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A FINITE ELEMENT SOLUTION FOR UNILATERAL CONTACT PROBLEMS

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Abstract. In this paper we study the finite element approximation of the Signorini problem with friction. We propose an algorithm of Bensoussan-Lions type for which we prove the convergence. An error estimate is derived and numerical results are given.

Résumé. On étudie dans cet article l'approximation par éléments finis du problème de Signorini avec frottement. On propose un algorithme du type Bensoussan-Lions pour lequel on démontre la convergence. Une estimation de l'erreur est obtenue et des résultats numériques sont présentés.

INTRODUCTION

The present work is concerned with the numerical analysis of unilateral contact problem known as Signorini problem with friction (see [4], [5]).

Results on the existence of the solutions of the quasivariational inequality involved by this problem have been obtained for the first time by Duvaut [5] for a non-local friction law where sufficient conditions for uniqueness have been also given and by Nečas, Jarušek and Haslinger [9] for a local friction law in a particular case.

Regularity properties have been given in [3].

Finite element analysis of the Signorini problem with friction have been studied in [7], [8] in the case of prescribed normal forces on the contact boundary and in [10] where an abstract error estimate is derived.

In this paper we use an algorithm of Bensoussan-Lions type for obtaining the numerical solution of the quasivariational inequality formulated in Section 1 and which describes the Signorini problem. The convergence of this algorithm is proved in Section 2 where we also derive an error estimate of the finite element approximation with respect to mesh parameter h.

An analysis of numerical results is made in Section 3.

1. A VARIATIONAL FORMULATION OF SIGNORINI PROBLEM

We shall consider the problem of finding the field of displacements in a linearly elastic body which is in unilateral contact with a rigid support following a non-local friction law (see [5], [10]).

In order to give a variational formulation of this problem, let Ω be the bounded Lipschitzian domain in R^p, $p=2,3$, occupied by the body in the initial unstressed state. Let us denote by Γ the boundary of Ω and let $\Gamma_0, \Gamma_1, \Gamma_2$ be open and disjoint parts of Γ such that $\Gamma = \bar{\Gamma}_0 \cup \bar{\Gamma}_1 \cup \bar{\Gamma}_2$ and $\Gamma_2 \neq \emptyset$. Suppose that $\Gamma_2 \in C^2$.

We introduce the following space:

$$V = \{ \underset{\sim}{v} \in [H^1(\Omega)]^p ;\ \underset{\sim}{v} = 0 \text{ a.e. on } \Gamma_0 \} \tag{1.1}$$

which is a Hilbert space with the scalar product of $[H^1(\Omega)]^p$. We shall denote by $\|\cdot\|$ its associated norm.

We shall use the following notations for the normal and tangential components of the displacements and of the stress vector, respectively

$$u_n = u_i n_i, \quad u_{ti} = u_i - u_n n_i ,$$
$$\sigma_n = \sigma_{ij} n_i n_j, \quad \sigma_{ti} = \sigma_{ij} n_j - \sigma_n n_i ,$$

where $n = (n_i)$ is the outward normal unit vector on Γ.

If we denote by K the following non-empty closed convex subset of V:

$$K = \{ \underset{\sim}{v} \in V;\ v_n \leq 0 \quad \text{a.e. on } \Gamma_2 \} \tag{1.2}$$

then it is known (see [5], [10]) that a variational formulation of the Signorini problem with non-local friction law is as follows:

find $u \in K$ such that

$$a(\underset{\sim}{u}, \underset{\sim}{v} - \underset{\sim}{u}) + j_0(\underset{\sim}{u}, \underset{\sim}{v}) - j_0(\underset{\sim}{u}, \underset{\sim}{u}) \geq L(\underset{\sim}{v} - \underset{\sim}{u}), \ \forall\ \underset{\sim}{v} \in K \tag{1.3}$$

where:

$$a(\underset{\sim}{u},\underset{\sim}{v})=\int_{\Omega}\sigma_{ij}(\underset{\sim}{u})\,\varepsilon_{ij}(\underset{\sim}{v})\,d\underset{\sim}{x}, \tag{1.4}$$

$$j_o(\underset{\sim}{u},\underset{\sim}{v})=\int_{\Gamma_2}\mu\,|\mathcal{Q}(\sigma_n(\underset{\sim}{u}))|\,|\underset{\sim}{v}_t|\,d\underset{\sim}{s}, \tag{1.5}$$

$$L(\underset{\sim}{v})=\int_{\Omega}f_i v_i\,d\underset{\sim}{x}+\int_{\Gamma_1}t_i v_i\,d\underset{\sim}{s}, \tag{1.6}$$

where $\varepsilon_{ij}(\underset{\sim}{v})=\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)$ is the strain tensor, $\sigma_{ij}(\underset{\sim}{v})$ is the stress tensor related to $\varepsilon_{ij}(\underset{\sim}{v})$ by means of the generalized Hooke's law:

$$\sigma_{ij}(\underset{\sim}{v}) = a_{ijkh}\varepsilon_{kh}(\underset{\sim}{v}), \quad \text{in } \Omega,$$

$\mu\in L^{\infty}(\Gamma_2)$ is the coefficient of friction such that $\mu\geqslant 0$ a.e. on Γ_2, $\mathcal{Q}$ is a linear and continuous mapping from $H^{-1/2}(\Gamma_2)$ to $L^2(\Gamma_2)$, $\underset{\sim}{f}\in[L^2(\Omega)]^p$ is the body force and $\underset{\sim}{t}\in[L^2(\Gamma_1)]^p$ is the prescribed surface traction.

We have used the summation convention.

Suppose that the elasticity coefficients of the body a_{ijkh} satisfy the symmetry condition:

$$a_{ijkh} = a_{jikh} = a_{khij},$$

and that $a_{ijkh}\in C^1(\bar{\Omega})$ where $C^1(\bar{\Omega})$ denotes the once continuously differentiable functions on $\bar{\Omega}$.

Suppose in addition that the bilinear and continuous form $a(.,.)$ is V-elliptic i.e. $\exists\,\alpha>0$ such that

$$a(\underset{\sim}{v},\underset{\sim}{v})\geqslant\alpha\|\underset{\sim}{v}\|^2, \quad \forall\, \underset{\sim}{v}\in V. \tag{1.7}$$

Remark 1.1 : Note that if $\text{mes}(\Gamma_0)>0$ then (1.7) holds by

Korn's inequality. In the case $mes(\Gamma_o)=0$ then (1.7) is satisfied if, for example, $K \cap D = \{\underset{\sim}{0}\}$ where D is the set of rigid displacements (see e.g. [2]).

In order to justify the application of an algorithm of Bensoussan-Lions type to the quasivariational inequality (1.3) we shall prove the following existence and uniqueness theorem.

THEOREM 1.1: Let V, K, a, j_o, L be defined by (1.1), (1.2), (1.4)-(1.6). Then there exists $\mu_1 > 0$ such that for every μ with $\|\mu\|_{L^\infty(\Gamma_2)} \leq \mu_1$ the inequality (1.3) has an unique solution.

Proof: It is easy to verify that if there exists a solution $\underset{\sim}{u}$ of (1.3) then $\underset{\sim}{u} \in K_o$ where:

$$K_o = \{\underset{\sim}{v} \in V;\ a(\underset{\sim}{v}, \underset{\sim}{\varphi}) = \underset{\sim}{f}(\underset{\sim}{\varphi}),\ \forall \underset{\sim}{\varphi} \in [\mathcal{D}(\Omega)]^p\},\quad f(\underset{\sim}{\varphi}) = \int_\Omega f_i \varphi_i \, d\underset{\sim}{x}.$$

Also, let us observe that $\sigma_n(\underset{\sim}{v}) \in H^{-1/2}(\Gamma_2), \forall\, \underset{\sim}{v} \in K_o$ so that we may take $j_o : K_o \times K \longrightarrow R$.

Let $\bar{S}$ be the function which associates to every $\underset{\sim}{w} \in K_o$, the element $\bar{S}\underset{\sim}{w} \in K_o$ such that:

$$a(\bar{S}\underset{\sim}{w}, \underset{\sim}{v} - \bar{S}\underset{\sim}{w}) + j_o(\underset{\sim}{w}, \underset{\sim}{v}) - j_o(\underset{\sim}{w}, \bar{S}\underset{\sim}{w}) \geq L(\underset{\sim}{v} - \bar{S}\underset{\sim}{w}),\ \forall\, \underset{\sim}{v} \in K. \tag{1.8}$$

Taking into account that for $\underset{\sim}{w} \in K_o$ given, the functional $j_o(\underset{\sim}{w}, .) : K \rightarrow R$ is , convex and lower semicontinuous on K, it follows that the variational inequality (1.8) has an unique solution $\bar{S}\underset{\sim}{w} \in K$. In addition $\bar{S}\underset{\sim}{w} \in K_o$ so that the mapping $\bar{S} : K_o \longrightarrow K_o$ is well-defined.

We remark that the set of fixed points of $\bar{S}$ coincides with the set of solutions of the inequality (1.3).

Therefore, the question of the existence and uniqueness of solutions of (1.3) reduces to the existence and uniqueness of fixed points of $\bar{S}$.

Now we show that for μ sufficiently small, $\bar{S}$ is a contraction. Indeed, for $w_1, w_2 \in K_o$ arbitrarily, from (1.8) we obtain:

$$\alpha \| \bar{S}w_1 - \bar{S}w_2 \|^2 \leq | j_o(w_1, \bar{S}w_2) + j_o(w_2, \bar{S}w_1) - j_o(w_1, \bar{S}w_1) - j_o(w_2, \bar{S}w_2) | . \quad (1.9)$$

It is clear that σ_n is a continuous operator from K_o in $H^{-1/2}(\Gamma_2)$ from which we obtain that:

$$| j_o(u_1, v_2) + j_o(u_2, v_1) - j_o(u_1, v_1) - j_o(u_2, v_2) | \leq c \|\mu\|_{L^\infty(\Gamma_2)} \| u_1 - u_2 \| \cdot$$
$$\cdot \| v_1 - v_2 \| \forall u_1, u_2 \in K_o, \ \forall v_1, v_2 \in K. \quad (1.10)$$

If we take $\mu_1 < \frac{\alpha}{c}$ then for every μ with $\|\mu\|_{L^\infty(\Gamma_2)} \leq \mu_1$, we obtain from (1.9) and (1.10):

$$\| \bar{S}w_1 - \bar{S}w_2 \| \leq k \| w_1 - w_2 \| \ , \ \forall \ w_1, w_2 \in K_o$$

where $k = \dfrac{c\|\mu\|_{L^\infty(\Gamma_2)}}{\alpha} < 1$.

Therefore, the mapping $\bar{S}$ has an unique fixed point hence there exists an unique solution of (1.3).

Formulation (1.3) is not suitable for approximation; the reason for this is that $j_o(\cdot, v)$ is defined on K_o which is difficult to approximate. To avoid this, we proceed as follows.

We shall consider, for simplicity, the mapping $\mathcal{R}$ given by:

$$\mathcal{R}(z)(x) = \langle z, \omega^x \rangle \ , \forall \ z \in H^{-1/2}(\Gamma_2), \forall \ x \in \Gamma_2,$$

where $\langle \cdot, \cdot \rangle$ denotes duality pairing on $H^{-1/2}(\Gamma_2) x H^{1/2}(\Gamma_2)$, $\omega^x(y) = \omega(|x-y|), \forall \ y \in \bar{\Omega}$ with $\omega \in \mathcal{D}(-\delta, \delta)$ ($\delta \in R$, $\delta > 0$) such that

$\omega \geq 0$ and

$$\int_{-\delta}^{\delta} \omega \, dt = 1.$$

For every $\underset{\sim}{x} \in \Gamma_2$ we shall consider $w^x \in [H^1(\Omega)]^p$ defined by $w^x = \omega^x \underset{\sim}{N}$ where $\underset{\sim}{N} \in [H^1(\Omega)]^p$ such that $\gamma(\underset{\sim}{N}) = \underset{\sim}{n}$ a.e. on Γ_2 where $\gamma : [(H^1(\Omega)]^p \longrightarrow [H^{1/2}(\Gamma)]^p$ is the trace operator.

With the above notations we define the application $Q : [H^1(\Omega)]^p \longrightarrow L^2(\Gamma_2)$ by

$$Q(\underset{\sim}{v})(\underset{\sim}{x}) = a(\underset{\sim}{v}, \underset{\sim}{w}^x) - f(\underset{\sim}{w}^x), \ \forall \ \underset{\sim}{v} \in [H^1(\Omega)]^p, \forall \ \underset{\sim}{x} \in \Gamma_2 .$$

In the following we denoteby u the solution (that there exists and is unique) of (1.3). Using Green's formula, it results that:

$$\mathcal{Q}(\sigma_n(\underset{\sim}{u})) = Q(\underset{\sim}{u}) \qquad \text{on } \Gamma_2.$$

It is easy to show that the problem (1.3) is equivalent with

$$a(\underset{\sim}{u}, \underset{\sim}{v} - \underset{\sim}{u}) + j(\underset{\sim}{u}, \underset{\sim}{v}) - j(\underset{\sim}{u}, \underset{\sim}{u}) \geq L(\underset{\sim}{v} - \underset{\sim}{u}), \ \forall \ \underset{\sim}{v} \in K \tag{1.11}$$

where $j : [H^1(\Omega)]^p \times [H^1(\Omega)]^p \longrightarrow R$ is defined by

$$j(\underset{\sim}{u}, \underset{\sim}{v}) = \int_{\Gamma_2} \mu |Q(\underset{\sim}{u})| \, |\underset{\sim}{v}_t| \, d\underset{\sim}{s} \tag{1.12}$$

Further on, instead of (1.3), we shall approximate the problem (1.11).

LEMMA 1.1: The mapping $j : [H^1(\Omega)]^p \times [H^1(\Omega)]^p \longrightarrow R$ defined by (1.12) satisfies:

$$|j(u_1,v_2)+j(u_2,v_1)-j(u_1,v_1)-j(u_2,v_2)|\leq C_0\|\mu\|_{L^\infty(\Gamma_2)}\|u_1-u_2\|\,\|v_1-v_2\|,$$

$$\forall\, u_1,u_2,v_1,v_2\in[H^1(\Omega)]^p, \qquad (1.13)$$

with C_0 a positive constant depending on Γ_2 and ω.

Proof: Let $u_1,v_1,u_2,v_2\in[H^1(\Omega)]^p$. From the definition (1.12) of j we have

$$|j(u_1,v_2)+j(u_2,v_1)-j(u_1,v_1)-j(u_2,v_2)| \leq$$

$$\leq \int_{\Gamma_2}\mu\,|Q(u_1)-Q(u_2)|\,|v_{2t}-v_{1t}|\,ds = \int_{\Gamma_2}\mu\,|a(u_1-u_2,w^x)|\,|v_{2t}-v_{1t}|\,ds \leq$$

$$\leq C\|\mu\|_{L^\infty(\Gamma_2)}\|u_1-u_2\|\,\|v_1-v_2\|\int_{\Gamma_2}\|w^x\|\,ds\leq C_0\|\mu\|_{L^\infty(\Gamma_2)}\|u_1-u_2\|\,\|v_1-v_2\|,$$

with $C_0=C\cdot \mathrm{mes}(\Gamma_2)\max_{x\in\Gamma_2}\|w^x\|$ and where Schwarz's inequality, the continuity of the bilinear form a and the trace theorem where used.

Let us denoteby S the mapping $S:K\longrightarrow K$ which associates to every $w\in K$ the solution (unique) of the following problem:

$$a(Sw,v-Sw)+j(w,v)-j(w,Sw)\geq L(v-Sw),\ \forall\ v\in K.$$

We define the sequence: for $u^0\in K$ chosen arbitrarily we put $u^n=Su^{n-1}$ i.e. u^n satisfies the inequality:

$$a(u^n,v-u^n)+j(u^{n-1},v)-j(u^{n-1},u^n)\geq L(v-u^n),\ \forall\ v\in K. \qquad (1.14)$$

By similar arguments as in the proof of Theorem 1.1 and using Lemma 1.1 it results that, for any $\mu\geq 0$ with $\|\mu\|_{L^\infty(\Gamma_2)}<\frac{\alpha}{C_0}$, S is a contraction. Therefore we obtain:

$$\| \underset{\sim}{u}^n - \underset{\sim}{u} \| \leq k^n \| \underset{\sim}{u}^o - \underset{\sim}{u} \| \leq C k^n \qquad (1.15)$$

where C and k are positive constarts independent of n with

$$k = \frac{C_o \|\mu\|_{L^\infty(\Gamma_2)}}{\alpha} < 1.$$

2. FINITE ELEMENT APPROXIMATION OF THE PROBLEM

We shall give a finite element approximation of the variational inequality (1.11).

Following the standard procedure in the finite element method we consider a family $(V_h)_h$ of finite dimensional subspaces of V (see [1]).

Let $(K_h)_h$ be a family of non-empty closed convex subsets of V_h which approximate K in the sence that:

(i) $\forall \underset{\sim}{v} \in K, \exists \underset{\sim}{v}_h \in K_h$ such that $\underset{\sim}{v}_h \longrightarrow \underset{\sim}{v}$ in V,

(ii) $\forall \underset{\sim}{v}_h \in K_h$ with $\underset{\sim}{v}_h \longrightarrow \underset{\sim}{v}$ in V then $\underset{\sim}{v} \in K$.

Now, we formulate the following discrete problem:

$$\text{find } \underset{\sim}{u}_h \in K_h \text{ such that}$$
$$a(\underset{\sim}{u}_h, \underset{\sim}{v}_h - \underset{\sim}{u}_h) + j(\underset{\sim}{u}_h, \underset{\sim}{v}_h) - j(\underset{\sim}{u}_h, \underset{\sim}{u}_h) \geqslant L(\underset{\sim}{v}_h - \underset{\sim}{u}_h), \ \forall \underset{\sim}{v}_h \in K_h \qquad (2.1)$$

Applying similar arguments as in §1, it results that the mapping $S_h : K_h \longrightarrow K_h$ which associates to every $\underset{\sim}{w}_h \in K_h$ the element $S_h \underset{\sim}{w}_h \in K_h$ defined by:

$$a(S_h \underset{\sim}{w}_h, \underset{\sim}{v}_h - S_h \underset{\sim}{w}_h) + j(\underset{\sim}{w}_h, \underset{\sim}{v}_h) - j(\underset{\sim}{w}_h, S_h \underset{\sim}{w}_h) \geqslant L(\underset{\sim}{v}_h - S_h \underset{\sim}{w}_h), \ \forall \underset{\sim}{v}_h \in K_h$$

is a contraction for any μ with $\|\mu\|_{L^\infty(\Gamma_2)} < \frac{\alpha}{C_o}$.

Therefore we have the following result.

THEOREM 2.1: Suppose that $\|\mu\|_{L^\infty(\Gamma_2)} < \frac{\alpha}{C_o}$. Then the problem (2.1) has an unique solution.

Further on we assume that the condition of Theorem 2.1 holds

Let $\{u_h^o\}_h$ be an uniformly bounded sequence such that $u_h^o \in K_h$. From Theorem 2.1 it follows that we may define the sequence: for $u_h^o \in K_h$ we put

$$u_h^n = S_h u_h^{n-1} \,, \tag{2.2}$$

hence, we have:

$$\|u_h^n - u_h\| \leq k^n \|u_h^o - u_h\| \leq Ck^n \tag{2.3}$$

where C is a positive constant independent of h and n and where $k = C_o \|\mu\|_{L^\infty(\Gamma_2)}/\alpha$.

We shall now establish the convergence of $\{u_h\}_h$ to u without any regularity assumption on the solution u of the problem (1.3).

In order to obtain this result we define an auxiliary sequence of problems: for $w_h^o \in K_h$ given such that $\{w_h^o\}_h$ is uniformly bounded, we denote by $w_h^n \in K_h$, the solution, that there exists and is unique, of the problem:

$$a(w_h^n, v_h - w_h^n) + j(u^{n-1}, v_h) - j(u^{n-1}, w_h^n) \geqslant L(v_h - w_h^n), \forall\, v_h \in K_h, \tag{2.4}$$

where $u^{n-1} \in K$ is defined by (1.14).

The relationship between $\{w_h^n\}_h$ and u^n is made clear by the following result:

PROPOSITION 2.1: The sequence $\{\underset{\sim}{w}_h^n\}_h$ defined by (2.4) approximates the solution $\underset{\sim}{u}^n$ of (1.14) in the sense:

$$\underset{\sim}{w}_h^n \longrightarrow \underset{\sim}{u}^n \quad \text{as } h \longrightarrow 0.$$

Proof: We first show that the sequence $\{\underset{\sim}{w}_h^n\}_h$ is uniformly bounded in h. For this, using the continuity of a, that $\{\underset{\sim}{u}^n\}_n$ is bounded and the property (i) of K_h, we derive from (1.7) and (2.4):

$$\alpha\|\underset{\sim}{w}_h^n\|^2 \leq a(\underset{\sim}{w}_h^n, \underset{\sim}{w}_h^n) \leq a(\underset{\sim}{w}_h^n, \underset{\sim}{w}_h^n) + j(\underset{\sim}{u}^{n-1}, \underset{\sim}{w}_h^n) \leq a(\underset{\sim}{w}_h^n, \underset{\sim}{v}_h) + j(\underset{\sim}{u}^{n-1}, \underset{\sim}{v}_h) - L(\underset{\sim}{v}_h - \underset{\sim}{w}_h^n) \leq C_1\|\underset{\sim}{w}_h^n\| + C_2, \quad \forall\, \underset{\sim}{v}_h \in K_h, \tag{2.5}$$

C_1, C_2 being positive constants independent of n and h. Hence the sequence $\{\underset{\sim}{w}_h^n\}_h$ is uniformly bounded in h and, passing to a sequence which we still denote by $\{\underset{\sim}{w}_h^n\}_h$, it follows that $\underset{\sim}{w}_h^n \rightharpoonup \underset{\sim}{w}^n$ as $h \to 0$. By condition (ii) we also have $\underset{\sim}{w}^n \in K$.

Let $\underset{\sim}{v} \in K$. Taking in (2.5) a sequence $\{\underset{\sim}{v}_h\}_h$ with $\underset{\sim}{v}_h \in K_h$ such that $\underset{\sim}{v}_h \longrightarrow \underset{\sim}{v}$ (whose existence is insured by condition (i)) we obtain:

$$a(\underset{\sim}{w}^n, \underset{\sim}{w}^n) + j(\underset{\sim}{u}^{n-1}, \underset{\sim}{w}^n) \leq \liminf_{h \to 0} \left[a(\underset{\sim}{w}_h^n, \underset{\sim}{w}_h^n) + j(\underset{\sim}{u}^{n-1}, \underset{\sim}{w}_h^n)\right] \leq$$
$$\leq a(\underset{\sim}{w}^n, \underset{\sim}{v}) + j(\underset{\sim}{u}^{n-1}, \underset{\sim}{v}) - L(\underset{\sim}{v} - \underset{\sim}{w}^n), \quad \forall\, \underset{\sim}{v} \in K.$$

Therefore, from the uniqueness of solution of (1.14) we have $\underset{\sim}{w}^n = \underset{\sim}{u}^n$.

Let us show that $\underset{\sim}{w}_h^n \longrightarrow \underset{\sim}{u}^n$ in V. We observe that we have:

$$\alpha\|\underset{\sim}{w}_h^n - \underset{\sim}{u}^n\|^2 + j(\underset{\sim}{u}^{n-1}, \underset{\sim}{w}_h^n) \leq a(\underset{\sim}{w}_h^n - \underset{\sim}{u}^n, \underset{\sim}{w}_h^n - \underset{\sim}{u}^n) + j(\underset{\sim}{u}^{n-1}, \underset{\sim}{w}_h^n) \leq$$
$$\leq a(\underset{\sim}{w}_h^n, \underset{\sim}{v}_h) + j(\underset{\sim}{u}^{n-1}, \underset{\sim}{v}_h) - L(\underset{\sim}{v}_h - \underset{\sim}{w}_h^n) + a(\underset{\sim}{u}^n, \underset{\sim}{u}^n) - 2a(\underset{\sim}{w}_h^n, \underset{\sim}{u}^n), \quad \forall\, \underset{\sim}{v}_h \in K_h \tag{2.6}$$

Taking in (2.6) a sequence $\{\underset{\sim}{v}_h\}_h$ converging strongly to $\underset{\sim}{u}^n$, whose existence is insured by condition (i), we obtain, passing to the limit as $h \to 0$:

$$j(\underset{\sim}{u}^{n-1},\underset{\sim}{u}^n) \leq \liminf_{h\to 0} j(\underset{\sim}{u}^{n-1}, \underset{\sim}{w}_h^n) \leq$$

$$\leq \limsup_{h\to 0}\left[\alpha\| \underset{\sim}{w}_h^n-\underset{\sim}{u}^n\|^2+j(\underset{\sim}{u}^{n-1},\underset{\sim}{w}_h^n)\right] \leq j(\underset{\sim}{u}^{n-1},\underset{\sim}{u}^n)$$

from which we conclude

$$\liminf_{h\to 0} j(\underset{\sim}{u}^{n-1},\underset{\sim}{w}_h^n)=j(\underset{\sim}{u}^{n-1},\underset{\sim}{u}^n)$$

and

$$\lim_{h\to 0} \| \underset{\sim}{w}_h^n - \underset{\sim}{u}^n \| = 0 .$$

We are now prepared to prove the main result of this paper.

THEOREM 2.2: Let $\underset{\sim}{u}$ and $\underset{\sim}{u}_h$ be the solutions of (1.3) and (2.1), respectively. Then,

$$\underset{\sim}{u}_h \to \underset{\sim}{u} \quad \text{as } h \to 0$$

Proof: We observe that we have:

$$\|\underset{\sim}{u}_h-\underset{\sim}{u}\| \leq \|\underset{\sim}{u}_h-\underset{\sim}{u}_h^n\| + \|\underset{\sim}{u}_h^n-\underset{\sim}{u}^n\| + \|\underset{\sim}{u}^n-\underset{\sim}{u}\| \tag{2.7}$$

In order to estimate the second term in the right-hand side of (2.7) we first deduce from the definitions of $\underset{\sim}{u}_h^n$ and $\underset{\sim}{w}_h^n$ that

$$\alpha \| \underset{\sim}{u}_h^n - \underset{\sim}{w}_h^n\|^2 \leq a(\underset{\sim}{w}_h^n - \underset{\sim}{u}_h^n, \underset{\sim}{w}_h^n - \underset{\sim}{u}_h^n) \leq$$

$$\leq j(\underset{\sim}{u}^{n-1},\underset{\sim}{u}_h^n)+j(\underset{\sim}{u}_h^{n-1},\underset{\sim}{w}_h^n)-j(\underset{\sim}{u}^{n-1},\underset{\sim}{w}_h^n)-j(\underset{\sim}{u}_h^{n-1},\underset{\sim}{u}_h^n),$$

from which, using Lemma 1.1, we obtain:

$$\| \underset{\sim}{u}_h^n - \underset{\sim}{w}_h^n \| \leq \| \underset{\sim}{u}^{n-1} - \underset{\sim}{u}_h^{n-1} \| \tag{2.8}$$

By choosing $\underset{\sim}{w}_h^o = \underset{\sim}{u}_h^o$, we shall prove by reccurence that:

$$\| \underset{\sim}{u}_h^n - \underset{\sim}{u}^n \| \leq \sum_{i=0}^{n} \| \underset{\sim}{w}_h^i - \underset{\sim}{u}^i \| \qquad \forall n \geqslant 0 \tag{2.9}$$

Indeed, for n=0 the result is evident. If we suppose that (2.9) holds for n-1 then we have:

$$\| \underset{\sim}{u}_h^n - \underset{\sim}{u}^n \| \leq \| \underset{\sim}{u}_h^n - \underset{\sim}{w}_h^n \| + \| \underset{\sim}{w}_h^n - \underset{\sim}{u}^n \| \leq$$

$$\leq \| \underset{\sim}{u}_h^{n-1} - \underset{\sim}{u}^{n-1} \| + \| \underset{\sim}{w}_h^n - \underset{\sim}{u}^n \| \leq \sum_{i=0}^{n} \| \underset{\sim}{w}_h^i - \underset{\sim}{u}^i \|$$

where we have used (2.8). It follows that (2.9) holds for every n, $n \geqslant 0$.

By (2.3) and (1.15), for $\varepsilon > 0$ given, there exists $N_\varepsilon > 0$ such that:

$$\| \underset{\sim}{u}_h^n - \underset{\sim}{u}_h \| + \| \underset{\sim}{u}^n - \underset{\sim}{u} \| \leq \frac{\varepsilon}{2}, \ \forall \ n \geqslant N_\varepsilon . \tag{2.10}$$

Choosing $n=N_\varepsilon$ in (2.7) we have by using (2.9) and (2.10):

$$\| \underset{\sim}{u}_h - \underset{\sim}{u} \| \leq \varepsilon/2 + \sum_{i=0}^{N_\varepsilon} \| \underset{\sim}{w}_h^i - \underset{\sim}{u}^i \| . \tag{2.11}$$

But from Proposition 2.1 it results that, for every i, there exists $H_\varepsilon^i > 0$ such that:

$$\| \underset{\sim}{w}_h^i - \underset{\sim}{u}^i \| \leq \frac{\varepsilon}{2(N_\varepsilon+1)}, \ \forall \ h \leq H_\varepsilon^i \tag{2.12}$$

Concluding, from (2.11) and (2.12), for $\varepsilon > 0$ given, there exists $H_\varepsilon = \min\left\{H_\varepsilon^i,\ i=0,1,\dots,N_\varepsilon^i\right\}$ such that:

$$\|\underset{\sim}{u}_h - \underset{\sim}{u}\| \leq \varepsilon, \quad \forall\ h < H_\varepsilon,$$

hence $u_h \longrightarrow u$ as $h \longrightarrow 0$.

It now remains to derive an error estimate for the finite element approximation (2.1).

Let us begin by making additional assumptions about Γ_2, V_h and K_h. Assume that $\Gamma_2 \in C^\infty$.

Suppose that there exists an operator $\pi_h : V \rightarrow V_h$ such that:

$$\|\pi_h \underset{\sim}{v} - \underset{\sim}{v}\|_r \leq Ch^{2-r} \|\underset{\sim}{v}\|_2, \quad r=0,1, \tag{2.13}$$

$$\|\pi_h \underset{\sim}{v} - \underset{\sim}{v}\|_{-1/2,\Gamma_2} \leq Ch^2 \|\underset{\sim}{v}\|_2, \tag{2.14}$$

for every $\underset{\sim}{v} \in [H^2(\Omega)]^p \cap V$ where $\|\cdot\|_{-1/2,\Gamma_2}$ and $\|\cdot\|_2$ denote the norms on $\left[H^{-1/2}(\Gamma_2)\right]^p$ and $\left[H^2(\Omega)\right]^p$, respectively.

We also suppose that

$$\pi_h \underset{\sim}{u} \in K_h \tag{2.15}$$

where $\underset{\sim}{u}$ is the solution of (1.3).

Remark 2.1: It is known that if V_h is defined by:

$$V_h = \left\{\underset{\sim}{v} \in V \cap [C^0(\bar{\Omega})]^p\ ;\ \underset{\sim}{v}|_T \in [P_k]^p,\ \forall\ T \in \mathcal{T}_h\right\},$$

where $\mathcal{T}_h$ is a regular triangulation of $\bar{\Omega}$ such that $\bar{\Omega} = \bigcup_{T \in \mathcal{T}_h} T$ and P_k is the space of all polynomials of degree $\leq k$ in the variables $x_1,\dots,x_p$ with $k \geq 1$, then (2.13), (2.14) are satisfied if $\pi_h \underset{\sim}{v}$ denotes, as usual, the V_h - interpolant of the function $\underset{\sim}{v} \in V$ (see [1]), Next, defining K_h as in [8] (2.15) holds.

THEOREM 2.3: Suppose that the condition (2.13)-(2.15) hold and that $K_h \subset K$. If $\underset{\sim}{u} \in [H^2(\Omega)]^p \cap K$ then:

$$\|\underset{\sim}{u}_h - \underset{\sim}{u}\| \leq Ch\|\underset{\sim}{u}\|_2 \tag{2.16}$$

where C is a positive constant independent of h.

Proof: Taking $\underset{\sim}{v}=\underset{\sim}{u}_h$ in (1.11) and $\underset{\sim}{v}_h=\pi_h\underset{\sim}{u}$ in (2.1) we obtain:

$$\alpha\|\underset{\sim}{u}_h-\underset{\sim}{u}\|^2 \leq a(\underset{\sim}{u}-\underset{\sim}{u}_h,\underset{\sim}{u}-\underset{\sim}{u}_h) \leq a(\underset{\sim}{u}_h-\underset{\sim}{u},\pi_h\underset{\sim}{u}-\underset{\sim}{u})+a(\underset{\sim}{u},\pi_h\underset{\sim}{u}-\underset{\sim}{u})-L(\pi_h\underset{\sim}{u}-\underset{\sim}{u})+$$
$$+j(\underset{\sim}{u},\underset{\sim}{u}_h)+j(\underset{\sim}{u}_h,\pi_h\underset{\sim}{u})-j(\underset{\sim}{u}_h,\underset{\sim}{u}_h)-j(\underset{\sim}{u},\pi_h\underset{\sim}{u})+j(\underset{\sim}{u},\pi_h\underset{\sim}{u})-j(\underset{\sim}{u},\underset{\sim}{u}). \tag{2.17}$$

Taking into account that $\underset{\sim}{u} \in [H^2(\Omega)]^p$ it follows that:

$$a(\underset{\sim}{u},\underset{\sim}{v})-L(\underset{\sim}{v})=\int_{\Gamma_2}\sigma_{ij}(\underset{\sim}{u})n_j v_i\,ds \leq C_1\|\underset{\sim}{u}\|_2\|\underset{\sim}{v}\|_{-1/2,\Gamma_2} \quad ,\forall\, \underset{\sim}{v}\in V \tag{2.18}$$

where we have used Green's formula, and that:

$$j(\underset{\sim}{u},\pi_h\underset{\sim}{u})-j(\underset{\sim}{u},\underset{\sim}{u}) \leq \int_{\Gamma_2}\mu\,|Q(\underset{\sim}{u})|\,|(\pi_h\underset{\sim}{u}-\underset{\sim}{u})_t|\,ds \leq$$
$$\leq C_2\|\underset{\sim}{u}\|_2\|\pi_h\underset{\sim}{u}-\underset{\sim}{u}\|_{-1/2,\Gamma_2}. \tag{2.19}$$

Substitution of (2.18), (2.19) and (1.13) into (2.17) yields:

$$\alpha\|\underset{\sim}{u}-\underset{\sim}{u}_h\|^2 \leq C'\|\underset{\sim}{u}_h-\underset{\sim}{u}\|\|\pi_h\underset{\sim}{u}-\underset{\sim}{u}\| + C''\|\underset{\sim}{u}\|_2\|\pi_h\underset{\sim}{u}-\underset{\sim}{u}\|_{-1/2,\Gamma_2} \tag{2.20}$$

by the continuity of $a(\cdot,\cdot)$.

Using Young's inequality:

$$ab \leq \varepsilon a^2/2 + b^2/2\varepsilon \qquad \forall\,\varepsilon>0,\ \forall\, a,b\in R,$$

for $\varepsilon = \alpha/C'$, we obtain from (2.20)

$$\frac{\alpha}{2}\|u_h - u\|^2 \leq \frac{(C')^2}{2}\|\pi_h u - u\|^2 + C''\|u\|_2 \|\pi_h u - u\|_{-1/2,\Gamma_2}$$

Therefore, by (2.13) and (2.14), the estimate (2.16) follows.

3. NUMERICAL EXAMPLES

Let us consider a plane linear elastic body which in the initial unstressed state occupies the domain $\Omega = (0,16)x(0,8)$. For comparison purpose we have considered problems for the same with different values of tractions and coefficients of friction.

The decomposition of the boundary Γ into Γ_o, Γ_1 and Γ_2 is given by

$\Gamma_o = \{(x_1,x_2) \in R^2;\ 0 < x_2 < 8,\ x_1 = 0\}$,

$\Gamma_1 = \Gamma_1' \cup \Gamma_1''$ where:

$\Gamma_1' = \{(x_1 x_2) \in R^2;\ 0 < x_1 < 16,\ x_2 = 8\}$,

$\Gamma_1'' = \{(x_1,x_2) \in R^2;\ 0 < x_2 < 8,\ x_1 = 16\}$,

$\Gamma_2 = \{(x_1,x_2) \in R^2;\ 0 < x_1 < 16,\ x_2 = 0\}$.

We suppose that the body is homogeneous isotropic and is characterized by a Young's modulus of $E = 10^6$ and a Poisson's ratio of $\nu = 0,3$. We have considered plane stress problems.

Let $(\mathcal{T}_h)_h$ a regular family of triangulations of $\bar{\Omega}$ such that $\bar{\Omega} = \bigcup_{T \in \mathcal{T}_h} T$.

Let V_h, K_h be the finite-element approximations of the space V and of the convex set K, respectively, defined by:

$$V_h=\left\{ \underset{\sim}{v}_h=(\underset{\sim}{v}_h^1,\underset{\sim}{v}_h^2)\in[C^o(\bar{\Omega})]^2;\ \underset{\sim}{v}_h(\underset{\sim}{a}_i)=0,\forall\, \underset{\sim}{a}_i\in\Gamma_o\cap\Sigma'_h,\right.$$
$$\left.\underset{\sim}{v}_h|_T\in[P_1]^2\ ,\forall\, T\in\mathcal{T}_h \right\},$$

$$K_h=\left\{ \underset{\sim}{v}_h=(\underset{\sim}{v}_h^1,\underset{\sim}{v}_h^2)\in V_h;\ \underset{\sim}{v}_h^2(\underset{\sim}{a}_i)\geq 0,\forall\, \underset{\sim}{a}_i\in\Gamma_2\cap\Sigma'_h \right\}.$$

In order to solve the discrete problem (2.1) which approximates the given problem (1.3) it suffices, as we have seen in § 2, to solve the following sequence of discrete variational inequalities:

$$\underset{\sim}{u}_h^n\in K_h$$
$$a(\underset{\sim}{u}_h,\underset{\sim}{v}_h-\underset{\sim}{u}_h)+j(\underset{\sim}{u}_h^{n-1},\underset{\sim}{v}_h)-j(\underset{\sim}{u}_h^{n-1},\underset{\sim}{u}_h)\geq L(\underset{\sim}{v}_h-\underset{\sim}{u}_h),\forall\, \underset{\sim}{v}_h\in K_h \qquad (3.1)$$

for $\underset{\sim}{u}_h^o\in K_h$ given.

We remark that (3.1) is a Signorini problem with "given friction" which is equivalent with an optimisation problem for a non-differentiable functional. For this reason it is advantageous to use the following saddle point formulation (see e.g.[8]):

$$(\underset{\sim}{u}^n,p^n)\in K\times\Lambda^n$$
$$\mathcal{L}^n(\underset{\sim}{u}^n,q)\leq\mathcal{L}^n(\underset{\sim}{u}^n,p^n)\leq\mathcal{L}^n(\underset{\sim}{v},p^n)\quad ,\forall\, \underset{\sim}{v}\in K,\forall q\in\Lambda^n \qquad (3.2)$$

where

$$\mathcal{L}^n(\underset{\sim}{v},q) = \frac{1}{2}\, a(\underset{\sim}{v},\underset{\sim}{v})-L(\underset{\sim}{v})+\int_{\Gamma_2} q g^n \underset{\sim}{v}_t\, ds\ ,$$

$$g^n=\mu|\mathcal{R}(\sigma_n(\underset{\sim}{u}^{n-1}))|\ ,$$

$$\Lambda^n=\left\{ q\in L^2(\Gamma_2);\ |q|\leq 1 \text{ a.e. on supp } g^n,\ q=0 \text{ on } \Gamma_2 \text{ supp } g^n \right\}.$$

For simplicity we have omitted the subscript h.

We have applied Uzawa's algorithm to solve the problem (3.2).

Three numerical examples have been solved by the finite element approximation discussed in above, assuming the absence of body forces i.e. $\underset{\sim}{f}=\underset{\sim}{0}$. In the first example we take the traction $\underset{\sim}{t}$ defined by $\underset{\sim}{t}=(0,0)$ on Γ_1' and $\underset{\sim}{t}=(500,0)$ on Γ_1''. In the second example we consider $\underset{\sim}{t}=(0,-300)$ on Γ_1' and $\underset{\sim}{t}=(500,0)$ on Γ_1'' and in the last example $\underset{\sim}{t}=(500,-300)$ on Γ_1' and $\underset{\sim}{t}=(0,0)$ on Γ_1''.

We decomposed $\bar{\Omega}$ in 64 triangular finite elements as is shown in figure 1.

fig.1. The finite element mesh

In all these examples we are particularly interested in showing the influence of the friction's coefficient on the tangential displacements on Γ_2 as is illustrated in figures 2-4.

Fig.2. The tangential displacements in example 1.

Fig.3. The tangential displacements in example 2

Fig.4. The tangential displacements in example 3

For this purpose the coefficients of friction were taken equal to 0.2, 0.4 and 0.6 respectively.

To initialize the process defined by (3.1) we have take $\underset{\sim}{u}_h^o$ as being the unique solution of the classical Signorini problem:

$$a(\underset{\sim}{u}_h,\underset{\sim}{v}_h-\underset{\sim}{u}_h)\geqslant L(\underset{\sim}{v}_h-\underset{\sim}{u}_h) \quad ,\forall\ \underset{\sim}{v}_h\in K_h \ ,$$

which corresponds to the case $\mu = 0$.

According to expectation, the tangential displacements obtained in example 2 are smaller than those obtained in example 1. Also, in figure 4 one can see the influence of combined tractions on Γ_1' on the tangential displacements on Γ_2.

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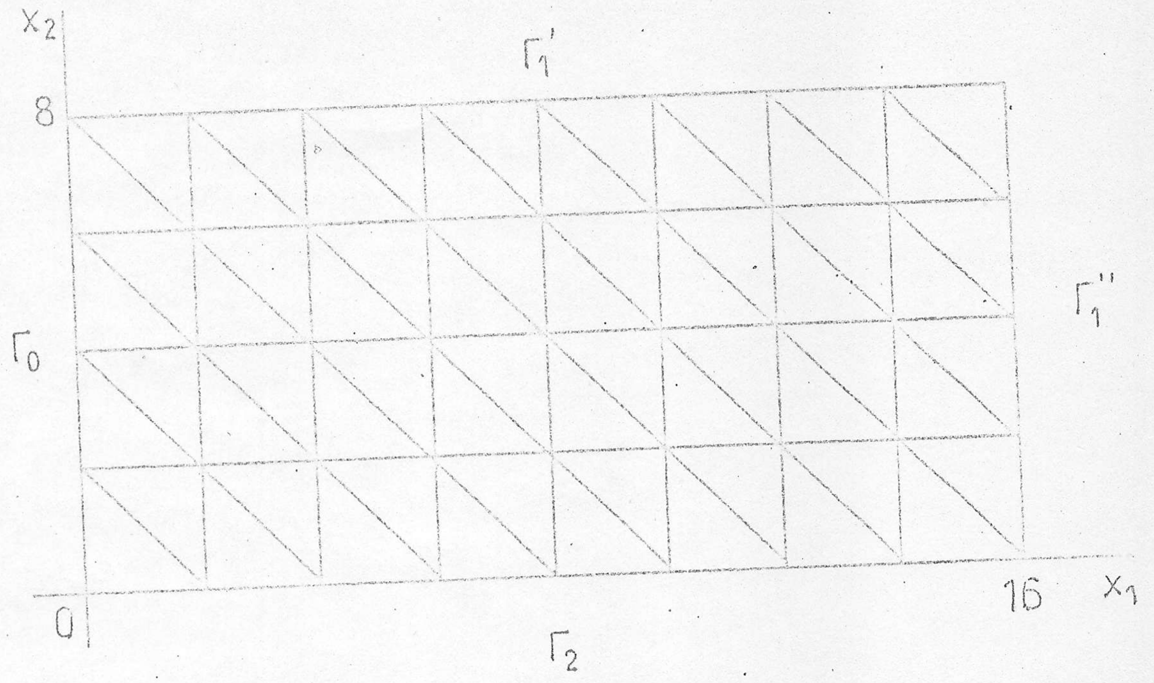
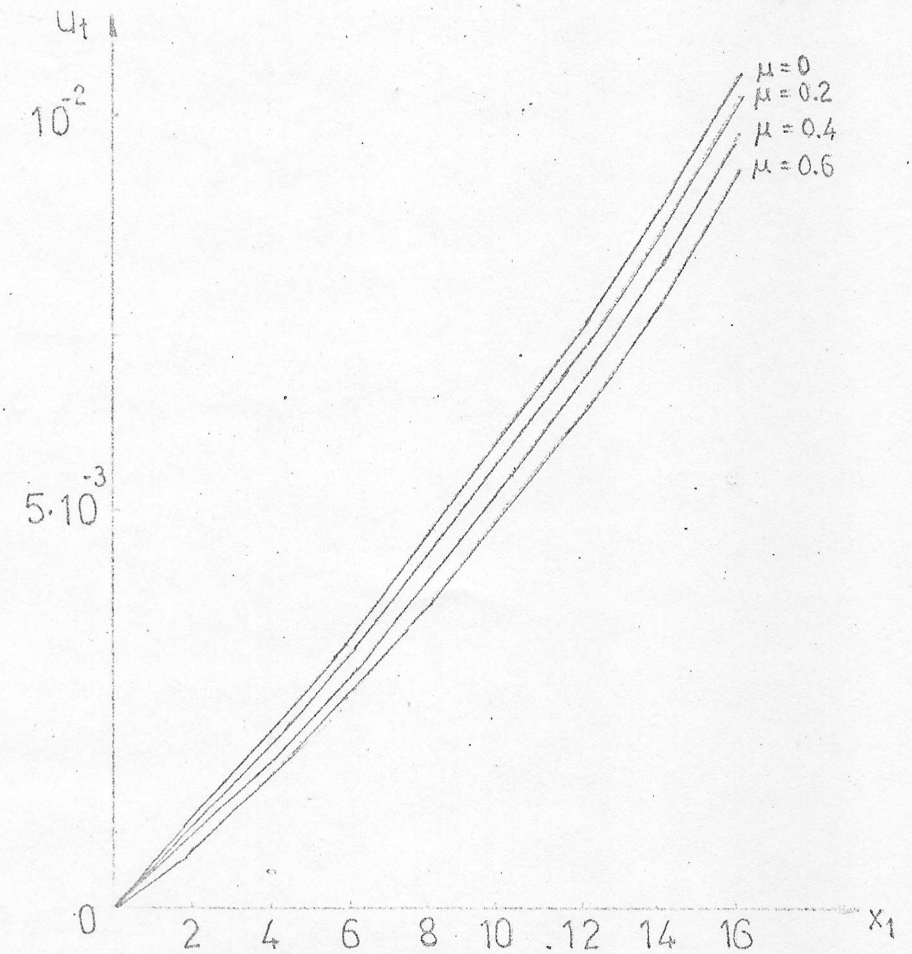


x_2
Γ_1'
8
Γ_1''
Γ_0
0
16
x_1
Γ_2

Fig. 1



u_t
10^{-2}
$5 \cdot 10^{-3}$
$\mu = 0$
$\mu = 0.2$
$\mu = 0.4$
$\mu = 0.6$
0
2
4
6
8
10
12
14
16
x_1

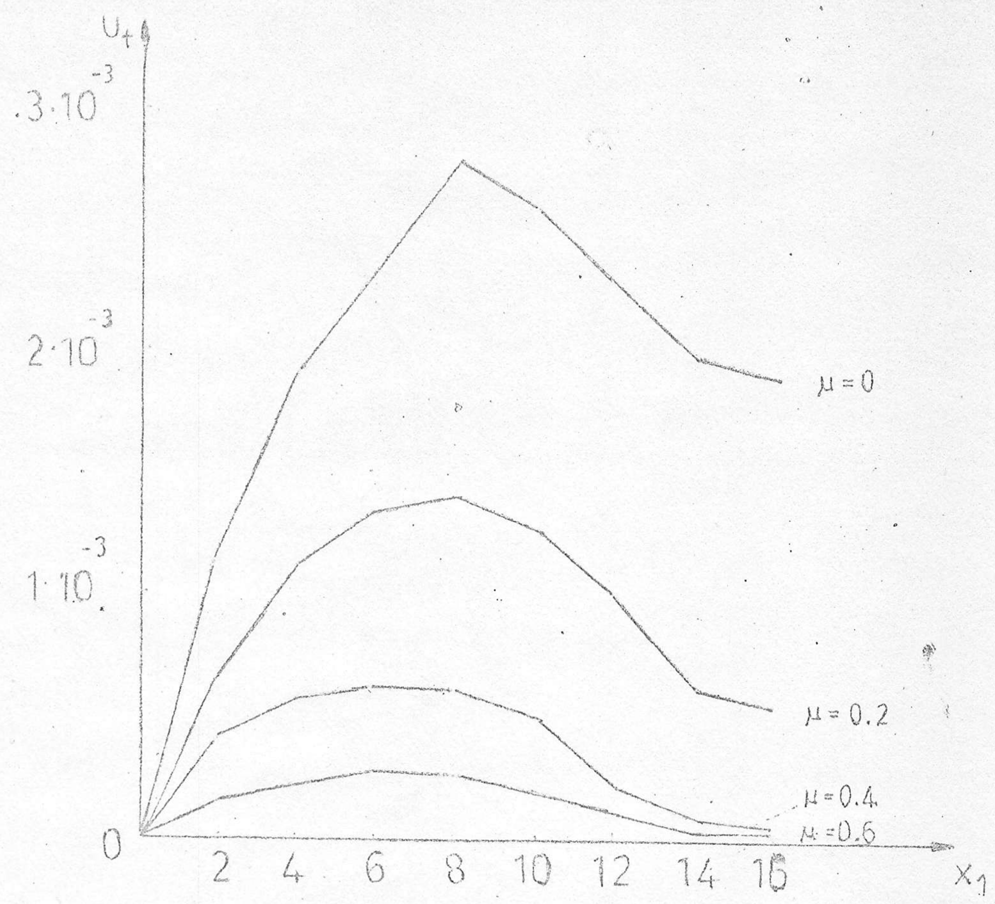
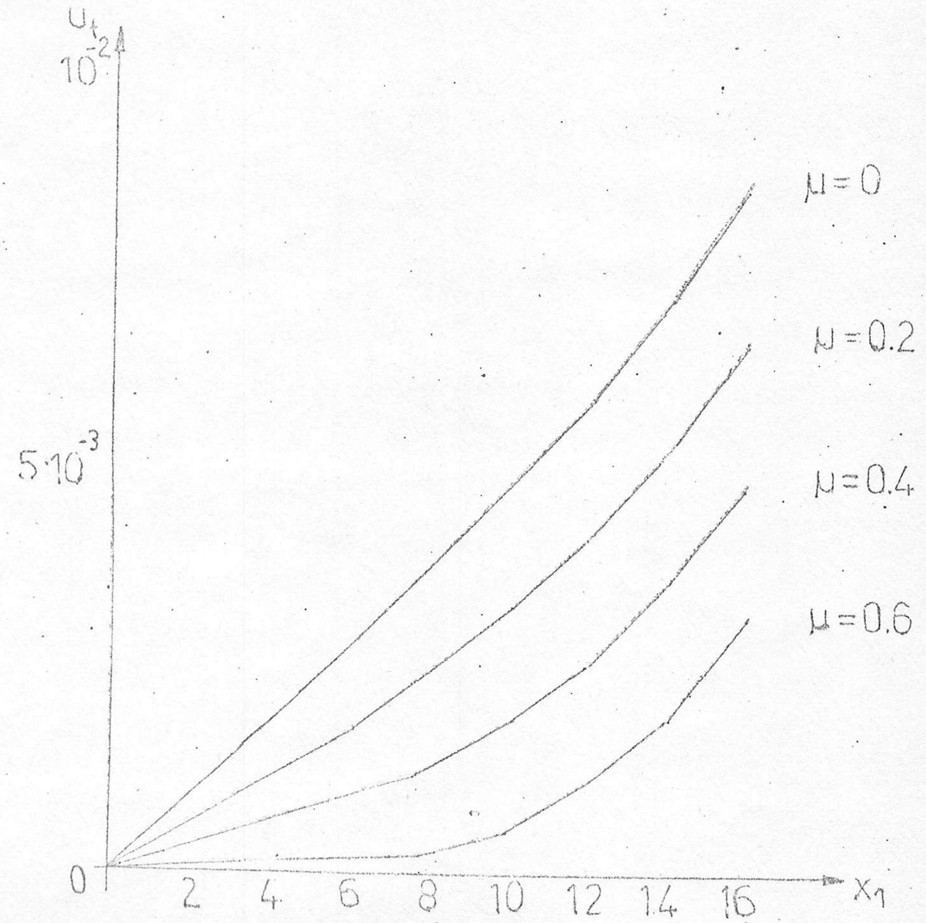

u_t
$3\cdot10^{-3}$
$2\cdot10^{-3}$
$1\cdot10^{-3}$
0
$\mu=0$
$\mu=0.2$
$\mu=0.4$
$\mu=0.6$
2 4 6 8 10 12 14 16
x_1

Fig. 4


u_t
10^{-2}
$5\cdot10^{-3}$
0
$\mu=0$
$\mu=0.2$
$\mu=0.4$
$\mu=0.6$
2 4 6 8 10 12 14 16
x_1