=}N3\rangle \longrightarrow \langle N\#N1{*}N2{=}N1N3\rangle$

π_7) $\langle N\#N1{*}{=}N3\rangle \longrightarrow \langle N\#N3\rangle$

Rule π_5 serve to initiate the computation of f(n).g(n) and also test if f and g are computed in the same $n\in\omega$.

Rule π_6 serve for the multiplication of the value f(n) (N1) g(n) times (N2) and rule π_7 serve at the end of multiplication.

Like above, G_3 is rg to f.g.

Moreover, define $n_o^3=\max\{n_o^1,n_o^2\}$ and for any $n\geq n_o^3$, add the hyperrule $\langle i^n\# i^{f(n)g(n)}\rangle \longrightarrow a$

A standard derivation in G_3 is:

$$z_3 \overset{*}{\underset{G_3}{\Longrightarrow}} \langle i^n\# i^{f(n)}\$ i^n\# i^{g(n)}\rangle \overset{*}{\underset{\pi_5,\pi_4}{\Longrightarrow}} \langle i^n\# i^{f(n)}{*}i^{g(n)}{=}\rangle$$

$$\overset{g(n)}{\underset{\pi_6}{\Longrightarrow}} \langle i^n\# i^{f(n)}{*}{=}i^{f(n)g(n)}\rangle \underset{\pi_7}{\Longrightarrow} \langle i^n\# i^{f(n)g(n)}\rangle \Longrightarrow a$$

Note that π_6 is used only g(n) times and therefore $K_\ell^3=\{a\}$ for $\ell=c_1^1 f(n) + c_2^1 + (c_1^2+1)g(n)+c_2^2 + 5$.

We define $d=\max\{c_1^1,c_1^2+1\}$, $c_2^3=c_2^1 + c_2^2 + 5$ and thus $\ell\leq d(f(n)+g(n))+c_2^3\leq 2df(n)g(n)+c_2^3$ (because $f(n)>1$ and $g(n)>1$).

Therefore, for $c_1^3=2d$, we obtain that $f.g\in A_m$.

If g is a constant function, $g(n)=t$, then $f.g=\underbrace{f+f\ldots+f}_{t}$ and because A_m is closed under sum result that $f.g\in A_m$.

Thus, in any case f.g A_m.

A_m is closed under exponent. Let $f, g \in A_m$ be two non-constant functions. Consider again the construction of the grammar G_3 which is defined for sum, with next changes: M_3 contains also N4, R_M^3 contains also $N4 \to X$, for any $X \in M_1 \cup M_2$ V_3 contains also the symbols $\uparrow$, $\star$, $\equiv$ and π_3 is replaced with the next five hyperrules:

π_8) $\langle N\#N1\$N2\#N3\rangle \to \langle N\#N1\uparrow N3\$i\rangle\langle N?N2\rangle$

π_9) $\langle N\#N1\uparrow iN2\$N3\rangle \to \langle N\#N2\$N3\star N1\equiv\rangle$

π_{10}) $\langle N\#N2\$N1\star iN3\equiv N4\rangle \to \langle N\#N2\$N1\star N3\equiv N1N4\rangle$

π_{11}) $\langle N\#N2\$N1\star\equiv N3\rangle \to \langle N\#N1\uparrow N2\$N3\rangle$

π_{12}) $\langle N\#N1\uparrow\$N3\rangle \to \langle N\#N3\rangle$

Rule π_8 serve to initiate the computation of $f(n)^{g(n)}$. Rule π_9-π_{11} serve to compute the value of $f(n)^{g(n)}$ by repeated multiplications. Rule π_{12} serve to finish the computation.

Assume that $f(n)^{g(n)}=m$, then in G_3:

$$z_3 \overset{*}{\underset{G_3}{\Longrightarrow}} \langle i^n\#i^{f(n)}\$i^n\#i^{g(n)}\rangle \overset{*}{\underset{\pi_4,\pi_8}{\Longrightarrow}} \langle i^n\#i^{f(n)}\uparrow i^{g(n)}\$i\rangle \underset{\pi_9}{\Longrightarrow}$$

$$\underset{\pi_9}{\Longrightarrow} \langle i^n\#i^{g(n)-1}\$i\star i^{f(n)}\equiv\rangle \overset{*}{\underset{\pi_{10}}{\Longrightarrow}} \langle i^n\#i^{g(n)-1}\$i\star\equiv i^{f(n)}\rangle$$

$$\underset{\pi_{11}}{\Longrightarrow} \langle i^n\#i^{f(n)}\uparrow i^{g(n)-1}\$i^{f(n)}\rangle \underset{\pi_9}{\Longrightarrow} \langle i^n\#i^{g(n)-2}\$i^{f(n)}\star i^{f(n)}\equiv\rangle$$

$$\overset{*}{\underset{\pi_{10}}{\Longrightarrow}} \langle i^n\#i^{g(n)-2}\uparrow i^{f(n)}\star\equiv i^{f(n)}\rangle \underset{\pi_{11}}{\Longrightarrow} \langle i^n\#i^{f(n)}\uparrow i^{g(n)-2}\$i^{f^2(n)}\rangle$$

$$\overset{*}{\underset{\pi_9-\pi_{11}}{\Longrightarrow}} \langle i^n\#i^{f(n)}\uparrow\$i^{f(n)^{g(n)}}\rangle \underset{\pi_{12}}{\Longrightarrow} \langle i^n\#i^{f(n)^{g(n)}}\rangle = \langle i^n\#i^m\rangle$$

Conversely, if $z_3 \overset{*}{\underset{G_3}{\Longrightarrow}} \langle i^n\#i^m\rangle$, then it is easy to prove that $f(n)^{g(n)}=m$ and thus G_3 is rg to f^g.

We put $n_o^3=\max\{n_o^1,n_o^2\}$ and add the hyperrule $\langle i^{n}{}_{\#i}{}^{f(n)}$
$\to a$.

Remark that the derivation $\langle i^{n}{}_{\#i}{}^{f(n)}{}_{\Uparrow i}{}^{g(n)-j}{}_{\$ i}{}^{f(n)\,j}\rangle$

$\xRightarrow[\pi_9-\pi_{11}]{*}\langle i^{n}{}_{\#i}{}^{f(n)}{}_{\Uparrow i}{}^{g(n)-j-1}{}_{\$ i}{}^{f(n)\,j+1}\rangle$ is of length $f(n)+2$ and i

must be done for $j=1,2,\dots,g(n)$.

Result that $\langle i^{n}{}_{\#i}{}^{f(n)}{}_{\$ i}{}^{n}{}_{\#i}{}^{g(n)}\rangle\xRightarrow{*}a$ is of length $g(n)(f(n)+2)+4$.

We deduce that $K^3_\ell=\{a\}$ for $\ell=c^1_1f(n)+c^1_2+c^2_1g(n)+c^2_2+g(n)(f$
$+2)+4$.

Note that $f(n)\geq 2$ and $g(n)\geq 2$ for $n\geq n_o^3$ and therefore the
exists c_1^3 and c_2^3 such that $\ell\leq c_1^3f(n)^{g(n)}+c_2^3$ and thus $f^g\in A_m$.

If g is constant, $g(n)=t$, then $f^g=\underbrace{f\cdot\ldots\cdot f}_{t}$ and result that $f^g\in A_m$ (A_m is closed under product).

Assume that $f(n)=t$, for any $n\in\omega$, with $t\in\omega$, $t\geq 2$ and g is a nonconstant function.

Let G_2 be the W grammar rg to g and define G_3 such that:

$M_3=M_2\cup\{N,N1,N2\}$, $V_3=V_2\cup\{\text{¢},\$\}$, $R^3_M=R^2_M\cup\bigcup_{X\in M_2}\{N\to X,\ N1\to X,$
$N2\to X\}$, $R^3_H=\tilde{R}^2_H\cup\{\pi_1,\dots,\pi_4\}$, where $\tilde{R}^2_H$ is obtained from R^2_H by replacing all hypernotions $\langle h\rangle\in H_2$ with $\langle\text{¢}h\rangle$ and $\pi_1,\dots,\pi_4$ are:

π_1) $z_3\to\langle\text{¢}v_2\rangle$, iff $z_2=\langle v_2\rangle$

π_2) $\langle\text{¢}N\#N1\rangle\to\langle N\#N1\$ i\rangle$

π_3) $\langle N\#iN1\$ N2\rangle\to\langle N\#N1\$\underbrace{N2\ldots N2}_{t}\rangle$

π_4) $\langle N\#\$ N2\rangle\to\langle N\#N2\rangle$

It is not difficult to obverse that G_3 is rg to $f(n)^{g(n)}=t^{g(n)}$.

Moreover, if we add the hyperrule $\langle i^{n}*i^{t^{g(n)}}\rangle \rightarrow a$ for a fixed $n \geqslant n_0^3$, then $K_\ell^3=\{a\}$ for $\ell=c_1^2 g(n)+c_2^2+g(n)+3$. Define $c_1^3=t^{c_1^2+1}$, $c_2^3=c_2^2+4$ and therefore $\ell \leqslant c_1^3 t^{g(n)}+c_2^3$.

Thus, result that $f^g \in A_m$.

From theorem 8 and exemple 5 result:

Corollary 9. For any $K \geqslant 1$, K fixed, the functions: $f(n)=n^K$ and $g(n)=K^n$ are TW self bounded.

Also we can deduce:

Corollary 10. Any polynom $p(n)=a_0 n^K+a_1 n^{K-1}+\ldots+a_K$, where $a_j \in \omega$, $j=0,\ldots,K$, $K \in \omega$ is a TW self bounded function.

Theorem 11

(i) If $f \in A_m$, $f:A \rightarrow B$, $(A,B \subseteq R_+)$, f bijective monotone increasing with $f(\omega) \subseteq \omega$ and if in condition of definition 3, $K_\ell=\{a\}$ for $\ell=c_1 n+c_2$ then $f^{-1} \in A_m$.

(ii) If $f,g \in A_m$, $f:B \rightarrow C$, $g:A \rightarrow B$ $(A,B,C \subseteq R_+)$ and if $f(n) \geqslant n$, $g(n) \geqslant n$ for any $n \in \omega$, then the composite function $f \circ g \in A_m$, where $(f \circ g)(n)=f([g(n)])$, for any $n \in \omega$.

Proof. (i) Let $G=((M,V,R_H),(H,\Sigma,R_H),z)$ be the W grammar rg to f and define the W grammar $G_1=((M_1,V_1,R_M^1),(H_1,\Sigma,R_H^1),z_1)$, where:

$M_1=M \cup \{N1,N2,N3,N4,Y\}$, $V_1=V \cup \{\not c,\uparrow,\downarrow,?\}$,

$R_M^1=R_M \cup \{A \rightarrow B \mid A \in \{N1,N2,N3,N4\}, B \in M\} \cup \{Y \rightarrow \uparrow,\ Y \rightarrow \downarrow\}$

$R^1_H = \tilde{R}_H \cup \{h_1, h_2, h_3, h_4, h_5\}$ where $\tilde{R}_H$ is obtained from R_H by replacing all hypernotions $\langle h \rangle \in H$ with $\langle N1YN2 \varepsilon h \rangle$ and:

$h_1)$ $z_1 \to \langle N1 \# N2 \rangle \langle N2 \Downarrow N1 \varepsilon v \rangle \langle N2i \Uparrow N1 \varepsilon v \rangle$ if $z = \langle v \rangle$

$h_2)$ $\langle N1YN2 \varepsilon N3 \# N4 \rangle \to \langle N4YN2 \rangle \langle N1?N3 \rangle$

$h_3)$ $\langle N1 \Downarrow N1N2 \rangle \to \lambda$

$h_4)$ $\langle N1iN2 \Uparrow N1 \rangle \to \lambda$

$h_5)$ $\langle N1?N1 \rangle \to \lambda$

Firstly, h_1 introduce three hypernotions and begin the process to compute the values of f (using $\tilde{R}_H$) for two values n_1 and n_2:

$$z_1 \underset{h_1}{\Rightarrow} \langle i^n \# i^m \rangle \langle i^m \Downarrow i^n \varepsilon v \rangle \langle i^{m+1} \Uparrow i^n \varepsilon v \rangle \overset{*}{\underset{\tilde{R}_H}{\Longrightarrow}}$$

$$\overset{*}{\underset{\tilde{R}_H}{\Longrightarrow}} \langle i^n \# i^m \rangle \langle i^m \Downarrow i^n \varepsilon i^{n_1} \# i^{f(n_1)} \rangle \langle i^{m+1} \Uparrow i^n \varepsilon i^{n_2} \# i^{f(n_2)} \rangle \quad (1)$$

The values n and m are checked if they satisfy: $[f^{-1}(n)] = m$.

Because f is bijective, monotone increasing and $f(\omega) \subseteq \omega$ we have the equivalences:

$$[f^{-1}(n)] = m \text{ iff } m \leq f^{-1}(n) < m+1 \text{ iff } f(m) \leq n < f(m+1)$$

In G_1 is checked if $f(m) \leq n < f(m+1)$.

Derivation (1) continue with:

$$(1) \overset{2}{\underset{h_2}{\Longrightarrow}} \langle i^n \# i^m \rangle \langle i^{f(n_1)} \Downarrow i^n \rangle \langle i^m ? i^{n_1} \rangle \langle i^{f(n_2)} \Uparrow i^n \rangle \langle i^{m+1} ? i^{n_2} \rangle$$

If $m=n_1$ and $m+1=n_2$ then $\langle i^m ? i^{n_1}\rangle$ and $\langle i^{m+1} ? i^{n_2}\rangle$ derive in λ. Moreover, $\langle i^{f(m)} \downarrow i^n\rangle$ and $\langle i^{f(m+1)} \uparrow i^n\rangle$ derive in λ iff $f(m)\leq n$ and $n<f(m+1)$. Therefore $z_1 \overset{*}{\underset{G_1}{\Rightarrow}} \langle i^n \# i^m\rangle$ iff $\lfloor f^{-1}(n)\rfloor = m$.

Consider $n_o^1 = f(n_o)$, let $n > n_o^1$ be a natural number and add the hyperrule $\langle i^n \# i^m\rangle \to a$ where $m=\lfloor f^{-1}(n)\rfloor$.

It is easy to observe that $K_{G_1}^r = \{a\}$ for $r=c_1(m+1)+ +c_2+3=c_1\lfloor f^{-1}(n)\rfloor + c_1+c_2+3$ and therefore $f^{-1}\in A_m$.

II. This part of the theorem is not important for this paper. We consider only the construction of G. If G_i, $i=1,2$, are the W grammar rg to f, respective g then define G rg to $f\circ g$:

$M=M_1 \cup M_2 \cup \{N, N1, N2, N3\}$, $V=V_1 \cup V_2 \cup \{¢, \$, ?\}$

$R_M=R_M^1 \cup R_M^2 \cup \{A \to B \mid A\in\{N, N1, N2, N3\},\ B\in M_1 \cup M_2\}$

$R_H=\tilde{R}_H^1 \cup \tilde{R}_H^2 \cup \{h_1, h_2, h_3, h_4\}$ where $\tilde{R}_H^2$ is obtained from R_H^2 by replacing $\langle h\rangle$ with $\langle ¢h\rangle$ and $\tilde{R}_H^1$ is obtained from R_H^1 by replacing $\langle h\rangle$ with $\langle N\#N1\$h\rangle$ and:

$h_1)$ $z \to \langle ¢v_2\rangle$ iff $z_2=\langle v_2\rangle$

$h_2)$ $\langle ¢N\#N1\rangle \to \langle N\#N1\$v_1\rangle$ iff $z_1=\langle v_1\rangle$

$h_3)$ $\langle N\#N1\$N2\#N3\rangle \to \langle N\#N3\rangle\langle N1?N2\rangle$

$h_4)$ $\langle N?N\rangle \to \lambda$

It is easy to observe that $f\circ g\in A_m$.

<u>Proposition 12.</u> For every $K\in\omega$, $K>2$, the function $f(n)=\sqrt[K]{n}$ and $g(n)=\log_K n$ are in A_m.

Proof. Let f_1 be the function $f_1(n)=n^K$, $f_1:R_+ \to R_+$. Obviously, f_1 is bijective, monotone increasing and $f_1(\omega)\subset\omega$. From Corollary 9, f_1 is in A_m. Moreover, from exemple 5, the function $h(n)=n$ satisfy definition 3 for $f=n$. From the proof of theorem 8 (A_m is closed under product) result that $h_1(n)=h(n).h(n)=n^2$ satisfy the condition $K_f=\{a\}$ for $f=C_1f(n)+c_2^1+(c_1^2+1)g(n)+c_2^2+5$ and thus, in our case $f=3n+5$. By induction on K we may prove that there exists $c_1,c_2\in\omega$ such that $f_1(n)=n^K$ satisfy the condition $K_f=\{a\}$ for $f=c_1n+c_2$.

Therefore are satisfied the conditions of theorem 11, i) and result that $f_1^{-1}(n)=\sqrt[K]{n}\in A_m$.

Analogous, $g_1(n)=K^n$, $g_1:R_+\to[1,+\infty)$ is bijective, monotone increasing, $g_1(\omega)\subset\omega$ and $g_1\in A_m$. Again, from the proof of theorem 8 result that g_1 satisfy $K_f=\{a\}$ for $f=(c_1^2+1)g(n)+c_2^2+4$. (A_m closed under exponent with f constant and g non-constant). Thus, from theorem 11 i), $\log_K n\in A_m$.

§5.2. The TW-separation property

If $\{x_n\}_{n\in\omega}$ is a sequence, then we denote by $\inf\limits_{n\to\infty} x_n$ the limit as $n\to\infty$ of the greatest lower bound of $x_n, x_{n+1}, x_{n+2}, \ldots$.

The next result is important.

Theorem 13 (The TW-separation property)

If $f\in A_m$, $f(n)\geq\log_2 n$ for every $n\geq n_0$, with n_0 fixed, and if $g:A\to B$, $A,B\subseteq R_+$ is a function such that:

$$\inf_{n\to\infty} \frac{g(n)}{f(n)} = 0,$$

then there exists a language L with the property that, for the functional criterion "a^n", stands $L\in TIMEW(f(n))$ and $L\notin TIMEW(g(n))$.

<u>Proof</u>. Consider the language $L=\{a^n b^m \mid m=2^{f(n)}, n\in\omega\}$. We prove that $L\in TIMEW(f(n))$. Because $f\in A_m$, there exists a W grammar G rg to f and the constants n_o, c_1, $c_2\in\omega$ such that is satisfied definition 3 relative to f. We define a new W grammar G' such that L(G')=L. Consider: $M'=M\cup\{N,N1\}$, $V'=V\cup\{\alpha,\beta\}$, $\Sigma'=\Sigma\cup\{a,b\}$, $R'_M=R_M\cup\{A\to B \mid A\in\{N,N1\}, B\in M\}$, z'=z and $R'_H=R_H\cup\{h_1,\dots,h_6\}$ where:

$h_1)$ $\langle N\#N1\rangle \to \langle\alpha N\rangle\langle\beta N1\rangle$

$h_2)$ $\langle\alpha NN\rangle \to \langle\alpha N\rangle\langle\alpha N\rangle$

$h_3)$ $\langle\alpha NNi\rangle \to a\langle\alpha N\rangle\langle\alpha N\rangle$

$h_4)$ $\langle\alpha\rangle \to \lambda$

$h_5)$ $\langle\beta iN\rangle \to \langle\beta N\rangle\langle\beta N\rangle$

$h_6)$ $\langle\beta\rangle \to b$

Note that, a derivation in G' it is of the form:

$$z_1 \overset{*}{\underset{R_H}{\Longrightarrow}} \langle i^n\# i^{f(n)}\rangle \underset{h_1}{\Longrightarrow} \langle\alpha i^n\rangle\langle\beta i^{f(n)}\rangle \overset{*}{\underset{h_2-h_6}{\Longrightarrow}} a^n b^{2^{f(n)}} \qquad (1)$$

Obviously, L(G')=L and we remark that after $\ell_1=\log_2 n+1$ iterations, $K'_{\ell_1}(\langle\alpha i^n\rangle)=\{a^n\}$. Moreover, after $\ell_2=f(n)+1$ iterations, $K'_{\ell_2}(\langle\beta i^{f(n)}\rangle)=\{b^{2^{f(n)}}\}$.

If $\ell_3=\max\{\ell_1,\ell_2\}+1$ then $K'_{\ell_3}(\langle i^n \# i^{f(n)}\rangle)=\{a^n b^{2^n}\}$.

Note that $f(n)\geq \log_2 n$ and therefore $\ell_3=f(n)+2$. For every $n\geq n_0$ we remark that $K'_\ell=\{a^n b^{2^n}\}$, where $\ell=(c_1+1)f(n)+c_2+2$.

We deduce that $L\in TIMEW(f(n))$.

Now, we prove that $L\notin TIMEW(g(n))$.

Assume the contrary, that means $L\in TIMEW(g(n))$.

Therefore there exists a W grammar G such that $L(G)=L$ and $TW_G(n)\leq g(n)$ a.e. Define the constants:

$$j_1=\max\{|\beta_0\ldots\beta_\ell| \mid \langle u_0\rangle\to\beta_0\langle u_1\rangle\beta_1\ldots\langle u_\ell\rangle\beta_\ell\in R_H\}$$

$$j_2=\max\{|\alpha| \mid \langle u\rangle\to\alpha\in R_H, \alpha\in\Sigma^*\}$$

$$j_3=\max\{\ell \mid \langle u_0\rangle\to\beta_0\langle u_1\rangle\beta_1\ldots\langle u_\ell\rangle\beta_\ell\in R_H\}$$

We prove that for every $n\geq 1$, for every strict hypernotion $\langle h\rangle$ and for every $v\in K_n(\langle h\rangle)$ stands:

$$|v|\leq j_3^{n-1}j_2 + j_1\sum_{i=0}^{n-2}j_3^i \qquad (2)$$

For $n=1$ and $n=2$ the inequality (2) is obviously true. For the inductive step assume (2) and let v be in $K_{n+1}(\langle h\rangle)$. There exists a strict rule $\langle h\rangle\to\beta_0\langle h_1\rangle\ldots\langle h_q\rangle\beta_q$ and the words $v_i\in K_n(\langle h_i\rangle)$, $i=1,\ldots,q$ such that $v=\beta_0 v_1\ldots v_q\beta_q$. We deduce that

$$|v|\leq q(j_3^{n-1}j_2+j_1\sum_{i=0}^{n-2}j_3^i)+|\beta_0\ldots\beta_q|\leq j_3(j_3^{n-1}j_2+j_1\sum_{i=0}^{n-2}j_3^i)+$$

$$+j_1 = j_3^n j_2 + j_1\sum_{i=0}^{n-1}j_3^i$$ and therefore (2) is true.

If $j=\max\{j_1,j_2,j_3\}$, then using (2), result:

$$|v| \leqslant j^n + j\sum_{i=0}^{n-2} j^i = \sum_{i=1}^{n} j^i \leqslant nj^n \leqslant 2^n j^n = (2j)^n.$$

We denote $s=2j$ and therefore for every $v \in K_n$, $|v| \leqslant s^n$. Thus for every $v \in Kg(n)$ stands that $|v| \leqslant s^{g(n)}$.

But $TW_G(n) \leqslant g(n)$ a.e. and therefore $K_{g(n)} \cap C_n \neq \emptyset$ a.e. Thus $a^n b^{2^{f(n)}} \in Kg(n)$ a.e. and we deduce that:

$$2^{f(n)} < |a^n b^{2^{f(n)}}| \leqslant s^{g(n)} \quad \text{a.e.} \tag{3}$$

From (3) result that $\log_s 2 \leqslant \frac{g(n)}{f(n)}$ a.e. which is in contradiction with $\inf\limits_{n\to\infty} \frac{g(n)}{f(n)} = 0$.

Thus, the language L cannot be generated by a W grammar G with $TW_G(n) \leqslant g(n)$ a.e. and therefore $L \notin TIMEW(g(n))$.

§5.3. Hierarchies of recursive enumerable sets

We consider again the GBS , $\mathcal{G} = (\Sigma, W, TW, "a^n")$ and define:

$$POLTW = \bigcup_{K \geqslant 1} TIMEW(n^K)$$

$$EXPTW = \bigcup_{K \geqslant 2} TIMEW(K^n)$$

Theorem 14. If $\mathcal{G}$ is the above GBS, then:

i) $TIMEW(\log_2 n) \subsetneq TIMEW(n)$

ii) $TIMEW(n^K) \subsetneq TIMEW(n^{K+1})$ for all $K \geqslant 1$

iii) $POLTW \subsetneq EXPTW$.

Proof. To show i) and ii) we may use directly theorem 13. To prove iii) note that: POLTW $\subseteq$ TIMEW$(2^n) \subsetneq$ TIMEW$(3^n) \subseteq$ $\subseteq$ EXPTW. Moreover, one may prove that POLTW $\subsetneq$ TIMEW(2^n) (see the proof of theorem 13).

Theorem 15 (A dense hierarchy). If $\mathcal{G}$ is the above GBS, then: for every $\alpha, \beta \in R_+$, if $\alpha < \beta$ then TIMEW$(n^\alpha) \subsetneq$ TIMEW(n^β).

Proof. Obviously, TIMEW$(n^\alpha) \subseteq$ TIMEW(n^β).

Let $\frac{p}{q} \in Q$ be such that $\alpha < \frac{p}{q} < \beta$. The function $f(n) = [\sqrt[q]{n}]^p$ is in A_m. We remark that $\frac{n^\alpha}{[\sqrt[q]{n}]^p} \leq \frac{n^\alpha}{(\sqrt[q]{n}-1)^p}$ for all $n \geq 2$. There exists $m \in \omega$ such that $[\sqrt[q]{n}]^p \geq [\log_2 n]$ for every $n \geq m$.

We remark also that:

$$\inf_{n\to\infty} \frac{n^\alpha}{[\sqrt[q]{n}]^p} = \lim_{n\to\infty} \frac{n^\alpha}{[\sqrt[q]{n}]^p} \leq \lim_{n\to\infty} \frac{n^\alpha}{(\sqrt[q]{n}-1)^p} = 0$$

From theorem 13 result that:

$$\text{TIMEW}(n^\alpha) \subsetneq \text{TIMEW}([\sqrt[q]{n}]^p) \qquad (1)$$

But $[\sqrt[q]{n}]^p \leq [\sqrt[q]{n^p}] \leq n^\beta$ and therefore

$$\text{TIMEW}([\sqrt[q]{n}]^p) \subseteq \text{TIMEW}(n^\beta) \qquad (2)$$

From (1) and (2) result theorem 15.

Comments. In (16) p.297 is proved the next result:

" Theorem 12.8. If $S_2(n)$ is a fully space-constructible function,

$$\inf_n \frac{S_1(n)}{S_2(n)} = 0$$

and $S_1(n)$ and $S_2(n)$ are each at least $\log_2 n$, then there is a language in DSPACE $(S_2(n))$ not in DSPACE$(S_1(n))$".

Our theorem 13 (the TW-separation property) is like the above theorem but we don't know yet if this is a coincidence or not!

FINAL REMARKS

In this paper is presented a variant of chapters 1-5 of (24) (which contains 9 chapters).

In chapter 6 of (24) is efectively defined an universal W grammar.

Chapter 7 of (24) is devoted to study the posibility to bound the generative complexity.

We present here some results without proofs.

Theorem 3. If $\mathcal{G}=(\Sigma,W,TW,C)$ is a GBS such that for every $n\in\omega$, $C_n\neq\emptyset$ and $C_n\cap C_m=\emptyset$ for all $n,m\in\omega$, $n\neq m$, then for any recursive function f, $f:\omega\to\omega$, there exists an infinite re language $L\subset\Sigma^*$ such that $L\notin TIMEW(f(n))$.

Corollary 4 i) In the GBS, $\mathcal{G}=(\Sigma,W,TW,"=n")$ for any recursive function f, $f:\omega\to\omega$, there exists an infinite re language $L\subset\Sigma^*$ such that $L\notin TIMEW(f(n))$.

ii) In the GBS, $\mathcal{G}=(\Sigma,W,TW, "a^n")$ for any recursive function f, $f:\omega\to\omega$, there exists an infinite re language $L\subset\Sigma^*$ such that $L\notin TIMEW(f(n))$.

Theorem 5. If $\mathcal{G}=(\Sigma,W,TW,C)$ is a GBS such that card$(\Sigma)\geq 2$, for every $n\in\omega$, $C_n\neq\emptyset$ and for every $u\in C_n$, $|u|\geq n$, then for any recursive function f, $f:\omega\to\omega$, there exists an infinite re language $L\subset\Sigma^*$ such that $L\notin TIMEW(f(n))$.

Corollary 6. In the GBS, $\mathcal{G}=(\Sigma,W,TW, "\geq n")$ with card$(\Sigma)\geq 2$

for any recursive function f, $f:\omega \to \omega$ there exists an infinite re language $L \subset \Sigma^*$ such that $L \notin TIMEW(f(n))$.

Chapter 8 of (24) is devoted to study some restrictions on the W grammars such that these grammars generate only recursive language.

In chapter 9 of (24) we suggest some directions to study.

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